Thue-Morse sequences of squares in compact groups

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joint work with Michael Drmota

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Summary

- Thue-Morse sequence
- Generalized Thue-Morse sequence and main result
- Sketch of the proof
- Applications
  - La somme des chiffres des carrés
  - Automatic sequences

★ Thue-Morse sequence

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0110100110010110100101100110100 1

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$$\boxed{t_n = s_2(n) \bmod 2}$$

$$n = \sum_{i=0}^{\ell-1} \varepsilon_i(n) q^i \qquad \varepsilon_i(n) \in \{0, 1, \ldots, q-1\}, \qquad s_q(n) = \sum_{i=0}^{\ell-1} \varepsilon_i(n)$$

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Mauduit and Rivat (2009):

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★ Generalized Thue-Morse sequences and main results

- H ... compact (Hausdorff) group
- $q \geqslant 2$ and $g_0, g_1, \ldots, g_{q-1} \in H$ with $g_0 = e$ (identity element)
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$H = \langle \mathbb{Z}/2\mathbb{Z}, + \rangle$, $q = 2$, $g_0 = 0$, $g_1 = 1$: $\quad T(n) = s_2(n) \bmod 2 = t_n$

Theorem (Drmota and Morgenbesser, 2010)

There exists a positive integer $m = m(q, g_0, \ldots, g_{q-1})$ such that the following holds: Set

$$\mathrm{d}\nu = \sum_{v=0}^{m} \mathbf{1}_{g_v U} \cdot Q(v, m)\, \mathrm{d}\mu,$$

where

- $\mu \ldots$ *Haar measure on G,*
- $U = \mathrm{cl}(\{T(mn) : n \geqslant 0\}) \ldots$ *normal subgroup of G of index m,*
- $Q(v, m) = \#\{0 \leqslant n < m : n^2 \equiv v \bmod m\}$.

Then $(T(n^2))_{n \geqslant 0}$ is ν-uniformly distributed in G, that is,

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Then $(T(n^2))_{n \geqslant 0}$ is ν-uniformly distributed in G, that is,

$$\frac{1}{N} \sum_{n=0}^{N-1} f(T(n^2)) \to \int_G f\, \mathrm{d}\nu.$$

for all continuous functions $f : G \to \mathbb{C}$.

A **unitary group representation** is a continuous homomorphism $D : G \to U_n$ for some $n \geqslant 1$.

$U_n \ldots$ group of unitary $n \times n$ matrices over $\mathbb{C}$

D is irreducible if there is no proper subspace W of $\mathbb{C}^n$ with $D(x)W \subseteq W$ for all $x \in G$

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Lemma

Let G be a compact group and ν a regular normed Borel measure on G. Then a sequence $(x_n)_{n \geqslant 0}$ is ν-uniformly distributed in G iff

$$\frac{1}{N}\sum_{n=0}^{N-1} D(x_n) \to \int_G D \, \mathrm{d}\nu$$

holds for all irreducible unitary representations D of G.

Remarks:

- The *characteristic integer* m is the largest integer such that $m \mid q-1$ and such that there exists a representation D of G with

$$D_1(g_u) = e^{-2\pi i \frac{u}{m}} \qquad \text{for all } u \in \{0, 1, \dots, q-1\}.$$

- $(T(n^2))_{n \geqslant 0}$ is uniformly distributed in G (i.e., $\nu = \mu$) iff $m \leqslant 2$.

- If G is connected, then $T(n^2)$ is uniformly distributed in G.

★ Sketch of the proof

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- $D_0, \ldots, D_{m-1}$:

$$D_k(g_u) = e^{-2\pi i \frac{k}{m} u} \qquad \text{for all } 0 \leqslant u < q \text{ and } 0 \leqslant k < m$$

- all other irreducible unitary representations. . .

