Medietic numeration systems

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Arithmetic mean

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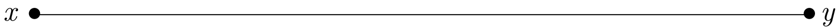

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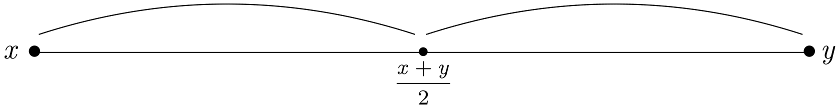

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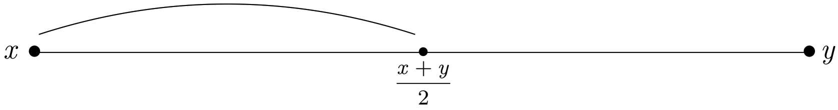

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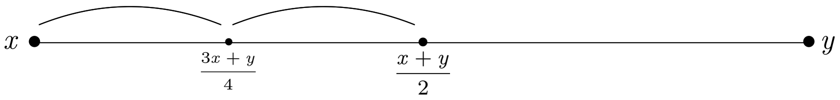

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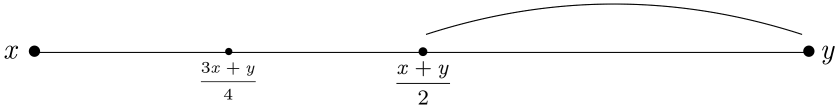

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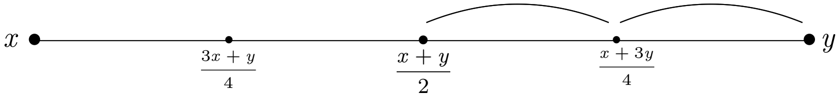

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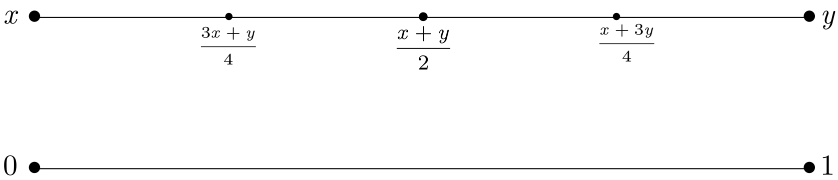

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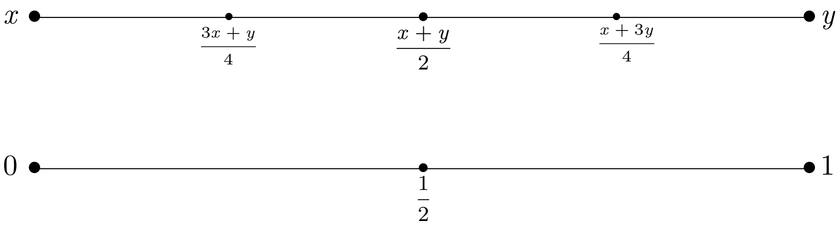

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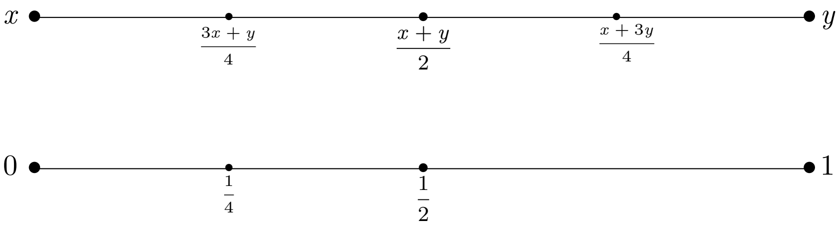

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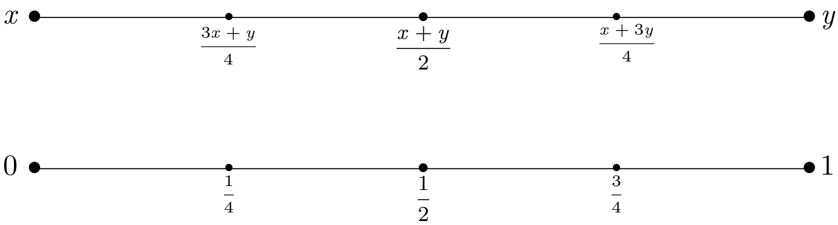

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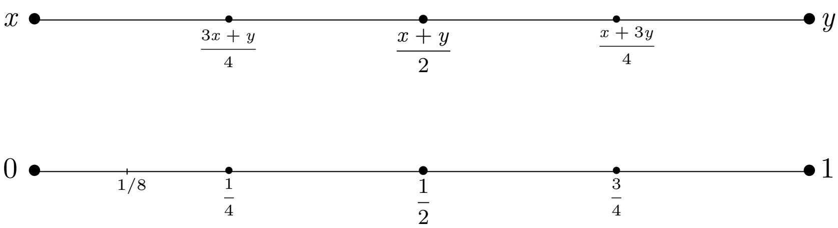

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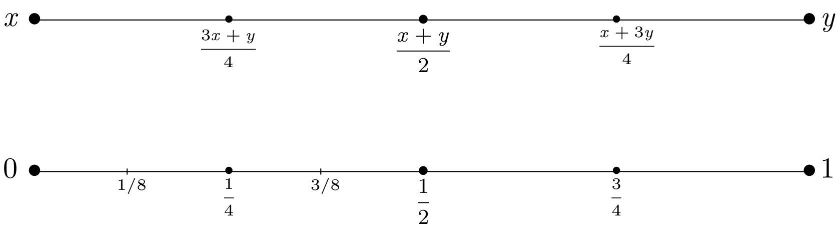

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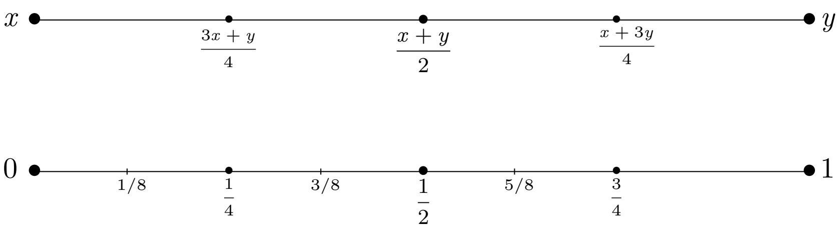

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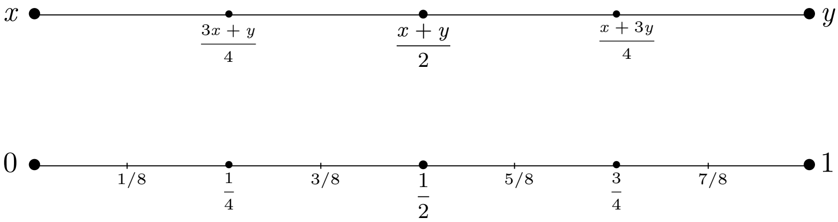

