Beta - conjugates of algebraic numbers

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Motivation

What is a beta-conjugate ?

The present work takes its origin in :
1) Boyd ’96 : introduction of

“beta-conjugates",

2) Boyd ’77 and Bertin/Boyd ’95 : introduction of

a second variable “t"

to obtain new families of polynomials $Q(z,t)$; Generalization of Salem’s construction of Salem polynomials.

in 2) : no dynamics.

Boyd '96

Assume β is a Parry number, with preperiod m and period p,

$$P_n(x) := x^n - c_1 x^{n-1} - \ldots - c_n$$

where

$$T_\beta^n(1) = \beta^n - c_1 \beta^{n-1} - \ldots - c_n.$$

Then β satisfies the polynomial equation $R(\beta) = 0$, where

$R(x) = P_{m+p}(x) - P_m(x)$ if $m > 0$,
and $R(x) = P_p(x)$ if $m = 0$.

(Characteristic polynomial of the beta-number β in Parry '60)

The zeros of $R(x)$ lie in Solomyak's fractal Ω and are of modulus $\leq \frac{1+\sqrt{5}}{2}$.
Zeros of this polynomial, which are not Galois conjugates of β, are called beta-conjugates.

Boyd ’77

Theorem (Salem ’45)

Every P.V. number β is a limit point of numbers of the class (T) on both sides.

with Salem’s construction : if β is not a quadratic Pisot unit, $P_\beta(X)$ its minimal polynomial, consider

$$Q_m(z) := z^m P_\beta(z) + P_\beta^*(z)$$

or

$$W_m(z) = (z^m P_\beta(z) - P_\beta^*(z))/(X - 1)$$

$\rightarrow$ cancel at Salem numbers when m large enough. Not irreducible in general : are = a Salem polynomial $\times$ a product of cyclotomic polynomials.

Boyd's new families : m large enough, $\epsilon = \pm 1$, t varying

$$Q_m(z,t) := z^m P_\beta(z) + \epsilon \, t \, P^*_\beta(z)$$

$$\in \mathbb{Q}[z,t].$$

“it is profitable to add a new variable..." : branches, varying collection of zeros, lying within the open unit disc, on $|z| = 1$, ...

$m = 1, q \geq 2.$

Theorem (Bertin Boyd '95)

Let τ be a Salem number with minimal polynomial P_τ. Then τ is such that $\epsilon = -\mathrm{sgn} P_\beta(0)$ and $|P_\beta(0)| = q$ if and only if there is a cyclotomic polynomial K with simple roots and $K(1) \neq 0$ and a reciprocal polynomial L with the following properties :

(a) $L(0) = q - 1$,

(b) $deg(L) = deg(KP_\tau) - 1$,

(c) $L(1) \geq -K(1)P_\tau(1)$,

(d) L has all its zeros on $|z| = 1$ and they interlace the zeros of KP_τ on $|z| = 1$ in the following sense : let $e^{i\psi_1}, \ldots, e^{i\psi_r}$ be the zeros of L with $Imz \geq 0$, excluding $z = -1$, with $0 < \psi_1 < \ldots < \psi_r < \pi$, and let $e^{i\phi_1}, \ldots, e^{i\phi_r}$ be the zeros of KP_τ on $|z| = 1$, $Imz \geq 0$, with $0 < \phi_1 < \ldots < \phi_r \leq \pi$, then

$$0 < \psi_1 < \phi_1 < \ldots < \psi_r < \phi_r.$$

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Problem Reformulation

Let $\beta > 1$. The Rényi β-expansion of 1

$$d_\beta(1) = 0.t_1 t_2 t_3 \ldots \qquad \text{and corresponds to} \qquad 1 = \sum_{i=1}^{+\infty} t_i \beta^{-i} \, ,$$

$t_1 = \lfloor \beta \rfloor, t_2 = \lfloor \beta \{\beta\} \rfloor = \lfloor \beta T_\beta(1) \rfloor, t_3 = \lfloor \beta\{\beta\{\beta\}\}\rfloor = \lfloor \beta T_\beta^2(1) \rfloor, \ldots$ The digits t_i belong to $\mathcal{A}_\beta := \{0, 1, 2, \ldots, \lceil \beta - 1 \rceil\}$.