Van der Corput type inequality:

$$\left\| \sum_{0 \leqslant n < N} D(T(n^2)) \right\|_{\mathbb{F}} \leqslant \left(\frac{dN}{R} \sum_{|r| < R} \left(1 - \frac{|r|}{R}\right) \left\| \sum_{\substack{0 \leqslant n \leqslant B \\ 0 \leqslant n+r \leqslant B}} D(T(n+r)^2) D(T(n^2))^H \right\|_{\mathbb{F}} \right)^{1/2} + \frac{f}{2} R$$

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$$T(n) = g_{\varepsilon_0(n)} g_{\varepsilon_1(n)} \cdots g_{\varepsilon_{\lambda-1}(n)} g_{\varepsilon_\lambda(n)} \cdots g_{\varepsilon_{\ell-1}(n)}$$

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$$(n+r)^2 = (\varepsilon_{\ell-1}\varepsilon_{\ell-2} \ldots \varepsilon_{\lambda} \ldots)_q, \qquad n^2 = (\varepsilon_{\ell-1}\varepsilon_{\ell-2} \ldots \varepsilon_{\lambda} \ldots)_q$$

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$$D(T((n+r)^2))D(T(n^2))^H$$

$$= D(T_{\lambda}((n+r)^2))D(g_{\varepsilon_{\lambda}})\cdots D(g_{\varepsilon_{\ell-1}})D(g_{\varepsilon_{\ell-1}})^H \cdots D(g_{\varepsilon_{\lambda}})^H D(T_{\lambda}(n^2))^H$$

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$$T_{\lambda}(n) = g_{\varepsilon_0(n)} g_{\varepsilon_1(n)} \cdots g_{\varepsilon_{\mu-1}(n)} g_{\varepsilon_\mu(n)} \cdots g_{\varepsilon_{\lambda-1}(n)}$$

Fourier terms:

$$F_{\lambda}(h) = \frac{1}{q^{\lambda}} \sum_{0 \leqslant u < q^{\lambda}} e^{-2\pi i \frac{hu}{q^{\lambda}}} D(T_{\lambda}(u))$$

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$$\cdot \sum_{\substack{0\leqslant h_1,h_2,h_3,h_4<q^{\lambda}\\ (h_1+h_2+h_3+h_4,q^{\lambda})=d\\ d|2r(h_1+h_2)+2sq^{\mu}(h_2+h_3)+\ell}} \|F_{\mu,\lambda}(h_1)\|_{\mathbb{F}}\,\|F_{\mu,\lambda}(h_2)\|_{\mathbb{F}}\,\|F_{\mu,\lambda}(h_3)\|_{\mathbb{F}}\,\|F_{\mu,\lambda}(h_4)\|_{\mathbb{F}}$$

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The analogue of this expression appears in Mauduit and Rivat's work and can be estimated as in their case.

★ Applications

★ La somme des chiffres des carrés:

Theorem (Mauduit and Rivat, 2009)

Let $q, r \geqslant 2$ *and set* $m = \gcd(q-1, r)$. *Then we have*

(i) $\lim_{x\to\infty} \frac{1}{x} \# \{n \leqslant x : s_q(n^2) \equiv a \bmod r\} = \frac{1}{r} Q(a, m)$,

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- $H = \mathbb{R}/\mathbb{Z}, \quad g_j = \alpha j \bmod 1,\ 0 \leqslant j \leqslant q-1.$
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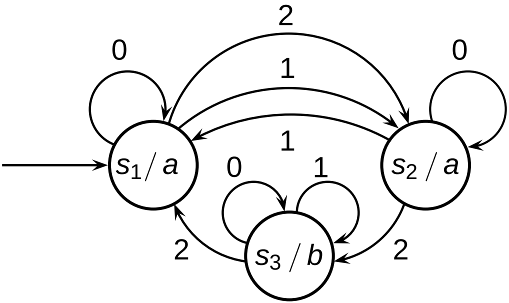
- $T(n) = \alpha s_q(n) \bmod 1$ ∎

★ Applications

★ Automatic sequences:

Definition

A sequence $(u_n)_{n\geqslant 0}$ is called a *q-automatic sequence*, if u_n is the output of an automaton when the input is the q-ary expansion of n.