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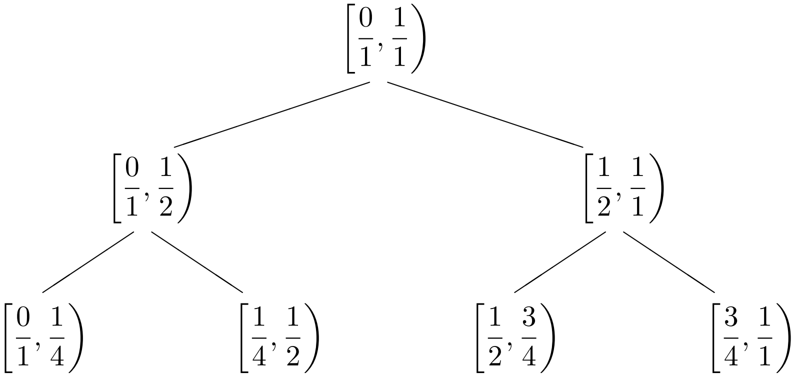

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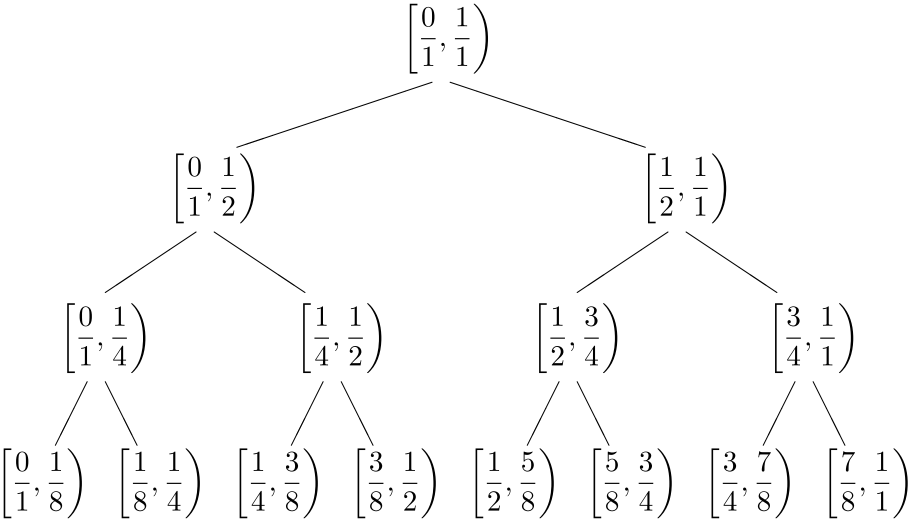

Arithmetic mean

Theorem

The codage of $x \in [0, 1)$ by the dichotomy algorithm gives an analytic expression of x, with the following rule: putting $x_n = 0$ (resp. $x_n = 1$) when the left subinterval is chosen at the n-th step, we have

$$x = \sum_{n \geq 1} \frac{x_n}{2^n}.$$

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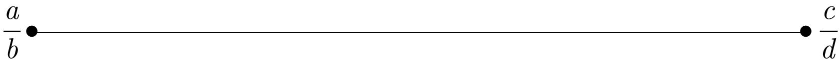

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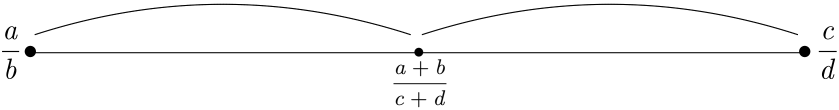

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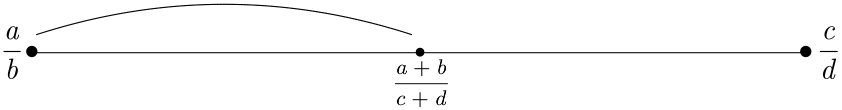

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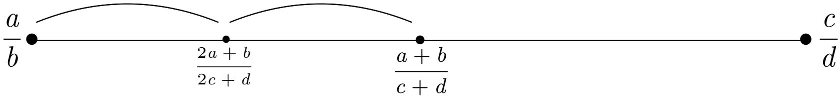

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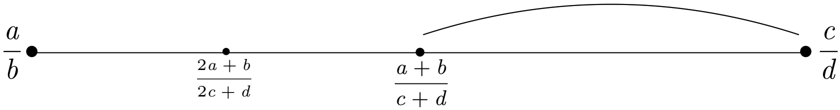

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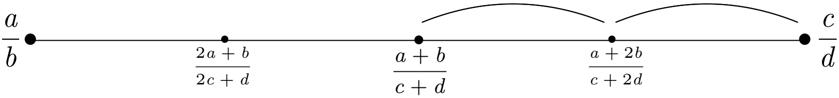

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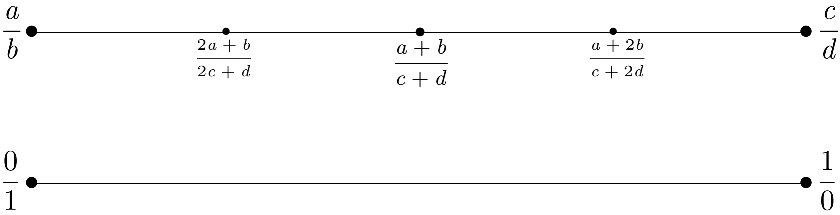

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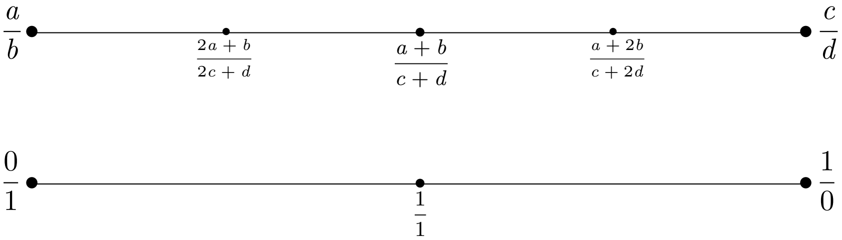

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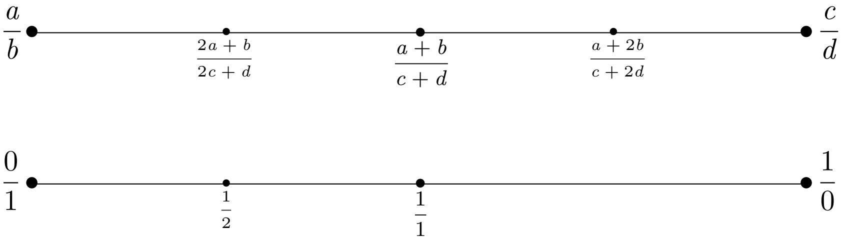

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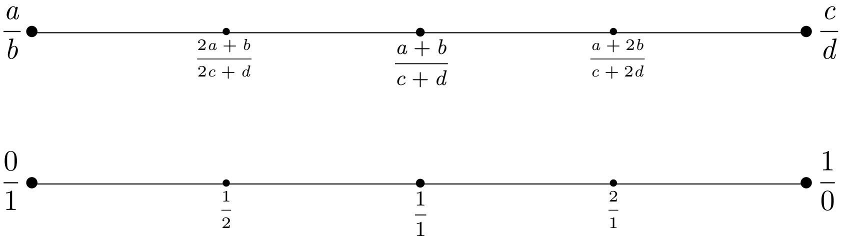

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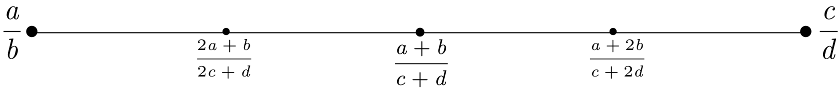

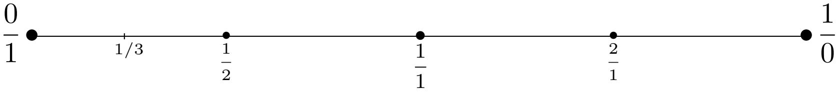

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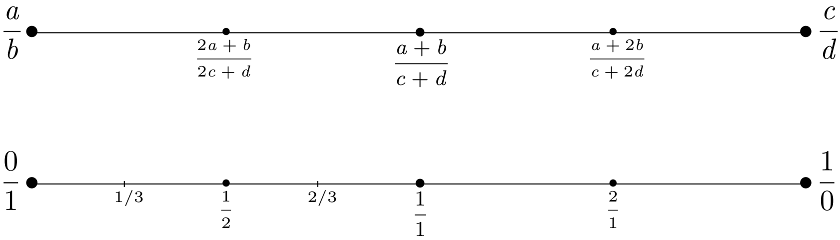

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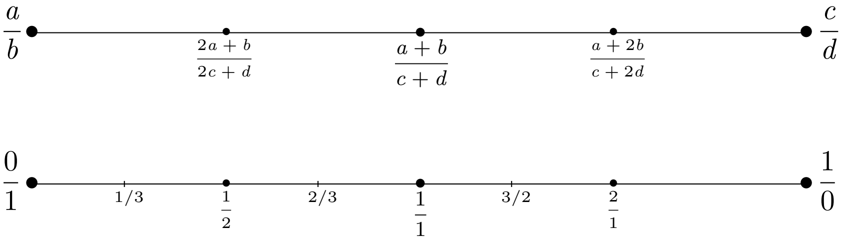

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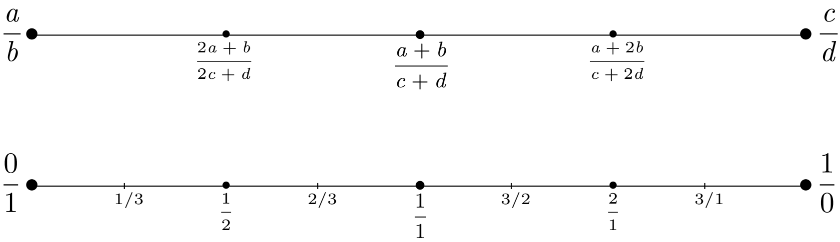

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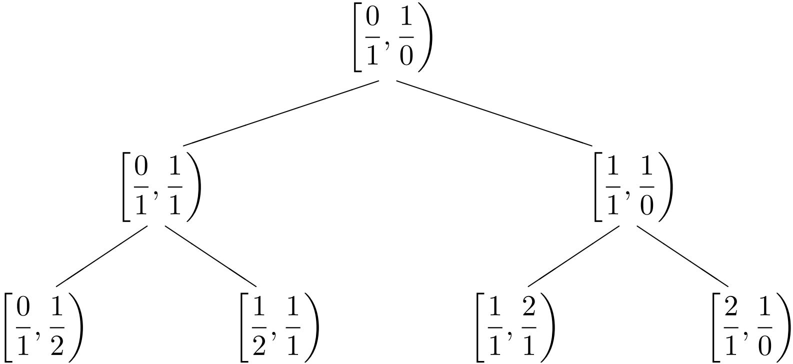
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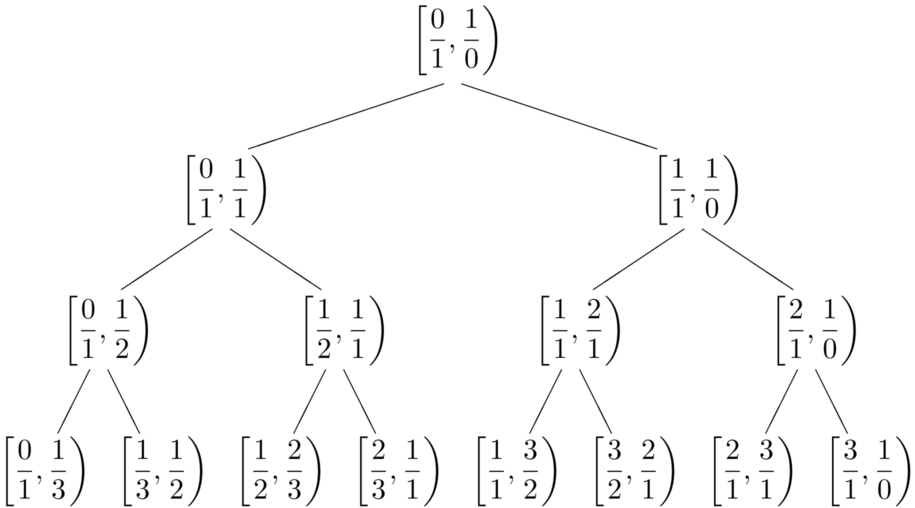
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Theorem

The codage of $x \in [0,1)$ by the Stern-Brocot algorithm gives an analytic expression of x, with the following rule: denoting by $R^{a_0} L^{a_1} R^{a_2} L^{a_3} \ldots$ the codage of x, where $a_0 \geq 0$ and $a_n > 0$ for $n > 0$, we have

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}.$$

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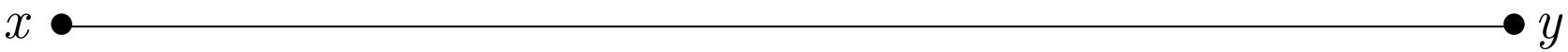

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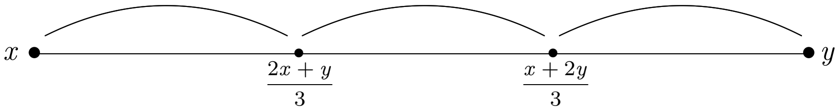

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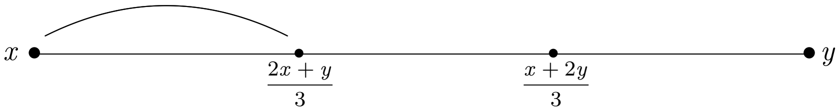