Beta-transformation : $T_\beta : [0,1] \rightarrow [0,1], \; x \rightarrow \{\beta x\}$.

Iterates : $T_\beta^0 =$Id, $\quad T_\beta^j = T_\beta(T_\beta^{j-1}), \quad j \geq 0$.

Parry number : $\beta > 1$ for which $d_\beta(1)$ eventually periodic.

For any $\beta > 1$, define the Parry upper function :

$$f_\beta(z) := -1 + \sum_{i=1}^{+\infty} t_i z^i ,$$

Since $f_\beta(z)$ is a rational fraction if and only if β is a Parry number (Szegő-Carlson-Polya Theorem), the meromorphic function $f_\beta(z)$ admits, as domain of definition $\mathcal{D}_\beta$, either

$$\mathbb{C} \qquad \text{for a Parry number,}$$

or

$$\mathbb{D} = \{z \mid |z| < 1\} \qquad \text{for a nonParry number}$$

$$(|z| = 1 \text{ as natural boundary}).$$

Definition : Let $\beta > 1$ be an algebraic number. A beta-conjugate of β, say ω, is a complex number which satisfies

$$f_\beta(\omega^{-1}) \;=\; 0$$

with $\omega^{-1} \in \mathcal{D}_\beta$ s.t. $\omega \neq$ a Galois conjugate of β (Recall : $f_\beta(\frac{1}{\beta}) = 0$).

Claim : this definition is equivalent to that of Boyd ’96 if β is a Parry number.

Objective : for $\beta > 1$ any algebraic number, characterize

$$\text{all zeros of } f_\beta(z) := -1 + \sum_{i=1}^{+\infty} t_i z^i$$

in its domain of definition $\mathcal{D}_\beta$.

Problem : express canonically $f_\beta(z)$ as a product, so that the zeros exactly arise from the factors of this product : expected by analogy with the Riemann zeta function developped into Euler product - recall that, if β is a nonsimple Parry number, a Theorem of Ito and Takahashi shows that $f_\beta(z) = -1/\zeta_\beta(z)$, where $\zeta_\beta(z)$ is the Artin-Mazur zeta function of the beta-transformation T_β.

Does this theory, analog to Euler products, exist ? How ?

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Fractional power series - Puiseux Expansions

Let $g \in \mathbb{C}[[x, y]]$. We are interested in solving for x the equation

$$g(x, y) = 0.$$

This question goes back to Newton. This means that we want to find some sort of series in y, say $x(y)$, such that

$$g(x(y), y) = 0,$$

$g(x(y), y)$ being the series in y obtained by substituting $x(y)$ for x in g. We need to deal with series in fractionary powers of y.

Denote $\mathbb{C}((x))$ the field of the formal Laurent series

$$\sum_{i=d}^{\infty} a_i x^i, \qquad d \in \mathbb{Z}, a_i \in \mathbb{C}.$$

An element of $\mathbb{C}((x^{1/n}))$ has the form

$$s = \sum_{i \geq r} a_i x^{i/n}.$$

The field of fractionary power series is denoted by $\mathbb{C} << x >>$ and by definition is the direct limit of the system

$$\left\{ \mathbb{C}((x^{1/n})), \phi_{n,n'} \right\},$$

where, for n dividing n' (with $n' = dn$),

$$\phi_{n,n'} : \mathbb{C}((x^{1/n})) \to \mathbb{C}((x^{1/n'}))$$

maps

$$\sum a_i x^{i/n} \text{ to } \sum a_i x^{di/dn'}.$$

A Puiseux series is by definition a fractionary power series

$$s = \sum_{i \geq r} a_i x^{i/n}$$

for which the order in x

$$o_x(s) := \frac{\min\{i \mid a_i \neq 0\}}{n}$$

is > 0. Choice : n and $\gcd\{i \mid a_i \neq 0\}$ have no common factor. Then n is called the ramification index, or the polydromy order, of s.

If $s \in \mathbb{C}((x^{1/n}))$ is a Puiseux series, the series $\sigma_\epsilon(s), \epsilon^n = 1$, will be called the conjugates of s. Then

$$\sigma_\epsilon(s) = \sum_{i \geq r} \epsilon^i a_i z^{i/n}.$$

The set of all conjugates of s is called the conjugacy class of s. The number of different conjugates of s is denoted by $\nu(s)$.

Newton polygon of g : let

$$g = \sum_{\alpha > 0, j > 0} A_{\alpha,j} x^{\alpha} y^{j} \qquad \in \mathbb{C}[[x, y]]$$

and obtain the discrete set of points with nonnegative integral coefficients

$$\Delta(g) := \{(\alpha, j) \mid A_{\alpha,j} \neq 0\},$$

called the Newton diagram of g. Take

$$\Delta'(g) := \Delta(g) + (\mathbb{R}^{+})^{2}.$$

Then the convex hull of $\Delta'(g)$ admits a border which is composed of two half-lines (vertical/ horizontal, coordinate axes) and a polygonal line, called the Newton polygon of g, joining them (denoted $N(g)$).

The height $h(N(g))$ of g is by definition the maximal ordinate of the vertices of the Newton polygon $N(g)$.

A branch of s is the set of Puiseux series which compose a given conjugacy class of s.

If y is a Puiseux series, write $g_y = \prod_{i=1}^{\nu(y)}(X - y_i(Y))$, the $y_i, i = 1, \ldots, \nu(y)$ being the conjugates of y.

Theorem

For any $g(X, Y) \in \mathbb{C}[[X, Y]]$,

(i) *there are Puiseux series $y_1, y_2, \ldots, y_m, m \geq 0$, so that g decomposes in the form*

$$g = u\, Y^r\, g_{y_1}\, g_{y_2} \ldots g_{y_m}$$

where $r \in \mathbb{N}$, and u is an invertible series in $\mathbb{C}[[X, Y]]$,

(ii) *the height of the Newton polygon of g is*

$$h(N(g)) = \nu(y_1) + \nu(y_2) + \ldots + \nu(y_m)$$

and the X-roots of g are the conjugates of the $y_j(Y), j = 1, \ldots, m$.

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Beta-conjugates as Puiseux series

Let $\mathbb{K}$ be a commutative field and $g(X, Y) \neq 0$ an element of $\mathbb{K}[[X, Y]]$ such that $g(0, 0) = 0$.

Definition

A parametrization of g is a couple $[\alpha(T), \gamma(T)]$ of elements of $\mathbb{K}[[T]]$ which satisfies

(i) α and γ are not simultaneously identically zero,

(ii) $\alpha(0) = \gamma(0) = 0$,

(iii) $g(\alpha(T), \gamma(T)) = 0 \in \mathbb{K}[[T]]$.

Let $\beta > 1$ be an algebraic number.

key observation : the three functions

$$z - 1/\beta$$

$$P^*_\beta(z)$$

$$f_\beta(z)$$

cancel at $1/\beta \in (0, 1) \in \mathbb{D}$.

Change our vision into a view of a germ of analytical function over a surface, and decompose it according to Puiseux.

Let $\beta > 1$ be an algebraic number, and $\mathbb{K}_\beta$ the Galois closure of $\mathbb{Q}(\beta)$. Here the parametrization

$$[X - \frac{1}{\beta}, P^*_\beta(X)]$$

is fixed.

Origin in $\mathbb{C}^2$, for the germ :

$$(1/\beta, 0) \text{ in } \mathbb{C}^2.$$

The class

$$\left\{ g(X,Y) \in \mathbb{K}_\beta[[X,Y]] \mid g(X - \frac{1}{\beta}, P^*_\beta(X)) = f_\beta(X) \right\}$$

is not empty (by identification of coefficients).

Rk : sufficient to consider a representant g of this class in

$$\mathbb{K}_\beta[[Y]][X]$$

with $\deg_X(g) <$deg(β) (the Euclidean division of $(X - \frac{1}{\beta})^k$, $k >$ deg(β), by $P^*_\beta(X)$ provides a remainder of degree less than deg(β)).