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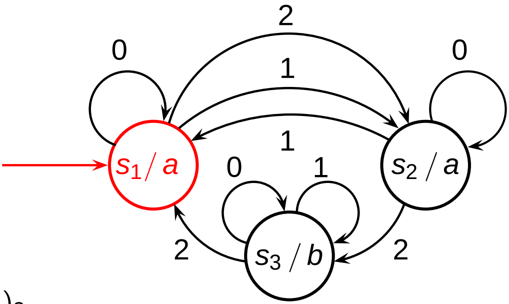

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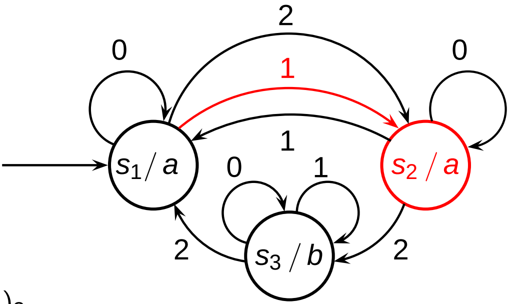

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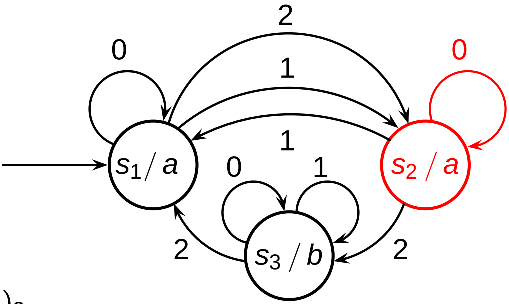

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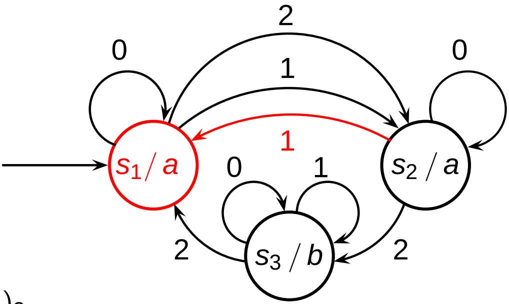

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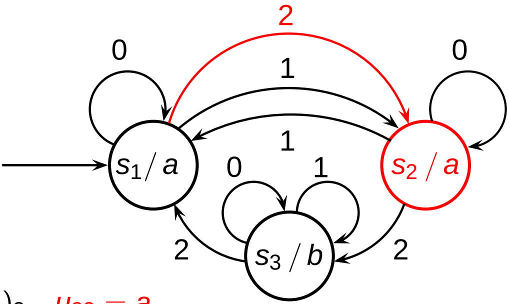

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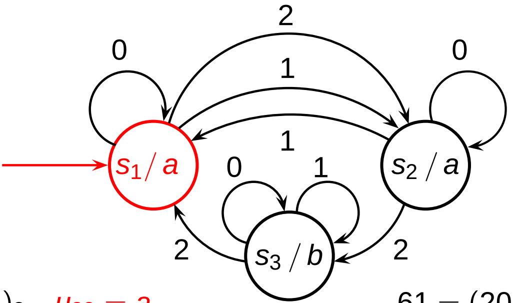

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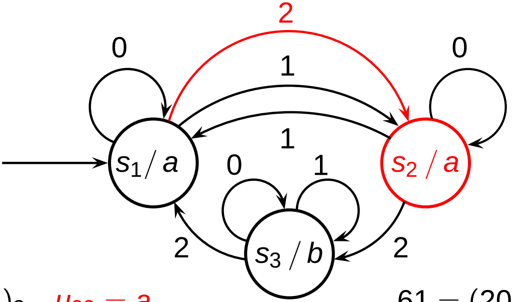

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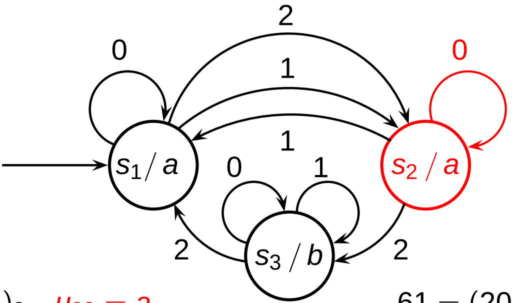