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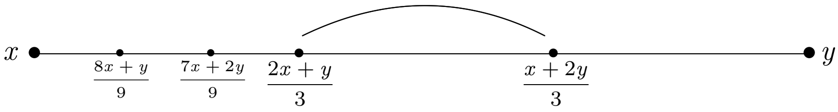

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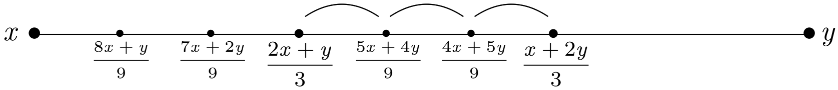

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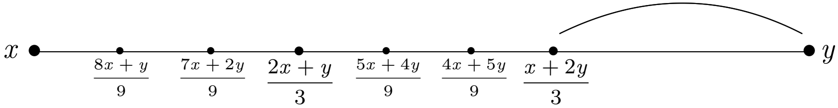

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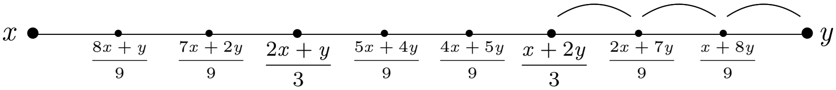

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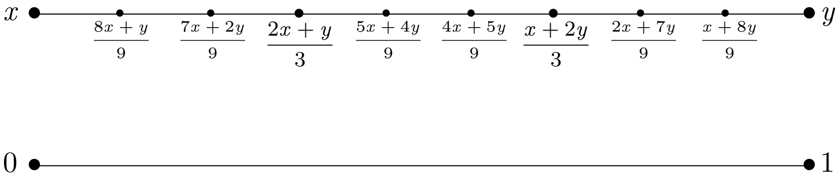

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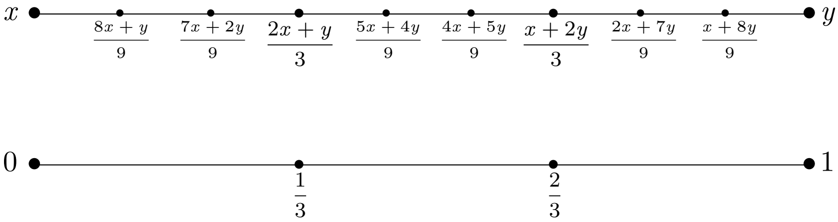

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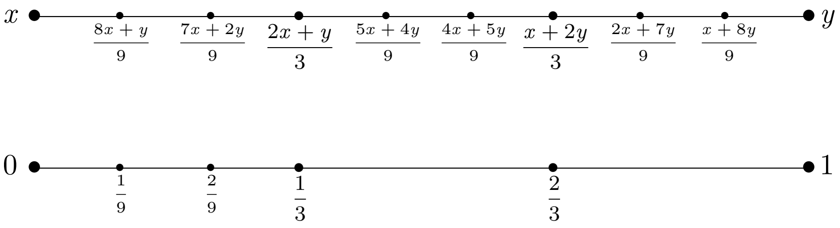

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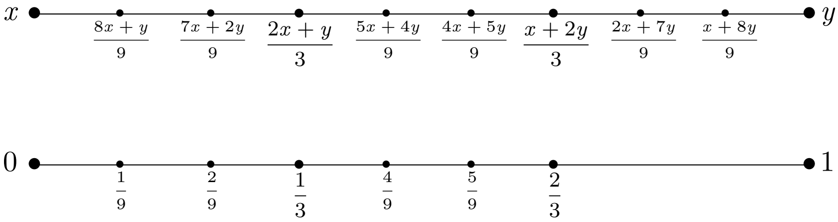

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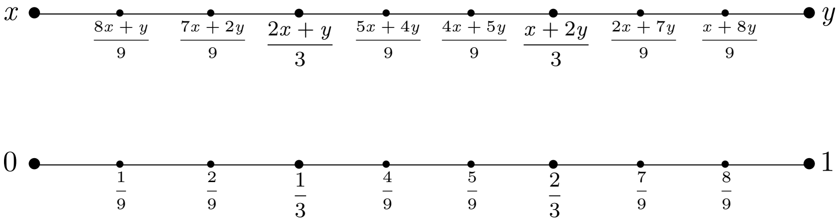

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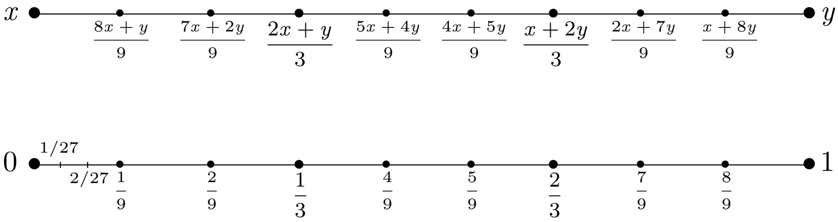

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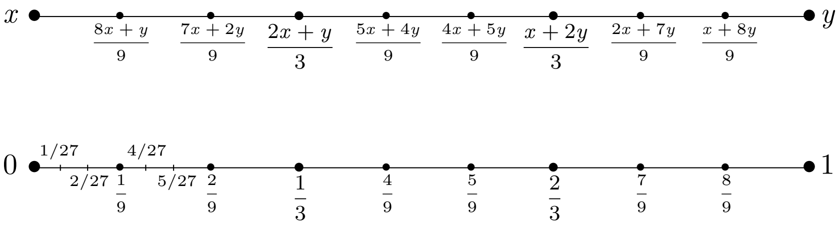

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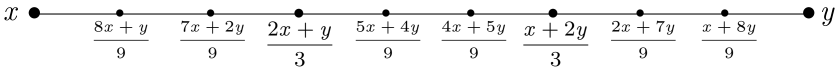
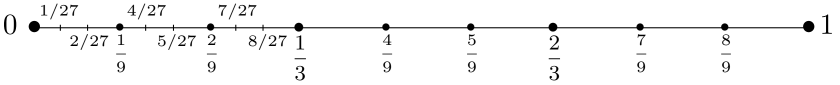
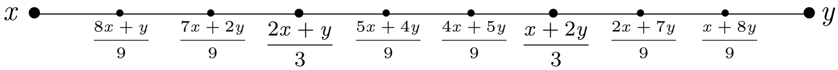
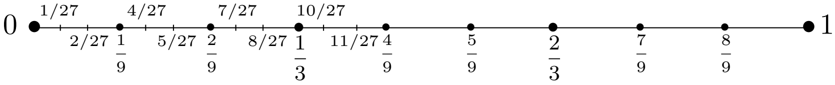




In base 3

Cutting rule: $\quad x,\ y \quad \longrightarrow \quad \dfrac{2x+y}{3} \quad \underline{\textit{and}} \quad \dfrac{x+2y}{3}$





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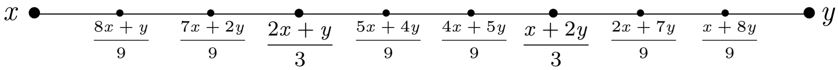

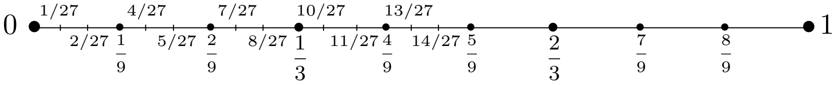

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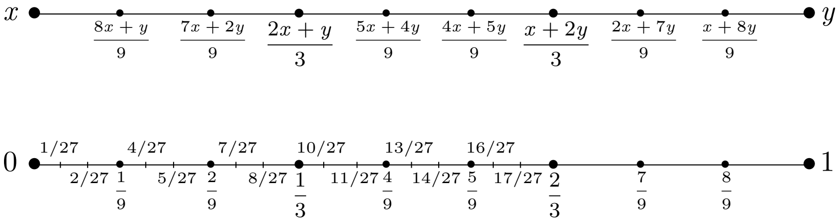

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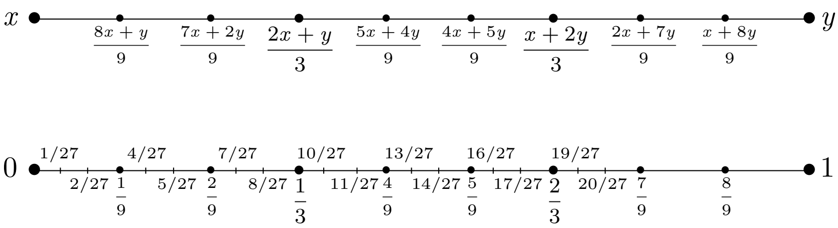
Cutting rule: $x, y \longrightarrow \dfrac{2x+y}{3}$ <u>*and*</u> $\dfrac{x+2y}{3}$



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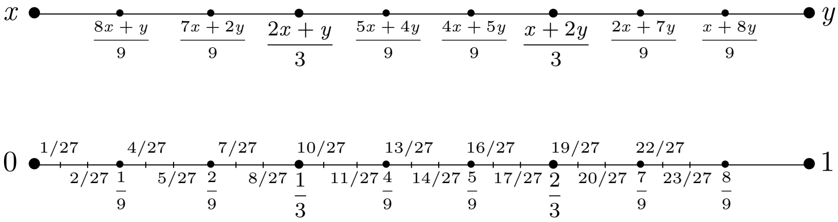

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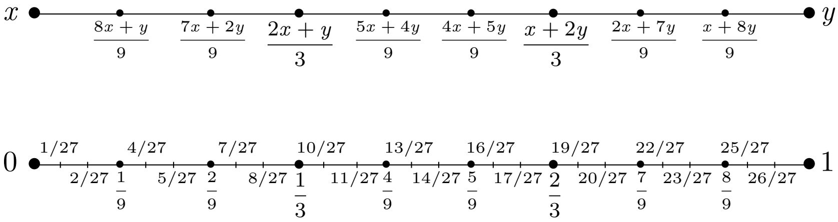

In base 3

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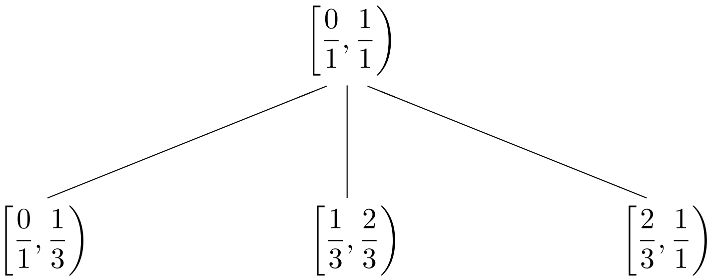
$$\left[\frac{0}{1}, \frac{1}{1}\right)$$

In base 3


$\left[\frac{0}{1}, \frac{1}{1}\right)$
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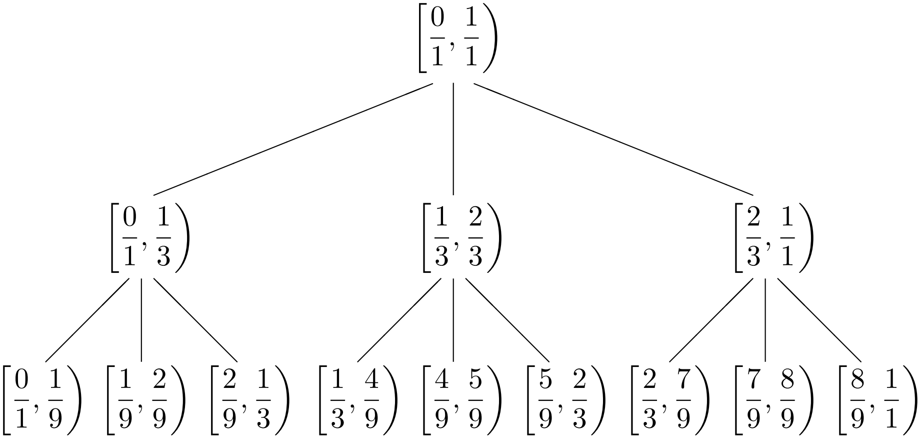

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$\left[\frac{2}{3}, \frac{1}{1}\right)$
$\left[\frac{0}{1}, \frac{1}{9}\right)$
$\left[\frac{1}{9}, \frac{2}{9}\right)$
$\left[\frac{2}{9}, \frac{1}{3}\right)$
$\left[\frac{1}{3}, \frac{4}{9}\right)$
$\left[\frac{4}{9}, \frac{5}{9}\right)$
$\left[\frac{5}{9}, \frac{2}{3}\right)$
$\left[\frac{2}{3}, \frac{7}{9}\right)$
$\left[\frac{7}{9}, \frac{8}{9}\right)$
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In base 3

Theorem

The codage of $x \in [0,1)$ by the trichotomy algorithm gives an analytic expression of x, with the following rule: putting $x_n = 0$ (resp. $x_n = 1$, $x_n = 2$) when the left (resp. central, right) subinterval is chosen at the n-th step, we have

$$x = \sum_{n \geq 1} \frac{x_n}{3^n}.$$

“Continued fractions with three letters”

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Cutting rule: $\frac{a}{b}, \frac{c}{d} \longrightarrow \frac{s}{t}$ $\underline{and}$ $\frac{u}{v}$

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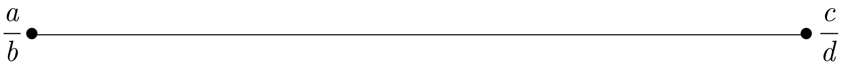

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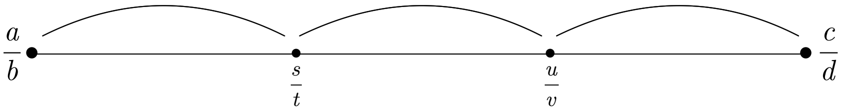