Since

$$\mathbb{K}_\beta[[X, Y]] \subset \mathbb{C}[[X, Y]]$$

decompose g according to Puiseux's Theorem :

$$g = u\, y^r\, g_{y_1}\, g_{y_2} \ldots g_{y_m}$$

with $g_{y_j} = \prod_{i=1}^{\nu(y_j)} (X - y_{i,j}(Y))$, the $y_{i,j},\ i = 1, \ldots, \nu(y_j)$ being the conjugates of y_j.

Then $\quad g(X - \frac{1}{\beta}, P^*_\beta(X)) =$

$$u\,(P^*_\beta(X))^r \prod_{i=1}^{\nu(y_1)} (X - \frac{1}{\beta} - y_{i,1}(P^*_\beta(X))) \ldots \prod_{i=1}^{\nu(y_m)} (X - \frac{1}{\beta} - y_{i,m}(P^*_\beta(X))) =$$

$$f_\beta(X) = -1 + \sum_{j \geq 1} t_j X^j.$$

Rk : (i) $f_\beta(\frac{1}{\beta}) = 0$ and $f'_\beta(\frac{1}{\beta}) > 0$ imply : $r = 0$ or $r = 1$,
(ii) this identity provides the exhaustive list of zeros of $f_\beta(z)$, and an alternate definition of beta-conjugates.

Definition :
A beta-conjugate of β is a complex number ω which satisfies

$$\omega = \frac{1}{\beta} + \sum_{i \geq 0} a_i (P_\beta^*(\omega))^{i/n}$$

for all Puiseux series deduced computed from g, i.e. from f_β.

Rationality questions

(i) Computation of the branches $y_{i,j}(X)$ from $g(X, Y)$ using the Newton-Puiseux algorithm (from the Newton polygon of g).

D. Duval "Rational Puiseux expansions" : Puiseux expansions have coefficients in $\mathbb{K}_\beta$.

(ii) Computation of the ramification indices (polydromy orders) : integers ≥ 1.

Factorization of the Parry polynomial

If β is a Parry number, with $p =$ period in $d_\beta(1)$ if β is not simple,

$$f_\beta(X) = -1 + \sum_{j \geq 1} t_j X^j \qquad \text{is a rational fraction}$$

written

$$f_\beta(X) = -\frac{1}{1 - X^p} P^*_{p,\beta}(X)$$

where $P_{p,\beta}(z)$ is the Parry polynomial of β (characteristic polynomial of the beta-number β, in Parry '60).
If β is simple, $f_\beta(X)$ is the polynomial formed from the preperiod in d_β.

We will make reference to

$$x(y) = \sum_{k=0}^{\infty} a_k \left(y^{1/e}\right)^k$$

which is one of the Puiseux series given above.
Let ζ_e denote a primitive e-th root of unity. The branch of the series $x(y)$ is the set of series

$$B(x(y)) = \left\{ \sum_{k \geq 0} a_k \left(\zeta_e^j y^{1/e}\right)^k \mid j = 0, 1, \ldots, e-1 \right\}.$$

$B(x(y))$ contains precisely e distinct series. Let $L = \mathbb{Q}(a_0, a_1, a_2, \ldots), s = [L : \mathbb{Q}]$ and $\sigma_1, \sigma_2, \ldots, \sigma_v$ the v embeddings of L into $\overline{\mathbb{Q}}$. We have : $L = \mathbb{K}_\beta$ and $s = \deg(\beta)$.