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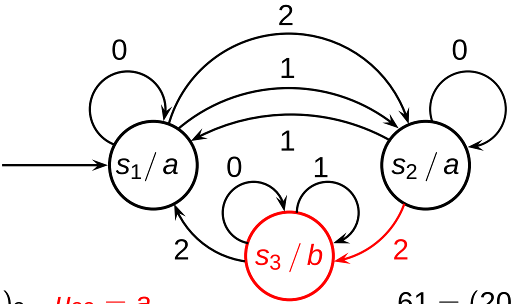

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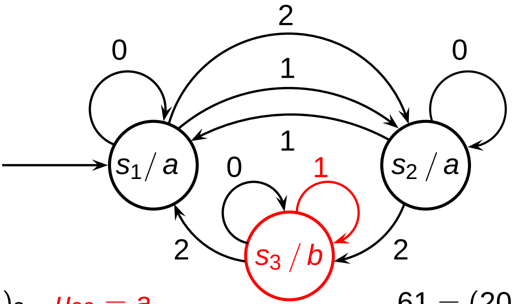

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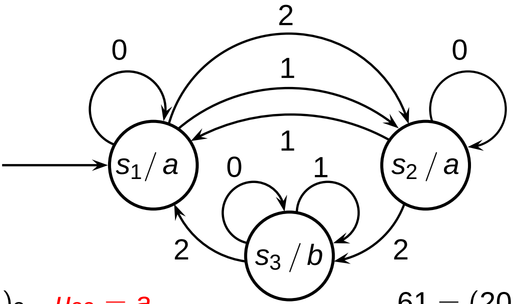
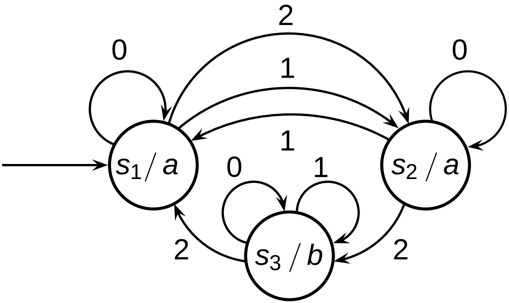
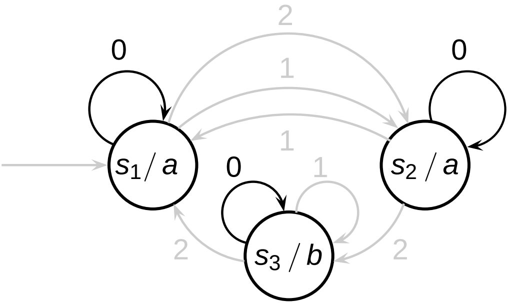


$32 = (1012)_3 \quad u_{32} = a,$ $\qquad 61 = (2021)_3 \quad u_{61} = b$

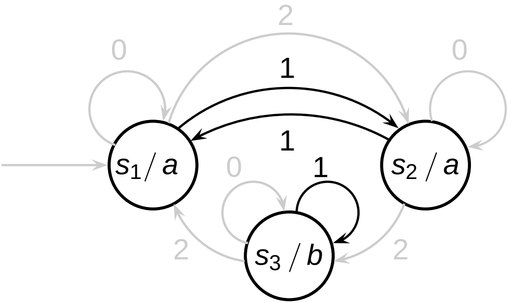
$(u_n)_{n\geqslant 0}$: *aaaaabaabaabaaabbaaabaaabbaaabaaabbaaaaaaba* . . .



2
0
1
0
1
s_1 / a
0
1
s_2 / a
2
s_3 / b
2



$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



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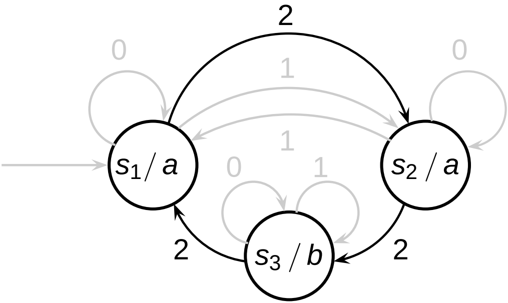