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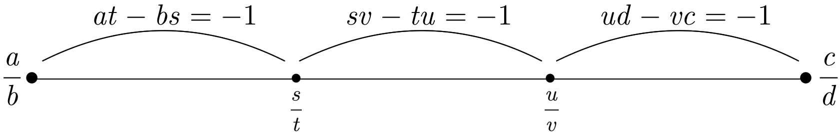

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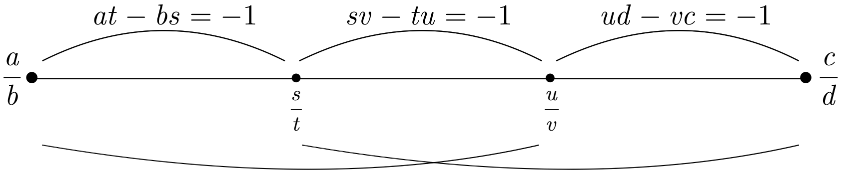

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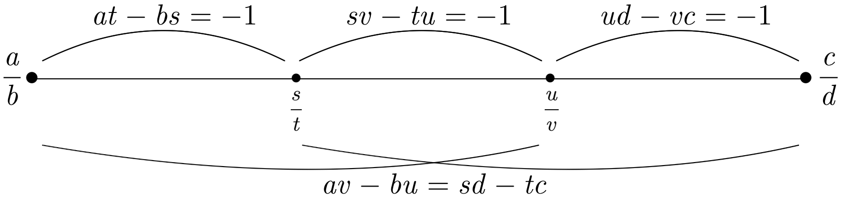
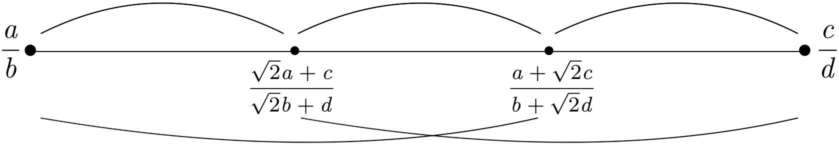


“Continued fractions with three letters”

Cutting rule: $\dfrac{a}{b}, \dfrac{c}{d} \longrightarrow \dfrac{\sqrt{2}a + c}{\sqrt{2}b + d}$ <u>*and*</u> $\dfrac{a + \sqrt{2}c}{b + \sqrt{2}d}$



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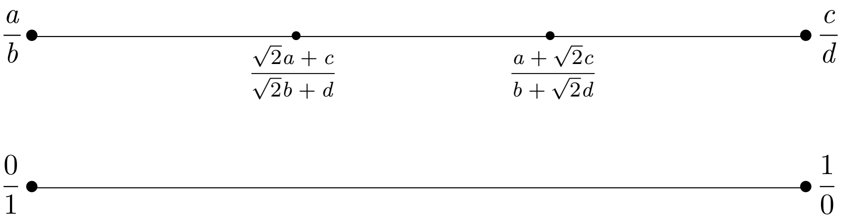

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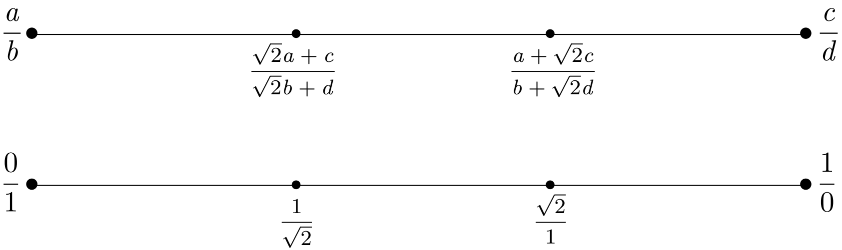

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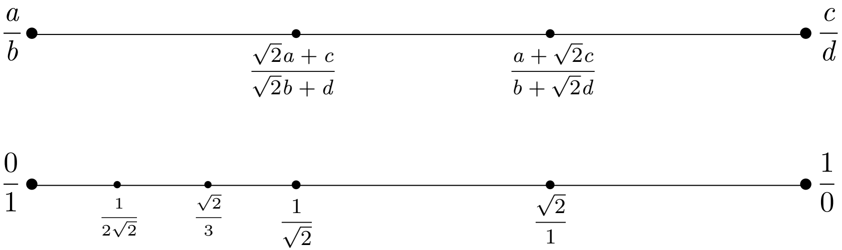

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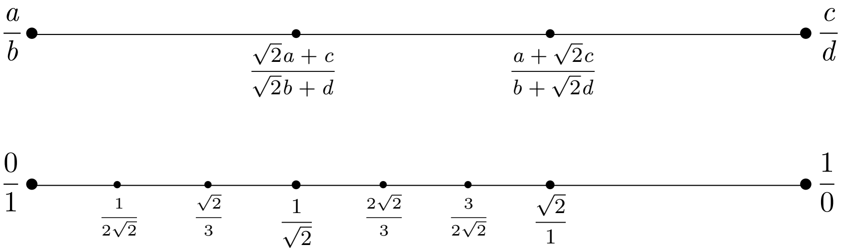

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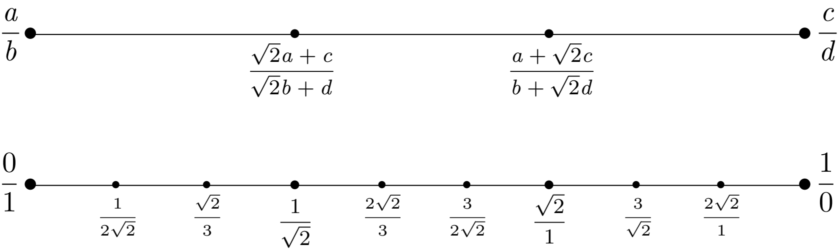

“Continued fractions with three letters”

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$$\left[\frac{0}{1}, \frac{1}{0}\right)$$

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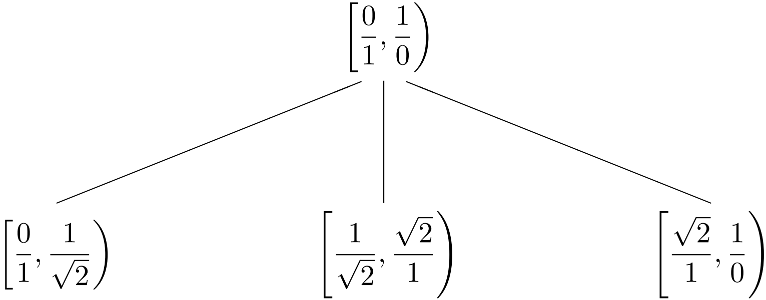

“Continued fractions with three letters”

- $\left[\frac{0}{1}, \frac{1}{0}\right)$
  - $\left[\frac{0}{1}, \frac{1}{\sqrt{2}}\right)$
    - $\left[\frac{0}{1}, \frac{1}{2\sqrt{2}}\right)$
    - $\left[\frac{1}{2\sqrt{2}}, \frac{\sqrt{2}}{3}\right)$
    - $\left[\frac{\sqrt{2}}{3}, \frac{1}{\sqrt{2}}\right)$
  - $\left[\frac{1}{\sqrt{2}}, \frac{\sqrt{2}}{1}\right)$
    - $\left[\frac{1}{\sqrt{2}}, \frac{2\sqrt{2}}{3}\right)$
    - $\left[\frac{2\sqrt{2}}{3}, \frac{3}{2\sqrt{2}}\right)$
    - $\left[\frac{3}{2\sqrt{2}}, \frac{\sqrt{2}}{1}\right)$
  - $\left[\frac{\sqrt{2}}{1}, \frac{1}{0}\right)$
    - $\left[\frac{\sqrt{2}}{1}, \frac{3}{\sqrt{2}}\right)$
    - $\left[\frac{3}{\sqrt{2}}, \frac{2\sqrt{2}}{1}\right)$
    - $\left[\frac{2\sqrt{2}}{1}, \frac{1}{0}\right)$

“Continued fractions with three letters”

Theorem

The codage of $x \in [0,1)$ by the “3-Stern-Brocot” algorithm gives an (explicit) analytic expression of x in $\sqrt{2}$-continued fraction:

$$x = a_0\sqrt{2} + \cfrac{1}{a_1\sqrt{2} + \cfrac{1}{a_2\sqrt{2} + \cfrac{1}{a_3\sqrt{2} + \cdots}}}.$$

Rosen continued fractions

Rosen continued fractions

Definition

Les $k \geq 3$ be an integer, let $\lambda := 2\cos(\pi/k)$. An expansion of $x \in \mathbb{R}$ in λ-*continued fraction* is an expression of the form

$$x = a_0\lambda + \cfrac{1}{a_1\lambda + \cfrac{1}{a_2\lambda + \cdots}} =: [a_0, a_1, a_2, \ldots]_\lambda,$$

where the a_n are integers, all different from 0 apart, possibly, a_0.

Rosen continued fractions

Theorem

In the interval $[0, 2)$, the values $\lambda = 2\cos(\pi/k)$ are the only ones for which the subgroup of homographies of $\mathbb{H}^2$ generated by

$$z \longmapsto z + \lambda \quad \text{et} \quad z \longmapsto -\frac{1}{z}$$

is discrete.

A combinatorial presentation

A combinatorial presentation

Definition

Let $k \geq 3$, let $\lambda := 2\cos(\pi/k)$. Put $\alpha_0 := 0$, $\alpha_1 := 1$ and, for any i between 2 and $k-2$:

$$\alpha_i := \lambda\alpha_{i-1} - \alpha_{i-2}.$$

A combinatorial presentation

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Let $k \geq 3$, let $\lambda := 2\cos(\pi/k)$. Put $\alpha_0 := 0$, $\alpha_1 := 1$ and, for any i between 2 and $k-2$:

$$\alpha_i := \lambda\alpha_{i-1} - \alpha_{i-2}.$$

The *mediants* of $\frac{a}{b}$ and $\frac{c}{d}$ (with $ad - bc = -1$) are the values

$$\frac{\alpha_1 a + \alpha_0 c}{\alpha_1 b + \alpha_0 d} \qquad \frac{\alpha_2 a + \alpha_1 c}{\alpha_2 b + \alpha_1 d} \qquad \cdots \qquad \frac{\alpha_{k-2} a + \alpha_{k-3} c}{\alpha_{k-3} b + \alpha_{k-3} d}.$$

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Example

For $k = 5$ ($\lambda = \varphi = (1 + \sqrt{5})/2$).

A combinatorial presentation

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$$\text{Mediants of } \frac{a}{b} \text{ and } \frac{c}{d} \quad : \quad \frac{\varphi a + c}{\varphi b + d}, \quad \frac{\varphi a + \varphi c}{\varphi b + \varphi d}, \quad \frac{a + \varphi c}{b + \varphi d}.$$

A combinatorial presentation

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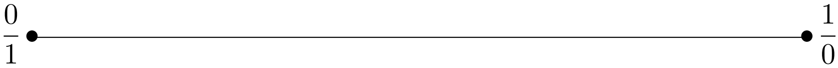

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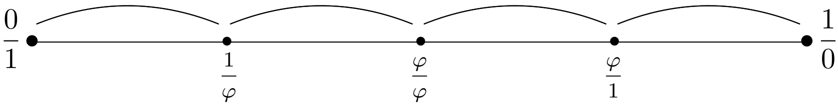

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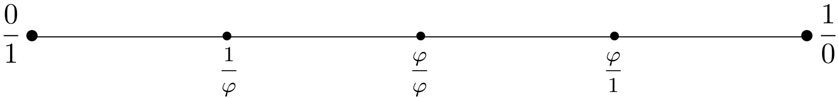

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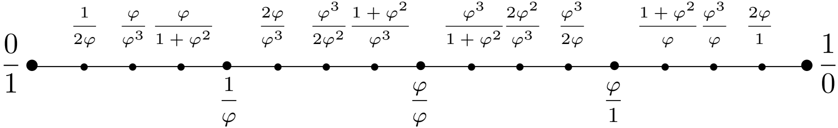


A combinatorial presentation

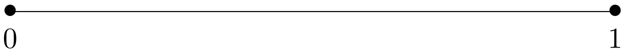
Theorem

There exists a natural correspondence between the codage with $(k-1)$ letters of the "Rosen-Stern-Brocot" algorithm and a $2\cos(\pi/k)$-continued fraction expansion.

After Rosen

Euclidean base	Hyperbolic base
2	$1 = 2\cos(\pi/3)$
3	$\sqrt{2} = 2\cos(\pi/4)$
4	$\varphi = 2\cos(\pi/5)$
5	$\sqrt{3} = 2\cos(\pi/6)$
$\vdots$	$\vdots$

In base golden ratio



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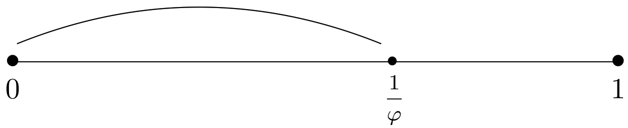

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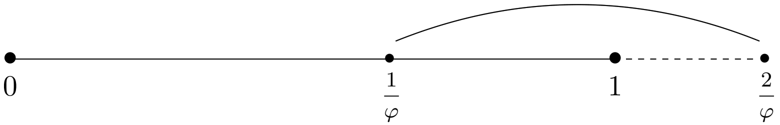

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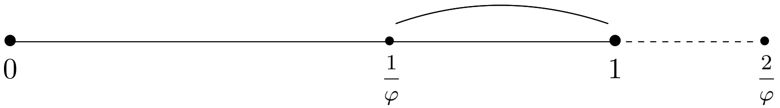

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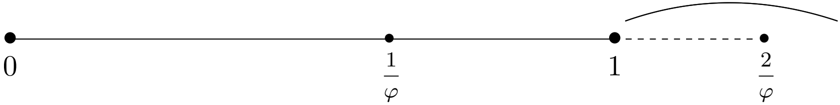

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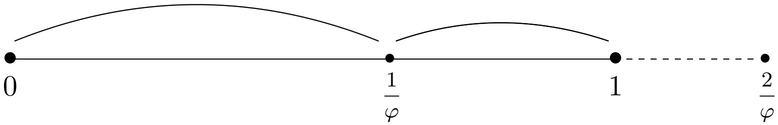

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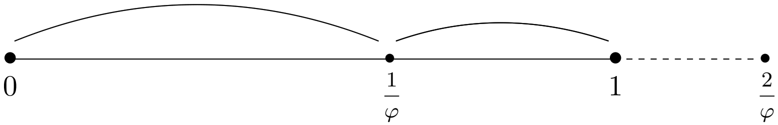
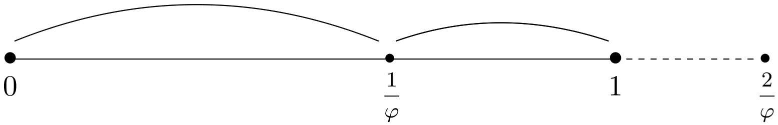


The codage of a number x by a sequence $(x_n)_n$ of 0 (when we go to the left) and 1 (when we go to the right) satisfies

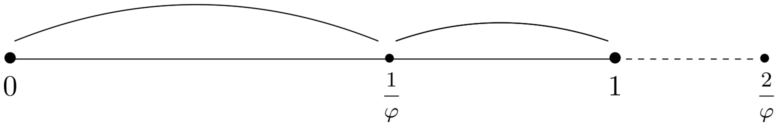
In base golden ratio



The codage of a number x by a sequence $(x_n)_n$ of 0 (when we go to the left) and 1 (when we go to the right) satisfies

- we do never see the pattern 11 ;

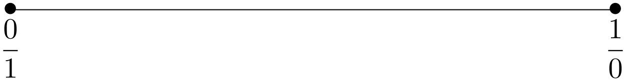
In base golden ratio



The codage of a number x by a sequence $(x_n)_n$ of 0 (when we go to the left) and 1 (when we go to the right) satisfies

- we do never see the pattern 11 ;
- $x = \sum_{n \geq 1} \frac{x_n}{\varphi^n}$.

“Continued fractions without 11”



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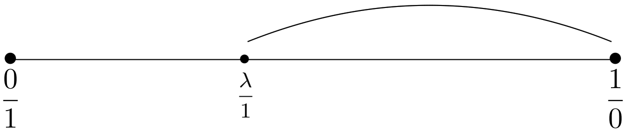

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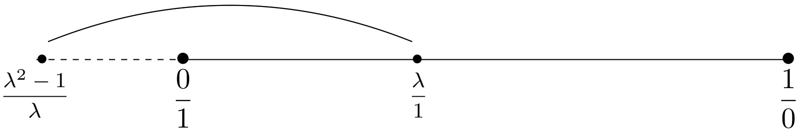

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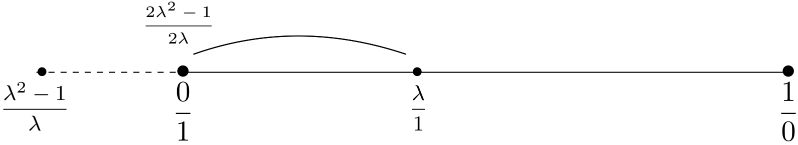
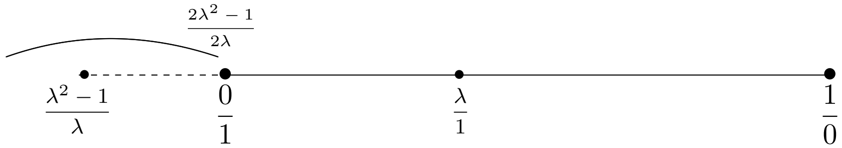
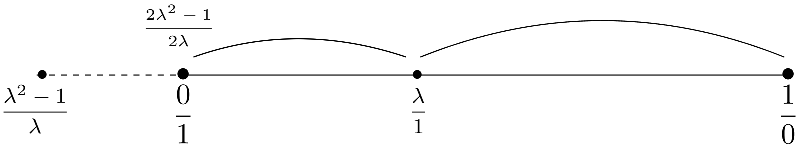


“Continued fractions without 11”



“Continued fractions without 11”



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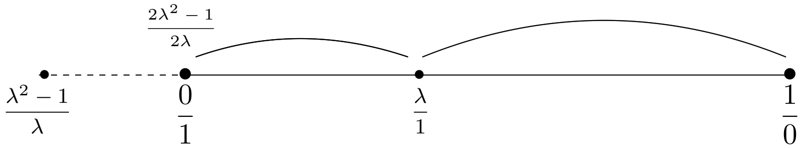

To get a convenient link, we need

$$\frac{2\lambda^2-1}{2\lambda} = \frac{0}{1}$$

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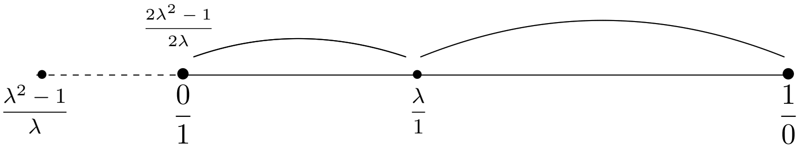
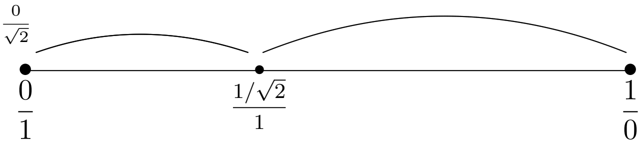


To get a convenient link, we need

$$\frac{2\lambda^2-1}{2\lambda} = \frac{0}{1}$$

soit $\lambda = 1/\sqrt{2}$.

“Continued fractions without 11”



Theorem

The codage of $x \in [0, +\infty]$ by this algorithm corresponds to a $1/\sqrt{2}$-continued fraction expansion of x, *via* the natural variant of the Euclidean algorithm associated to $(1/\sqrt{2})$-partial quotients.

After Rosen

base euclidienne	base hyperbolique
φ	$1/\sqrt{2}$
2	$1 = 2\cos(\pi/3)$
3	$\sqrt{2} = 2\cos(\pi/4)$
4	$\varphi = 2\cos(\pi/5)$
5	$\sqrt{3} = 2\cos(\pi/6)$
$\vdots$	$\vdots$

After Rosen

base euclidienne	base hyperbolique
φ	$1/\sqrt{2}$
2	$1 = 2\cos(\pi/3)$
$1+\sqrt{2}$	$\sqrt{3/2}$
3	$\sqrt{2} = 2\cos(\pi/4)$
4	$\varphi = 2\cos(\pi/5)$
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$\dots$	$\dots$
4	$\varphi = 2\cos(\pi/5)$
$\dots$	$\dots$
$\dots$	$\dots$
5	$\sqrt{3} = 2\cos(\pi/6)$
$\vdots$	$\vdots$