An extension of Kleene's and Ochmański's theorems to infinite traces[1]

Paul Gastin
Université Paris 6
LITP, Institut Blaise Pascal
4, place Jussieu
F-75252 Paris Cedex 05

Antoine Petit
Université Paris Sud
LRI, URA CNRS 410
Bât. 490
F-91405 Orsay Cedex

Wiesław Zielonka
Université de Bordeaux 1
LaBRI, URA CNRS 1304
351, cours de la Libération
F-33405 Talence Cedex

Abstract

As was noted by Mazurkiewicz, traces constitute a convenient tool for describing finite behaviour of concurrent systems. Extending in a natural way Mazurkiewicz's original definition, infinite traces have been recently introduced enabling to deal with infinite behaviour of non-terminating concurrent systems. In this paper we examine the basic families of recognizable sets and of rational sets of infinite traces. The seminal Kleene characterization of recognizable subsets of the free monoid and its subsequent extensions to infinite words due to Büchi and to finite traces due to Ochmański are the cornerstones of the corresponding theories. The main result of our paper is an extension of these characterizations to the domain of infinite traces. Using recognizing and weakly recognizing morphisms, as well as a generalization of the Schützenberger product of monoids, we prove various closure properties of recognizable trace languages. Moreover, we establish normal form representations for recognizable and rational sets of infinite traces.

Résumé

Mazurkiewicz a montré que le monoïde des traces forme un modèle tout à fait adapté à la description des comportements des systèmes concurrents. En étendant de façon très naturelle la définition originale de Mazurkiewicz, les traces infinies ont été récemment introduites afin de modéliser les comportements infinis des systèmes concurrents qui ne s'arrêtent pas. Ce papier est consacré à l'étude des familles de langages reconnaissables et de langages rationnels de traces infinies. Le théorème de Kleene, son extension aux mots infinis par Büchi et son extension aux traces finies par Ochmański sont des résultats fondamentaux de ces théories. Le résultat principal de cet article étend ce théorème aux langages de traces infinies. En utilisant la notion de morphismes reconnaissants et faiblement reconnaissants ainsi qu'une généralisation du produit de Schützenberger pour les monoïdes, on prouve des propriétés de clôture de la famille des langages reconnaissables de traces infinies. De plus, on établit des formes normales permettant de représenter les langages rationnels et reconnaissables de traces infinies.

[1]This research has been supported by the ESPRIT Basic Research Actions No. 3166 ASMICS and No. 3148 DEMON.

1 Introduction

The characteristic property of asynchronous distributed systems is the absence of any kind of centralized control mechanism. The actions executed by separate components are causally independent and different external observers can witness different time ordering of their execution in the same computation. Thus, when we specify or examine the behaviour of a parallel system, the order in which independent actions are executed seems irrelevant and even impossible to precise. For these reasons, Mazurkiewicz [28] proposed to identify two sequential behaviours if they differ only in the order of independent actions. In this way an equivalence relation over the set of sequences of actions is induced and the term traces was coined by Mazurkiewicz to name its equivalence classes. For a fixed independence relation, traces form a monoid known as free partially commutative monoid. These monoids were first considered by combinatorists [6], but since traces describe in a natural way the behaviour of concurrent asynchronous systems, they have also been studied intensively in relation to concurrency theory in the last years, see for instance the monograph [10] or surveys [1, 30, 36], where extensive bibliographies of the subject are given.

The traces introduced originally by Mazurkiewicz are in fact what we call finite traces, they represent only finite behaviours of concurrent systems. However, since many concurrent systems, as for example operating systems, are non-terminating by their very nature, we are often much more interested in their infinite rather than finite behaviours. These infinite behaviours can be described at some level of abstraction by infinite traces. The theory of infinite traces was initiated only recently but it attracts growing attention and is developing rapidly. Implicit definitions of infinite traces can be found in [16] in relation with problems of infinite serializabilities and in [3] in relation with Petri Nets. Some ideas concerning infinite dependence graphs are also presented in [30]. Nevertheless the first explicit definitions of infinite traces were proposed independently by Gastin [18] and Kwiatkowska [26]. Subsequently infinite traces were examined intensively — we could note here papers related to topological properties [4, 19], to PoSet properties [23, 26], to connections with event structures [23]. In order to obtain a uniformly continuous concatenation, Diekert proposed a nice generalization of infinite traces to "complex traces" [11]. Note that the traces we deal with in this paper are called real traces in [11] and [12].

The simplest concurrent systems are composed of finite state processes and their behaviour is represented by recognizable sets of traces which constitute therefore one of the basic family of trace languages. While recognizable sets of finite traces are well examined and several deep results characterizing this family are known, recognizable languages of infinite traces remain largely unexplored and only some preliminary facts were established [19, 20].

Our paper undertakes a more systematic study of recognizability of sets of infinite traces. In Section 2 we introduce (finite and infinite) traces and their representation by means of dependence graphs. We define also the concatenation of traces and the main operations on trace languages. Section 3 opens with the definition of recognizability of trace languages. One of the first results on infinite words establishes the closure of recognizable sets under complement [5]. To demonstrate this fact Büchi proved a key result which, reformulated in the abstract setting of recognizability by means of monoids, states that the notion of recognizability and the notion of weak recognizability yield in fact the same family of languages of infinite words. The proof that weakly recognizable and recognizable morphisms define the same family of languages of infinite words makes use of the Schützenberger product of monoids [35, 37, 38]. In Section 3 we generalize the Schützenberger product and define a diamond product of monoids which is appropriately tailored to cope with peculiarities of trace multiplication. Using this product we show that recognizable and weakly recognizable sets coincide also for infinite traces. An important corollary, obtained as a by-product of this result, states that for a set T of finite traces the recognizability of T^* implies the recognizability of T^ω. This result is interesting by itself as numerous sufficient conditions are known that ensure recognizability of T^* for a recognizable trace language T [25, 31, 32, 34] and now they can be applied to ensure recognizability of the infinite iteration T^ω of T. There are however other interesting consequences of this fact. First, this result enables to set up a kind of normal form theorem for recognizable languages of infinite traces (Theorem 3.22). The second consequence is presented in Section 4. Although the family of recognizable trace languages forms a boolean algebra for the operations of union, intersection and complementation and is closed under concatenation, it is not closed, in general, neither under finite iteration nor under infinite iteration. Thus Kleene's and Büchi's characterizations of recognizable sets of respectively finite and infinite words that identify these sets with rational languages does not hold for traces. However similar characterizations are of great interest since they allow to construct from single actions all recognizable languages in a simple and systematic way by means of a few basic operations. For finite traces a suitable characterization of recognizable sets was given by Ochmański [33]. He introduced a new operation on trace languages — c-iteration — and defined the family of c-rational trace languages which are obtained as rational languages with c-iteration replacing iteration. A result of Métivier [31], which was also independently announced by Ochmański [33] and by Clerbout and Latteux in the more general framework of semi-commutations [7], states that recognizable finite trace languages are closed under this new operation. As a consequence, the family of c-rational trace languages is included in the family of recognizable trace languages. The major breakthrough due to Ochmański is his elegant proof of the inverse inclusion establishing the equality of the two families. In Section 4 we show that this result extends to infinite traces — the families of recognizable and c-rational sets of infinite traces are equal and included in the family of rational sets. Section 4 terminates with a theorem presenting a normal form characterization for rational languages of infinite traces.

A preliminary form of our paper appeared in [22], we should note however that the present final version differs in fact considerably from [22] and contains some additional results. Some initial results concerning in particular closure properties can also be found in [20], they are included here for the sake of completeness and with sometimes more elegant proofs. In our work we were constantly inspired by the remarkable treatise [37] of Perrin and Pin on infinite words, which results in our attempts to present a uniform approach to recognizability by means of monoid morphisms.

2 Traces — definitions and basic properties

We begin by fixing notations. For a finite alphabet A, A^* and A^ω will denote the sets of finite and infinite words respectively. Then $A^\infty = A^* \cup A^\omega$ is the set of all finite and infinite words. The empty word, as well as the unit element of any monoid, will be denoted by **1**. The concatenation over A^∞ is defined by

$$\forall x, y \in A^\infty, \; x \cdot y = \begin{cases} xy & \text{if } x \in A^* \\ x & \text{otherwise} \end{cases}$$

where xy is the word obtained by appending y to x. Concatenation yields a monoid structure over A^∞. The length of a word x is denoted by $\mid x \mid$, while if $a \in A$ then $\mid x \mid_a$ is the number of occurrences of the letter a in x. By $alph(x)$ we shall denote the set $\{a \in A \mid \mid x \mid_a > 0\}$ of letters occurring in a word x, while $alph_\omega(x)$ is the set $\{a \in A \mid \mid x \mid_a = \infty\}$ of letters that occur infinitely often in x.

Let $I \subseteq A \times A$ be a symmetrical and irreflexive relation over a finite alphabet A. The letters of A can be viewed as actions in a distributed system. Then $(a,b) \in I$ means intuitively that the actions a, b are independent and can be executed in any order or even in parallel. The relation I is called independence relation while its complement $D = (A \times A) \backslash I$ is the corresponding dependence relation.

There are numerous different equivalent definitions of traces. Here we introduce them by means of occurrence graphs [29]. This definition allows uniform approach to finite and infinite traces. Moreover, occurrence graphs illustrate intuitive ideas lying behind the notion of trace.

Let $u \in A^\infty$. Then

$$V_u = \{(a,i) \mid a \in A \text{ and } 0 \leq i < \mid u \mid_a\}$$

Intuitively, V_u is the set of action occurrences in u, (a,i) being the $(i+1)$st occurrence of a in u.

Let I be an independence relation and let D be the corresponding dependence relation. By $\Gamma_D(u)$ we shall denote the oriented graph (V_u, E_u) with V_u as the set of vertices and $E_u \subseteq V_u \times V_u$ as the set of edges, where $((a,i),(b,j))$ is in E_u if $(a,b) \in D$ and the occurrence (a,i) precedes the occurrence (b,j) in u, i.e., more formally, if there exists a prefix v of u such that $(a,i) \in V_v$ while $(b,j) \notin V_v$.

Now, we define an equivalence relation $\sim_I$ over the set of words by setting

$$\forall u, v \in A^\infty, \; u \sim_I v \qquad \text{if} \qquad \Gamma_D(u) = \Gamma_D(v)$$

The equivalence classes of $\sim_I$ are called traces. By $M^\infty(A,I)$ we shall denote the set of all traces, $M^\infty(A,I) = A^\infty / \sim_I$. This set is partitioned on two sets: the set $M(A,I) = A^* / \sim_I$ of finite traces, which are equivalence classes of finite words, and the set $M^\omega(A,I) = A^\omega / \sim_I$ of infinite traces, which are equivalence classes of infinite words.

Throughout the paper by $\varphi : A^\infty \longrightarrow M^\infty(A,I)$ we shall denote the canonical mapping that maps each word $x \in A^\infty$ to its equivalence class under $\sim_I$. The equivalence class

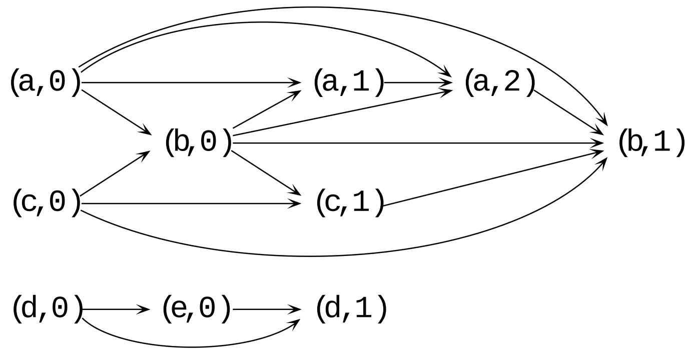

Figure 1: A *finite* dependence graph

$\varphi(\mathbf{1})$ of the empty word $\mathbf{1}$ is called the empty trace; it will also be denoted by $\mathbf{1}$. Let $t \in M^\infty(A, I)$ be a trace and let x be any word such that $\varphi(x) = t$. Then $\Gamma_D(t) = \Gamma_D(x)$ is the occurrence graph of t and $alph(t) = alph(x)$, $alph_\omega(t) = alph_\omega(x)$ are respectively the set of letters occurring in t and the set of letters occurring infinitely often in t.

The interest raised by traces results from their interpretation as a causal relation between events . Let $t \in M^\infty(A, I)$ and $\Gamma_D(t) = (V_t, E_t)$. Let E_t^+ be the transitive closure of the relation E_t. Suppose that $(a, i), (b, j) \in V_t$. Then $((a, i), (b, j)) \in E_t^+$ means that in t the $(i + 1)$st occurrence of action a causally precedes the $(j + 1)$st occurrence of action b. If neither $((a, i), (b, j)) \in E_t^+$ nor $((b, j), (a, i)) \in E_t^+$ then these two occurrences of a and b are causally independent.

Example 2.1 Let a—b—c d—e be the graph of the dependence relation, i.e. $A = \{a, b, c, d, e\}$ and $D = \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, b), (b, a), (b, c), (c, b), (d, e), (e, d)\}$. Let $u = acdbaeacbd$ and $t = \varphi(u)$. Figure 1 represents the occurrence graph $\Gamma_D(t) = \Gamma_D(u)$.

We see for instance that in $\Gamma_D(t)$ the first occurrence $(a, 0)$ of a precedes the second occurrence $(c, 1)$ of c, while the second occurrence $(a, 1)$ of a is independent of $(c, 1)$. Moreover, occurrences of a, b, c are independent of occurrences of d, e.

Let us note that usually $\sim_I$ is defined in a slightly different way for finite words [28]. In that classical definition $\sim_I$ is the reflexive and transitive closure of the relation $\sim$ defined below:

$$\forall x, y \in A^*,\ x \sim y \qquad \text{if} \qquad \exists u, v \in A^*,\ \exists (a, b) \in I,\ x = uabv \text{ and } y = ubav$$

In other words, $x \sim_I y$ if we can obtain x from y by a finite number of transpositions of neighbouring independent letters. As it is well-known for finite words this definition of $\sim_I$ is equivalent with the previous one [29]. Nevertheless, it turns out that this classical definition cannot be extended directly to infinite words, for instance we have $(ab)^\omega \sim_I (ba)^\omega$ if $(a, b) \in I$ but it is impossible to obtain the word $(ba)^\omega$ from the word $(ab)^\omega$ by a finite number of transpositions of letters. Note that $\varphi((ab)^\omega)$ is the trace representing the behaviour where both the action a and the action b are executed independently infinitely many times.

Remark 2.1 *i) Let $x_1, x_2, y_1, y_2 \in A^\infty$ be such that $x_1 \sim_I x_2$ and $y_1 \sim_I y_2$. Then $x_1y_1 \sim_I x_2y_2$.*

ii) Let $x_0, x_1, x_2, \ldots$, $y_0, y_1, y_2, \ldots$ be two infinite sequences of words of A^∞ such that $x_i \sim_I y_i$ for all $i \in \mathbb{N}$. Then $x_0x_1x_2\ldots \sim_I y_0y_1y_2\ldots$.

This remark shows that $\sim_I$ is a congruence over the set A^∞ of words inducing the monoid structure over $M^\infty(A, I)$ with the multiplication given by

$$\forall x, y \in A^\infty,\ \varphi(x) \cdot \varphi(y) = \varphi(xy) \tag{1}$$

Moreover, the second part of the remark shows that $\sim_I$ is a congruence for infinite concatenation over A^∞. Thus we could define the infinite multiplication of traces by setting for any sequence $x_0, x_1, x_2, \ldots$ of elements of A^∞

$$\varphi(x_0)\varphi(x_1)\varphi(x_2)\ldots = \varphi(x_0x_1x_2\ldots) \tag{2}$$

However, while the formulae (1) and (2) yield consistent and formally correct definitions of finite and infinite multiplications of traces, they are not satisfactory in general. In fact (1) does not seem to be adequate if $x \in A^\omega$, i.e. if we multiply on the left by an infinite trace. To illustrate the problem let us consider the multiplication of independent traces.

Two traces $t, r \in M^\infty(A, I)$ are said to be *independent*, which is denoted by $(t, r) \in I$, if $alph(t) \times alph(r) \subseteq I$.

Suppose that $t, r \in M(A, I)$ and $(t, r) \in I$. Then $V_t \cap V_r = \emptyset$ and we have simply $\Gamma_D(tr) = \Gamma_D(rt) = (V_t \cup V_r, E_t \cup E_r)$, where $\Gamma_D(t) = (V_t, E_t)$ and $\Gamma_D(r) = (V_r, E_r)$. This fact reflects the intuitive idea that if t and r are independent then executing t and next r amounts to executing t and r in parallel. It is reasonable to require that the same property holds for all traces of $M^\infty(A, I)$.

For instance, let $(a, b) \in I$. Then for all $0 \le n, m < \infty$ we have

$$\varphi(a^n)\varphi(b^m) = \varphi(a^nb^m) = \varphi(b^ma^n) = \varphi(b^m)\varphi(a^n)$$

and the occurrence graph of $\varphi(a^nb^m)$ is the union of the occurrence graphs of $\varphi(a^n)$ and $\varphi(b^m)$. On the other hand, let us consider two traces $t = \varphi(a^\omega)$ and $r = \varphi(b^\omega)$. These two traces are independent and the union $\Gamma_D(t) \cup \Gamma_D(r) = (V_t \cup V_r, E_t \cup E_r)$ of their occurrence graphs is the occurrence graph of the trace $\varphi((ab)^\omega)$. Thus for any "reasonable" definition of trace multiplication we should have

$$\varphi(a^\omega) \cdot \varphi(b^\omega) = \varphi((ab)^\omega) \neq \varphi(a^\omega) = \varphi(a^\omega b^\omega)$$

Since our interest in traces stems mainly from the interpretation of their occurrence graphs as behaviours of concurrent systems, it is natural to expect that an adequate definition of trace multiplication should be formulated in terms of occurrence graphs. Actually this definition is a bit easier to formulate in terms of dependence graphs which are labelled counterparts of occurrence graphs.

In the sequel we shall consider oriented graphs with vertices labelled by elements of the alphabet A. Two such graphs (V_i, E_i, λ_i), $i = 1, 2$, where $\lambda_i : V_i \longrightarrow A$ is the labelling mapping, are isomorphic if there exists an isomorphism f of unlabeled graphs (V_1, E_1) and (V_2, E_2) preserving the labellings, i.e. such that $\forall v \in V_1,\ \lambda_1(v) = \lambda_2(f(v))$. By labelled graphs we shall mean the corresponding isomorphism classes.

If (V, E) is an unlabeled graph and $\lambda : V \longrightarrow A$ a vertex labelling then the corresponding labelled graph will be denoted by $[V, E, \lambda]$. Recall that E^+ stands for the transitive closure of E.

A labelled graph $[V, E, \lambda]$ is a *dependence graph* over (A, I) if the following conditions hold:
(D0) the underlying unlabeled graph (V, E) is oriented and acyclic,
(D1) the set V of vertices is countable,
(D2) $\forall \alpha, \beta \in V,\ (\lambda(\alpha), \lambda(\beta)) \in D \iff \alpha = \beta$ or $(\alpha, \beta) \in E$ or $(\beta, \alpha) \in E$,
(D3) $\forall \alpha \in V,\ card(\{\beta \in V \mid (\beta, \alpha) \in E^+\}) < \infty$.

The set of dependence graphs over (A, I) will be denoted by $\mathcal{DG}(A, I)$. There is a natural bijection Λ between the sets $M^\infty(A, I)$ and $\mathcal{DG}(A, I)$ that is defined in the following way. Let $t \in M^\infty(A, I)$, $\Gamma_D(t) = (V_t, E_t)$. By $\lambda_t : V_t \longrightarrow A$ we shall denote the natural labelling defined as $\lambda_t((a, i)) = a$ for $(a, i) \in V_t$. Then we set

$$\Lambda(t) = [V_t, E_t, \lambda_t]$$

It is clear that $\Lambda(t)$ verifies (D0)–(D3), i.e. the mapping Λ is well defined. Now we show that Λ is injective. Let $t_1, t_2 \in M^\infty(A, I)$. Suppose that $\Gamma_D(t_i) = (V_i, E_i)$ and $\lambda_i : V_i \longrightarrow A$ are natural labellings of V_i, $i = 1, 2$. Moreover, let f be an isomorphism of (V_1, E_1, λ_1) and (V_2, E_2, λ_2). Since f preserves labellings, $f((a, i))$ is equal (a, j) for some $j \in \mathbb{N}$. But both in $\Gamma_D(t_1)$ and $\Gamma_D(t_2)$ the following condition holds:

$$((a, k), (a, l)) \in E_i \text{ iff } k < l \qquad (i = 1, 2)$$

Thus, since f is an isomorphism of unlabeled graphs $\Gamma_D(t_1)$ and $\Gamma_D(t_2)$, we should have in fact $f((a, i)) = (a, i)$, i.e. $\Gamma_D(t_1) = \Gamma_D(t_2)$, which implies in turn that $t_1 = t_2$.

It remains to prove that Λ is surjective. To this end we shall use the so-called Foata normal form of traces [4, 6, 19]. Let $[V, E, \lambda]$ be a dependence graph. We shall define inductively two sequences $V_0, V_1, V_2, \ldots$ and $U_0, U_1, U_2, \ldots$ of subsets of V:

- $U_0 = V$, $V_0 = \{\alpha \in U_0 \mid \neg\exists\beta \in U_0, (\beta, \alpha) \in E\}$
- $U_{i+1} = V \backslash (\bigcup_{j=0}^{i} V_j)$, $V_{i+1} = \{\alpha \in U_{i+1} \mid \neg\exists\beta \in U_{i+1}, (\beta, \alpha) \in E\}$

It is easy to prove inductively the following two facts characterizing the sequence $V_0, V_1, V_2, \ldots$:

$$\text{for each } \alpha \in V, \text{ if } card(\{\beta \in V \mid (\beta, \alpha) \in E^+\}) = n_\alpha \text{ then } \alpha \in \bigcup_{i=0}^{n_\alpha} V_i \qquad (3)$$

and

$$\text{if } (\alpha,\beta)\in E \text{ and } \alpha\in V_i,\ \beta\in V_j \text{ then } i<j \tag{4}$$

The first of these facts shows that

$$V=\bigcup_i V_i \tag{5}$$

Moreover, if $\alpha,\beta\in V_i$ then $(\alpha,\beta)\notin E$, i.e. $(\lambda(\alpha),\lambda(\beta))\in I$. Thus all elements of V_i have different labels and $card(V_i)\le card(A)$. Let $x=x_0x_1x_2\ldots$ be any word such that each x_i is a list of labels (without omissions and repetitions) of elements of V_i. Directly from (4) and (5) it follows that $[V,E,\lambda]$ and $[V_x,E_x,\lambda_x]$, where $\Gamma_D(x)=(V_x,E_x)$ and $\lambda_x:V_x\longrightarrow A$ is the natural labelling, represent the same labelled graph. Hence Λ is surjective and we have proved that $M^\infty(A,I)$ is in bijection with $\mathcal{DG}(A,I)$.

There exists a natural multiplication operation over the set $\mathcal{DG}(A,I)$ of dependence graphs which reflects well the intuitive idea of the sequential composition of traces. This operation is defined in the following way. Multiplying two dependence graphs $[V_0,E_0,\lambda_0]$ and $[V_1,E_1,\lambda_1]$ we first take their disjoint union and next add edges joining vertices $v_0\in V_0$ and $v_1\in V_1$ whenever $(\lambda_0(v_0),\lambda_1(v_1))\in D$. Formally, let $[V_0,E_0,\lambda_0],[V_1,E_1,\lambda_1]\in\mathcal{DG}(A,I)$, where without loss of generality we can assume that $V_0\cap V_1=\emptyset$. Then

$$[V_0,E_0,\lambda_0]\cdot[V_1,E_1,\lambda_1]=[V,E,\lambda],$$

where

$$\begin{aligned}
V &= V_0\cup V_1,\\
E &= E_0\cup E_1\cup\{(v_0,v_1)\mid v_0\in V_0,\ v_1\in V_1 \text{ and } (\lambda_0(v_0),\lambda_1(v_1))\in D\},\\
\lambda &= \lambda_0\cup\lambda_1
\end{aligned}$$

Unfortunately, in general the resulting labelled graph $[V,E,\lambda]$ is not necessarily a dependence graph, whereas it always satisfies (D0)–(D2) it does not satisfy (D3) unless $\{(a,b)\in A^2\mid card(\lambda_0^{-1}(a))=\omega \text{ and } \lambda_1^{-1}(b)\neq\emptyset\}\subseteq I$. Thus, although this operation seems to be more natural than the one defined by formula (1), it has the drawback of inducing the trace multiplication that is only partial:
for $t,r\in M^\infty(A,I)$,

$$t\cdot r=\begin{cases} u & \text{if } alph_\omega(t)\times alph(r)\subseteq I \text{ and } u\in M^\infty(A,I) \text{ is}\\ & \text{such that } \Lambda(t)\cdot\Lambda(r)=\Lambda(u)\\ \text{undefined} & \text{otherwise}\end{cases} \tag{6}$$

In this paper this formula is used in the sequel as the definition of trace multiplication. Now we shall give some elementary properties of this operation. First let us note that the multiplication defined by (6) is associative:

Remark 2.2 *For all $t,r,u\in M^\infty(A,I)$ we have $t(ru)=(tr)u$, where this equality means that the left-hand side exists iff the right-hand side exists and then both sides are equal.*

Previous remark concerning the closure of $\mathcal{DG}(A,I)$ under multiplication yields directly the following fact:

Remark 2.3 *Let $t_0, \cdots, t_k$ be traces of $M^\infty(A,I)$, then $t_0 \cdots t_k$ exists if for all i,j such that $0 \leq i < j \leq k$ we have $alph_\omega(t_i) \times alph(t_j) \subseteq I$.*

The multiplication defined by (6) does not satisfy the condition (1) in general, however (1) remains valid if we multiply by a finite trace on the left:

Remark 2.4 *For all $x \in A^*$ and $y \in A^\infty$ the following equality holds in $M^\infty(A,I)$:*

$$\varphi(x) \cdot \varphi(y) = \varphi(xy)$$

Proof: Let $\Gamma_D(x) = (V_x, E_x)$, $\Gamma_D(y) = (V_y, E_y)$ and let $\lambda_x : V_x \longrightarrow A$ and $\lambda_y : V_y \longrightarrow A$ be the natural labellings. Then it is easy to see that

$$\Lambda(\varphi(xy)) = [V_x, E_x, \lambda_x] \cdot [V_y, E_y, \lambda_y]$$

which yields the thesis. □

When dealing with infinite traces we need also the notion of infinite product of traces. As in the case of the finite concatenation of traces, this operation can be expressed naturally in terms of dependence graphs. Let $t_0, t_1, t_2, \ldots$ be an infinite sequence of traces of $M^\infty(A,I)$ and let $\Lambda(t_i) = [V_i, E_i, \lambda_i]$ for all $i \in \mathbb{N}$, where again without loss of generality we can assume that all sets V_i are pairwise disjoint. Now we set

$$[V_0, E_0, \lambda_0] \cdot [V_1, E_1, \lambda_1] \cdot [V_2, E_2, \lambda_2] \cdots = [V, E, \lambda]$$

where

$$\begin{array}{l} V = \bigcup_{i=0}^\infty V_i, \\ E = \bigcup_{i=0}^\infty E_i \cup \bigcup_{i,j=0}^\infty \{(v', v'') \mid v' \in V_i,\ v'' \in V_j,\ i < j,\ (\lambda_i(v'), \lambda_j(v'')) \in D\} \\ \lambda = \bigcup_{i=0}^\infty \lambda_i \end{array}$$

As previously, the resulting labelled graph always satisfies (D0)-(D2), whereas it satisfies (D3) if and only if for all i,j such that $0 \leq i < j$ we have $alph_\omega(t_i) \times alph(t_j) \subseteq I$. This allows to define a partial infinite product of traces as follows. Let $t_0, t_1, t_2, \ldots$ be a sequence of elements of $M^\infty(A,I)$, then

$$t_0 t_1 t_2 \ldots = \begin{cases} u & \text{if } alph_\omega(t_i) \times alph(t_j) \subseteq I, \text{ for all } i < j \text{ and } u \in M^\infty(A,I) \text{ is} \\ & \text{such that } \Lambda(t_0) \cdot \Lambda(t_1) \cdot \Lambda(t_2) \cdots = \Lambda(u) \\ \text{undefined} & \text{otherwise} \end{cases}$$

Note that, in particular, if all traces t_i are finite then their infinite product always exists. In a similar way, although the formula (2) is not correct in general for the infinite product of traces defined above, it remains valid as long as we multiply finite traces only:

Remark 2.5 *Let $x_0, x_1, x_2, \ldots$ be a sequence of finite words of A^*. Then*

$$\varphi(x_0)\varphi(x_1)\varphi(x_2)\ldots = \varphi(x_0 x_1 x_2 \ldots)$$

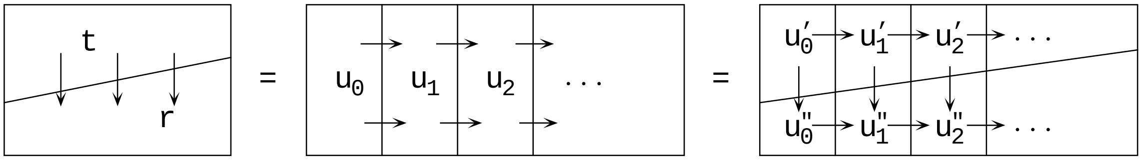

Figure 2: $t \cdot r = u_0 u_1 u_2 \ldots$

One of the basic tools enabling elegant approach to finite and infinite trace concatenation is the following factorization lemma. A special case of this lemma was previously given independently by Cori, Perrin [9] and Mazurkiewicz [29].

Lemma 2.6 *Let $t, r \in M^\infty(A, I)$. Let $u_0, u_1, u_2, \ldots$ be a (finite or infinite) sequence of finite traces of $M(A, I)$. Then*

$$t \cdot r = u_0 u_1 u_2 \ldots$$

if and only if there exist traces $u'_0, u'_1, u'_2, \ldots, u''_0, u''_1, u''_2, \ldots$ such that

- *i) for each i, $u_i = u'_i \cdot u''_i$,*
- *ii) $t = u'_0 u'_1 u'_2 \ldots$, $r = u''_0 u''_1 u''_2 \ldots$,*
- *iii) for all i, j, if $i < j$ then $(u''_i, u'_j) \in I$*

Proof: Let $u = tr = u_0 u_1 u_2 \ldots$ (cf. Figure 2).

Let $\Lambda(u) = [V, E, \lambda]$. Since $u = tr$, the set V can be partitioned on two subsets V_t, and V_r, $V_t \cap V_r = \emptyset$, such that $\Lambda(t) = [V_t, E_t, \lambda_t]$, $\Lambda(r) = [V_r, E_r, \lambda_r]$, where $E_t = E \cap (V_t \times V_t)$, $E_r = E \cap (V_r \times V_r)$, $\lambda_t = \lambda|_{V_t}$, $\lambda_r = \lambda|_{V_r}$ and

$$E = E_t \cup E_r \cup \{(\alpha, \beta) \in V_t \times V_r \mid (\lambda(\alpha), \lambda(\beta)) \in D\}. \tag{7}$$

Intuitively, V_t corresponds to the set of actions of the prefix t of u while V_r is the set of actions of the suffix r of u.

Similarly, since $u = u_0 u_1 u_2 \ldots$, V can be partitioned on a family of pairwise disjoint sets V_i, $\bigcup_i V_i = V$, such that $\Lambda(u_i) = [V_i, E_i, \lambda_i]$, where $\lambda_i = \lambda|_{V_i}$, $E_i = E \cap (V_i \times V_i)$ and

$$E = \bigcup_i E_i \cup \{(\alpha, \beta) \in V_i \times V_j \mid i < j \text{ and } (\lambda(\alpha), \lambda(\beta)) \in D\}. \tag{8}$$

Again, V_i corresponds here to the set of actions of the factor u_i of the trace u. Let $V'_i = V_i \cap V_t$, $V''_i = V_i \cap V_r$, $E'_i = (V'_i \times V'_i) \cap E$, $E''_i = (V''_i \times V''_i) \cap E$. Let λ'_i, λ''_i be restrictions of the mapping λ on the sets V'_i, V''_i respectively. It is clear that $[V'_i, E'_i, \lambda'_i]$ and $[V''_i, E''_i, \lambda''_i]$ satisfy the conditions (D0)–(D3), i.e. they are dependence graphs. Let u'_i, u''_i be traces such that $\Lambda(u'_i) = [V'_i, E'_i, \lambda'_i]$ and $\Lambda(u''_i) = [V''_i, E''_i, \lambda''_i]$. From (7) and (8) it follows that

$$E_i = E'_i \cup E''_i \cup \{(\alpha, \beta) \in V'_i \times V''_i \mid (\lambda(\alpha), \lambda(\beta)) \in D\},$$

which shows that

$$[V_i', E_i', \lambda_i'] \cdot [V_i'', E_i'', \lambda_i''] = [V_i, E_i, \lambda_i],$$

i.e. $u_i'u_i'' = u_i$ and (i) holds.

Similarly it is clear that

$$[V_0', E_0', \lambda_0'] \cdot [V_1', E_1', \lambda_1'] \cdot [V_2', E_2', \lambda_2'] \cdots = [V_t, E_t, \lambda_t]$$

and

$$[V_0'', E_0'', \lambda_0''] \cdot [V_1'', E_1'', \lambda_1''] \cdot [V_2'', E_2'', \lambda_2''] \cdots = [V_r, E_r, \lambda_r],$$

which proves (ii).

Suppose that (iii) does not hold and for some $i, j,\ i < j$, there exist $a \in alph(u_i'')$ and $b \in alph(u_j')$ such that $(a, b) \in D$. Let α be any occurrence of a in $[V_i'', E_i'', \lambda_i'']$, i.e. $\alpha \in V_i''$, $\lambda(\alpha) = a$. Similarly, let β be any occurrence of b in $[V_j', E_j', \lambda_j']$, i.e. $\beta \in V_j'$, $\lambda(\beta) = b$. From (8) it follows that $(\alpha, \beta) \in E$. On the other hand, $\alpha \in \bigcup_i V_i'' = V_r$ and $\beta \in \bigcup_j V_j' = V_t$, thus we get $(\alpha, \beta) \in E \cap (V_r \times V_t)$. But by (7), $E \cap (V_r \times V_t) = \emptyset$, a contradiction.

If the conditions (i),(ii),(iii) are satisfied then we can verify immediately that $\Lambda(tr) = \Lambda(u_0u_1u_2\ldots)$. □

As in the case of words, the definition of trace multiplication extends directly to sets of traces.
If $T_1, T_2 \subseteq M^\infty(A, I)$ then

$$T_1 \cdot T_2 = \{t_1t_2 \mid t_1 \in T_1 \text{ and } t_2 \in T_2\}$$

Similarly, if $T \subseteq M^\infty(A, I)$ then the finite iteration T^* of T is defined as

$$T^* = \bigcup_{i=0}^{\infty} T^i, \qquad \text{where} \qquad T^0 = \{\mathbf{1}\} \qquad \text{and} \qquad T^{i+1} = T^i \cdot T,$$

while

$$T^\omega = \{t_0t_1t_2\ldots \mid \forall i \in \mathbb{N},\ t_i \in T\}$$

is the infinite iteration of T.

We would like to end this section with a remark concerning trace multiplication. The reader may wonder if it is really "reasonable" to use a multiplication which is only partially defined. In fact other multiplication operations were proposed previously [11, 19, 27]. Let us note that the multiplication defined by Kwiatkowska is totally defined but not associative, on the other hand the multiplications defined by Diekert and Gastin are associative and totally defined but, in general, only on sets that strictly contain the set $M^\infty(A, I)$. However, there are three major cases where no controversy exists and where all definitions give the same result:

M1 multiplication $t_1 \cdot t_2$, where $t_1 \in M(A, I)$ and $t_2 \in M^\infty(A, I)$, i.e. the multiplication by a finite trace on the left,

M2 multiplication $t_1 \cdot t_2$ of independent traces, $(t_1, t_2) \in I$.

M3 infinite multiplication $t_0 t_1 t_2 \dots$ of finite traces, $\forall i \in \mathbb{N},\ t_i \in M(A, I)$.

On the other hand, as we shall see in Section 3, to examine recognizable subsets of $M^\infty(A, I)$ it suffices in fact to consider these three cases. Therefore, as far as recognizability is concerned, all definitions mentioned above are "reasonable" and we have adopted the simplest associative one.

3 Recognizable sets of traces

3.1 Definitions and basic properties

We begin this section with the definition of recognizable sets of traces.

Definition 3.1 *A morphism $\eta : M(A, I) \longrightarrow S$ into a finite monoid S recognizes a subset T of $M^\infty(A, I)$ if for each infinite sequence $t_0, t_1, t_2, \dots$ of traces from $M(A, I)$ the following implication holds*

$$t_0 t_1 t_2 \dots \in T \Longrightarrow \eta^{-1}(\eta(t_0))\eta^{-1}(\eta(t_1))\eta^{-1}(\eta(t_2)) \dots \subseteq T$$

A subset T of $M^\infty(A, I)$ is recognizable if it is recognized by some morphism into a finite monoid.

The definition of recognizable morphism given above is usually used to define recognizable subsets of infinite words [37], therefore it is reasonable to apply it in order to define recognizable subsets of the set $M^\omega(A, I)$ of infinite traces. It is less evident why we can apply it as well for subsets of $M(A, I)$ (or even generally to subsets of $M^\infty(A, I)$), especially in view of the fact that $M(A, I)$ is a monoid and so the classical definition of recognizability in monoids [15] readily applies and in fact is usually used in this case [9, 10, 31].

Let us recall that according to the classical definition a subset T of $M(A, I)$ is recognized by a morphism $\eta : M(A, I) \longrightarrow S$ into a finite monoid S if $\eta^{-1}(\eta(T)) = T$. It turns out that the two definitions are in fact equivalent for subsets of $M(A, I)$. First of all let us note that if $T \subseteq M(A, I)$ and $t_0 t_1 t_2 \dots \in T$ then almost all traces in the sequence $t_0, t_1, t_2, \dots$ are empty since otherwise $t_0 t_1 t_2 \dots$ would be an infinite trace. Thus there exists $k \in \mathbb{N}$ such that $t_0 t_1 t_2 \dots = t_0 \dots t_k$ and $t_i = \mathbf{1}$ for $i > k$.

Now suppose that η recognizes $T \subseteq M(A, I)$ in the sense of Definition 3.1. We shall prove that $\eta^{-1}(\mathbf{1}) = \{\mathbf{1}\}$. Indeed let $t_0 t_1 t_2 \dots \in T$ and let k be such that $t_i = \mathbf{1}$ for $i > k$. If $\eta^{-1}(\mathbf{1}) \neq \{\mathbf{1}\}$ then the set $\eta^{-1}(\eta(t_0))\eta^{-1}(\eta(t_1))\eta^{-1}(\eta(t_2)) \dots \eta^{-1}(\eta(t_{k+1}))\eta^{-1}(\eta(t_{k+2})) \dots$ would contain at least one infinite trace, i.e. could not be included in T. Thus, for $t \in T$ we have $\eta^{-1}(\eta(t)) = \eta^{-1}(\eta(t))\eta^{-1}(\eta(\mathbf{1}))\eta^{-1}(\eta(\mathbf{1})) \dots \subseteq T$, and we see that $\eta^{-1}(\eta(T)) \subseteq T$, i.e. η recognizes T in the sense of the classical definition.

On the other hand suppose that η is a morphism such that $\eta^{-1}(\eta(T)) = T$. Without loss of generality we can assume that $\eta^{-1}(\mathbf{1}) = \{\mathbf{1}\}$. (If η does not satisfy this assumption then we take the morphism $\eta' : M(A,I) \longrightarrow S^u$, where $S^u = S \cup \{u\}$ and $u \notin S$ is a new unit element, i.e. multiplication in S is completed by $s \cdot u = u \cdot s = s$ for $s \in S^u$, and $\eta'(t) = \eta(t)$ if $t \neq \mathbf{1}$, $\eta'(\mathbf{1}) = u$.) Then the fact that η recognizes T in the sense of Definition 3.1 follows immediately from the formula $\eta^{-1}(\eta(t_0)) \ldots \eta^{-1}(\eta(t_k)) \subseteq \eta^{-1}(\eta(t_0 \ldots t_k))$ which is valid for all $t_0, \ldots, t_k \in M(A,I)$.

As we have seen in the discussion above, Definition 3.1 is rather artificial when applied to subsets of $M(A,I)$ since it involves infinite factorizations of finite traces, where, in any case, almost all factors are in fact empty traces and so the classical definition is much more natural. However there is one advantage of adopting Definition 3.1 in our paper: it allows a uniform approach to finite and infinite traces so that we can avoid clumsy case analysis in several proofs. Let us finish these remarks with again one observation. The sets $M(A,I)$ and $M^\omega(A,I)$ are recognizable subsets of $M^\infty(A,I)$. In fact, they are recognized by any morphism η such that $\eta^{-1}(\mathbf{1}) = \{\mathbf{1}\}$.

Remark 3.2 *A morphism $\eta : M(A,I) \longrightarrow S$ recognizes a subset T of $M^\infty(A,I)$ iff for each infinite sequence $t_0, t_1, t_2, \ldots$ of traces of $M(A,I)$ the following implication holds*

$$\eta^{-1}(\eta(t_0))\eta^{-1}(\eta(t_1))\eta^{-1}(\eta(t_2)) \ldots \cap T \neq \emptyset \Longrightarrow \eta^{-1}(\eta(t_0))\eta^{-1}(\eta(t_1))\eta^{-1}(\eta(t_2)) \ldots \subseteq T$$

To prove this remark, note that if η recognizes T and $\eta^{-1}(\eta(t_0))\eta^{-1}(\eta(t_1))\eta^{-1}(\eta(t_2)) \ldots \cap T \neq \emptyset$ then there exist traces $r_0, r_1, r_2, \ldots$ of $M(A,I)$ such that $r_0 r_1 r_2 \ldots \in T$ and $\forall i,\ \eta(r_i) = \eta(t_i)$. But then

$$\eta^{-1}(\eta(t_0))\eta^{-1}(\eta(t_1))\eta^{-1}(\eta(t_2)) \ldots = \eta^{-1}(\eta(r_0))\eta^{-1}(\eta(r_1))\eta^{-1}(\eta(r_2)) \ldots \subseteq T$$

Conversely, if η verifies the condition given by Remark 3.2 then it is clear that η recognizes T since $t_0 t_1 t_2 \ldots \in \eta^{-1}(\eta(t_0))\eta^{-1}(\eta(t_1))\eta^{-1}(\eta(t_2)) \ldots$.

In the sequel, the family of recognizable subsets of M, where M stands for one of the sets $M(A,I)$, $M^\omega(A,I)$, $M^\infty(A,I)$, will be denoted by Rec_M.

Let M be any monoid and U a subset of M. We recall that the syntactic congruence $\sim_U$ of U is defined in the following way: for all $m_1, m_2 \in M$, $m_1 \sim_U m_2$ if the following condition holds

$$\forall v, w \in M,\ v m_1 w \in U \Longleftrightarrow v m_2 w \in U$$

As it is well-known (cf. [15]), a subset U of M is recognizable iff its syntactic congruence $\sim_U$ has a finite index (i.e. has a finite number of equivalence classes).

Let $T \subseteq M(A,I)$ and $L = \varphi^{-1}(T) \subseteq A^*$, where $\varphi : A^* \longrightarrow M(A,I)$ is the canonical morphism. Then it can easily be verified that

$$\forall x, y \in A^*,\ x \sim_L y \text{ iff } \varphi(x) \sim_T \varphi(y)$$

where $\sim_L$ and $\sim_T$ are the syntactic congruences of L and T respectively. This fact implies directly the following well-known result:

Remark 3.3 *A subset T of $M(A,I)$ is recognizable iff $\varphi^{-1}(T)$ is a recognizable subset of A^*.*

The syntactic congruence $\sim_L$ for subsets L of A^ω was defined by Arnold [2]. This definition can directly be extended to subsets of $M^\omega(A,I)$. Let T be a subset of $M^\omega(A,I)$. Then the *syntactic congruence* $\sim_T$ of T is the equivalence relation over $M(A,I)$ defined in the following way:
for $t,r \in M(A,I)$, $t \sim_T r$ if

$$\forall u,v,w \in M(A,I), \quad \begin{array}{l}(utv)w^\omega \in T \iff (urv)w^\omega \in T \text{ and} \\ u(vtw)^\omega \in T \iff u(vrw)^\omega \in T\end{array}$$

It is easy to verify that $\sim_T$ defined above is really a congruence over the monoid $M(A,I)$.

To each congruence $\sim$ over $M(A,I)$ is canonically associated a morphism from $M(A,I)$ into $M(A,I)/\sim$ mapping each trace t into its equivalence class $[t]_\sim$. A congruence is said to *recognize* a subset T of $M^\omega(A,I)$ if the associated morphism recognizes T. In other words, a congruence $\sim$ over $M(A,I)$ recognizes T if for any two infinite sequences $u_0,u_1,u_2,\ldots$ and $v_0,v_1,v_2,\ldots$ of elements of $M(A,I)$ the following condition holds

$$(\forall i \in \mathbb{N}, u_i \sim v_i) \Longrightarrow (u_0u_1u_2\ldots \in T \iff v_0v_1v_2\ldots \in T)$$

On the other hand, if $\eta : M(A,I) \longrightarrow S$ is a monoid morphism recognizing a subset T of $M^\infty(A,I)$ then the relation $\sim$ defined as:

$$\forall u,v \in M(A,I), \ u \sim v \text{ iff } \eta(u) = \eta(v)$$

is a congruence recognizing T.
Thus a subset T of $M^\omega(A,I)$ is recognizable iff there exists a congruence $\sim$ of a finite index over $M(A,I)$ recognizing T.

The following proposition due to Arnold [2] characterizes recognizable subsets of A^ω by means of their syntactic congruences.

Proposition 3.4 ([2]) *Let $L \subseteq A^\omega$. Then L is a recognizable subset of A^ω iff the syntactic congruence $\sim_L$ has a finite index and recognizes L.*
Moreover, if L is a recognizable subset of A^ω then $\sim_L$ is the coarsest congruence recognizing L.

As it is known [2, 37] there exist non-recognizable subsets L of A^ω for which the syntactic congruence $\sim_L$ has a finite index (but it does not recognize L). Thus the condition that $\sim_L$ recognizes L cannot be omitted in Proposition 3.4.

Proposition 3.5 ([20]) *Let $T \subseteq M^\omega(A,I)$. Then the following conditions are equivalent:*

i) T is a recognizable subset of $M^\omega(A,I)$, *ii)* $\varphi^{-1}(T)$ *is a recognizable subset of* A^ω*,*

iii) *the syntactic congruence* $\sim_T$ *of* T *has a finite index and recognizes* T*.*

Moreover, if $T \in Rec\_M^\omega(A, I)$ *then* $\sim_T$ *is the coarsest congruence recognizing* T*.*

Proof: Let $L = \varphi^{-1}(T) \subseteq A^\omega$. We prove three claims which yield directly the thesis.

CLAIM 1. Let $\sim_L$ and $\sim_T$ be the syntactic congruences of T and L respectively. Then

$$\forall x, y \in A^* \ , \ x \sim_L y \text{ iff } \varphi(x) \sim_T \varphi(y)$$

This claim results from the following equivalences:

$$\begin{array}{lll} & x \sim_L y & \\ \text{iff} & \forall u, v, w \in A^*, & (uxv)w^\omega \in L \Longleftrightarrow (uyv)w^\omega \in L \text{ and} \\ & & u(vxw)^\omega \in L \Longleftrightarrow u(vyw)^\omega \in L \\ \text{iff} & \forall u, v, w \in A^*, & (\varphi(u)\varphi(x)\varphi(v))\varphi(w)^\omega \in T \Longleftrightarrow (\varphi(u)\varphi(y)\varphi(v))\varphi(w)^\omega \in T \text{ and} \\ & & \varphi(u)(\varphi(v)\varphi(x)\varphi(w))^\omega \in T \Longleftrightarrow \varphi(u)(\varphi(v)\varphi(y)\varphi(w))^\omega \in T \\ \text{iff} & \varphi(x) \sim_T \varphi(y) & \end{array}$$

CLAIM 2. Let $\sim_w$ and $\sim_t$ be congruences over A^* and $M(A, I)$ respectively and suppose that they verify the following condition

$$\forall x, y \in A^*, \ x \sim_w y \text{ iff } \varphi(x) \sim_t \varphi(y) \tag{1}$$

Then $\sim_w$ recognizes L iff $\sim_t$ recognizes T.
Moreover, the quotient monoids $A^*/\sim_w$ and $M(A, I)/\sim_t$ are isomorphic and the congruences $\sim_w$ and $\sim_t$ have the same index.

Direct verification shows that if $\sim_w$ and $\sim_t$ verify (1) then the mapping $\Psi : A^*/\sim_w \longrightarrow M(A, I)/\sim_t$ defined by

$$\forall x \in A^*, \ \Psi([x]_{\sim_w}) = [\varphi(x)]_{\sim_t}$$

is an isomorphism between the quotient monoids $A^*/\sim_w$ and $M(A, I)/\sim_t$. Since the indices of the congruences $\sim_t$ and $\sim_w$ are equal to the number of elements in the corresponding quotient monoids, the congruences $\sim_t$ and $\sim_w$ have the same index. Moreover, since for each sequence $x_0, x_1, x_2, \ldots$ of elements of A^*, $x_0 x_1 x_2 \ldots \in L$ iff $\varphi(x_0)\varphi(x_1)\varphi(x_2) \ldots \in T$, $\sim_w$ recognizes L iff $\sim_t$ recognizes T.

CLAIM 3. Let $T \in Rec\_M^\omega(A, I)$ and let $\sim$ be a congruence recognizing T. Then the syntactic congruence $\sim_T$ is coarser than $\sim$.

Let $t \sim r$ for $t, r \in M(A, I)$. Since $\sim$ recognizes T it is clear that for all $u, v, w \in M(A, I)$

$$(utv)w^\omega \in T \Longleftrightarrow (urv)w^\omega \in T$$

and

$$u(vtw)^\omega \in T \Longleftrightarrow u(vrw)^\omega \in T,$$

that is $t \sim_T r$.

Now we can proceed to the proof of the proposition.
(i) $\Longrightarrow$ (ii) Let $\sim_t$ be a congruence of a finite index recognizing T. Then we define a congruence $\sim_w$ over A^* in the following way

$$\forall x, y \in A^*,\ x \sim_w y \text{ if } \varphi(x) \sim_t \varphi(y)$$

By Claim 2 $\sim_w$ has a finite index and recognizes L, hence L is a recognizable subset of A^ω.
(ii) $\Longrightarrow$ (iii) If L is recognizable then by Proposition 3.4 the syntactic congruence $\sim_L$ recognizes L and it has a finite index. Moreover, it is the coarsest congruence recognizing L. Then by Claims 1 and 2, $\sim_T$ recognizes T, has a finite index and by Claim 3 it is the coarsest congruence recognizing T.
(iii) $\Longrightarrow$ (i) Obvious. □

3.2 Weakly recognizable trace languages

One of the basic results concerning recognizable sets of infinite words states that the families of recognizable and weakly recognizable subsets of A^ω coincide. The aim of this section is to prove that this result holds for infinite traces as well.

For a finite monoid S we set $P(S) = \{(s,e) \in S \times S \mid s \cdot e = s \text{ and } e \cdot e = e\}$. Following the terminology of Perrin,Pin [37] we call elements of $P(S)$ linked pairs. Moreover, we recall that an idempotent is an element $e \in S$ such that $e \cdot e = e$.

Definition 3.6 *Let $\eta : M(A,I) \longrightarrow S$ be a morphism into a finite monoid S and let $T \subseteq M^\omega(A,I)$. Then η weakly recognizes T if*

$$T = \bigcup_{(s,e) \in P_T(S)} \eta^{-1}(s)(\eta^{-1}(e))^\omega$$

where $P_T(S) = \{(s,e) \in P(S) \mid \eta^{-1}(s)(\eta^{-1}(e))^\omega \subseteq T\}$.
A subset T of $M^\omega(A,I)$ is weakly recognizable if it is weakly recognizable by some morphism into a finite monoid.

Although the proof that each recognizable subset of $M^\omega(A,I)$ is weakly recognizable is much the same as for subsets of A^ω [35, 37], we give it in detail for the sake of completeness. This proof is based on the following lemma, which is given in [37], where also a simple direct proof is presented.

Lemma 3.7 ([37]) *Let Z be a finite or infinite alphabet, let S be a finite set and let f be a mapping from Z^+ in S. Then for each infinite sequence $z_0, z_1, z_2, \ldots$ of words of Z^+ there exist an infinite increasing sequence of integers $0 < i_0 < i_1 < i_2 \ldots$ and an element $e \in S$ such that*

$$f(z_{i_k} \cdots z_{i_m - 1}) = e \quad \textit{for each} \quad k \geq 0 \textit{ and } m > k$$

Proof: This lemma can also be deduced from the following infinite version of the classical theorem due to Ramsey [24]:

> Let X be an infinite set, $\mathcal{P}_k(X)$ the family of k-element subsets of X and $\chi : \mathcal{P}_k(X) \longrightarrow S$ a "coloring" mapping associating with each k-subset of X a "color" from a finite set S of colors. Then there exists an infinite subset Y of X such that all k-subsets of Y have the same color:
>
> $$\forall Z_1, Z_2 \subseteq Y,\ |Z_1|=|Z_2|= k \Longrightarrow \chi(Z_1) = \chi(Z_2)$$

In our case, the coloring mapping $\chi : \mathcal{P}_2(\mathbb{N}) \longrightarrow S$ is defined by

$$\chi(\{i,j\}) = f(z_i \cdots z_{j-1}) \text{ for all } 0 \leq i < j$$

By Ramsey's theorem there exist an infinite subset Y of $\mathbb{N}$ and an element e of S such that $\chi(\{i,j\}) = e$ for all $i, j \in Y,\ i < j$. The elements of Y arranged in increasing order constitute the required sequence. □

Lemma 3.8 *Let $\eta : M(A,I) \longrightarrow S$ be a morphism into a finite monoid S and let $t_0t_1t_2 \ldots \in M^\omega(A,I)$, where each $t_i,\ i \geq 0$, is in $M(A,I)$. Then there exists a linked pair $(s,e) \in P(S)$ and an infinite increasing sequence of integers $0 < i_0 < i_1 < i_2 \ldots$ such that*

$$\eta(t_0 \ldots t_{i_1-1}) = s$$
$$\eta(t_{i_k} \ldots t_{i_{k+1}-1}) = e \textit{ for all } k \geq 0$$
$$\eta^{-1}(\eta(t_0))\eta^{-1}(\eta(t_1))\eta^{-1}(\eta(t_2)) \ldots \subseteq \eta^{-1}(s)(\eta^{-1}(e))^\omega$$

Proof: From Lemma 3.7 applied to the sequence $t_0, t_1, t_2, \ldots$ and to the morphism η we deduce that there exists an infinite sequence $0 < i_0 < i_1 < i_2 \ldots$ such that $\eta(t_{i_k} \ldots t_{i_m-1}) = e$ for all $k \geq 0$ and $m > k$. Let $s = \eta(t_0 \ldots t_{i_1-1})$. Observe that (s,e) is a linked pair since

$$e \cdot e = \eta(t_{i_0} \ldots t_{i_1-1}) \cdot \eta(t_{i_1} \ldots t_{i_2-1}) = \eta(t_{i_0} \ldots t_{i_2-1}) = e$$

and

$$\begin{aligned} s \cdot e &= \eta(t_0 \ldots t_{i_1-1}) \cdot e = (\eta(t_0 \ldots t_{i_0-1}) \cdot \eta(t_{i_0} \ldots t_{i_1-1})) \cdot e = \eta(t_0 \ldots t_{i_0-1}) \cdot (e \cdot e) \\ &= \eta(t_0 \ldots t_{i_0-1}) \cdot e = \eta(t_0 \ldots t_{i_0-1}) \cdot \eta(t_{i_0} \ldots t_{i_1-1}) = \eta(t_0 \ldots t_{i_1-1}) \\ &= s \end{aligned}$$

Now it suffices to observe that

$$\eta^{-1}(\eta(t_0)) \cdots \eta^{-1}(\eta(t_{i_1-1})) \subseteq \eta^{-1}(\eta(t_0 \cdots t_{i_1-1})) = \eta^{-1}(s)$$

and

$$\eta^{-1}(\eta(t_{i_k})) \cdots \eta^{-1}(\eta(t_{i_{k+1}-1})) \subseteq \eta^{-1}(\eta(t_{i_k} \cdots t_{i_{k+1}-1})) = \eta^{-1}(e) \text{ for all } k \geq 1$$

□

Proposition 3.9 ([20]) *Let $\eta : M(A,I) \longrightarrow S$ be a morphism into a finite monoid S and let $T \subseteq M^\omega(A,I)$. Then the following conditions are equivalent:*

i) *η recognizes T,*

ii) $\forall s_1, s_2 \in S, \quad \eta^{-1}(s_1)(\eta^{-1}(s_2))^\omega \cap T \neq \emptyset \Longrightarrow \eta^{-1}(s_1)(\eta^{-1}(s_2))^\omega \subseteq T,$

iii) $\forall (s,e) \in P(S), \quad \eta^{-1}(s)(\eta^{-1}(e))^\omega \cap T \neq \emptyset \Longrightarrow \eta^{-1}(s)(\eta^{-1}(e))^\omega \subseteq T.$

Proof:
(i)$\Longrightarrow$(ii) Let t_1, t_2 be elements of $M(A,I)$ such that $t_1 \in \eta^{-1}(s_1)$ and $t_2 \in \eta^{-1}(s_2)$. Then

$$\eta^{-1}(s_1)(\eta^{-1}(s_2))^\omega = \eta^{-1}(\eta(t_1))\eta^{-1}(\eta(t_2))\eta^{-1}(\eta(t_2))\ldots$$

and it suffices to apply Remark 3.2.
(ii)$\Longrightarrow$(iii) Obvious.
(iii)$\Longrightarrow$(i) Suppose that η verifies (iii) and let $t_0, t_1, t_2, \ldots$ be a sequence of traces of $M(A,I)$ such that $t_0t_1t_2\ldots \in T$. First observe that by Lemma 3.8

$$\eta^{-1}(\eta(t_0))\eta^{-1}(\eta(t_1))\eta^{-1}(\eta(t_2))\ldots\ldots \subseteq \eta^{-1}(s)(\eta^{-1}(e))^\omega \qquad (1)$$

for some linked pair $(s,e) \in P(S)$. Hence, we have $\eta^{-1}(s)(\eta^{-1}(e))^\omega \cap T \neq \emptyset$ and since η verifies (iii):

$$\eta^{-1}(s)(\eta^{-1}(e))^\omega \subseteq T \qquad (2)$$

From (1) and (2) we get finally:

$$\eta^{-1}(\eta(t_0))\eta^{-1}(\eta(t_1))\eta^{-1}(\eta(t_2))\ldots\ldots \subseteq T$$

and therefore η recognizes T. □

Now we are able to achieve the proof that each recognizable subset of $M^\omega(A,I)$ is weakly recognizable.

Lemma 3.10 ([20]) *Let $\eta : M(A,I) \longrightarrow S$ be a morphism into a finite monoid S recognizing a subset T of $M^\omega(A,I)$. Then η weakly recognizes T.*

Proof: To get the thesis it suffices to show that if η recognizes T then for each $t \in T$ there exists a linked pair (s,e) such that

$$t \in \eta^{-1}(s)(\eta^{-1}(e))^\omega \subseteq T \qquad (1)$$

Let $t = t_0t_1t_2\ldots \in T$, where all $t_i,\ i \geq 0$, belong to $M(A,I)$. Then by Lemma 3.8,

$$t \in \eta^{-1}(\eta(t_0))\eta^{-1}(\eta(t_1))\eta^{-1}(\eta(t_2))\ldots \subseteq \eta^{-1}(s)(\eta^{-1}(e))^\omega$$

for some linked pair $(s,e) \in P(S)$ and by Proposition 3.9, since $t \in \eta^{-1}(s)(\eta^{-1}(e))^\omega \cap T \neq \emptyset$, we obtain (1). □

It can be shown that the converse of Lemma 3.10 does not hold even in the case of infinite words; a morphism η can weakly recognize a subset L of A^ω without recognizing it. Nevertheless, it turns out (cf. [35, 37]) that given a morphism $\eta : A^* \to S$ weakly recognizing a subset L of A^ω the Schützenberger product [38] $\eta \Diamond \eta : A^* \longrightarrow S \Diamond S$ recognizes L. We will extend this result to trace languages by generalizing the Schützenberger product $\Diamond$ to the diamond product $\Diamond_I$.

Let S and Q be two monoids. The diamond product $\Diamond_I(S, Q)$ of S and Q is a monoid defined in the following way. Elements of $\Diamond_I(S, Q)$ are subsets of the set $S \times \mathcal{P}(A) \times Q \times \mathcal{P}(A)$. The multiplication in $\Diamond_I(S, Q)$ is specified by the following formula:

$$R_1 \circ R_2 = \{(s_1 s_2, \alpha_1 \cup \alpha_2, q_1 q_2, \beta_1 \cup \beta_2) \mid (s_1, \alpha_1, q_1, \beta_1) \in R_1,\ (s_2, \alpha_2, q_2, \beta_2) \in R_2 \text{ and } \beta_1 \times \alpha_2 \subseteq I\}$$

for $R_1, R_2 \subseteq S \times \mathcal{P}(A) \times Q \times \mathcal{P}(A)$.

To see that the multiplication in $\Diamond_I(S, Q)$ is really associative one can verify easily that both $(R_1 \circ R_2) \circ R_3$ and $R_1 \circ (R_2 \circ R_3)$ consist of all quadruples of the form

$$(s_1 s_2 s_3, \alpha_1 \cup \alpha_2 \cup \alpha_3, q_1 q_2 q_3, \beta_1 \cup \beta_2 \cup \beta_3)$$

such that

$$\forall 1 \le i \le 3,\ (s_i, \alpha_i, q_i, \beta_i) \in R_i \text{ and } \forall 1 \le i < j \le 3,\ \ \beta_i \times \alpha_j \subseteq I$$

If $\eta_1 : M(A, I) \longrightarrow S$, $\eta_2 : M(A, I) \longrightarrow Q$ are morphisms into the monoids S and Q then

$$\Diamond_I(\eta_1, \eta_2) : M(A, I) \longrightarrow \Diamond_I(S, Q)$$

is a morphism defined by

$$\Diamond_I(\eta_1, \eta_2)(t) = \{(\eta_1(u), alph(u), \eta_2(v), alph(v)) \mid u, v \in M(A, I) \text{ are such that } t = uv\}$$

We shall show that $\Diamond_I(\eta_1, \eta_2)$ is really a morphism from $M(A, I)$ into $\Diamond_I(S, Q)$. Let $t = t_1 t_2 \in M(A, I)$. Then $\Diamond_I(\eta_1, \eta_2)(t) = \{(\eta_1(u), alph(u), \eta_2(w), alph(w)) \mid t_1 t_2 = uw\}$. By Lemma 2.6, $t_1 t_2 = uw$ if and only if there exist traces z_0, z_1, z_2, z_3 of $M(A, I)$ such that

$$t_1 = z_0 z_1,\ \ t_2 = z_2 z_3,\ \ u = z_0 z_2,\ \ w = z_1 z_3 \text{ and } (z_1, z_2) \in I$$

Therefore

$$\begin{aligned}
&\Diamond_I(\eta_1, \eta_2)(t) \\
&= \{(\eta_1(z_0 z_2), alph(z_0 z_2), \eta_2(z_1 z_3), alph(z_1 z_3)) \mid t_1 = z_0 z_1,\ t_2 = z_2 z_3 \text{ and } (z_1, z_2) \in I\} \\
&= \{(\eta_1(z_0)\eta_1(z_2), alph(z_0) \cup alph(z_2), \eta_2(z_1)\eta_2(z_3), alph(z_1) \cup alph(z_3)) \mid \\
&\qquad\qquad t_1 = z_0 z_1,\ t_2 = z_2 z_3 \text{ and } (z_1, z_2) \in I\} \\
&= \{(\eta_1(z_0), alph(z_0), \eta_2(z_1), alph(z_1)) \mid t_1 = z_0 z_1\} \circ \{(\eta_1(z_2), alph(z_2), \eta_2(z_3), alph(z_3)) \mid t_2 = z_2 z_3\} \\
&= \Diamond_I(\eta_1, \eta_2)(t_1) \circ \Diamond_I(\eta_1, \eta_2)(t_2)
\end{aligned}$$

The diamond product being defined, we now prove an auxiliary lemma.

Lemma 3.11 *Let $\eta : M(A,I) \longrightarrow S$ be a morphism into a finite monoid S, $\gamma = \diamond_I(\eta,\eta)$ and $u \in M^\omega(A,I)$. Then for each linked pair $(s,e) \in P(S)$ such that*

$$u \in \eta^{-1}(s)(\eta^{-1}(e))^\omega$$

and for any sequence $u_0, u_1, u_2, \ldots$ of traces of $M(A,I)$ such that $u = u_0u_1u_2\ldots$ the following inclusion holds

$$\gamma^{-1}(\gamma(u_0))\gamma^{-1}(\gamma(u_1))\gamma^{-1}(\gamma(u_2))\ldots \subseteq \eta^{-1}(s)(\eta^{-1}(e))^\omega$$

Proof: Let $v = v_0v_1v_2\ldots$ be an infinite trace of $M^\omega(A,I)$ such that

$$\forall i \in \mathbb{N},\ v_i \in M(A,I) \text{ and } \diamond_I(\eta,\eta)(v_i) = \diamond_I(\eta,\eta)(u_i) \tag{1}$$

To prove the thesis we should show that $v \in \eta^{-1}(s)(\eta^{-1}(e))^\omega$.
Since $u \in \eta^{-1}(s)(\eta^{-1}(e))^\omega$ there exist traces $w_0 \in \eta^{-1}(s)$ and $w_i \in \eta^{-1}(e)$, $i \geq 1$, such that

$$u_0u_1u_2\ldots = u = w_0w_1w_2\ldots \tag{2}$$

We construct inductively two infinite strictly increasing sequences of integers $i_0, i_1, i_2, \ldots$ and $j_0, j_1, j_2, \ldots$:

- set $i_0 = 0$ and $j_0 = 0$
- if j_k is defined then i_{k+1} is the least integer greater than i_k and such that
$$w_0 \cdots w_{j_k} \leq u_0 \cdots u_{i_{k+1}-1} \tag{3}$$
- if i_{k+1} is defined then j_{k+1} is the least integer greater than j_k such that
$$u_0 \cdots u_{i_{k+1}-1} \leq w_0 \cdots w_{j_{k+1}} \tag{4}$$

Equation (2) ensures that all inductive steps of this construction are always feasible. Now we set $z_0 = w_0$ and, for $k \geq 0$, $z_{k+1} = w_{j_k+1}\ldots w_{j_{k+1}}$ and $y_k = u_{i_k}\ldots u_{i_{k+1}-1}$. Conditions (3) and (4) imply that

$$z_0\ldots z_k \leq y_0\ldots y_k \leq z_0\ldots z_kz_{k+1}, \quad \text{for } k \geq 0 \tag{5}$$

Now we have

$$\eta(z_0) = \eta(w_0) = s$$

and since e is an idempotent

$$\eta(z_{k+1}) = \eta(w_{j_k+1})\ldots\eta(w_{j_{k+1}}) = e\ldots e = e \quad \text{for} \quad k \geq 0$$

Therefore we get

$$z_0 \in \eta^{-1}(s) \text{ and } z_k \in \eta^{-1}(e), \quad \text{for } k \geq 1 \tag{6}$$

From (5) there exist two sequences $y'_0, y'_1, y'_2, \ldots$ and $y''_0, y''_1, y''_2, \ldots$ of traces of $M(A, I)$ such that:

$$y'_0 = z_0, y_0 \cdots y_k y'_{k+1} = z_0 \cdots z_{k+1} \text{ and } z_0 \cdots z_k y''_k = y_0 \cdots y_k \text{ for } \quad k \geq 0$$

We shall prove that $y_k = y'_k y''_k$ for $k \geq 0$ and $z_k = y''_{k-1} y'_k$ for $k \geq 1$.

Indeed we have $y'_0 y''_0 = z_0 y''_0 = y_0$ and for $k \geq 0$, $(y_0 \cdots y_k y'_{k+1}) y''_{k+1} = (z_0 \cdots z_{k+1}) y''_{k+1} = y_0 \cdots y_k y_{k+1}$, which after canceling $y_0 \cdots y_k$ on the left gives $y'_{k+1} y''_{k+1} = y_{k+1}$. Similarly, since $(z_0 \cdots z_k y''_k) y'_{k+1} = y_0 \cdots y_k y'_{k+1} = z_0 \cdots z_k z_{k+1}$ we get $z_{k+1} = y''_k y'_{k+1}$.

Now we set $x_k = v_{i_k} \ldots v_{i_{k+1}-1}$ for $k \geq 0$. From (1) and from the definition of x_k and y_k it follows that $\diamond_I(\eta, \eta)(x_k) = \diamond_I(\eta, \eta)(y_k)$. Hence there exist two sequences $x'_0, x'_1, x'_2, \ldots$ and $x''_0, x''_1, x''_2, \ldots$ of traces of $M(A, I)$ such that:

$$x_k = x'_k x''_k, \eta(x'_k) = \eta(y'_k) \text{ and } \eta(x''_k) = \eta(y''_k), \forall k \geq 0$$

Now we have $v_0 v_1 v_2 \ldots = x_0 x_1 x_2 \ldots = x'_0 (x''_0 x'_1)(x''_1 x'_2) \ldots$ and $\eta(x'_0) = \eta(y'_0) = \eta(z_0)$, and $\eta(x''_k x'_{k+1}) = \eta(y''_k y'_{k+1}) = \eta(z_{k+1})$ for $k \geq 0$. Therefore $v_0 v_1 v_2 \ldots \in \eta^{-1}(s)(\eta^{-1}(e))^\omega$ and the lemma is proved. □

Corollary 3.12 *Let T be a subset of $M^\omega(A, I)$ weakly recognized by a morphism $\eta : M(A, I) \longrightarrow S$ into a finite monoid S. Then the morphism $\diamond_I(\eta, \eta) : M(A, I) \longrightarrow \diamond_I(S, S)$ recognizes T.*

Proof: Let $\gamma = \diamond_I(\eta, \eta)$ and suppose that $u = u_0 u_1 u_2 \ldots \in T$, where $\forall i \in \mathbb{N}$, $u_i \in M(A, I)$. Since η weakly recognizes T there exists a linked pair $(s, e) \in P(S)$ such that

$$u \in \eta^{-1}(s)(\eta^{-1}(e))^\omega \subseteq T$$

But by Lemma 3.11 we get

$$\gamma^{-1}(\gamma(u_0))\gamma^{-1}(\gamma(u_1))\gamma^{-1}(\gamma(u_2)) \ldots \subseteq \eta^{-1}(s)(\eta^{-1}(e))^\omega,$$

i.e. $\gamma^{-1}(\gamma(u_0))\gamma^{-1}(\gamma(u_1))\gamma^{-1}(\gamma(u_2)) \ldots \subseteq T$. □

As a corollary to Lemma 3.10 and Corollary 3.12 we obtain the following result.

Theorem 3.13 *A subset T of $M^\omega(A, I)$ is weakly recognizable if and only if it is recognizable.*

The rest of this subsection is devoted to an extended discussion of the structure of diamond product. Since this discussion does not affect the other sections, it can be skipped at the first reading.

An equivalence relation $\simeq$ over $M(A, I)$ is said to be *independence invariant* if it satisfies the following condition for all $t_1, t_2, r_1, r_2 \in M(A, I)$

$$\text{if} \quad t_1 \simeq t_2 \quad \text{and} \quad r_1 \simeq r_2 \quad \text{then} \quad (t_1, r_1) \in I \Leftrightarrow (t_2, r_2) \in I$$

Note that this definition says simply that if X_1 and X_2 are equivalence classes of $\simeq$ then either all elements of X_1 are independent of all elements of X_2 or all elements of X_1 depend on all elements of X_2.

A relation $\simeq$ over $M(A,I)$ is an *independence invariant congruence* if it is an independence invariant and a congruence over $M(A,I)$, i.e. it satisfies the previous condition and moreover for all $t_1, t_2, r_1, r_2 \in M(A,I)$

$$\text{if} \quad t_1 \simeq t_2 \quad \text{and} \quad r_1 \simeq r_2 \quad \text{then} \quad t_1 r_1 \simeq t_2 r_2$$

Let $\simeq$ be an independence invariant congruence over $M(A,I)$ and let $H = M(A,I)/\simeq$ be the quotient monoid of $M(A,I)$ by $\simeq$. By $\xi : M(A,I) \longrightarrow H$ we shall denote the canonical morphism mapping each trace $t \in M(A,I)$ to its equivalence class under $\simeq$. Now we can generalize the construction of diamond product $\diamond_I(S,Q)$ by defining the monoid $\diamond_I(S,Q,H)$, which elements are subsets of $S \times H \times Q \times H$ and for $R_1, R_2 \subseteq S \times H \times Q \times H$, the multiplication $R_1 \circ R_2$ yields the following set:

$$\{(s_1 s_2, g_1 g_2, q_1 q_2, h_1 h_2) \mid (s_i, g_i, q_i, h_i) \in R_i,\ i = 1,2, \text{ and } \forall t_1 \in \xi^{-1}(h_1), \forall t_2 \in \xi^{-1}(g_2), (t_1, t_2) \in I\}$$

In consequence, the diamond product $\diamond_I(\eta_1, \eta_2)$ of morphisms, where $\eta_1 : M(A,I) \longrightarrow S$ and $\eta_2 : M(A,I) \longrightarrow Q$, generalizes to the morphism

$$\diamond_I(\eta_1, \eta_2, \xi) : M(A,I) \longrightarrow \diamond_I(S,Q,H)$$

defined by

$$\diamond_I(\eta_1, \eta_2, \xi)(t) = \{(\eta_1(u), \xi(u), \eta_2(w), \xi(w)) \mid u, w \in M(A,I) \text{ are such that } t = uw\}$$

It is straightforward to verify that the definitions of $\diamond_I(S,Q,H)$ and $\diamond_I(\eta_1,\eta_2,\xi)$ are sound and that all proofs involving the diamond product can be carried out without any substantial modifications with this generalized product. In fact, the diamond product defined previously is a special case of the generalized diamond product. To see it, let us note that the equivalence relation $\simeq$ defined by:

$$\forall t_1, t_2 \in M(A,I), \quad t_1 \simeq t_2 \quad \text{if} \quad alph(t_1) = alph(t_2)$$

is an independence invariant congruence and the quotient monoid $H = M(A,I)/\simeq$ is then isomorphic with the monoid $(\mathcal{P}(A), \cup, \emptyset)$ of all subsets of A, where the monoid operation is the set union and the neutral element is the empty set. Thus we see that $\diamond_I(S,Q) = \diamond_I(S,Q,\mathcal{P}(A))$ and $\diamond_I(\eta_1,\eta_2) = \diamond_I(\eta_1,\eta_2,alph)$, where $alph : M(A,I) \longrightarrow \mathcal{P}(A)$ is the morphism mapping each trace t to its alphabet $alph(t)$.

Let $\simeq_1$ and $\simeq_2$ be two independence invariant congruences such that $\simeq_1 \subseteq \simeq_2$ and let $H_i = M(A,I)/\simeq_i$, $i = 1,2$. Then the monoid H_2 is a quotient of H_1 and similarly $\diamond_I(S,Q,H_2)$ is a quotient of $\diamond_I(S,Q,H_1)$. This remark indicates that it would be interesting to find the greatest independence invariant congruence, since, if it exists, it would induce the smallest diamond product which is a quotient of all other diamond products, so is the best in some sense.

Let $\simeq_g$ be the equivalence relation over $M(A,I)$ defined in the following way:

$$t_1 \simeq_g t_2 \quad \text{if} \quad \forall r \in M(A,I), \ \ (t_1, r) \in I \Longleftrightarrow (t_2, r) \in I$$

This relation is the greatest independence invariant congruence. Indeed, immediately from the definition it follows that $\simeq_g$ is the greatest independence invariant whereas the fact that $\simeq$ is also a congruence over $M(A,I)$ follows from the following observation

$$\forall t_1, t_2, r \in M(A,I), \ (t_1 t_2, r) \in I \ \text{ iff } \ (t_1, r) \in I \text{ and } (t_2, r) \in I$$

To express $\simeq_g$ in a more explicit way note that for $t, r \in M(A,I)$, $(t,r) \in I$ iff $(t, \varphi(a)) \in I$ for each $a \in alph(r)$. Thus

$$t_1 \simeq_g t_2 \quad \text{iff } \forall a \in A, \ (t_1, \varphi(a)) \in I \Longleftrightarrow (t_2, \varphi(a)) \in I$$

But for each trace t, $\{a \in A \mid (t, \varphi(a)) \in I\} = A \backslash D(alph(t))$, where $D(\beta) = \{c \in A \mid \exists b \in \beta, (b,c) \in D\}$ for $\beta \subseteq A$. Thus we get

$$t_1 \simeq_g t_2 \quad \text{iff} \quad D(alph(t_1)) = D(alph(t_2))$$

The quotient monoid $H_g = M(A,I)/\simeq_g$ is isomorphic with the monoid $\mathcal{FD}_D = (\{D(\beta) \mid \beta \subseteq A\}, \cup, \emptyset)$, whose elements are subsets of A of the form $D(\beta)$, where β ranges over $\mathcal{P}(A)$, the monoid operation is set union and the empty set $\emptyset = D(\emptyset)$ is the neutral element. Indeed, the previous condition implies that the mapping f such that $f([t]_{\simeq_g}) = D(alph(t))$ is a bijection between H_g and $\mathcal{FD}_D$ and moreover we have

$$f([t_1 t_2]_{\simeq_g}) = D(alph(t_1 t_2)) = D(alph(t_1)) \cup D(alph(t_2)) = f([t_1]_{\simeq_g}) \cup f([t_2]_{\simeq_g})$$

Thus

$$\diamondsuit_I(\eta_1, \eta_2, \xi_g) : M(A,I) \longrightarrow \diamondsuit_I(S, Q, \mathcal{FD}_D)$$

is the best of all diamond products, where $\xi_g : M(A,I) \longrightarrow \mathcal{FD}_D$ is the morphism defined by $\xi_g(t) = D(alph(t))$ for $t \in M(A,I)$. We can also define multiplication over $\diamondsuit_I(S, Q, \mathcal{FD}_D)$ in a more explicit way, without any reference to the mapping ξ_g. First let us note the following fact

Lemma 3.14 *Let $t_1, t_2 \in M(A,I)$, $\alpha_1 = D(alph(t_1))$, $\alpha_2 = D(alph(t_2))$. Then t_1 and t_2 are independent iff $D^{-1}(\alpha_1) \times D^{-1}(\alpha_2) \subseteq I$, where $D^{-1}(\beta) = \{a \in A \mid D(a) \subseteq \beta\}$.*

Proof: First note that $alph(t_i) \subseteq D^{-1}(\alpha_i)$, $i = 1, 2$. Thus $D^{-1}(\alpha_1) \times D^{-1}(\alpha_2) \subseteq I$ yields $alph(t_1) \times alph(t_2) \subseteq I$, i.e. $(t_1, t_2) \in I$.

Now suppose that $D^{-1}(\alpha_1) \times D^{-1}(\alpha_2)$ is not contained in I and $(a,b) \in D \cap (D^{-1}(\alpha_1) \times D^{-1}(\alpha_2))$. Since $b \in D^{-1}(\alpha_2)$ and $(a,b) \in D$ we have $a \in D(b) \subseteq \alpha_2 = D(alph(t_2))$, i.e. there exists $c \in alph(t_2)$ such that $(a,c) \in D$. From this fact and from $a \in D^{-1}(\alpha_1)$ it results that $c \in D(a) \subseteq \alpha_1 = D(alph(t_1))$. Therefore for some $d \in alph(t_1)$, $(c,d) \in D$, hence $(d,c) \in D \cap (alph(t_1) \times alph(t_2))$ and t_1, t_2 are not independent. □

Using this lemma we obtain the following formula for the multiplication in $\diamondsuit_I(S, Q, \mathcal{FD}_D)$:

$$R_1 \circ R_2 = \{(s_1s_2, \alpha_1 \cup \alpha_2, q_1q_2, \beta_1 \cup \beta_2) \mid (s_i, \alpha_i, q_i, \beta_i) \in R_i,\ i = 1,2 \text{ and } D^{-1}(\beta_1) \times D^{-1}(\alpha_2) \subseteq I\}$$

for $R_1, R_2 \subseteq S \times \mathcal{FD}_D \times Q \times \mathcal{FD}_D$. As we have remarked previously, the diamond product $\Diamond_I$ plays the same role for traces as the Schützenberger product for words. But the relation between these two products is even more closer.

Let $I = \emptyset$. Then $\mathcal{FD}_D = \{\emptyset, A\}$ since $D(\emptyset) = \emptyset$ and $D(\beta) = A$ for each nonempty subset β of A. Note also that the canonical mapping $\varphi : A^* \longrightarrow M(A, \emptyset)$ is a monoid isomorphism, which allows us to identify these monoids. Moreover, it turns out that in fact the diamond product $\Diamond_\emptyset(\eta_1, \eta_2, \xi_g)$ in the case of empty independence relation and the Schützenberger product $\eta_1 \Diamond \eta_2$ of morphisms coincide.

3.3 Closure properties of recognizable trace languages

In this subsection we examine basic closure properties of various families of recognizable trace languages. Let us note that the closure under union, intersection, complementation and concatenation can also be proved by means of Remark 3.3 and Proposition 3.5 ([9, 20, 33]).

Proposition 3.15 *The families $Rec\_M(A,I)$, $Rec\_M^\omega(A,I)$ and $Rec\_M^\infty(A,I)$ are closed under the boolean operations of finite union, finite intersection and complement.*

Proof: Throughout this proof, M stands for one of the sets $M(A,I)$, $M^\omega(A,I)$ or $M^\infty(A,I)$. Let $\eta : M(A,I) \longrightarrow S$ be a morphism into a finite monoid recognizing a subset T of M. From Remark 3.2, for any sequence $t_0, t_1, t_2, \ldots$ of finite traces either

$$\eta^{-1}(\eta(t_0))\eta^{-1}(\eta(t_1))\eta^{-1}(\eta(t_2))\ldots \subseteq T$$

or

$$\eta^{-1}(\eta(t_0))\eta^{-1}(\eta(t_1))\eta^{-1}(\eta(t_2))\ldots \subseteq \overline{T},$$

i.e. η recognizes the complement $\overline{T}$ of T as well.

Now let $\eta_i : M(A,I) \longrightarrow S_i$ be morphisms into finite monoids recognizing subsets T_i, $i = 1,2$, of M. Let $\gamma : M(A,I) \longrightarrow S_1 \times S_2$ be the morphism into the direct product of S_1 and S_2 defined by $\gamma(t) = (\eta_1(t), \eta_2(t))$ for each trace t of $M(A,I)$. We shall show that γ recognizes $T_1 \cap T_2$. For any finite trace t we have

$$\gamma^{-1}(\gamma(t)) = \{u \in M \mid \eta_1(u) = \eta_1(t) \text{ and } \eta_2(u) = \eta_2(t)\} = \eta_1^{-1}(\eta_1(t)) \cap \eta_2^{-1}(\eta_2(t))$$

Assume that $t_0t_1t_2\ldots \in T_1 \cap T_2$, where each t_i belongs to $M(A,I)$. Then for $i = 1,2$,

$$\gamma^{-1}(\gamma(t_0))\gamma^{-1}(\gamma(t_1))\gamma^{-1}(\gamma(t_2))\ldots \subseteq \eta_i^{-1}(\eta_i(t_0))\eta_i^{-1}(\eta_i(t_1))\eta_i^{-1}(\eta_i(t_2))\ldots \subseteq T_i$$

and we see that γ recognizes $T_1 \cap T_2$. In this way we have proved that Rec_M is closed under complement and finite intersections. The closure under finite union follows immediately since $T_1 \cup T_2$ can be written as $\overline{\overline{T_1} \cap \overline{T_2}}$. □

Note that Proposition 3.15 implies that a subset T of $M^\infty(A,I)$ is recognizable if and only if the sets $T \cap M(A,I)$ and $T \cap M^\infty(A,I)$ are recognizable.

The closure of the family $Rec\_M(A,I)$ under multiplication was proved independently by Fliess [17], Cori,Perrin [9] and Ochmański [33]. We extend this result by proving that the family $Rec\_M^\infty(A,I)$ is closed under multiplication.

Proposition 3.16 ([20]) *If T_1 and T_2 are recognizable subsets of $M^\infty(A,I)$ then $T_1 \cdot T_2$ is also recognizable.*

Proof: Let η_1 and η_2 be morphisms into finite monoids recognizing T_1 and T_2 respectively. We shall prove that $\diamond_I(\eta_1,\eta_2)$ recognizes T_1T_2.

Let $t \in T_1T_2, t = t_1t_2$ for some $t_1 \in T_1$ and $t_2 \in T_2$, let $u_0, u_1, u_2, \ldots$ be a sequence of elements of $M(A,I)$ such that $t = u_0u_1u_2\ldots$. By Lemma 2.6 there exist sequences $u'_0, u'_1, u'_2, \ldots$ and $u''_0, u''_1, u''_2, \ldots$ of traces of $M(A,I)$ such that

$$u'_iu''_i = u_i \quad \text{for all} \quad i \geq 0 \tag{1}$$

$$u'_0u'_1u'_2\ldots = t_1 \quad \text{and} \quad u''_0u''_1u''_2\ldots = t_2 \tag{2}$$

$$(u''_i, u'_j) \in I \quad \text{for all} \quad i < j. \tag{3}$$

Let $v_0, v_1, v_2, \ldots$ be a sequence of elements of $M(A,I)$ such that

$$\forall i,\ \diamond_I(\eta_1,\eta_2)(v_i) = \diamond_I(\eta_1,\eta_2)(u_i) \tag{4}$$

To prove our claim we have to show that $v_0v_1v_2\ldots \in T_1T_2$. But (4) and (1) imply that for each i there exist traces v'_i, v''_i of $M(A,I)$ such that

$$v'_iv''_i = v_i,\ \eta_1(v'_i) = \eta_1(u'_i) \quad \text{and} \quad \eta_2(v''_i) = \eta_2(u''_i) \tag{5}$$

and

$$alph(v'_i) = alph(u'_i) \quad \text{and} \quad alph(v''_i) = alph(u''_i) \tag{6}$$

The last condition and (3) yield

$$(v''_i, v'_j) \in I \quad \text{for all} \quad i < j \tag{7}$$

Let $r_1 = v'_0v'_1v'_2\ldots$ and $r_2 = v''_0v''_1v''_2\ldots$. Since η_1 recognizes T_1 and $t_1 \in T_1$, (5) implies that $r_1 \in T_1$. Similarly, we obtain $r_2 \in T_2$. But from (7) we get by Lemma 2.6 that

$$v_0v_1v_2\ldots = (v'_0v''_0)(v'_1v''_1)(v'_2v''_2)\ldots = (v'_0v'_1v'_2\ldots)(v''_0v''_1v''_2\ldots) = r_1r_2 \in T_1T_2$$

□

As it is well-known the family $Rec\_M(A,I)$ (and hence $Rec\_M^\infty(A,I)$ as well) is not closed under finite iteration. The standard counterexample is the following

Example 3.1 Let $A = \{a,b\}$, $I = \{(a,b),(b,a)\}$ and $T = \{\varphi(ab)\} \in Rec\_M(A,I)$. Then $T^* = \{\varphi((ab)^n) \mid n \geq 0\}$ and $\varphi^{-1}(T^*) = \{x \in \{a,b\}^* \mid |x|_a = |x|_b\}$ is not a recognizable

subset of A^*, hence, by Proposition 3.3, T^* is not recognizable.
On the other hand, it is worth noticing that $T^\omega = \{\varphi((ab)^\omega)\}$ is a recognizable subset of $M^\omega(A, I)$ since $\varphi^{-1}(T^\omega) = \{x \in A^\omega \mid |x|_a = |x|_b = \infty\}$ is a recognizable subset of A^ω. The syntactic congruence of T^ω has four classes $\{\varphi(1)\}$, $\{\varphi(a^n) \mid n \geq 1\}$, $\{\varphi(b^n) \mid n \geq 1\}$ and $\{\varphi(a^k b^n) \mid k, n \geq 1\}$ and, as one can verify immediately, it recognizes T^ω.

The following example shows that $Rec\_M^\infty(A, I)$ is not closed under infinite iterations.

Example 3.2 Let $A = \{a, b, c\}$, $T = \{(a, b), (b, a)\}$ and $T = \{\varphi(ab), \varphi(c)\}$. Obviously T is recognizable. To show that $T^\omega = \varphi(\{ab, c\}^\omega)$ is not recognizable it suffices to observe that the syntactic congruence of T^ω has infinite index.
We set $U = T^\omega$. Then $\varphi(a^n) \sim_U \varphi(a^m)$ iff $n = m$. Indeed, for $n \neq m$ we have

$$\varphi(b^n)\varphi(a^m)\varphi(\mathbf{1})(\varphi(c))^\omega = \varphi(a^m b^n c^\omega) \notin U$$

while

$$\varphi(b^n)\varphi(a^n)\varphi(\mathbf{1})(\varphi(c))^\omega = \varphi((ab)^n c^\omega) \in U$$

These two examples raise the question about when a finite or infinite iteration of a subset T of $Rec\_M(A, I)$ is actually recognizable. As for the finite iteration, this problem was studied extensively (cf. [31, 34, 32, 25]) and many different sufficient conditions are known. The most useful of them is the one given by Métivier [31] (and also announced independently by Ochmański [33]). In order to formulate this condition we introduce important notions of connected traces and trace components.

A non-empty subset β of A is said to be non-connected if there exist non-empty sets β_1, β_2 such that $\beta_1 \cup \beta_2 = \beta$ and $\beta_1 \times \beta_2 \subseteq I$; otherwise β is connected. For each $\emptyset \neq \beta \subseteq A$ there exists a unique family $\{\beta_1, \ldots, \beta_k\}$ of non-empty connected sets such that $\bigcup_{i=1}^k \beta_i = \beta$ and $\beta_i \times \beta_j \subseteq I$ for $i \neq j$. The sets $\beta_1, \ldots, \beta_k$ are the connected components of β.

A non-empty trace t of $M^\infty(A, I)$ is *connected* if $alph(t)$ is connected. If t is non-connected then there exists a family $\{t_1, \ldots, t_k\}$ of pairwise independent connected non-empty traces such that $t = t_1 \cdots t_k$. The traces $t_1, \ldots, t_k$ are called the connected components of t. Note that then $\{alph(t_1), \ldots, alph(t_k)\}$ is the family of connected components of $alph(t)$. This concept has a natural interpretation in terms of dependence graphs. A trace t is connected iff its dependence graph $\Lambda(t)$ is connected. Otherwise $\Lambda(t)$ consists of several connected components $[V_i, E_i, \lambda_i]$, $1 \leq i \leq k$, all of them being dependence graphs, and the traces $t_i = \Lambda^{-1}([V_i, E_i, \lambda_i])$ are precisely the connected components of t.

Let us consider for instance the trace t from Example 2.1. Then $alph(t) = A$ has two connected components $\beta_1 = \{a, b, c\}$ and $\beta_2 = \{d, e\}$. Similarly the trace t has two connected components, $t_1 = \varphi(acbaacb)$ and $t_2 = \varphi(ded)$.

Proposition 3.17 ([31]) *Let T be a recognizable subset of $M(A, I)$ such that each trace t of T is connected. Then T^* is recognizable.*

3.4 Normal form of recognizable languages of infinite traces

In this subsection we present a normal form theorem for recognizable subsets of $M^\omega(A,I)$ — Theorem 3.22. This theorem shows that a subset T of $M^\omega(A,I)$ is recognizable iff it can be constructed from specific recognizable subsets of $M(A,I)$.

We begin with a lemma providing an interesting sufficient condition ensuring the recognizability of the infinite iteration T^ω of subsets T of $M(A,I)$.

Proposition 3.18 *Let T be a recognizable subset of $M(A,I)$ such that $T \cdot T \subseteq T$. Then T^ω is recognizable.*

Note that the condition $T \cdot T \subseteq T$ is obviously equivalent to $T = T^+$.

Proof: Let $\eta : M(A,I) \longrightarrow S$ be a morphism into a finite monoid S recognizing T such that $\eta^{-1}(\mathbf{1}) = \{\mathbf{1}\}$. (As we have seen in the discussion following Definition 3.1 there always exists such a morphism for any recognizable subset of $M(A,I)$). We shall show that η weakly recognizes the set $T^\omega \cap M^\omega(A,I)$.

Let $t \in T^\omega \cap M^\omega(A,I)$ and let $t_0, t_1, t_2, \ldots$ be a sequence of traces of $T \setminus \{\mathbf{1}\}$ such that $t = t_0 t_1 t_2 \ldots$. Applying Lemma 3.8 to η and sequence $t_0, t_1, t_2, \ldots$ we obtain an infinite sequence $0 < i_1 < i_2 < i_3 < \ldots$ of integers and a linked pair $(s,e) \in P(S)$ such that

$$\eta(t_0 \cdots t_{i_1-1}) = s \quad \text{and} \quad \eta(t_{i_k} \cdots t_{i_{k+1}-1}) = e \text{ for } k \geq 1 \tag{1}$$

Now note that since $T \cdot T \subseteq T$, all the traces $(t_0 \cdots t_{i_1-1})$ and $(t_{i_k} \cdots t_{i_{k+1}-1})$, $k \geq 1$, belong to T, thus $s, e \in \eta(T)$. But η recognizes T, hence $\eta^{-1}(\eta(T)) = T$, i.e. in particular $\eta^{-1}(s) \subseteq T$ and $\eta^{-1}(e) \subseteq T$, which together with (1) implies that

$$t = t_0 t_1 t_2 \ldots \in \eta^{-1}(s)(\eta^{-1}(e))^\omega \subseteq T^\omega \cap M^\omega(A,I)$$

which proves that η weakly recognizes $T^\omega \cap M^\omega(A,I)$. If $T^\omega \cap M(A,I)$ is not empty then $\mathbf{1} \in T$ and we have clearly $T^\omega \cap M(A,I) = T^+$. Since $T^+ = T$ we deduce that η recognizes $T^\omega \cap M(A,I)$. □

Corollary 3.19 *Let $T \subseteq M(A,I)$. If $T^* \in Rec\_M(A,I)$ then $T^\omega \in Rec\_M^\infty(A,I)$.*

Proof: Follows directly from Proposition 3.18 since $T^* \cdot T^* \subseteq T^*$. □

Note that Example 3.1 shows that the inverse of Corollary 3.19 does not hold, recognizability of T and T^ω does not imply that T^* is recognizable. The interest of Corollary 3.19 stems from the fact that, as we have mentioned previously, numerous sufficient conditions ensuring the recognizability of the finite iteration T^* of a subset T of $M(A,I)$ were discovered recently and all of them may now be applied to the infinite iteration as well. In particular from Proposition 3.17 we get the following condition that will be used extensively in the sequel.

Corollary 3.20 *Let T be a recognizable subset of $M(A,I)$ such that all traces of T are connected. Then T^ω is a recognizable subset of $M^\infty(A,I)$.*

Sometimes it is useful to consider the set $\mathcal{P}(A)$ as a commutative monoid with the set union $\cup$ as the monoid operation. Then the mapping $alph : M(A,I) \longrightarrow \mathcal{P}(A)$ associating with each trace t its alphabet $alph(t)$ is a monoid morphism.

A morphism $\eta : M(A,I) \longrightarrow S$ is said to be *alphabetic* if for all traces t_1, t_2 of $M(A,I)$, $\eta(t_1) = \eta(t_2)$ implies $alph(t_1) = alph(t_2)$. Let $\eta : M(A,I) \longrightarrow S$ be a morphism and let $\overline{\eta} : M(A,I) \longrightarrow S \times \mathcal{P}(A)$ be the direct product of the morphisms η and $alph$, $\forall t \in M(A,I)$, $\overline{\eta}(t) = (\eta(t), alph(t))$. For each trace $t \in M(A,I)$ we have

$$\overline{\eta}^{-1}(\overline{\eta}(t)) = \eta^{-1}(\eta(t)) \cap \{u \in M(A,I) \mid alph(u) = alph(t)\} \subseteq \eta^{-1}(\eta(t))$$

Thus if η recognizes a subset T of $M^{\infty}(A,I)$ then $\overline{\eta}$ recognizes this set as well. Moreover, $\overline{\eta}$ is alphabetic. Thus we obtain the following result:

Lemma 3.21 *A subset T of $M^{\infty}(A,I)$ is recognizable iff it is recognized by an alphabetic morphism into a finite monoid.*

And now we can pass to the main result of this subsection.

Theorem 3.22 *A subset T of $M^{\omega}(A,I)$ is recognizable if and only if T is a finite union of languages of the form*

$$UV_1^{\omega} \cdots V_k^{\omega}$$

where $U, V_1, \ldots, V_k$ are recognizable non-empty subsets of $M(A,I)$ such that there exist non-empty connected subsets $B_1, \ldots, B_k$ of A verifying $\forall i,\ 1 \le i \le k,\ \forall t \in V_i,\ alph(t) = B_i$ and $\forall i, j,\ 1 \le i < j \le k,\ B_i \times B_j \subseteq I$.

Proof: Corollary 3.20 and the fact that the family $Rec\_M^{\infty}(A,I)$ is closed under finite union and multiplication show that each trace language of the form described by the thesis is recognizable.

Now suppose that $T \in Rec\_M^{\omega}(A,I)$. Let $\eta : M(A,I) \longrightarrow S$ be a morphism into a finite monoid S recognizing T. From Lemma 3.21, we can assume without loss of generality that η is alphabetic. Moreover from Lemma 3.10 η weakly recognizes T and thus

$$T = \bigcup_{(s,e) \in P_T(S)} \eta^{-1}(s)(\eta^{-1}(e))^{\omega}$$

where $P_T(S) = \{(s,e) \in P(S) \mid \emptyset \neq \eta^{-1}(s)(\eta^{-1}(e))^{\omega} \subseteq T\}$ and it suffices to show that each of the sets $\eta^{-1}(s)(\eta^{-1}(e))^{\omega}$ has the required form.

In the sequel, we denote by R the set $\eta^{-1}(e)$. Let $(s,e) \in P_T(S)$ and let B be a subset of A such that

$$\forall t \in R,\ alph(t) = B \tag{1}$$

Let $B_1, \ldots, B_k$ be the connected components of B. Let $I' = I \cap (B \times B)$ and $I_j = I \cap (B_j \times B_j)$ be the restrictions of I to the sets B and B_j respectively. It is easy to prove (see e.g. [10]) that the mapping χ from $N = M(B_1, I_1) \times \ldots \times M(B_k, I_k)$ into $M(B, I')$

defined by $\chi(t_1, \ldots, t_k) = t_1 \ldots t_k$ is an isomorphism. Then $R \subseteq M(B, I')$ and $\chi^{-1}(R)$ is a recognizable subset of N. By Mezei's theorem (Proposition 12.2, page 68, of [15]) there exist an integer n and languages $R_{i,j} \in Rec\_M(B_j, I_j)$, for $1 \leq i \leq n$ and $1 \leq j \leq k$, such that

$$\chi^{-1}(R) = \bigcup_{1 \leq i \leq n} R_{i,1} \times \ldots \times R_{i,k}$$

Let $R_i = \chi(R_{i,1} \times \ldots \times R_{i,k}) = R_{i,1} \ldots R_{i,k}$, we obtain

$$R = \bigcup_{1 \leq i \leq n} R_i$$

We claim that

$$R^\omega = \bigcup_{1 \leq i,j \leq n} R_i R_j^\omega$$

One inclusion is obvious. Using Lemma 3.7 on the existence of Ramsey's factorizations we show the opposite inclusion. First note that $R^+ = R$ since $R \cdot R = \eta^{-1}(e) \cdot \eta^{-1}(e) \subseteq \eta^{-1}(e \cdot e) = \eta^{-1}(e) = R$. Now let f be any mapping from R into the set $\{1, \ldots, n\}$ such that for each $r \in R$, $r \in R_{f(r)}$. Let $r_0, r_1, r_2, \ldots$ be any infinite sequence of elements of R. From Lemma 3.7 it follows that there exists l, $1 \leq l \leq n$, and an infinite sequence $0 < i_0 < i_1 < i_2 \ldots$ such that $f(r_{i_k} \ldots r_{i_{k+1}-1}) = l$ for all $k \geq 0$. Thus $r_0 r_1 r_2 \ldots \in R_m R_l^\omega$, where $m = f(r_0 \cdots r_{i_0 - 1})$ and the opposite direction is proved.
Hence we obtain

$$\eta^{-1}(s)(\eta^{-1}(e))^\omega = \bigcup_{1 \leq i,j \leq n} \eta^{-1}(s) R_i R_{j,1}^\omega \ldots R_{j,k}^\omega$$

Moreover from $alph(r) = B$ for all $r \in R$, we deduce that $alph(r) = B_i$ for all $r \in R_{i,j}$ and therefore the theorem is proved. □

4 Rational, c-rational and sc-rational sets of traces

The aim of this section is to study the relationship between recognizable subsets of $M^\infty(A, I)$ and the languages which can be generated starting from atomic actions and applying modular operators (union, concatenation and iterations). We introduce three families of trace languages — c–rational, sc–rational and rational trace languages and prove that the first two of these families coincide with the family of recognizable trace languages and are included in the family of rational sets of traces. Moreover, we show a kind of normal form theorem for rational trace languages.

In the sequel, for any trace $t \in M^\infty(A, I)$ we shall denote by $C(t)$ the set of *connected components* of t (cf. Subsection 3.3). Moreover, for a subset T of $M^\infty(A, I)$ we set $C(T) = \bigcup_{t \in T} C(t)$.

We begin with some auxiliary lemmas.

Lemma 4.1 *If T is a recognizable subset of $M^\infty(A, I)$ then $C(T)$ is recognizable as well.*

Proof: Suppose that T is recognized by a morphism $\eta : M(A,I) \longrightarrow S$. Let $\overline{\eta} : M(A,I) \longrightarrow S \times \mathcal{P}(A)$ be the alphabetic morphism defined by $\overline{\eta}(t) = (\eta(t), alph(t))$ (cf. Subsection 3.4). We shall show that $\overline{\eta}$ recognizes $C(T)$.
Let $u_0, u_1, u_2, \ldots$ and $v_0, v_1, v_2, \ldots$ be two sequences of finite traces such that $\forall i, \overline{\eta}(u_i) = \overline{\eta}(v_i)$. This implies that

$$\forall i, \qquad \eta(u_i) = \eta(v_i) \tag{1}$$

and

$$\forall i, \qquad alph(u_i) = alph(v_i) \tag{2}$$

Let $u = u_0u_1u_2\ldots$, $v = v_0v_1v_2\ldots$ and suppose that $u \in C(T)$. Then there exists a trace $r \in M^\infty(A,I)$ such that $(u,r) \in I$ and $ur \in T$. Let $r_0, r_1, r_2, \ldots$ be any sequence of elements of $M(A,I)$ such that $r = r_0r_1r_2\ldots$. Since $(u,r) \in I$, we have

$$(u_i, r_j) \in I \qquad \text{for all } i, j \tag{3}$$

Therefore $ur = (u_0u_1u_2\ldots)(r_0r_1r_2\ldots) = (u_0r_0)(u_1r_1)(u_2r_2)\ldots$. Now by (1) we get for all i, $\eta(u_ir_i) = \eta(v_ir_i)$. Hence, as $ur \in T$ and η recognizes T, we obtain that

$$(v_0r_0)(v_1r_1)(v_2r_2)\ldots \in T$$

But (2) and (3) imply that $(v_i, r_j) \in I$ for all i, j. Thus, we have

$$vr = (v_0v_1v_2\ldots)(r_0r_1r_2\ldots) = (v_0r_0)(v_1r_1)(v_2r_2)\ldots \in T.$$

Now observe that (2) implies $alph(u) = alph(v)$, thus v is connected iff u is connected. Summarizing we see that v is a connected component of the trace $vr \in T$, i.e. $v \in C(T)$ and $\overline{\eta}$ really recognizes $C(T)$. □

Lemma 4.2 *Let $T \subseteq M^\infty(A,I)$, $T_f = T \cap M(A,I)$, $T_i = T \cap M^\omega(A,I)$ and $n = card(A)$. Then*

$$T^* = \bigcup_{k=0}^{n} (T_f^* T_i)^k T_f^*$$

and

$$T^\omega = \bigcup_{k=0}^{n} (T_f^* T_i)^k T_f^\omega$$

Proof: Let $t \in T^*$, $t = t_0 \cdots t_m$, where each t_i belongs to T. Since for all $u \in M^\omega(A,I)$ and $v \in M^\infty(A,I)$, $u \cdot v$ is defined iff $alph_\omega(u) \times alph(v) \subseteq I$, the sequence $t_0, \ldots, t_m$ can contain at most $card(A)$ traces of $M^\omega(A,I)$. Thus there exists k, $0 \le k \le card(A)$, such that $t \in (T_f^*T_i)^k T_f^*$. This proves the inclusion

$$T^* \subseteq \bigcup_{k=0}^{n} (T_f^* T_i)^k T_f^*$$

The inverse inclusion is obvious. The proof of the second formula is similar. □

Corollary 4.3 *Let T be a recognizable subset of $M^\infty(A,I)$ such that all traces of T are connected. Then T^* and T^ω are recognizable.*

Proof: Let $T_f = T \cap M(A,I)$ and $T_i = T \cap M^\omega(A,I)$. By Proposition 3.17 and Corollary 3.20 T_f^* and T_f^ω are recognizable. Now the thesis follows directly from Lemma 4.2 since the family of recognizable sets is closed under finite union and concatenation. □

Now we are ready to introduce and examine two new operations on trace languages. Let $T \subseteq M^\infty(A,I)$. The *finite and infinite concurrent iterations* of T are defined in the following way :

$$T^{c-*} = (C(T))^* \qquad \text{and} \qquad T^{c-\omega} = (C(T))^\omega.$$

As a direct consequence of Lemma 4.1 and Corollary 4.3 we obtain the following result.

Theorem 4.4 *If T is a recognizable subset of $M^\infty(A,I)$ then T^{c-*} and $T^{c-\omega}$ are recognizable.*

We shall now present three important families of trace languages: rational, c-rational and sc-rational trace languages.

The family $Rat\_M(A,I)$ of *rational* subsets of $M(A,I)$ is the smallest family $\mathcal{F}$ of trace languages such that

(R1) $\emptyset \in \mathcal{F}$ and $\forall a \in A, \{\varphi(a)\} \in \mathcal{F}$,
(R2) if $T_1, T_2 \in \mathcal{F}$ then $T_1 \cup T_2 \in \mathcal{F}$ and $T_1 \cdot T_2 \in \mathcal{F}$,
(RAT) if $T \in \mathcal{F}$ then $T^* \in \mathcal{F}$.

The family $Rat\_M^\infty(A,I)$ of rational subsets of $M^\infty(A,I)$ is the smallest family $\mathcal{F}$ of trace languages verifying (R1),(R2),(RAT) and the following condition

(RAT_ω) if $T \in \mathcal{F}$ then $T^\omega \in \mathcal{F}$.

Recall that the concatenation is only a partial operation on $M^\infty(A,I)$.

The family $sc\_Rat\_M(A,I)$ of *sc-rational* subsets of $M(A,I)$ is the smallest family $\mathcal{F}$ of trace languages verifying (R1),(R2) and the following condition

(sc_RAT) if $T \in \mathcal{F}$ and all traces of T are connected then $T^* \in \mathcal{F}$.

Similarly, the family $sc\_Rat\_M^\infty(A,I)$ of sc-rational subsets of $M^\infty(A,I)$ is the smallest family $\mathcal{F}$ satisfying (R1),(R2),(sc_RAT) and

(sc_RAT_ω) if $T \in \mathcal{F}$ and all traces of T are connected then $T^\omega \in \mathcal{F}$.

Finally, the families $c\_Rat\_M(A,I)$ and $c\_Rat\_M^\infty(A,I)$ of *c-rational* trace languages are defined as the families $Rat\_M(A,I)$ and $Rat\_M^\infty(A,I)$ with (RAT) and (RAT_ω) replaced respectively by the following conditions:

(c_RAT) if $T \in \mathcal{F}$ then $T^{c-*} \in \mathcal{F}$,
(c_RAT_ω) if $T \in \mathcal{F}$ then $T^{c-\omega} \in \mathcal{F}$.

Let us note that the following inclusions obviously hold:

$$sc\_Rat\_M(A,I) \subseteq Rat\_M(A,I)$$

and

$$sc\_Rat\_M(A,I) \subseteq c\_Rat\_M(A,I)$$

Moreover, from Propositions 3.15 and 3.16 and Theorem 4.4, it follows that

$$c\_Rat\_M(A, I) \subseteq Rec\_M(A, I)$$

Ochmański [33] has proved that for each recognizable subset of $M(A, I)$ there exists a rational expression defining this set such that all finite iterations occurring in this expression are applied only to sets that contain exclusively connected traces. Using our notation this fact is expressed by the following inclusion

$$Rec\_M(A, I) \subseteq sc\_Rat\_M(A, I)$$

Let us note that while Ochmański distinguished the family $c\_Rat\_M(A, I)$ he did not introduce explicitly the family of sc-rational sets. Summarizing all these inclusions we get the following result.

Theorem 4.5 ([33])

$$sc\_Rat\_M(A, I) = c\_Rat\_M(A, I) = Rec\_M(A, I) \subseteq Rat\_M(A, I)$$

As it turns out this theorem generalizes to subsets of $M^\infty(A, I)$:

Theorem 4.6

$$sc\_Rat\_M^\infty(A, I) = c\_Rat\_M^\infty(A, I) = Rec\_M^\infty(A, I) \subseteq Rat\_M^\infty(A, I)$$

Proof: Again the inclusions

$$sc\_Rat\_M^\infty(A, I) \subseteq Rat\_M^\infty(A, I)$$

and

$$sc\_Rat\_M^\infty(A, I) \subseteq c\_Rat\_M^\infty(A, I)$$

are obvious. From Propositions 3.15 and 3.16 and Theorem 4.4, we get that

$$c\_Rat\_M^\infty(A, I) \subseteq Rec\_M^\infty(A, I)$$

To accomplish the proof it suffices to show that

$$Rec\_M^\infty(A, I) \subseteq sc\_Rat\_M^\infty(A, I)$$

Let $T \in Rec\_M^\infty(A, I)$ and $T_f = T \cap M(A, I)$, $T_i = T \cap M^\omega(A, I)$. By Theorem 4.5

$$T_f \in Rec\_M(A, I) = sc\_Rat\_M(A, I) \subseteq sc\_Rat\_M^\infty(A, I) \tag{1}$$

By Theorem 3.22, T_i is a finite union of sets of the form $UV_1^\omega \cdots V_k^\omega$, where $U, V_1, \ldots, V_k$ belong to $Rec\_M(A, I)$=$sc\_Rat\_M(A, I)$ and all traces of each of the sets V_i are connected. Therefore

$$T_i \in sc\_Rat\_M^\infty(A, I) \tag{2}$$

as well. These two facts (1) and (2) show that $T \in sc\_Rat\_M^{\infty}(A, I)$. □

Classical Kleene theorem states the equality of recognizable and rational subsets of the free monoid A^*. Similar fact holds also for subsets of A^ω — recognizable and ω-rational subsets of A^ω coincide [37]. The equalities $Rec\_M(A, I) = c\_Rat\_M(A, I)$ and $Rec\_M^{\infty}(A, I) = c\_Rat\_M^{\infty}(A, I)$ can be interpreted as counterparts of the Kleene theorem with finite and infinite iterations replaced by c-iterations.

We end this section with a characterization of rational subsets of $M^{\infty}(A, I)$. Let $T \subseteq M^{\infty}(A, I)$, $\alpha \subseteq A$. Then $P_\alpha(T)$ will denote the set $\{t \in T \mid alph(t) = \alpha\}$. First, we establish two auxiliary lemmas.

Lemma 4.7 *Let $T \in Rat\_M(A, I)$. Then $\forall \alpha \subseteq A, P_\alpha(T) \in Rat\_M(A, I)$.*

Proof: The proof is carried out by structural induction on rational expressions.
(R1) $P_\alpha(\emptyset) = \emptyset \in Rat\_M(A, I)$, $P_{\{a\}}(\{\varphi(a)\}) = \{\varphi(a)\} \in Rat\_M(A, I)$ and if $\alpha \neq \{a\}$ then $P_\alpha(\{\varphi(a)\}) = \emptyset \in Rat\_M(A, I)$.
(R2) Let $T_1, T_2 \subseteq Rat\_M(A, I)$ and $\alpha \subseteq A$. Then

$$P_\alpha(T_1 \cup T_2) = P_\alpha(T_1) \cup P_\alpha(T_2)$$

and

$$P_\alpha(T_1 \cdot T_2) = \bigcup_{\alpha_1 \cup \alpha_2 = \alpha} P_{\alpha_1}(T_1) \cdot P_{\alpha_2}(T_2)$$

the thesis follows from the induction hypothesis for T_1 and T_2.
(RAT) Let $T \subseteq M(A, I)$ and $\alpha \subseteq A$. We set $Q_\alpha(T) = \{t \in T \mid alph(t) \subseteq \alpha\} = \cup_{\beta \subseteq \alpha} P_\beta(T)$. Then

$$P_\alpha(T^*) = \bigcup (Q_\alpha(T))^* \cdot P_{\alpha_1}(T) \cdot (Q_\alpha(T))^* \cdot P_{\alpha_2}(T) \cdot (Q_\alpha(T))^* \cdots P_{\alpha_k}(T) \cdot (Q_\alpha(T))^*$$

where the union is taken over over all sequences $\alpha_1, \ldots, \alpha_k$ of subsets of α such that $\alpha_1 \cup \cdots \cup \alpha_k = \alpha$ and $\alpha_i \neq \alpha_j$ for $i \neq j$ and now the thesis follows from the induction hypothesis for T. □

Lemma 4.8 *Let $S, T \subseteq M(A, I)$. The following formulas hold:*

$$S^\omega \cdot T = \bigcup_{\alpha, \beta, \gamma \subseteq A \text{and } \beta \times \gamma \subseteq I} P_\alpha(S^*) \cdot P_\gamma(T) \cdot \left(P_\beta(S^*)\right)^\omega$$

$$S^\omega \cdot T^\omega = \bigcup_{\alpha, \beta, \gamma, \delta \subseteq A \text{and } \beta \times (\gamma \cup \delta) \subseteq I} P_\alpha(S^*) \cdot P_\gamma(T^*) \cdot \left(P_\beta(S^*) \cdot P_\delta(T^*)\right)^\omega$$

Proof: Observe first that the right hand side is obviously included in the left hand side. Conversely, let $t \in S^\omega \cdot T$, then there exist $r_0, r_1, r_2, \ldots$ in S and s in T such that $t = r \cdot s$ where $r = r_0 r_1 r_2 \ldots$. Let $\alpha = alph(r)$, $\beta = alph_\omega(r)$ and $\gamma = alph(s)$. Since the product $r \cdot s$ is well defined, we have $\beta \times \gamma \subseteq I$. Now there exists an increasing sequence

$i_0, i_1, i_2, \ldots$ of integers such that $alph(r_0 \cdots r_{i_0-1}) = \alpha$ and $alph(r_{i_{k-1}} \cdots r_{i_k-1}) = \beta$ for all $k > 0$. Finally, we obtain

$$t = r \cdot s = \left(r_0 \cdots r_{i_0-1}\right) \cdot s \cdot \left(r_{i_0} \cdots r_{i_1-1}\right) \cdot \left(r_{i_1} \cdots r_{i_2-1}\right) \cdots \in P_\alpha(S^*) \cdot P_\gamma(T) \cdot \left(P_\beta(S^*)\right)^\omega$$

which proves the first formula.
The proof for the second formula is similar. □

Theorem 4.9 *A trace language $T \subseteq M^\infty(A, I)$ is rational if and only if T is a finite union of sets of the form $R \cdot S^\omega$, where $R, S \in Rat\_M(A, I)$.*

Proof: Let $\mathcal{F}$ be the family of languages which are finite union of sets of the form $R \cdot S^\omega$, where $R, S \in Rat\_M(A, I)$. Clearly, the family $\mathcal{F}$ is contained in $Rat\_M^\infty(A, I)$. We shall show that the family $\mathcal{F}$ satisfies the conditions (R1), (R2), (RAT) and (Rat$_\omega$) which proves the converse inclusion.
(R1) More generally, $Rat\_M(A, I) \subseteq \mathcal{F}$ since for $R \in Rat\_M(A, I)$ we have $R = R \cdot \{\mathbf{1}\}^\omega \in \mathcal{F}$.

(R2) The family $\mathcal{F}$ is clearly closed under finite union. The closure under concatenation follows from Lemma 4.7 and 4.8.

(RAT) and (RAT$_\omega$) Let $T \in \mathcal{F}$, there exist an integer n and rational sets $R_1, S_1, \ldots, R_n, S_n \in Rat\_M(A, I)$ such that $T = \cup_{1 \le i \le n} R_i S_i^\omega$. We may assume that, for some $m \in \{0, \ldots, n\}$, we have $\mathbf{1} \in S_i$ if and only if $1 \le i \le m$. As in Lemma 4.2, we set $T_f = T \cap M(A, I)$ and $T_i = T \cap M^\omega(A, I)$. Then we have $T_f = \cup_{1 \le i \le m} R_i S_i^*$ and $T_i = \cup_{1 \le i \le n} R_i (S_i \setminus \{\mathbf{1}\})^\omega$. Therefore, $T_f \in Rat\_M(A, I)$ and we obtain $T_f^*, T_f^\omega, T_i \in \mathcal{F}$. Using Lemma 4.2 and the closure of $\mathcal{F}$ under finite union and concatenation we deduce that T^* and T^ω are in $\mathcal{F}$.

□

As a direct consequence of this normal form, we obtain a tight link between rational word languages and rational trace languages. Note that this generalizes the corresponding well-known result for finite traces.

Corollary 4.10 *A language $T \subseteq M^\infty(A, I)$ is rational if and only if there exists a rational language $L \subseteq A^\infty$ such that $T = \varphi(L)$.*

5 Conclusion

In this paper, we have investigated properties of the families of rational and recognizable sets of infinite traces.

A normal form for the rational languages of infinite traces (Theorem 4.9) has been given.

Several characterizations of the family of recognizable languages of infinite traces have also been proposed. Extending the Schützenberger product to the diamond product, we have proved that the families of recognizable languages and weakly recognizable languages coincide (Theorem 3.13). Then, we have shown that the infinite iteration T^ω of a recognizable language T of finite traces is recognizable if its finite iteration T^* is recognizable (Corollary 3.19). This fact has several interesting consequences. First, a kind of normal form theorem (Theorem 3.22) for recognizable languages of infinite traces is derived from this result. Second, Ochmański's theorem on finite traces and Büchi's theorem on infinite words are extended to infinite traces. More precisely, we prove that the families of recognizable and c-rational languages of infinite traces are equal (Theorem 4.6).

We would like to point out some recent advances in the theory. Using in particular one of our main results (Theorem 4.6), a characterization of recognizable languages of infinite traces as languages definable in monadic second order logic is proposed in [14]. In [21] finite state asynchronous automata, initially constructed to work on finite traces [8, 39], are adapted to infinite traces and it is proved that they accept exactly all recognizable sets of infinite traces. Note however that the automata of [21] generalize Büchi nondeterministic automata on infinite words. The problem of how to construct for infinite trace languages a suitable counterpart of Muller deterministic automata has been solved recently in [13].

Finally, in [12] recognizability questions are addressed in the framework of complex traces. Some results presented in our paper are basic in that theory.

References

[1] I.J. Aalbersberg and G. Rozenberg. Theory of traces. *Theoret. Comput. Sci.*, 60:1–82, 1988.

[2] A. Arnold. A syntactic congruence for rational ω-languages. *Theoret. Comput. Sci.*, 39:333–335, 1985.

[3] E. Best and R. Devillers. Sequential and concurrent behaviour in Petri net theory. *Theoret. Comput. Sci.*, 55:87–136, 1987.

[4] P. Bonizzoni, G. Mauri, and G. Pighizzini. About infinite traces. In *Proceedings of the ASMICS workshop on Free Partially Commutative Monoids*, Tech. Rep. TUM-I9002, pages 1–10. Technical University of Munich (Germany), 1990.

[5] R. Büchi. On a decision method in restricted second order arithmetic. In *Proceedings of the International Congress on Logic, Methodology and Philosophy*, pages 1–11. Standford University Press, 1962.

[6] P. Cartier and D. Foata. *Problèmes combinatoires de commutation et réarrangements.* Lecture Notes in Mathematics 85. Springer Verlag, 1969.

[7] M. Clerbout and M. Latteux. Semi-Commutations. *Information and Computation*, 73:59–74, 1987.

[8] R. Cori, Y. Métivier, and W. Zielonka. Asynchronous mappings and asynchronous cellular automata. Tech. Rep. LaBRI 89-97, Université Bordeaux I (France), 1990. to appear in *Information and Computation.*

[9] R. Cori and D. Perrin. Automates et commutations partielles. *R.A.I.R.O.-Informatique Théorique et Applications*, 19:21–32, 1985.

[10] V. Diekert. *Combinatorics on Traces.* Lecture Notes in Computer Science 454. Springer Verlag, 1990.

[11] V. Diekert. On the concatenation of infinite traces. In *Proceedings of the 8th Annual Symposium on Theoretical Aspects of Computer Science (STACS'91)*, Lecture Notes in Computer Science 480, pages 105–117. Springer Verlag, 1991.

[12] V. Diekert, P. Gastin, and A. Petit. Recognizable complex trace languages. In *Proceedings of the 16th Symposium on Mathematical Foundations of Computer Science (MFCS'91)*, Lecture Notes in Computer Science 520, pages 131–140. Springer Verlag, 1991.

[13] V. Diekert and A. Muscholl. Deterministic asynchronous automata for infinite traces. In *Proceedings of the 10th Annual Symposium on Theoretical Aspects of Computer Science (STACS'93)*, Lecture Notes in Computer Science ???, pages ???–??? Springer Verlag, 1992.

[14] W. Ebinger. On logical definability of ω-trace languages. In *Proceedings of the ASMICS workshop on Infinite Traces*, Tech. Rep. University of Stuttgart (Germany), 1992.

[15] S. Eilenberg. *Automata, Languages and Machines*, volume A. Academic Press, New York, 1974.

[16] M.P. Flé and G. Roucairol. Maximal serializability of iterated transactions. *Theoret. Comput. Sci.*, 38:1–16, 1985.

[17] M. Fliess. Matrices de Hankel. *J. Math. Pures et Appl.*, 53:197–224, 1974.

[18] P. Gastin. *Un modèle distribué.* PhD thesis, LITP, Université Paris 7 (France), 1987.

[19] P. Gastin. Infinite traces. In *Proceedings of the Spring School of Theoretical Computer Science on Semantics of Systems of Concurrent Processes*, Lecture Notes in Computer Science 469, pages 277–308. Springer Verlag, 1990.

[20] P. Gastin. Recognizable and rational languages of finite and infinite traces. In *Proceedings of the 8th Annual Symposium on Theoretical Aspects of Computer Science (STACS'91)*, Lecture Notes in Computer Science 480, pages 89–104. Springer Verlag, 1991.

[21] P. Gastin and A. Petit. Asynchronous cellular automata for infinite traces. In *Proceedings of the 19th International Colloquium on Automata, Languages and Programming (ICALP'92)*, Lecture Notes in Computer Science. Springer Verlag, 1992. Also available as Tech. Rep. LITP 91-68, Université Paris 6 (France), 1991.

[22] P. Gastin, A. Petit, and W. Zielonka. A Kleene theorem for infinite trace languages. In *Proceedings of the 18th International Colloquium on Automata, Languages and Programming (ICALP'91)*, Lecture Notes in Computer Science 510, pages 254–266. Springer Verlag, 1991.

[23] P. Gastin and B. Rozoy. The poset of infinitary traces. Tech. Rep. LITP 91.24, Université Paris 6 (France), 1991. To appear in *Theoret. Comput. Sci.*

[24] R.L. Graham, B.L. Rothschild, and J.H. Spencer. *Ramsey Theory.* John Wiley & Sons, New York, 1980.

[25] K. Hashigushi. Recognizable closures and submonoids of free partially commutative monoids. *Theoret. Comput. Sci.*, 86:233–241, 1991.

[26] M. Kwiatkowska. *Fairness for non-interleaving concurrency.* PhD thesis, University of Leicester (England), 1989.

[27] M. Kwiatkowska. On the domain of traces and sequential composition. In *Proceedings of the 16th Colloquium on Trees in Algebra and Programming (CAAP'91)*, Lecture Notes in Computer Science 493, pages 42–56. Springer Verlag, 1991.

[28] A. Mazurkiewicz. Concurrent program schemes and their interpretations. Tech. rep. DAIMI PB 78, Aarhus University, 1977.

[29] A. Mazurkiewicz. Traces, histories, graphs: Instances of a process monoid. In *Proceeding of the 11th Symposium on Mathematical Foundations of Computer Science (MFCS'84)*, Lecture Notes in Computer Science 176, pages 115–133. Springer Verlag, 1984.

[30] A. Mazurkiewicz. Trace theory. In *Advances in Petri Nets'86*, Lecture Notes in Computer Science 255, pages 279–324. Springer Verlag, 1987.

[31] Y. Métivier. On recognizable subsets of free partially commutative monoids. *Theoret. Comput. Sci.*, 58:201–208, 1988.

[32] Y. Métivier and B. Rozoy. On the star operation in free partially commutative monoids. *International Journal of Foundations of Computer Science*, 2(3):257–265, 1991.

[33] E. Ochmański. Regular behaviour of concurrent systems. *Bulletin of the European Association for Theoretical Computer Science (EATCS)*, 27:56–67, Oct 1985.

[34] E. Ochmański. Notes on a star mystery. *Bulletin of the European Association for Theoretical Computer Science (EATCS)*, 40:252–257, Feb 1990.

[35] D. Perrin. An introduction to finite automata on infinite words. In *Proceedings of the Spring School of Theoretical Computer Science on Automata on Infinite Words*, Lecture Notes in Computer Science 192, pages 2–17. Springer Verlag, 1984.

[36] D. Perrin. Partial commutations. In *Proceedings of the 16th International Colloquium on Automata, Languages and Programming (ICALP'89)*, Lecture Notes in Computer Science 372, pages 637–651. Springer Verlag, 1989.

[37] D. Perrin and J.E. Pin. Mots Infinis. Tech. Rep., LITP 91.06, Université Paris 7 (France), 1991.

[38] M.P. Schützenberger. A propos des relations rationnelles fonctionnelles. In *Automata, Languages and Programming*, pages 103–114. North-Holland, Amsterdam, 1972.

[39] W. Zielonka. Notes on finite asynchronous automata. *R.A.I.R.O.-Informatique Théorique et Applications*, 21:99–135, 1987.