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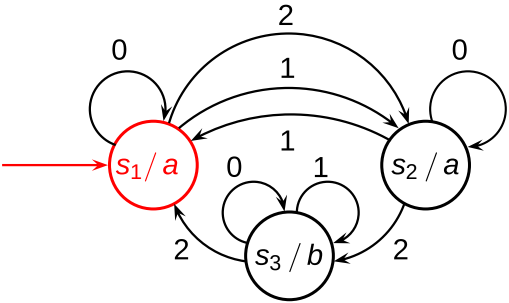

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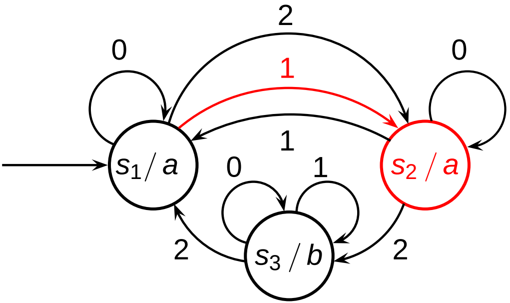

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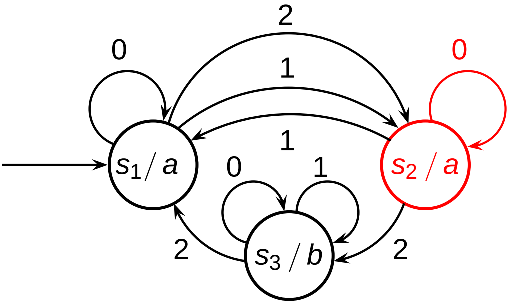
$32 = (1012)_3$:

$$M_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$



$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

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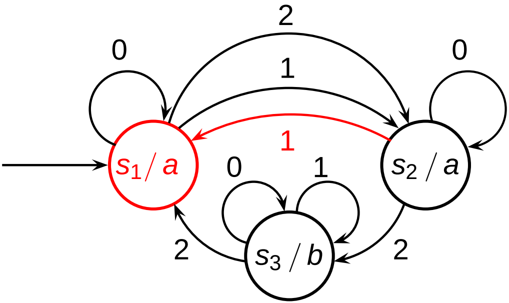

$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

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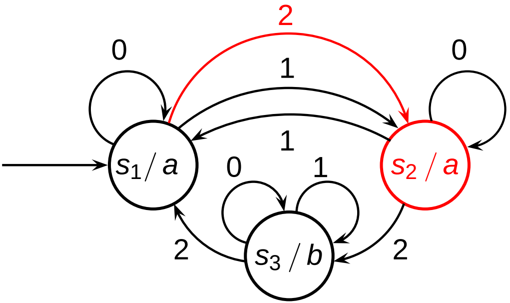

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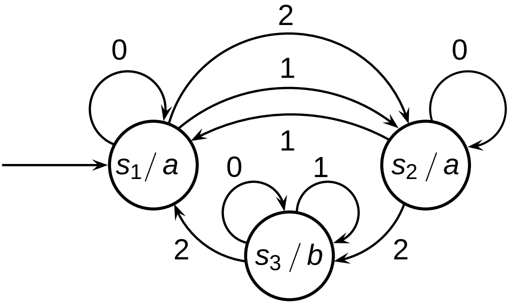
$$32 = (1012)_3 : \qquad M_2 \circ M_1 \circ M_0 \circ M_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$



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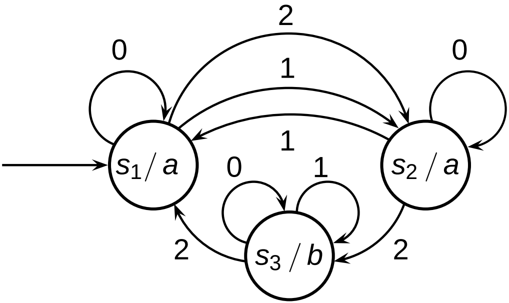
$$\boxed{S(n) := M_{\varepsilon_0(n)} M_{\varepsilon_1(n)} \cdots M_{\varepsilon_{\ell-1}(n)}}$$

$$\boxed{u_n = f\Big(S(n)\,\mathbf{e}_1\Big), \qquad \mathbf{e}_j^T S(n)\,\mathbf{e}_1 = \begin{cases} 1, & \text{output state } s_j \\ 0, & \text{otherwise.} \end{cases}}$$