The conjugacy class of $x(y)$ is

$$C(x(y)) = \left\{ \sum_{k \geq 0} \sigma_i(a_k) \left(\zeta_e^j y^{1/e}\right)^k \;\middle|\; i = 1, \ldots, v\,,\; j = 0, 1, \ldots, e-1 \right\}$$

Theorem (Walsh '94)

(i) The product $\prod_{B(x(y))}(x - x_i(y))$ *is irreducible in* $\overline{\mathbb{Q}}((y))[x]$*, of degree* e *in* x*,*
(ii) the product $\prod_{C(x(y))}(x - x_i(y))$ *is irreducible in* $\mathbb{Q}((y))[x]$ *of degree* $e(s/s_0)$ *in* x*, where*
$s_0 := \#\{\sigma : L \to \overline{\mathbb{Q}}; \exists t \in \mathbb{Z}$ *such that* $\sigma(a_k) = a_k \zeta_e^{tk}$ *for all* $k \geq 0\}$.

Substitute : x by $x - 1/\beta$, and y by $P_\beta^*(x)$ provides, by (ii), a polynomial in $\mathbb{K}_\beta[x]$, irreducible.

The irreducible factors which contain the beta-conjugates of β, in the factorization of the reciprocal of the Parry polynomial $P^*_{p,\beta}$, are irreducible over $\mathbb{K}_\beta$.

Then, the identification of these factors over $\mathbb{K}_\beta[x]$ provides a geometrical origin to these factors, by Walsh '94.

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Smallest Salem number known. Lehmer's polynomial (Lehmer '33) :

$$L(X) = X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1.$$

$\beta = 1.17628\ldots$, dominant root of $L(X)$, is a Parry number. We have :

$$d_\beta(1) = 0.10^{10}10^{18}(10^{12}10^{18}10^{22}10^{18})^\omega.$$

The Parry polynomial of β is

$$P_{p,\beta}(X) = L(X) \times [R(X) \times \Phi_2(X)\Phi_4(X)\Phi_{12}(X)\Phi_{22}(X)],$$

where $R(X)$ is a reciprocal polynomial of degree 48 of height 3. The Parry polynomial of β is of degree 75 and its height is 1.

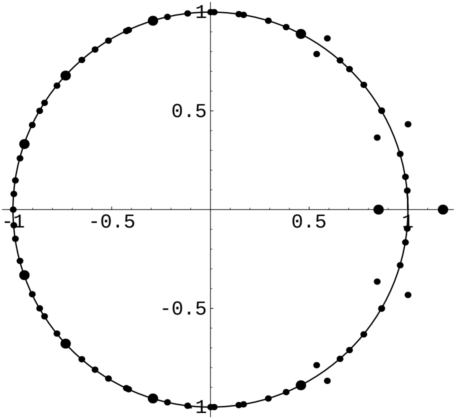

FIG.: Galois conjugates (big bullets) and beta-conjugates (small bullets) of Lehmer's number $\beta = 1.17628\ldots$, smallest Salem number known.

In the factorization of $P_{p,\beta}(X)$ the factors $R, \Phi_2, \Phi_4, \Phi_{12}, \Phi_{22}$ are irreducible over $\mathbb{Q}$, then over $\mathbb{K}_\beta$ (Galois closure of $\mathbb{Q}(\beta)$).

See them as polynomials in

$$\mathbb{K}_\beta[X].$$

Now, for the conjugacy class $C(x(y))$, Walsh'94 Theorem implies

$$\prod_{C(x(y))} (X - 1/\beta - x_i(P_\beta^*(X))) \in \mathbb{K}_\beta[X]$$

is irreducible in $\mathbb{Q}((P_\beta^*(X)))[X] = \mathbb{K}_\beta[X]$.

2 origins : identifying the irreducible factors in $\mathbb{K}_\beta[X]$ provides

- one irreducible factor in $P_{p,\beta}$ exactly arises from one conjugacy class of Puiseux series relative to the germ at $(1/\beta, 0)$, and its roots are the beta-conjugates relative to this conjugacy class,
- the number of factors of $P_{p,\beta}$, except P_β, is exactly the number of conjugacy classes of Puiseux expansions in the germ,
- the branches originate at $(1/\beta, 0)$ and stem in spiral close or over the unit circle ; each time they cross the complex plane, the junction is a beta-conjugate. What is their radius of convergence ? Do they intersect ?

Lehmer's number case : 5 classes except the Galois orbit of $1/\beta$ by the Galois group of $\mathbb{K}_\beta/\mathbb{Q}$.