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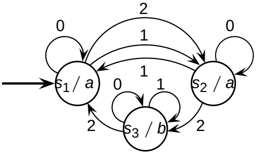
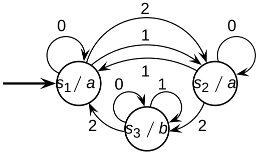
Definition

A q-automatic sequence is called *invertible* if there exists an automaton such that all transition matrices are invertible and M_0 is the identity matrix.



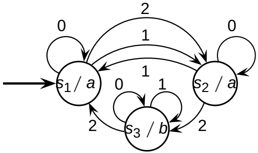
$(u_n)_{n\geqslant 0}$: *aaaaabaabaabaaabbaaabaaabbaaabaaabbaaaaaaba* . . .

Frequency of a in $(u_n)_{n\geqslant 0}$: $\lim_{N\to\infty} \frac{1}{N} \#\{0 \leqslant n < N : u_n = a\}$



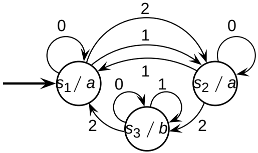
$(u_n)_{n\geqslant 0}$: $aaaaabaabaabaaabbaaabaaabbaaabaaabbaaaaaaba\ldots$

Frequency of a in $(u_{3n})_{n\geqslant 0}$: $\lim_{N\to\infty}\frac{1}{N}\#\{0\leqslant n<N : u_{3n}=a\}$



$(u_n)_{n\geqslant 0}$: $aaaaabaababaaabbaaabaaabbaaabaaabbaaaaaaba\ldots$

Frequency of a in $(u_{n^2})_{n\geqslant 0}$: $\lim_{N\to\infty}\frac{1}{N}\#\{0\leqslant n<N : u_{n^2}=a\} = ?$



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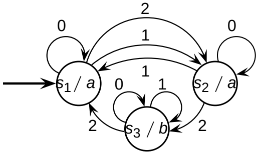
Frequency of a in $(u_{n^2})_{n\geqslant 0}$: $\quad \lim_{N\to\infty} \frac{1}{N} \#\{0 \leqslant n < N : u_{n^2} = a\} = ?$

$H \ldots$ group of permutation matrices, $\quad g_i = M_i$,



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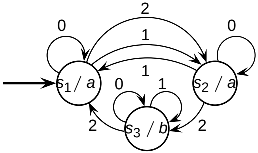

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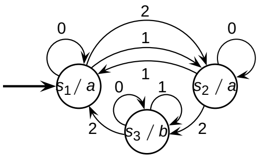

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Theorem (Drmota and Morgenbesser, 2010)

Let $q\geqslant 2$ and $(u_n)_{n\geqslant 0}$ be an invertible q-automatic sequence. Then the frequency of each letter of the subsequence $(u_{n^2})_{n\geqslant 0}$ exists.

Thank you!

1 LAUWERENS KUIPERS AND HARALD NIEDERREITER: *Uniform Distribution of Sequences*. Wiley-Interscience Publication, 1974

2 JEAN-PAUL ALLOUCHE AND JEFFREY SHALLIT: *Automatic sequences*. Cambridge University Press, 2003

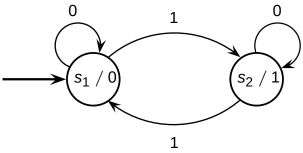
3 ALEKSANDR O. GELFOND: *Sur les nombres qui ont des propriétés additives et multiplicatives données*. Acta Arithmetica, 1968

4 CHRISTIAN MAUDUIT AND JOËL RIVAT: *La somme des chiffres des carrés*. Acta Mathematica, 2009

5 MICHAEL DRMOTA AND JOHANNES F. MORGENBESSER: *Generalized Thue-Morse Sequence of Squares*. submitted

Examples of automatic sequences

Thue-Morse sequence:



Rudin-Shapiro sequence: