ON THE JOINT DISTRIBUTION OF q-ADDITIVE FUNCTIONS ON POLYNOMIAL SEQUENCES

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ABSTRACT. The joint distribution of sequences $(f_\ell(P_\ell(n)))_{n\in\mathbb{N}}, \ell = 1,2,\dots,d$ and $(f_\ell(P_\ell(p)))_{p\in\mathbb{P}}$ respectively, where f_ℓ are q_ℓ-additive functions and P_ℓ polynomials with integer coefficients, is considered. A central limit theorem is proved for a larger class of q_ℓ and P_ℓ than by Drmota [3]. In particular, the joint limit distribution of the sum-of-digits functions $s_{q_1}(n), s_{q_2}(n)$ is obtained for arbitrary integers q_1, q_2. For strongly q-additive functions with respect to the same q, a central limit theorem is proved for arbitrary polynomials P_ℓ with the help of a joint representation of the digits of $P_\ell(n)$ by a Markov chain.

1. Introduction

For a given integer $q > 1$, every non-negative integer n has a unique *q-ary expansion*

$$n = \sum_{k\geq 0} \epsilon_{q,k}(n) q^k$$

with $\epsilon_{q,k}(n) \in \{0,1,\dots,q-1\}$ (where the index q will often be omitted). Then the *sum-of-digits function* is given by

$$s_q(n) = \sum_{k\geq 0} \epsilon_{q,k}(n).$$

This is a special case of a *q-additive function*, i.e. a real-valued function f defined on the non-negative integers which satisfies $f(0) = 0$ and

$$f(n) = \sum_{k\geq 0} f(\epsilon_{q,k}(n) q^k).$$

Such a function is said to be *strongly q-additive*, if

$$f(n) = \sum_{k\geq 0} f(\epsilon_{q,k}(n)).$$

Bassily and Kátai [1] proved the following central limit theorem.

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Theorem 1 (Bassily and Kátai [1]). *Let f be a q-additive function such that $f(bq^k) = \mathcal{O}(1)$ as $k \to \infty$ for all $b \in \{0, 1, \dots, q-1\}$. Assume $\frac{D(N)}{(\log N)^\eta} \to \infty$ as $N \to \infty$ for some $\eta > 0$ and let $P(n)$ be a polynomial with integer coefficients, degree r and positive leading term. Set*

$$\mu_k = \frac{1}{q} \sum_{b=0}^{q-1} f(bq^k), \qquad \sigma_k^2 = \frac{1}{q} \sum_{b=0}^{q-1} f(bq^k)^2 - \mu_k^2$$

and

$$M(N) = \sum_{k=0}^{[\log_q N]} \mu_k, \qquad D(N)^2 = \sum_{k=0}^{[\log_q N]} \sigma_k^2.$$

Then, as $N \to \infty$,

$$\frac{1}{N} \# \left\{ n < N \,\middle|\, \frac{f(P(n)) - M(N^r)}{D(N^r)} < x \right\} \to \Phi(x)$$

and

$$\frac{1}{\pi(N)} \# \left\{ p \in \mathbb{P}, p < N \,\middle|\, \frac{f(P(p)) - M(N^r)}{D(N^r)} < x \right\} \to \Phi(x),$$

where $\Phi(x)$ denotes the distribution function of the normal law.

This theorem was only stated for $\eta = \frac{1}{3}$. However, a short inspection of the proof shows that $\eta > 0$ is sufficient.

Drmota [3] generalised this theorem for certain joint distributions. From now on, denote by $\mu_{\ell,k}, \sigma_{\ell,k}, M_\ell, D_\ell$ the μ_k, σ_k, M, D of Theorem 1 with respect to f_ℓ.

Theorem 2 (Drmota [3]). *Let f_ℓ, $1 \le \ell \le d$, be q_ℓ-additive functions such that $f_\ell(bq_\ell^k) = \mathcal{O}(1)$ as $k \to \infty$ for all $b \in \{0, 1, \dots, q_\ell - 1\}$. Assume that $\frac{D_\ell(N)}{(\log N)^\eta} \to \infty$, as $N \to \infty$, for some $\eta > 0$ and let $P_\ell(x)$ be polynomials with integer coefficients of different degrees r_ℓ and positive leading terms, $1 \le \ell \le d$. Then, as $N \to \infty$,*

$$\frac{1}{N} \# \left\{ n < N \,\middle|\, \frac{f_\ell(P_\ell(n)) - M_\ell(N^{r_\ell})}{D_\ell(N^{r_\ell})} < x_\ell, 1 \le \ell \le d \right\} \to \Phi(x_1)\Phi(x_2)\cdots\Phi(x_d)$$

and

$$\frac{1}{\pi(N)} \# \left\{ p < N \,\middle|\, \frac{f_\ell(P_\ell(p)) - M_\ell(N^{r_\ell})}{D_{q_\ell}(N^{r_\ell})} < x_\ell, 1 \le \ell \le d \right\} \to \Phi(x_1)\Phi(x_2)\cdots\Phi(x_d).$$

Note that this theorem was stated only for coprime q_ℓ, but this assumption is not used in the proof and therefore not necessary.

The problem is the case of polynomials of the same degree. For $d = 2$, we show the following theorem.

Theorem 3. *Let $q_1, q_2 > 1$ be multiplicatively independent integers and let f_ℓ be q_ℓ-additive functions such that $f_\ell(bq_\ell^k) = \mathcal{O}(1)$ as $k \to \infty$ for all $b \in \{0, 1, \dots, q_\ell - 1\}$, $\ell = 1, 2$. Assume that $\frac{D_\ell(N)}{(\log N)^\eta} \to \infty$ as $N \to \infty$, for some $\eta > 0$ and let $P_\ell(n)$ be* *polynomials with integer coefficients of degree r and positive leading terms, $\ell = 1, 2$. Then, as $N \to \infty$,*

$$\frac{1}{N}\#\left\{n < N \left| \frac{f_\ell(P_\ell(n)) - M_\ell(N^r)}{D_\ell(N^r)} < x_\ell, \ell = 1, 2\right.\right\} \to \Phi(x_1)\Phi(x_2)$$

and

$$\frac{1}{\pi(N)}\#\left\{p < N \left| \frac{f_\ell(P_\ell(p)) - M_\ell(N^r)}{D_\ell(N^r)} < x_\ell, \ell = 1, 2\right.\right\} \to \Phi(x_1)\Phi(x_2).$$

The first convergence was shown by Drmota [3] for linear polynomials and coprime integers q_1, q_2. In [4], Drmota and the author stated this theorem, but still only for coprime integers. We will prove the case of multiplicatively independent integers in Section 3.

Furthermore, we solve the problem of equal degrees of the polynomials for strongly q-additive functions with respect to the same q in the following section. Note that this covers the case of multiplicatively dependent q_1, q_2 since q_1- and q_2-additive functions are q-additive, if $q_1^{s_1} = q_2^{s_2} = q$. Then the distributions clearly do not satisfy the independence relations of Theorems 2 and 3.

The main part of the proof of all theorems is a proposition similar to the following one (which proves Theorem 2).

Proposition 1 (Drmota [3]). *Let $P_\ell(n)$, $1 \le \ell \le d$, be polynomials of different degrees r_ℓ with integer coefficients and positive leading terms. Let $\lambda > 0$ be an arbitrary constant and h_ℓ, $1 \le \ell \le d$, non-negative integers. Then, as $N \to \infty$,*

$$\frac{1}{N}\#\left\{n < N \left| \epsilon_{q_\ell, k_j^{(\ell)}}(P_\ell(n)) = b_j^{(\ell)}, 1 \le j \le h_\ell, 1 \le \ell \le d\right.\right\} = \frac{1}{q_1^{h_1} q_2^{h_2} \cdots q_d^{h_d}} + \mathcal{O}\left((\log N)^{-\lambda}\right)$$

and

$$\frac{1}{\pi(N)}\#\left\{p < N \left| \epsilon_{q_\ell, k_j^{(\ell)}}(P_\ell(n)) = b_j^{(\ell)}, 1 \le j \le h_\ell, 1 \le \ell \le d\right.\right\} = \frac{1}{q_1^{h_1} q_2^{h_2} \cdots q_d^{h_d}} + \mathcal{O}\left((\log N)^{-\lambda}\right)$$

uniformly for integers

$$(\log N^{r_\ell})^\eta \le k_1^{(\ell)} < k_2^{(\ell)} < \cdots < k_{h_\ell}^{(\ell)} \le \log_{q_\ell} N^{r_\ell} - (\log N^{r_\ell})^\eta \quad (1 \le \ell \le d)$$

(with some $\eta > 0$) and $b_j^{(\ell)} \in \{0, 1, \ldots, q_\ell - 1\}$.

For a list of references of other results for q-additive functions, we refer to Drmota [3].

2. Strongly q-additive functions with respect to the same q

2.1. Results.

Theorem 4. *Let f_ℓ, $1 \le \ell \le d$, be strongly q-additive functions with $\sigma_\ell = \sigma_{\ell,k} > 0$ and $P_\ell(n) = g_{r_\ell}^{(\ell)} n^{r_\ell} + \cdots + g_1^{(\ell)} n + g_0^{(\ell)}$ polynomials with integer coefficients and positive leading terms. Then, as $N \to \infty$,*

$$\frac{1}{N} \# \left\{ n < N \,\middle|\, \frac{f_\ell(P_\ell(n)) - M_\ell(N^{r_\ell})}{D_\ell(N^{r_\ell})} < x_\ell, \ell = 1, \ldots, d \right\} \to \Phi_V(x_1, \ldots, x_d)$$

and

$$\frac{1}{\pi(N)} \# \left\{ p < N \,\middle|\, \frac{f_\ell(P_\ell(p)) - M_\ell(N^{r_\ell})}{D_\ell(N^{r_\ell})} < x_\ell, \ell = 1, 2, \ldots, d \right\} \to \Phi_V(x_1, \ldots, x_d)$$

where $\Phi_V(x_1, \ldots, x_d)$ denotes the distribution function of the d-dimensional normal law with covariance matrix $V = (v_{i,j})_{1 \le i,j \le d}$ given by

$$v_{i,j} = \begin{cases} 1 & \text{if } i = j \\ C_{i,j}\left(\frac{g_{r_i}^{(i)}}{(g_{r_i}^{(i)}, g_{r_j}^{(j)})}, \frac{g_{r_j}^{(j)}}{(g_{r_i}^{(i)}, g_{r_j}^{(j)})} \right) & \text{if } g_{r_j}^{(j)} P_i(n) \equiv g_{r_i}^{(i)} P_j(n) \\ \frac{r_i - \max\left\{ s \,\middle|\, g_{r_i}^{(i)} g_s^{(j)} \ne g_{r_j}^{(j)} g_s^{(i)} \right\}}{r_i} C_{i,j}\left(\frac{g_{r_i}^{(i)}}{(g_{r_i}^{(i)}, g_{r_j}^{(j)})}, \frac{g_{r_j}^{(j)}}{(g_{r_i}^{(i)}, g_{r_j}^{(j)})} \right) & \text{if } r_i = r_j \\ 0 & \text{else,} \end{cases}$$

where

$$\begin{aligned} C_{i,j}(g_i, g_j) = & \frac{1}{\sigma_i \sigma_j} \sum_{l=0}^{R_j - 1} \sum_{b_i=1}^{q-1} \sum_{b_j=1}^{q-1} \left(\pi_{b_i, b_j, g_i q^l, g_j} - \frac{1}{q^2} \right) f_i(b_i) f_j(b_j) \\ & + \frac{1}{\sigma_i \sigma_j} \sum_{l=1}^{R_i - 1} \sum_{b_i=1}^{q-1} \sum_{b_j=1}^{q-1} \left(\pi_{b_i, b_j, g_i, g_j q^l} - \frac{1}{q^2} \right) f_i(b_i) f_j(b_j) \end{aligned}$$

with R_ℓ such that $q | \frac{q^{R_\ell}}{(q^{R_\ell}, g_{r_\ell}^{(\ell)})}$ and

$$\pi_{b_i, b_j, g_i q^l, g_j} = \pi_{b_i, b_j, g, g'} = \frac{1}{q^2} - \frac{\left(\overline{(b_i+1)g'} - \overline{b_i g'} \right) \left(\overline{(b_j+1)g} - \overline{b_j g} \right)}{g g' q^2}$$
$$+ \frac{\min\left(\overline{b_i g'}, \overline{b_j g} \right) + \min\left(\overline{(b_i+1)g'}, \overline{(b_j+1)g} \right) - \min\left(\overline{(b_i+1)g'}, \overline{b_j g} \right) - \min\left(\overline{b_i g'}, \overline{b_j g} \right)}{g g' q}$$

where $g = \frac{g_i q^l}{(q^l, g_j)}$, $g' = \frac{g_j}{(q^l, g_j)}$ and $\overline{y}$ denotes the representative y' of $y' \equiv y\ (q)$ with $0 \le y' < q$. ($\pi_{b_i, b_j, g_i, g_j q^l}$ is given symmetrically.)

Remarks. If V is positive definite, we have, with $\mathbf{t} = (t_1, \ldots, t_d)$,

$$\Phi_V(x_1, \ldots, x_d) = \frac{1}{(2\pi)^{d/2} \sqrt{\det V}} \int_{-\infty}^{x_d} \cdots \int_{-\infty}^{x_1} e^{-\frac{1}{2} \mathbf{t} V^{-1} \mathbf{t}^t} dt_1 \ldots dt_d.$$

If $g_{r_\ell}^{(\ell)}$ is coprime to q, then we have $R_\ell = 1$.

$l \ge R_j$ implies $\pi_{b_i,b_j,g_iq^l,g_j} = \frac{1}{q^2}$ for all b_i, b_j.

The π_{b_i,b_j,g_iq^l,g_j} are the joint probabilities of digits $k+l$ and k of g_in and g_jn (which do not depend on k):

$$\pi_{b_i,b_j,g_iq^l,g_j} = \mathbf{Pr}[\epsilon_k(g_iq^ln) = b_i, \epsilon_k(g_j) = b_j] = \mathbf{Pr}[\epsilon_{k+l}(g_in) = b_i, \epsilon_k(g_j) = b_j].$$

Note that we need $C_{i,j}(g_i, g_j)$ only for coprime g_i, g_j.

The constant term of the polynomials plays no role.

Corollary 1. *Let $P_\ell(n) = g_{r_\ell}^{(\ell)} n^{r_\ell} + \cdots + g_1^{(\ell)} n + g_0^{(\ell)}$ be polynomials with integer coefficients and positive leading terms. Then, as $N \to \infty$,*

$$\frac{1}{N}\#\left\{n < N \,\middle|\, \frac{s_q(P_\ell(n)) - \frac{q-1}{2}\log_q N^{r_\ell}}{\sqrt{\frac{q^2-1}{12}\log_q N^{r_\ell}}} < x_\ell, \ell = 1, \ldots, d\right\}$$
$$\to \frac{1}{(2\pi)^{d/2}\sqrt{\det V}} \int_{-\infty}^{x_d} \cdots \int_{-\infty}^{x_1} e^{-\frac{1}{2}\mathbf{t}V^{-1}\mathbf{t}^t} dt_1 \ldots dt_d$$

with the positive definite matrix $V = (v_{i,j})_{1\le i,j\le d}$ given by

$$v_{i,j} = \begin{cases} 1 & \text{if } i = j \\ C_{i,j}\left(\frac{g_{r_i}^{(i)}}{(g_{r_i}^{(i)}, g_{r_j}^{(j)})}, \frac{g_{r_j}^{(j)}}{(g_{r_i}^{(i)}, g_{r_j}^{(j)})}\right) & \text{if } g_{r_j}^{(j)} P_i(n) \equiv g_{r_i}^{(i)} P_j(n) \\ \frac{r_i - \max\left\{s \,\middle|\, g_{r_i}^{(i)} g_s^{(j)} \ne g_{r_j}^{(j)} g_s^{(i)}\right\}}{r_i} C_{i,j}\left(\frac{g_{r_i}^{(i)}}{(g_{r_i}^{(i)}, g_{r_j}^{(j)})}, \frac{g_{r_j}^{(j)}}{(g_{r_i}^{(i)}, g_{r_j}^{(j)})}\right) & \text{if } r_i = r_j \\ 0 & \text{else,} \end{cases}$$

and

$$C_{i,j}(g_i, g_j) = \frac{q^2 - (q, g_i)^2 - (q, g_j)^2 + 1}{g_ig_j(q^2-1)}$$
$$+ \frac{1}{g_ig_j(q^2-1)}\left(\sum_{l=1}^{R_j-1} \frac{q^2 - \left(q, \frac{g_iq^l}{(q^l, g_j)}\right)}{q^l} + \sum_{l=1}^{R_i-1} \frac{q^2 - \left(q, \frac{g_jq^l}{(g_i, q^l)}\right)}{q^l}\right)$$

Remark. For monomials $P_\ell(n) = g_\ell n^r$ with $(g_\ell, q) = 1$ we just have

$$v_{i,j} = \frac{(g_i, g_j)^2}{g_ig_j}.$$

For $q = 2$ and $r = 1$, this was proved by W.M. Schmidt [6].

Furthermore, we can calculate the joint distribution of the sum-of-digits functions for multiplicatively dependent q_1, q_2.

Corollary 2. *For $q_1 = \tilde{q}^{s_1}, q_2 = \tilde{q}^{s_2}$ with positive integers $\tilde{q}, s_1, s_2$ and $(s_1, s_2) = 1$, we have, as $N \to \infty$,*

$$\frac{1}{N}\#\left\{n < N \,\middle|\, \frac{s_{q_1}(n) - \frac{q_1-1}{2}\log_{q_1} N}{\sqrt{\frac{q_1^2-1}{12}\log_{q_1} N}} < x_1, \frac{s_{q_2}(n) - \frac{q_2-1}{2}\log_{q_2} N}{\sqrt{\frac{q_2^2-1}{12}\log_{q_2} N}} < x_2\right\}$$
$$\to \frac{1}{2\pi\sqrt{1-C^2}} \int_{-\infty}^{x_2}\int_{-\infty}^{x_1} e^{-\frac{1}{2(1-C^2)}(t_1^2+t_2^2-2Ct_1t_2)} dt_1dt_2$$

with

$$C = \frac{\tilde{q}+1}{\tilde{q}-1}\sqrt{\frac{(q_1-1)(q_2-1)}{s_1 s_2 (q_1+1)(q_2+1)}}.$$

For general strongly q_ℓ-additive functions, similar statements can be derived easily. The case of multiplicatively independent q_1, q_2 is treated by Theorem 3.

2.2. A Markov chain and calculation of the covariance.

Define the polynomials

$$P_\ell^{(s)}(n) = g_{r_\ell}^{(\ell)} n^{r_\ell} + \cdots + g_s^{(\ell)} n^s \quad \text{for} \quad 1 \le s \le r = \max_{1\le\ell\le d} r_\ell.$$

and fix s in this subsection.

Furthermore, define vectors

$$\mathbf{w}_k^{(s)}(n) = (w_{k,s}, \ldots, w_{k,r}) = \left(\left\{\frac{n^s}{q^{k+1}}\right\}, \left\{\frac{n^{s+1}}{q^{k+1}}\right\}, \ldots, \left\{\frac{n^r}{q^{k+1}}\right\}\right)$$

for $0 \le n < N$, where $\{x\}$ denotes the fractional part of x and see, by Proposition 1, that they asymptotically form a net to the base q if $k \in \left[(\log N)^\eta, \log_q N^s - (\log N)^\eta\right]$ (but not for $k > \log_q N^s$). Proposition 1 gives rather bad error terms if we want to calculate the number of $\mathbf{w}_k^{(s)}(n)$ in an arbitrary set of $\mathbb{T}^{r-s+1}$. Nevertheless, this suggests that they are uniformly distributed and we use the Lebesgue measure as probability measure on $\mathbb{T}^{r-s+1}$.

We have $\epsilon_k(P_\ell^{(s)}(n)) = b$ if and only if

$$\left\{g_{r_\ell}^{(\ell)} w_{k,r_\ell} + \cdots + g_s^{(\ell)} w_{k,s}\right\} \in \left[\frac{b}{q}, \frac{b+1}{q}\right).$$

This means that, for each digit b, $\{\mathbf{w}_k^{(s)}(n) \mid \epsilon_k(P_\ell^{(s)}(n)) = b\}$ (as a set of $\mathbb{T}^{r-s+1}$) is contained in the stripe $S_{b,\ell}^{(s)}$ between the hyperplanes $g_{r_\ell}^{(\ell)} x_{r_\ell} + \cdots + g_s^{(\ell)} x_s = \frac{b}{q}$ (included) and $g_{r_\ell}^{(\ell)} x_{r_\ell} + \cdots + g_s^{(\ell)} x_s = \frac{b+1}{q}$ (excluded). If $P_\ell^{(s)}(n) \equiv 0$, set $S_{0,\ell}^{(s)} = \mathbb{T}^{r-s+1}$ and $S_{b,\ell}^{(s)} = \emptyset$ for $b \ne 0$.

Thus, each set $\{\mathbf{w}_k^{(s)}(n) \mid \epsilon_k(P_1^{(s)}(n)) = b_1, \ldots, \epsilon_k(P_d^{(s)}(n)) = b_d\}$ is contained in $S_{b_1,1}^{(s)} \cap \cdots \cap S_{b_d,d}^{(s)}$ and each of these intersections consists of a finite number of convex sets, the boundaries of which are the above hyperplanes. Let $(W_j^{(s)})_{1\le j\le\kappa_s}$ be the partition of $\mathbb{T}^r$ induced by these sets (or equivalently by the hyperplanes). Then $f_\ell|_{W_j^{(s)}}$ is constant for all ℓ, j.

Furthermore, we have $\epsilon_{k-j}(P_\ell^{(s)}(n)) = b$ if and only if $T^j(\mathbf{w}_k^{(s)}(n)) \in S_{b,\ell}^{(s)}$ with the map $T : \mathbb{T}^r \to \mathbb{T}^r$, $T(w_{k,s}, \ldots, w_{k,r}) = (q w_{k,s}, \ldots, q w_{k,r})$. Hence

$$\begin{aligned}\left\{n \,\middle|\, \epsilon_0(P_\ell^{(s)}(n)) = b_0^{(\ell)}, \ldots, \epsilon_k(P_\ell^{(s)}(n)) = b_k^{(\ell)}\right\}& \\ = \left\{n \,\middle|\, \mathbf{w}_k^{(s)}(n) \in T^{-k} S_{b_0^{(\ell)},\ell}^{(s)}, \ldots, \mathbf{w}_k^{(s)}(n) \in S_{b_k^{(\ell)},\ell}^{(s)}\right\}&\end{aligned}$$

and we define a sequence of random variables $(Y_k^{(s)})_{k\ge0}$ on $\{W_1^{(s)}, W_2^{(s)}, \ldots, W_{\kappa_s}^{(s)}\}$ by

$$\mathbf{Pr}[Y_0^{(s)} = W_{j_0}^{(s)}, \ldots, Y_k^{(s)} = W_{j_k}^{(s)}] = \lambda_{r-s+1}(T^{-k} W_{j_0}^{(s)} \cap \ldots T^{-1} W_{j_{k-1}}^{(s)} \cap W_{j_k}^{(s)})$$

for $1 \le j_i \le \kappa_s$, $0 \le i \le k$. (λ_n denotes the n-dimensional Lebesgue measure.)

Lemma 1. $\left(Y_k^{(s)}\right)_{k\geq 0}$ *is a Markov chain.*

Proof. Let U be the subspace of $\mathbb{R}^{r-s+1}$ spanned by the vectors $(g_s^{(\ell)},\dots,g_r^{(\ell)})$, $1\leq \ell\leq d$. If U has (full) rank $r-s+1$, then T is injective on each $W_j^{(s)}$, $1\leq j\leq \kappa_s$. Otherwise, $W_j^{(s)}$ contains with every point x all points $x+U^\perp$ and T is q^δ-to-one with $\delta = r-s+1-\operatorname{rank}(U)$. Furthermore, $TW_j^{(s)}$ is the (disjoint) union of sets $W_i^{(s)}$, since the image of the hyperplane $g_{r_\ell}^{(\ell)}x_{r_\ell}+\cdots+g_s^{(\ell)}x_s=\frac{b}{q}$ is the hyperplane $g_{r_\ell}^{(\ell)}x_{r_\ell}+\cdots+g_s^{(\ell)}x_s=0$. Hence we have

$$\begin{aligned}
&\mathbf{Pr}[Y_0^{(s)}=W_{j_0}^{(s)},\dots,Y_{k+1}^{(s)}=W_{j_{k+1}}^{(s)}] = \lambda_{r-s+1}(T^{-(k+1)}W_{j_0}^{(s)}\cap\cdots\cap W_{j_{k+1}}^{(s)})\\
&\quad=\frac{1}{q^\delta}\lambda_{r-s+1}(T^{-k}W_{j_0}^{(s)}\cap\cdots\cap W_{j_k}^{(s)}\cap TW_{j_{k+1}}^{(s)})\\
&\quad=\begin{cases}\frac{1}{q^\delta}\lambda_{r-s+1}(T^{-k}W_{j_0}^{(s)}\cap\cdots\cap W_{j_k}^{(s)}) & \text{if } W_{j_k}^{(s)}\subseteq TW_{j_{k+1}}^{(s)}\\ 0 & \text{else.}\end{cases}
\end{aligned}$$

Thus

$$\begin{aligned}
&\mathbf{Pr}[Y_{k+1}^{(s)}=W_{j_{k+1}}^{(s)}|Y_0^{(s)}=W_{j_0}^{(s)},\dots,Y_k^{(s)}=W_{j_k}^{(s)}]\\
&\qquad=\left\{\begin{matrix}\frac{1}{q^r} & \text{if } W_{j_k}^{(s)}\subseteq TW_{j_{k+1}}^{(s)}\\ 0 & \text{else}\end{matrix}\right\} = \mathbf{Pr}[Y_{k+1}^{(s)}=W_{j_{k+1}}^{(s)}|Y_k^{(s)}=W_{j_k}^{(s)}],
\end{aligned}$$

i.e. the Markov chain property is fulfilled.

As already noted, each f_ℓ is constant on each $W_j^{(s)}$ because of $W_j^{(s)}\subseteq S_{b_1,1}^{(s)}\cap\cdots\cap S_{b_d,d}$ for some b_i. Therefore we define the d-dimensional function f on $(W_j^{(s)})_{1\leq j\leq\kappa_s}$ by

$$f(W_j^{(s)})=\left(f_1(W_j^{(s)}),\dots,f_d(W_j^{(s)})\right)=(f_1(b_1),\dots,f_d(b_d)).$$

Before stating a central limit theorem for $f(Y_k^{(s)})$, we study the covariance $\mathbf{Cov}(f_i(Y_{k_i}^{(s)}),f_j(Y_{k_j}^{(s)}))$. To this effect, the following lemma, which will be proved together with Proposition 2, will be very useful. Note that $Y_k^{(s)}\subseteq S_{b,\ell}^{(s)}$ is equivalent to $f_\ell(Y_k^{(s)})=b$.

Lemma 2.

$$\mathbf{Pr}[Y_{k_i}^{(s)}\subseteq S_{b_i,i}^{(s)},Y_{k_j}^{(s)}\subseteq S_{b_j,j}^{(s)}]=\sum_{m_i,m_j:\frac{m_iP_i^{(s)}(n)}{q^{k_i}}+\frac{m_jP_j^{(s)}(n)}{q^{k_j}}\equiv 0} c_{m_i,b_i,q}c_{m_j,b_j,q}, \qquad (1)$$

where $c_{m,b,q}$ are the Fourier coefficients of $\mathbf{1}_{[b/q,(b+1)/q)}$

$$c_{0,b,q}=\frac{1}{q},\quad c_{m,b,q}=\frac{e\left(-\frac{mb}{q}\right)-e\left(-\frac{m(b+1)}{q}\right)}{2\pi i m}\ \textit{for } m\neq 0.$$

By Lemma 2, we have

$$\mathbf{Pr}[Y_{k_i}^{(s)}\subseteq S_{b_i,i}^{(s)},Y_{k_j}^{(s)}\subseteq S_{b_j,j}^{(s)}]=c_{0,b_i,q}c_{0,b_j,q}=\mathbf{Pr}[Y_{k_i}^{(s)}\subseteq S_{b_i,i}^{(s)}]\mathbf{Pr}[Y_{k_j}^{(s)}\subseteq S_{b_j,j}^{(s)}]$$

if the polynomials do not have the same degree or are not proportional. Then $\mathbf{Cov}(f_i(Y_{k_i}^{(s)}), f_j(Y_{k_j}^{(s)})) = 0$.

Now assume $r_i = r_j$ and that the polynomials are proportional. Furthermore, let w.l.o.g. $k_i \geq k_j$. Then the m_i in (1) must satisfy $m_i g_r^{(i)} \equiv 0\,(q^{k_i-k_j})$, i.e. $m_i \equiv 0\left(\frac{q^{k_i-k_j}}{(q^{k_i-k_j}, g_r^{(i)})}\right)$. If $k_i - k_j \geq R_i$, this implies $m_i \equiv 0\,(q)$. Hence we have $c_{m_i,b_i,q} c_{m_j,b_j,q} = 0$ for $(m_i, m_j) \neq (0,0)$ and

$$\mathbf{Cov}\left(f_i(Y_{k_i}^{(s)}), f_j(Y_{k_j}^{(s)})\right) = 0 \quad \text{ if } k_i - k_j \geq R_i \text{ or } k_j - k_i \geq R_j.$$

(For $k_j \geq k_i$, we get the result by the symmetry of the covariance.)

Since the Markov chain $(Y_k^{(s)})_{k\geq 0}$ is homogeneous, we obtain

$$\begin{aligned}
\mathbf{Cov}&\left(\sum_{k=A(N)}^{B(N)} f_i(Y_k^{(s)}), \sum_{k=A(N)}^{B(N)} f_j(Y_k^{(s)})\right) \\
&= \sum_{k=A(N)}^{B(N)} \sum_{l=\max(-R_i+1, A(N)-k)}^{\min(R_j-1, B(N)-k)} \mathbf{Cov}\left(f_i(Y_k^{(s)}), f_j(Y_{k+l}^{(s)})\right) \\
&= (B(N) - A(N)) \sum_{l=-R_i+1}^{R_j-1} \mathbf{Cov}\left(f_i(Y_k^{(s)}), f_j(Y_{k+l}^{(s)})\right) + \mathcal{O}(1)
\end{aligned}$$

for $A(N) = [(\log N)^\eta], B(N) = [\log_q N] - [(\log N)^\eta]$.

Now we can state the central limit theorem.

Proposition 2. *The sums of the random variables $f(Y_k^{(s)})$ satisfy a multidimensional central limit theorem with convergence of moments. More precisely, we have, for all $\mathbf{a} = (a_1, \ldots, a_d) \in \mathbb{R}^d$, as $N \to \infty$,*

$$\frac{\sum_{k=A(N)}^{B(N)} \sum_{\ell=1}^d \frac{a_\ell}{\sigma_\ell} f_\ell(Y_k^{(s)}) - \sum_{\ell=1}^d \frac{a_\ell}{\sigma_\ell} \overline{M}_\ell(N)}{\sqrt{B(N) - A(N)}} \to \mathcal{N}\left(0, \mathbf{a} V^{(s)} \mathbf{a}^t\right), \tag{2}$$

where the covariance matrix $V^{(s)} = \left(v_{i,j}^{(s)}\right)_{1\leq i,j\leq d}$ is given by

$$v_{i,j}^{(s)} = \frac{1}{\sigma_i \sigma_j} \sum_{l=-R_i+1}^{R_j-1} \mathbf{Cov}\left(f_i(Y_k^{(s)}), f_j(Y_{k+l}^{(s)})\right)$$

and for all integers $h_\ell \geq 0$ we have

$$\mathbf{E} \prod_{\ell=1}^d \left(\frac{\sum_{k=A(N)}^{B(N)} f_\ell(Y_k^{(s)}) - \overline{M}_\ell(N)}{\overline{D}_\ell(N)}\right)^{h_\ell} \to \int x_1^{h_1} \cdots x_d^{h_d} \, d\Phi_{V^{(s)}}(x_1, \ldots, x_d). \tag{3}$$

Proof. We have

$$\begin{aligned}
\mathbf{Var} \sum_{\ell=1}^d \sum_{k=A(N)}^{B(N)} \frac{a_\ell}{\sigma_\ell} f_\ell(Y_k^{(s)}) &= \sum_{i=1}^d \sum_{j=1}^d \mathbf{Cov}\left(\sum_{k=A(N)}^{B(N)} \frac{a_i}{\sigma_i} f_i(Y_k^{(s)}), \sum_{k=A(N)}^{B(N)} \frac{a_j}{\sigma_j} f_j(Y_k^{(s)})\right) \\
&= (B(N) - A(N)) \sum_{i=1}^d \sum_{j=1}^d \frac{a_i a_j}{\sigma_i \sigma_j} \sum_{l=-R_i+1}^{R_j-1} \mathbf{Cov}\left(f_i(Y_k^{(s)}), f_j(Y_{k+l}^{(s)})\right) + \mathcal{O}(1) \\
&= (B(N) - A(N)) \mathbf{a} V^{(s)} \mathbf{a}^t + \mathcal{O}(1).
\end{aligned}$$

If $\mathbf{a}V^{(s)}\mathbf{a}^t = 0$, then $\sum_{\ell=1}^{d}\sum_{k=A(N)}^{B(N)} a_\ell f_\ell(Y_k^{(s)}) = \mathcal{O}(1)$ and both sides in (2) are zero.

Otherwise, use the central limit theorem for stationary and homogeneous Markov chains or φ-mixing sequences (see e.g. Billingsley [2], p. 364) which holds if all states are recurrent and aperiodic. For $Y_k^{(s)}$, this condition is satisfied, since we clearly have an integer m such that $T^m W_j^{(s)} = \mathbb{T}^{r-s+1}$ for all $W_j^{(s)}$ and hence $\mathbf{Pr}[Y_{k+l}^{(s)} = W_{j_{k+l}}^{(s)} | Y_k^{(s)} = W_{j_k}^{(s)}] > 0$ for all $l \geq m$. This implies the φ-mixing property for $X_k = \sum_{\ell=1}^{d} a_\ell f_\ell(Y_k^{(s)})$ and the central limit theorem holds for X_k, too. (Note that X_k need not be a Markov chain, if $\sum_{\ell=1}^{d} a_\ell f_\ell$ is not injective.)

For the convergence of moments, it suffices to show that they exist. The onedimensional moments are

$$\mathbf{E}\left(\frac{\sum_{k=A(N)}^{B(N)} f_\ell(Y_k^{(s)}) - \overline{M}_\ell(N)}{\overline{D}_\ell(N)}\right)^{h_\ell} \sim \frac{1}{N}\sum_{n<N}\left(\frac{\sum_{k=A(N)}^{B(N)} f_\ell(\epsilon_k(n)) - \overline{M}_\ell(N)}{\overline{D}_\ell(N)}\right)^{h_\ell}$$

and converge therefore (cf. [1]). The multidimensional moments converge since $\mathbf{E}\left|X_N^r \tilde{X}_N^s\right| \leq \left(\mathbf{E}\, X_N^{2r}\right)^{\frac{1}{2}} \left(\mathbf{E}\, \tilde{X}_N^{2s}\right)^{\frac{1}{2}}$ holds for all random variables $X_N, \tilde{X}_N$. Thus the proposition is proved.

For the calculation of $\mathbf{Cov}(f_i(Y_k^{(s)}), f_j(Y_j^{(s)}))$, it suffices to consider $Y_k = Y_k^{(1)}$ and linear polynomials because of Lemma 2 and the succeeding remarks. For the sum-of-digits function, we even get explicit expressions.

Lemma 3. *Let $P_1(n) = g_1 n$, $P_2(n) = g_2 n$ and $f_1(n) = f_2(n) = s_q(n)$. Then the covariance of $f_1(Y_k)$ and $f_2(Y_k)$ is given by*

$$\mathbf{Cov}(f_1(Y_k), f_2(Y_k)) = \frac{(q^2 - d_1^2 - d_2^2 + 1)(g_1, g_2)^2}{12 g_1 g_2}, \tag{4}$$

where $d_1 = \left(q, \frac{g_1}{(g_1,g_2)}\right)$ and $d_2 = \left(q, \frac{g_2}{(g_1,g_2)}\right)$.

Proof. The covariance is given by

$$\begin{aligned}&\mathbf{Cov}(f_1(Y_k), f_2(Y_k)) \\ &\qquad = \sum_{b_1=0}^{q-1}\sum_{b_2=0}^{q-1} \mathbf{Pr}[\epsilon_k(g_1 n) = b_1, \epsilon_k(g_2 n) = b_2] b_1 b_2 - \mathbf{E}\, f_1(Y_k) \mathbf{E}\, f_2(Y_k).\end{aligned} \tag{5}$$

Because of Lemma 2, the digit probability does not change if we replace g_1, g_2 by $\frac{g_1}{(g_1,g_2)}, \frac{g_2}{(g_1,g_2)}$. Therefore assume $(g_1, g_2) = 1$. In order to get integers, set

$$\begin{aligned}a_{b_1,b_2} &= q g_1 g_2 \mathbf{Pr}[\epsilon_k(g_1 n) = b_1, \epsilon_k(g_2 n) = b_2] \\ &= \#\left\{x \in \{0,1,\ldots,q g_1 g_2 - 1\} \,\middle|\, \left[\frac{x}{g_2}\right] \equiv b_1\ (q), \left[\frac{x}{g_1}\right] \equiv b_2\ (q)\right\}.\end{aligned}$$

We study $A_{i,j} = \sum_{b_1=q-i}^{q-1}\sum_{b_2=q-j}^{q-1} a_{b_1,b_2}$ because of

$$\sum_{b_1=0}^{q-1}\sum_{b_2=0}^{q-1} a_{b_1,b_2} b_1 b_2 = \sum_{i=1}^{q-1}\sum_{b_1=q-i}^{q-1}\sum_{j=1}^{q-1}\sum_{b_2=q-j}^{q-1} a_{b_1,b_2}.$$

For every x in the set corresponding to a_{b_1,b_2}, $(qg_1g_2 - 1 - x)$ is in the set corresponding to $a_{q-1-b_1,q-1-b_2}$. Therefore we have $a_{b_1,b_2} = a_{q-1-b_1,q-1-b_2}$ and

$$
\begin{aligned}
A_{i,j} &= \sum_{b_1=0}^{i-1} \sum_{b_2=0}^{j-1} a_{b_1,b_2} \\
&= \#\{x \in \{0, \ldots, qg_1g_2 - 1\} \mid x \equiv 0, \ldots, ig_2 - 1 (qg_2), x \equiv 0, \ldots, jg_1 - 1 (qg_1)\}
\end{aligned}
$$

Since $(qg_1, qg_2) = q$, the system of congruences $x \equiv x_1\,(qg_2)$ and $x \equiv x_2\,(qg_1)$ has no solution x if $x_1 \not\equiv x_2\,(q)$ and a unique solution modulo qg_1g_2 for $x_1 \equiv x_2\,(q)$. If we denote the representative y' of $y' \equiv y\,(q)$ with $0 \le y' < q$ by $\overline{y}^{(q)}$, then

$$
\begin{aligned}
A_{i,j} &= ig_2 \frac{jg_1 - \overline{jg_1}^{(q)}}{q} + \overline{jg_1}^{(q)} \frac{ig_2 - \overline{ig_2}^{(q)}}{q} + \min(\overline{ig_2}^{(q)}, \overline{jg_1}^{(q)}) \\
&= \frac{ig_2 jg_1}{q} - \frac{\overline{ig_2}^{(q)} \overline{jg_1}^{(q)}}{q} + \min(\overline{ig_2}^{(q)}, \overline{jg_1}^{(q)}).
\end{aligned}
$$

Hence

$$
\sum_{i=1}^{q-1} \sum_{j=1}^{q-1} A_{i,j} = \frac{q(q-1)^2}{4} g_1 g_2 - \frac{q(q-d_1)(q-d_2)}{4} + d_1 d_2 \sum_{i=1}^{q''-1} \sum_{j=1}^{q'-1} \min(id_2, jd_1),
$$

where $q' = q/d_1$ and $q'' = q/d_2$. We have

$$
\begin{aligned}
&\sum_{i=1}^{q''-1} \sum_{j=1}^{q'-1} \min(id_2, jd_1) \\
&\quad = \sum_{i=1}^{q''-1} id_2 \left(q' - 1 - \left[\frac{id_2}{d_1} \right] \right) + \sum_{j=1}^{q'-1} jd_1 \left(q'' - 1 - \left[\frac{jd_1}{d_2} \right] \right) + \sum_{i=1}^{\frac{q''}{d_1}-1} id_1 d_2
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{i=1}^{q''-1} i \left(q' - 1 - \left[\frac{id_2}{d_1} \right] \right) &= (q'-1) \sum_{i=1}^{q''-1} i - \frac{d_2}{d_1} \sum_{i=1}^{q''-1} i^2 - \frac{1}{d_1} \sum_{i=1}^{q''-1} \overline{id_2}^{(d_1)} i \\
&= \frac{(q'-1)(q''-1)q''}{2} - \frac{q'(q''-1)(2q''-1)}{6} + \frac{1}{d_1} \sum_{j=0}^{\frac{q''}{d_1}-1} \sum_{i=1}^{d_1-1} (jd_1 + i) \overline{id_2}^{(d_1)} \\
&= \frac{q'(q''^2-1)}{6} + \frac{q''}{4} \left(-q'' - \frac{q''}{d_1} - d_1 + 3 \right) + \frac{q''}{d_1^2} \sum_{i=1}^{d_1-1} \overline{id_2}^{(d_1)} i.
\end{aligned}
$$

With

$$\begin{aligned}
\frac{d_2}{d_1}\sum_{i=1}^{d_1-1}\overline{id_2}^{(d_1)}i =&\frac{d_2}{d_1}\left(\sum_{i=1}^{d_1-1} d_2 i^2 - \sum_{i=\left[\frac{d_1}{d_2}\right]+1}^{\left[\frac{2d_1}{d_2}\right]} d_1 i - \cdots - \sum_{i=\left[\frac{(d_2-1)d_1}{d_2}\right]+1}^{d_1-1} (d_2-1)d_1 i\right)\\
=&d_2\left(\sum_{i=1}^{d_1-1}\frac{d_2}{d_1}i^2 - (d_2-1)\sum_{i=1}^{d_1-1} i + \sum_{i=1}^{\left[\frac{(d_2-1)d_1}{d_2}\right]} i + \cdots + \sum_{i=1}^{\left[\frac{d_1}{d_2}\right]} i\right)\\
=&\frac{d_2^2(d_1-1)(2d_1-1)}{6} - \frac{d_2(d_2-1)(d_1-1)d_1}{2}\\
&+\sum_{j=1}^{d_2-1}\frac{(jd_1-\overline{jd_1}^{(d_2)}+d_2)(jd_1-\overline{jd_1}^{(d_2)})}{2d_2}\\
=&\frac{d_1^2+d_2^2+1}{12}+\frac{d_1^2d_2+d_1d_2^2-3d_1d_2}{4}-\frac{d_1}{d_2}\sum_{j=1}^{d_2-1}\overline{jd_1}^{(d_2)}j
\end{aligned}$$

we obtain

$$\begin{aligned}
g_1g_2\mathbf{Cov}(f_1(Y_k),f_2(Y_k)) =&\frac{1}{q}\sum_{i=1}^{q-1}\sum_{j=1}^{q-1}A_{i,j} - g_1g_2\frac{(q-1)^2}{4}\\
=&-\frac{(q-d_1)(q-d_2)}{4}+\frac{q^2-d_2^2}{6}+\frac{-d_1q-q-d_1^2d_2+3d_1d_2}{4}\\
&+\frac{d_1^2+d_2^2+1}{12}+\frac{d_1^2d_2+d_1d_2^2-3d_1d_2}{4}\\
&+\frac{q^2-d_1^2}{6}+\frac{-d_2q-q-d_1d_2^2+3d_1d_2}{4}+\frac{q-d_1d_2}{2}\\
=&\frac{q^2-d_1^2-d_2^2+1}{12}
\end{aligned}$$

and the lemma is proved.

Clearly we have

$$\mathbf{Pr}[\epsilon_k(g_1n)=b_1,\epsilon_k(g_2n)=b_2]=\frac{A_{b_i+1,b_j+1}-A_{b_i,b_j+1}-A_{b_i+1,b_j}+A_{b_i,b_j}}{qg_1g_2}$$

for $(g_1,g_2)=1$. Thus

$$\mathbf{Pr}[\epsilon_k(g_1n)=b_1,\epsilon_k(g_2n)=b_2]=\pi_{b_1,b_2,g_1,g_2}$$

first for $(g_1,g_2)=1$, and, with Lemma 2, for general g_1,g_2. With the remarks succeeding Theorem 4, we get

$$v_{i,j}^{(s)}=\begin{cases} C_{i,j}\left(\frac{g_{r_i}^{(i)}}{(g_{r_i}^{(i)},g_{r_j}^{(j)})},\frac{g_{r_j}^{(j)}}{(g_{r_i}^{(i)},g_{r_j}^{(j)})}\right) & \text{if } g_{r_j}^{(j)}P_i^{(s)}(n)=g_{r_i}^{(i)}P_j^{(s)}(n)\\ 0 & \text{else.}\end{cases}$$

For $q_1 = \tilde{q}^{s_1}$ and $q_2 = \tilde{q}^{s_2}$, $f_1(n) = s_{q_1}(n)$ and $f_2(n) = s_{q_2}(n)$ are strongly q-additive functions with $q = q_1^{s_2} = q_2^{s_1}$. Then, for $P_1(n) = P_2(n) = n$, $(Y_k)_{k\geq 0}$ is clearly a sequence of independent random variables and

$$f_1(Y_k) = X_0 + \tilde{q}X_1 + \cdots + \tilde{q}^{s_1-1}X_{s_1-1} + X_{s_1} + \cdots + \tilde{q}^{s_1-1}X_{2s_1-1} + \cdots + \tilde{q}^{s_1-1}X_{s_1s_2-1},$$

$$f_2(Y_k) = X_0 + \tilde{q}X_1 + \cdots + \tilde{q}^{s_2-1}X_{s_2-1} + X_{s_2} + \cdots + \tilde{q}^{s_2-1}X_{2s_2-1} + \cdots + \tilde{q}^{s_2-1}X_{s_1s_2-1},$$

where $(X_j)_{0\leq j\leq s_1s_2-1}$ is a sequence of identically distributed independent random variables on $\{0, 1, \ldots, \tilde{q}-1\}$.

Hence we have

$$\mathbf{Cov}(f_1(Y_k), f_2(Y_k)) = \sum_{j=0}^{s_1s_2-1} c_j \mathbf{Var}\, X_j,$$

where c_j runs through $\{\tilde{q}^{ab} : 0 \leq a \leq s_1 - 1, 0 \leq b \leq s_2 - 1\}$ because of $(s_1, s_2) = 1$. This implies

$$\begin{aligned}\mathbf{Cov}(f_1(Y_k), f_2(Y_k)) &= \frac{\tilde{q}^2-1}{12}\left(1+\tilde{q}+\cdots+\tilde{q}^{s_1-1}\right)\left(1+\tilde{q}+\cdots+\tilde{q}^{s_2-1}\right)\\ &= \frac{(\tilde{q}+1)(\tilde{q}^{s_1}-1)(\tilde{q}^{s_2}-1)}{12(\tilde{q}-1)}.\end{aligned}$$

With $\sigma_1^2 = \mathbf{Var}\, f_1(Y_k) = s_2(q_1^2-1)/12$ and $\sigma_2^2 = \mathbf{Var}\, f_2(Y_k) = s_1(q_2^2-1)/12$, we get for the normalized covariance

$$\frac{\mathbf{Cov}(f_1(Y_k), f_2(Y_k))}{\sigma_1\sigma_2} = \frac{\tilde{q}+1}{\tilde{q}-1}\frac{(q_1-1)(q_2-1)}{\sqrt{s_1s_2(q_1^2-1)(q_2^2-1)}}.$$

2.3. Comparison of moments.

It remains to compare the moments of $f_\ell(P_\ell(n))$ to those in (3). We need the following proposition (cf. Proposition 1).

Proposition 3. *Let $P_\ell(x)$, $1 \leq \ell \leq d$, be integer polynomials with positive leading terms, $\lambda > 0$ an arbitrary constant and h_ℓ, $1 \leq \ell \leq d$, non-negative integers. Then for integers*

$$(\log N)^\eta \leq k_1^{(\ell)} < k_2^{(\ell)} < \cdots < k_{h_\ell}^{(\ell)} \leq \log_q N^{r_\ell} - (\log N)^\eta \quad (1 \leq \ell \leq d)$$

(with some $\eta > 0$) which satisfy

$$k_j^{(\ell)} \notin \left(\log_q N^s - (\log N)^\eta, \log_q N^s + (\log N)^\eta\right)$$

for all $1 \leq s \leq r_\ell - 1$, we have uniformly, as $N \to \infty$,

$$\begin{aligned}\frac{1}{N}\#\left\{n < N \,\middle|\, \epsilon_{k_j^{(\ell)}}(P_\ell(n)) = b_j^{(\ell)}, 1 \leq j \leq h_\ell, 1 \leq \ell \leq d\right\}\\ = \prod_{s=1}^{r} p^{(s)}_{k_1^{(1)},\cdots,k_{h_d}^{(d)},b_1^{(1)},\ldots,b_{h_d}^{(d)}} + \mathcal{O}\left((\log N)^{-\lambda}\right)\end{aligned}$$

and

$$\frac{1}{\pi(N)}\#\left\{p<N \,\middle|\, \epsilon_{k_j^{(\ell)}}(P_\ell(n))=b_j^{(\ell)}, 1\le j\le h_\ell, 1\le \ell\le d\right\}$$
$$=\prod_{s=1}^{r} p^{(s)}_{k_1^{(1)},\cdots,k_{h_d}^{(d)},b_1^{(1)},\dots,b_{h_d}^{(d)}}+\mathcal{O}\left((\log N)^{-\lambda}\right)$$

with

$$p^{(s)}_{k_1^{(1)},\cdots,k_{h_d}^{(d)},b_1^{(1)},\dots,b_{h_d}^{(d)}}=\begin{cases}\mathbf{Pr}\left[Y^{(s)}_{k_j^{(\ell)}}\subseteq S^{(s)}_{b_j^{(\ell)},\ell} \textit{ for all } (j,\ell)\in K_s\right] & \textit{if } K_s\neq\emptyset\\ 1 & \textit{else,}\end{cases}$$

where

$$K_s=\left\{(j,\ell)\,\middle|\,k_j^{(\ell)}\in\left[\log_q N^{s-1}+(\log N)^\eta,\log_q N^s-(\log N)^\eta\right]\right\}.$$

Proof. We follow the proofs of Lemma 5 in [1] and Proposition 1 in [3]. Let $\psi_{b,q,\Delta}(x)$ be defined by

$$\psi_{b,q,\Delta}(x)=\frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2}\mathbf{1}_{[b/q,(b+1)/q)}(\{x+z\})dz.$$

Its Fourier series $\sum_{m\in\mathbb{Z}} d_{m,b,q,\Delta}e(mx)$ is given by $d_{m,0,q,\Delta}=\frac{1}{q}$ and

$$d_{m,b,q,\Delta}=\frac{e\left(-\frac{mb}{q}\right)-e\left(-\frac{m(b+1)}{q}\right)}{2\pi im}\frac{e\left(\frac{m\Delta}{2}\right)-e\left(-\frac{m\Delta}{2}\right)}{2\pi im\Delta}\text{ for } m\neq 0.$$

Clearly we have

$$\psi_{b,q,\Delta}(x)=\begin{cases}1 & \text{if } x\in\left[\frac{b}{q}+\Delta,\frac{b+1}{q}-\Delta\right],\\ 0 & \text{if } x\in[0,1]\setminus\left[\frac{b}{q}-\Delta,\frac{b+1}{q}+\Delta\right].\end{cases}$$

If we set

$$t(y_1,\dots,y_d)=\prod_{\ell=1}^{d}\prod_{j=1}^{h_\ell}\psi_{b_j^{(\ell)},q_\ell,\Delta}\left(\frac{y_\ell}{q_\ell^{k_j^{(\ell)}+1}}\right),$$

then we get for $\Delta<1/(2q)$

$$\left|\#\left\{n<N\,\middle|\,\epsilon_{k_j^{(\ell)}}(P_\ell(n))=b_j^{(\ell)},1\le j\le h_\ell,1\le\ell\le d\right\}-\sum_{n<N}t(P_1(n),\dots,P_d(n))\right|$$
$$\le\sum_{\ell=1}^{d}\sum_{j=1}^{h_\ell}\#\left\{n<N\,\middle|\,\left\{\frac{P_\ell(n)}{q_\ell^{k_j^{(\ell)}+1}}\right\}\in U_{b_j^{(\ell)},q_\ell,\Delta}\right\}\ll\Delta N+N(\log N)^{-\lambda}$$

with $U_{b,q,\Delta}=[0,\Delta]\cup\bigcup_{b=1}^{q-1}\left[\frac{b}{q}-\Delta,\frac{b}{q}+\Delta\right]\cup[1-\Delta,1]$ and Lemma 4 of [1]. For primes, we get a similar statement.

Hence we have to consider the sums

$$\Sigma = \sum_{n<N} t(P_1(n),\ldots,P_d(n)) = \sum_{\mathbf{M}\in\mathcal{M}} T_{\mathbf{M}} \sum_{n<N} e\left(\mathbf{m}_1\cdot\mathbf{v}_1 P_1(n) + \cdots + \mathbf{m}_d\cdot\mathbf{v}_d P_d(n)\right),$$

where $\mathcal{M}$ is the set of all $(\mathbf{m}_1,\ldots,\mathbf{m}_d)$ with integer vectors $\mathbf{m}_\ell = (m_1^{(\ell)},\ldots,m_{h_\ell}^{(\ell)})$,

$$T_{\mathbf{M}} = \prod_{\ell=1}^{d}\prod_{j=1}^{h_\ell} d_{m_j^{(\ell)},b_j^{(\ell)},q,\Delta}$$

and $\mathbf{v}_\ell = \left(q_\ell^{-k_1^{(\ell)}-1},\ldots,q_\ell^{-k_{h_\ell}^{(\ell)}-1}\right)$.

First of all, set $\Delta = (\log N)^{-\delta}$ with an arbitrary (but fixed) constant $\delta > 0$. Then we can restrict to those $\mathbf{M}$ for which $|m_j^{(\ell)}| < (\log N)^{2\delta}$ for all j,ℓ because of

$$\sum_{\exists \ell,j:|m_j^{(\ell)}|\geq(\log N)^{2\delta}} |T_{\mathbf{M}}| \ll \left(\sum_{m=[(\log N)^{2\delta}]}^{\infty}\frac{1}{\Delta m^2}\right)\left(\sum_{m=0}^{\infty}\min\left(1,\frac{1}{m},\frac{1}{\Delta m^2}\right)\right)^{h-1}$$
$$\ll \frac{1}{\Delta}(\log N)^{-\delta}\left(\log\frac{1}{\Delta}\right)^{h-1} \ll (\log N)^{-\delta/2},$$

where $h = h_1+\cdots+h_d$. Furthermore, it is sufficient to consider just the case where $m_j^{(\ell)} \neq 0$ for all j,ℓ. (Otherwise, just reduce h_ℓ to a smaller value.)

Set

$$Q_{\mathbf{M}}(n) = \mathbf{m}_1\cdot\mathbf{v}_1 P_1(n) + \cdots + \mathbf{m}_d\cdot\mathbf{v}_d P_d(n).$$

We have to check whether $Q_{\mathbf{M}}(n)$ has degree r and satisfies the conditions of Lemmata 1 and 2 of [1] saying that

$$\frac{1}{N}\sum_{n<N} e(P(n)) = \mathcal{O}\left((\log N)^{-\tau_0}\right),$$
$$\frac{1}{\pi(N)}\sum_{p<N} e(P(p)) = \mathcal{O}\left((\log N)^{-\tau_0}\right),$$

as $N\to\infty$, hold if the the leading coefficient of $P(n)$ is $\frac{A}{H}$ with $(A,H)=1$ and

$$(\log N)^{\tau} < H < N^r(\log N)^{-\tau} \tag{6}$$

for some τ (depending on τ_0).

The coefficient of n^r is, if we set $k_{\max} = \max_\ell k_{h_\ell}^{(\ell)}$,

$$\frac{A_{\mathbf{M}}}{H_{\mathbf{M}}} = \sum_{(j,\ell)\in K_r}\frac{g_r^{(\ell)} m_j^{(\ell)} q^{k_{\max}-k_j^{(\ell)}}}{q^{k_{\max}}} + \sum_{(j,\ell)\notin K_r}\frac{g_r^{(\ell)} m_j^{(\ell)} q^{k_{\max}-k_j^{(\ell)}}}{q^{k_{\max}}} \tag{7}$$

with $(A_{\mathbf{M}},H_{\mathbf{M}}) = 1$. If $A_{\mathbf{M}} \neq 0$, then (6) is satisfied. If $A_{\mathbf{M}} = 0$, assume $k_{\max}\in K_r$. Then we obtain

$$\sum_{(j,\ell)\in K_r} g_r^{(\ell)} m_j^{(\ell)} q^{k_{\max}-k_j^{(\ell)}} \equiv 0 \left(q^{k_{\max}-(\log_q N^{r-1}-(\log N)^{\eta})}\right).$$

Because of $|m_j^{(\ell)}| < (\log N)^{2\delta}$, this implies $\sum_{(j,\ell)\in K_r} g_r^{(\ell)} m_j^{(\ell)} q^{k_{\max}-k_j^{(\ell)}} = 0$. Hence $A_{\mathbf{M}} = 0$ if and only if both sums in (7) are zero and we have

$$\begin{aligned}
&\frac{1}{N}\#\left\{n<N \,\middle|\, \epsilon_{k_j^{(\ell)}}(P_\ell(n)) = b_j^{(\ell)}, 1\le j\le h_\ell, 1\le \ell\le d\right\}\\
&=\frac{1}{N}\#\left\{n<N \,\middle|\, \epsilon_{k_j^{(\ell)}}(P_\ell(n)) = b_j^{(\ell)}, (j,\ell)\in K_r\right\}\\
&\quad\times\frac{1}{N}\#\left\{n<N \,\middle|\, \epsilon_{k_j^{(\ell)}}(P_\ell(n)) = b_j^{(\ell)}, (j,\ell)\notin K_r\right\} + \mathcal{O}\left((\log N)^{-\lambda}\right).
\end{aligned}$$

Now we can repeat the arguments for $(j,\ell)\in K_{r-1}$ and get inductively

$$\begin{aligned}
&\frac{1}{N}\#\left\{n<N \,\middle|\, \epsilon_{k_j^{(\ell)}}(P_\ell(n)) = b_j^{(\ell)}, 1\le j\le h_\ell, 1\le \ell\le d\right\}\\
&\qquad=\prod_{s=1}^{r}\frac{1}{N}\#\left\{n<N \,\middle|\, \epsilon_{k_j^{(\ell)}}(P_\ell(n)) = b_j^{(\ell)}, (j,\ell)\in K_s\right\} + \mathcal{O}\left((\log N)^{-\lambda}\right).
\end{aligned}$$

Hence we may assume from now on that all $k_j^{(\ell)}$ are contained in one set K_s for some $s\le r$.

If the degree of $Q_{\mathbf{M}}(n)$ is smaller than s, we have

$$|Q_{\mathbf{M}}(n)| \ll \frac{(\log N)^{2\delta} N^{s-1}}{q^{\log_q N^{s-1} + (\log N)^{\eta}}} = \frac{(\log N)^{2\delta}}{q^{(\log N)^{\eta}}}$$

for all $n<N$ and, with $e(y) = 1+\mathcal{O}(y)$,

$$\sum_{\left|m_j^{(\ell)}\right| < (\log N)^{2\delta}, \deg(Q_{\mathbf{M}}(n))<s} T_{\mathbf{M}}\left(\sum_{n<N} e(Q_{\mathbf{M}}(n)) - N\right) \ll \frac{N(\log N)^{2\delta(h+1)}}{q^{(\log N)^{\eta}}}.$$

Thus we can treat these $Q_{\mathbf{M}}(n)$ as if they were the zero polynomial and it suffices to regard the polynomials $P_\ell^{(s)}(n)$ and

$$Q_{\mathbf{M}}^{(s)}(n) = \mathbf{m}_1\cdot\mathbf{v}_1 P_1^{(s)}(n) + \cdots + \mathbf{m}_d\cdot\mathbf{v}_d P_d^{(s)}(n).$$

(6) is satisfied if and only if $Q_{\mathbf{M}}^{(s)}(n)\not\equiv 0$ and we obtain

$$\begin{aligned}
\Sigma = N\sum_{\mathbf{M}\in\mathcal{M}: Q_{\mathbf{M}}^{(s)}(n)\equiv 0} T_{\mathbf{M}} + \mathcal{O}&\left(N(\log N)^{-\tau_0}\sum_{\mathbf{M}\in\mathcal{M}: |m_j^{(\ell)}|<(\log N)^{2\delta}, Q_{\mathbf{M}}^{(s)}(n)\not\equiv 0} |T_{\mathbf{M}}|\right)\\
&+\mathcal{O}\left(N(\log N)^{-\delta/2}\right) + \mathcal{O}\left(N(\log N)^{-\lambda}\right).
\end{aligned}$$

Since the main term $\sum_{\mathbf{M}\in\mathcal{M}: Q_{\mathbf{M}}^{(s)}(n)\equiv 0} T_{\mathbf{M}}$ depends on Δ, we have to replace $T_{\mathbf{M}}$ by

$$T'_{\mathbf{M}} = \prod_{\ell=1}^{d}\prod_{j=1}^{h_\ell} c_{m_j^{(\ell)}, b_j^{(\ell)}, q}.$$

Hence we have to estimate the difference $\sum_{\mathbf{M}\in\mathcal{M}:Q_{\mathbf{M}}(n)\equiv 0}(T_{\mathbf{M}} - T'_{\mathbf{M}})$.

We clearly have

$$d_{m_j^{(\ell)},b_j^{(\ell)},q,\Delta} = c_{m_j^{(\ell)},b_j^{(\ell)},q}\left(1 + \mathcal{O}\left(m_j^{(\ell)}\Delta\right)\right)$$

as $\Delta \to 0$ and therefore

$$T_{\mathbf{M}} = T'_{\mathbf{M}}\left(1 + \mathcal{O}\left(\max_{j,\ell} m_j^{(\ell)}\Delta\right)\right). \tag{8}$$

First assume $|m_j^{(\ell)}| < (\log N)^{\delta/2}$ for all j,ℓ. From (8) and $c_{m_j^{(\ell)},b_j^{(\ell)},q} \le \min\left(1, \frac{1}{m_j^{(\ell)}}\right)$, we obtain

$$\begin{aligned}\sum_{\mathbf{M}\in\mathcal{M}:|m_j^{(\ell)}|<(\log N)^{\delta/2}} |T_{\mathbf{M}} - T'_{\mathbf{M}}| &\ll \sum_{\mathbf{M}\in\mathcal{M}:|m_j^{(\ell)}|<(\log N)^{\delta/2}} |T'_{\mathbf{M}}|(\log N)^{-\delta/2} \\ &\ll \left(\sum_{m=1}^{[(\log N)^{\delta/2}]}\frac{1}{m}\right)^h (\log N)^{-\delta/2} \le \frac{\left(\log(\log N)^{\delta/2}\right)^h}{(\log N)^{\delta/2}} \ll (\log N)^{-\delta/3}\end{aligned}$$

It remains to estimate the $T_{\mathbf{M}}$ and $T'_{\mathbf{M}}$ with $|m_j^{(\ell)}| > (\log N)^{\delta/2}$ for some j,ℓ which satisfy the equation $Q_{\mathbf{M}}^{(s)}(n) \equiv 0$, i.e.

$$\sum_{j,\ell} g_r^{(\ell)} q^{k_{\max} - k_j^{(\ell)}} m_j^{(\ell)} = 0.$$

By Lemma 14 of [4], we get

$$\sum_{\mathbf{M}\in\mathcal{M}:Q_{\mathbf{M}}^{(s)}(n)\equiv 0,|m_j^{(\ell)}|\ge(\log N)^{\delta/2}\text{ for some }j,\ell} T'_{\mathbf{M}} \ll (\log N)^{-\frac{\delta}{2(h-1)^2}}$$

and the same estimate for $T_{\mathbf{M}}$. Note that this lemma is stated for a linear equation where one of the coefficients is 1, but the proof can be easily adapted for general linear equations.

Hence

$$\sum_{\mathbf{M}\in\mathcal{M}:Q_{\mathbf{M}}^{(s)}(n)\equiv 0} T_{\mathbf{M}} = \tilde{p}^{(s)}_{k_1^{(1)},\ldots,k_{h_d}^{(d)},b_1^{(1)},\ldots,b_{h_d}^{(d)}} + \mathcal{O}\left((\log N)^{-\frac{\delta}{2(h-1)^2}}\right),$$

where

$$\tilde{p}^{(s)}_{k_1^{(1)},\ldots,k_{h_d}^{(d)},b_1^{(1)},\ldots,b_{h_d}^{(d)}} = \sum_{\mathbf{M}\in\mathcal{M}:Q_{\mathbf{M}}^{(s)}(n)\equiv 0} T'_{\mathbf{M}}$$

and we get

$$\Sigma = N\tilde{p}^{(s)}_{k_1^{(1)},\ldots,k_{h_d}^{(d)},b_1^{(1)},\ldots,b_{h_d}^{(d)}} + \mathcal{O}\left((\log N)^{-\lambda}\right),$$

for $\delta = 2(h-1)^2\lambda$ and $\tau_0 > \lambda$.

It remains to prove that the $\tilde{p}^{(s)}_{k_1^{(1)},\cdots,k_{h_d}^{(d)},b_1^{(1)},\ldots,b_{h_d}^{(d)}}$ are the probabilities defined by the Markov chain.

We have

$$\left\{n < N \,\middle|\, \epsilon_{k_j^{(\ell)}}(P_\ell^{(s)}(n)) = b_j^{(\ell)} \text{ for all } (j,l) \in K_s\right\}$$
$$= \left\{n < N \,\middle|\, \mathbf{w}^{(s)}_{k_{\max}}(n) \in \bigcap_{(j,\ell)\in K_s} T^{k_j^{(\ell)}-k_{\max}} S^{(s)}_{b_j^{(\ell)},\ell}\right\}$$

and this intersection consists of a finite number of convex sets, which can be arbitrarily well approximated by elementary rectangles

$$\prod_{i=s}^{r}\left[\sum_{j=1}^{J_i}\tilde{b}_j^{(i)}q^{-j}, \sum_{j=1}^{J_i}\tilde{b}_j^{(i)}q^{-j}+q^{-J_i}\right).$$

By Proposition 1, we get

$$\frac{1}{N}\#\left\{n < N \,\middle|\, \mathbf{w}^{(s)}_{k_{\max}}(n) \in \prod_{i=s}^{r}\left[\sum_{j=1}^{J_i}\tilde{b}_j^{(i)}q^{-j}, \sum_{j=1}^{J_i}\tilde{b}_j^{(i)}q^{-j}+q^{-J_i}\right)\right\}$$
$$= \frac{1}{N}\#\left\{n < N \,\middle|\, \epsilon_{k_{\max}-j+1}(n^i) = \tilde{b}_j^{(i)}, 1 \le j \le J_i, s \le i \le r\right\} \to \frac{1}{q^{J_s}\ldots q^{J_r}},$$

if $k_{\max} \le \log N - (\log N)^\eta$ and $J_i \le k_{\max} - (\log N)^\eta$. This means that the density in each of this rectangles converges to its Lebesgue measure. Since we do not change $\bigcap_{j,\ell} T^{k_j^{(\ell)}-k_{\max}} S^{(s)}_{b_j^{(\ell)},\ell}$ if we shift all $k_j^{(\ell)}$ and increase N, the J_i can be arbitrarily large. Therefore $\tilde{p}^{(s)}_{k_1^{(1)},\cdots,k_{h_d}^{(d)},b_1^{(1)},\ldots,b_{h_d}^{(d)}}$ must be its Lebesgue measure, which is just $p^{(s)}_{k_1^{(1)},\cdots,k_{h_d}^{(d)},b_1^{(1)},\ldots,b_{h_d}^{(d)}}$.

This also implies Lemma 2 ($d = 2, h_1 = h_2 = 1$).

Proposition 3 shows that we have to replace f_ℓ by $\overline{f}_\ell^{(N^{r_\ell})}$,

$$\overline{f}_\ell^{(N^{r_\ell})}(P_\ell(n)) = \sum_{s=1}^{r_\ell}\sum_{k=(s-1)\log_q N+A(N)}^{(s-1)\log_q N+B(N)} f_\ell(\epsilon_k(P_\ell(n))).$$

Note that $\overline{f}_\ell^{(N^{r_\ell})}(P_\ell(n)) = f_\ell(P_\ell(n)) + \mathcal{O}\left((\log N)^\eta\right)$. Similarly define $\overline{M}_\ell(N^{r_\ell})$ and $\overline{D}_\ell(N^{r_\ell})$ by taking the sum only over these k. Note that these definitions are slightly different from those in [3,4] (and [1], where $\overline{f}$ is denoted by f_1).

Corollary 3. *We have*

$$\frac{1}{N}\sum_{n<N}\prod_{\ell=1}^{d}\left(\frac{\overline{f}_\ell^{(N^{r_\ell})}(P_\ell(n)) - \overline{M}_\ell(N^{r_\ell})}{\overline{D}_\ell(N^{r_\ell})}\right)^{h_\ell}$$
$$- \mathbf{E}\prod_{\ell=1}^{d}\left(\frac{\sum_{s=1}^{r_\ell}\sum_{k=(s-1)\log_q N+A(N)}^{(s-1)\log_q N+B(N)} f_\ell\left(Y_k^{(s)}\right) - \overline{M}_\ell(N^{r_\ell})}{\overline{D}_\ell(N^{r_\ell})}\right)^{h_\ell} \to 0$$

and

$$\frac{1}{\pi(N)}\sum_{p<N}\prod_{\ell=1}^{d}\left(\frac{\overline{f}_\ell^{(N^{r_\ell})}(P_\ell(p))-\overline{M}_\ell(N^{r_\ell})}{\overline{D}_\ell(N^{r_\ell})}\right)^{h_\ell}$$
$$-\mathbf{E}\prod_{\ell=1}^{d}\left(\frac{\sum_{s=1}^{r_\ell}\sum_{k=(s-1)\log_q N+A(N)}^{(s-1)\log_q N+B(N)} f_\ell\left(Y_k^{(s)}\right)-\overline{M}_\ell(N^{r_\ell})}{\overline{D}_\ell(N^{r_\ell})}\right)^{h_\ell}\to 0,$$

where the $Y_k^{(s)}$ and $Y_{k'}^{(s')}$ are independent if $s\neq s'$.

Proof. The second terms are the sum over all integers

$$k_1^{(\ell)},\dots,k_{h_\ell}^{(\ell)}\in[A(N),\log_q N^{r_\ell}-A(N)]\setminus\bigcup_{s=1}^{r_\ell-1}[\log_q N^s-A(N),\log_q N^s+A(N)],$$

$1\le\ell\le d$, of

$$\mathbf{E}\prod_{\ell=1}^{d}\prod_{j=1}^{h_\ell}\frac{f_\ell\left(Y_{k_j^{(\ell)}}^{(s)}\right)-\mu_{\ell,k_j^{(\ell)}}}{D_\ell(N^{r_\ell})}$$
$$=\sum_{b_1^{(1)}=0}^{q-1}\cdots\sum_{b_{h_d}^{(d)}=0}^{q-1}\prod_{\ell=1}^{d}\prod_{j=1}^{h_\ell}\frac{f_\ell(b_j^\ell)-\mu_{\ell,k_j^{(\ell)}}}{\overline{D}_\ell(N^{r_\ell})}\mathbf{Pr}\left[Y_{k_j^{(\ell)}}^{(s)}\subseteq S_{b_j^{(\ell)}}^{(s)}\text{ for all } j,\ell\right],$$

where the s are such that $k_j^{(\ell)}\in K_s$. Since the $Y_{k_j^{(\ell)}}^{(s)}$ are independent for different s, we have

$$\mathbf{Pr}\left[Y_{k_j^{(\ell)}}^{(s)}\subseteq S_{b_j^{(\ell)}}^{(s)}\text{ for all }(j,\ell)\right]=\prod_{s=1}^{r}\mathbf{Pr}\left[Y_{k_j^{(\ell)}}^{(s)}\subseteq S_{b_j^{(\ell)}}^{(s)}\text{ for all }(j,\ell)\in K_s\right]$$

and, by Proposition 3, the corresponding first terms are the same up to an error term of $\mathcal{O}\left((\log N)^{-\lambda}\right)$. Hence the convergences are valid with error terms $\mathcal{O}\left((\log N)^{-\lambda+h-h\eta}\right)$.

Similarly to Corollary 2 of [3], we obtain

$$\frac{1}{N}\sum_{n<N}\prod_{\ell=1}^{d}\left(\frac{f_\ell(P_\ell(n))-M_\ell(N^{r_\ell})}{D_\ell(N^{r_\ell})}\right)^{h_\ell}$$
$$-\frac{1}{N}\sum_{n<N}\prod_{\ell=1}^{d}\left(\frac{\overline{f}_\ell^{(N^{r_\ell})}(P_\ell(n))-\overline{M}_\ell(N^{r_\ell})}{\overline{D}_\ell(N^{r_\ell})}\right)^{h_\ell}\to 0$$

and therefore, by the method of moments (see e.g. Billingsley [2], p. 390),

$$\frac{1}{N}\#\left\{n<N\,\middle|\,\frac{f_\ell(P_\ell(n))-M_\ell(N^{r_\ell})}{D_\ell(N^{r_\ell})}<x_\ell,\ell=1,2,\dots,d\right\}$$
$$\to\mathbf{Pr}\left[\frac{\sum_{s=1}^{r_\ell}\sum_{k=(s-1)\log_q N+A(N)}^{(s-1)\log_q N+B(N)} f_\ell\left(Y_k^{(s)}\right)-\overline{M}_\ell(N^{r_\ell})}{\overline{D}_\ell(N^{r_\ell})}<x_\ell,\ell=1,\dots,d\right].$$

Clearly we have $\overline{M}_\ell(N^{r_\ell}) = r_\ell \overline{M}_\ell(N)$, $\overline{D}_\ell(N^{r_\ell})^2 = r_\ell \overline{D}_\ell(N)^2$ and

$$\frac{\sum_{s=1}^{r_\ell}\sum_{k=(s-1)\log_q N+A(N)}^{(s-1)\log_q N+B(N)} f_\ell\left(Y_k^{(s)}\right) - \overline{M}_\ell(N^{r_\ell})}{\overline{D}_\ell(N^{r_\ell})}$$
$$= \frac{1}{\sqrt{r_\ell}}\sum_{s=1}^{r_\ell}\frac{\sum_{k=A(N)}^{B(N)} f_\ell\left(Y_k^{(s)}\right) - \overline{M}_\ell(N)}{\sigma_\ell\sqrt{B(N)-A(N)+1}} \to \frac{1}{\sqrt{r_\ell}}\left(Z_\ell^{(1)} + \cdots + Z_\ell^{(r)}\right)$$

by Proposition 2, where the $Z^{(s)} = (Z_1^{(s)}, \ldots, Z_d^{(s)})$ are independent normally distributed random vectors with covariance matrices $V^{(s)}$. (For $s > r_\ell$, we have $f_\ell(Y_k^{(s)}) = 0 = Z_\ell^{(s)}$ because of $P_\ell^{(s)}(n) \equiv 0$ and $S_{0,\ell}^{(s)} = \mathbb{T}^{r-s+1}$.)

Hence the sum is normally distributed and the elements of the covariance matrix V are given by

$$v_{i,j} = \frac{1}{\sqrt{r_i r_j}}\left(v_{i,j}^{(1)} + \cdots + v_{i,j}^{(r)}\right).$$

For $r_i \neq r_j$, all $v_{i,j}^{(s)}$ are zero, as well as for all $s > r_i$. If $g_{r_j}^{(j)} P_i(n) \equiv g_{r_i}^{(i)} P_j(n)$, then $v_{i,j}^{(1)} = \cdots = v_{i,j}^{(r_i)} = v_{i,j}$. If we just have $r_i = r_j$ and $g_{r_j}^{(j)} g_s^{(i)} = g_{r_i}^{(i)} g_s^{(j)}$ for all $s > s'$, then $v_{i,j}^{(s'+1)} = \cdots = v_{i,j}^{(r_i)}$ and $v_{i,j}^{(s)} = 0$ for $s \leq s'$. Therefore $v_{i,j} = \frac{r_i - s'}{r_i} v_{i,j}^{(r_i)}$ and the covariance matrix has the stated form.

The corresponding statements for primes are obtained similarly and this concludes the proof of Theorem 4.

3. Proof of Theorem 3

We have to prove the following proposition.

Proposition 4. *Let q_1, q_2 be multiplicatively independent integers and $P_1(n), P_2(n)$ integer polynomials with positive leading terms. Let $\lambda > 0$ be an arbitrary constant and h_1, h_2 non-negative integers. Then for integers*

$$(\log N^{r_\ell})^\eta \leq k_1^{(\ell)} < k_2^{(\ell)} < \cdots < k_{h_\ell}^{(\ell)} \leq \log_{q_\ell} N^{r_\ell} - (\log N^{r_\ell})^\eta \quad (\ell = 1, 2)$$

(with some $\eta > 0$), we have, as $N \to \infty$,

$$\frac{1}{N}\#\left\{n < N \,\middle|\, \epsilon_{q_1, k_j^{(1)}}(P_1(n)) = b_j^{(1)}, \epsilon_{q_2, k_j^{(2)}}(P_2(n)) = b_j^{(2)}, 1 \leq j \leq h_\ell\right\}$$
$$= \frac{1}{q_1^{h_1} q_2^{h_2}} + \mathcal{O}\left((\log N)^{-\lambda}\right)$$

and

$$\frac{1}{\pi(N)}\#\left\{p < N \,\middle|\, \epsilon_{q_1, k_j^{(1)}}(P_1(n)) = b_j^{(1)}, \epsilon_{q_2, k_j^{(2)}}(P_2(n)) = b_j^{(2)}, 1 \leq j \leq h_\ell\right\}$$
$$= \frac{1}{q_1^{h_1} q_2^{h_2}} + \mathcal{O}\left((\log N)^{-\lambda}\right)$$

uniformly for $b_j^{(\ell)} \in \{0, \ldots, q_\ell - 1\}$ and $k_j^{(\ell)}$ in the given range, where the implicit constant of the error term may depend on q_ℓ, P_ℓ, h_ℓ and λ.

For the proof we need the following three lemmata. The first one is a corollary to Baker's theorem on linear forms, in a version due to Waldschmidt [7].

Lemma 4 (Corollary 3 in [3]). *Let k_1, k_2 be positive integers, q_1, q_2 positive real numbers and m_1, m_2 real numbers such that $\frac{m_1}{q_1^{k_1}} + \frac{m_2}{q_2^{k_2}} \neq 0$. Then there exists a constant $C > 0$ such that*

$$\left| \frac{m_1}{q_1^{k_1}} + \frac{m_2}{q_2^{k_2}} \right| \geq \max\left(\frac{|m_1|}{q_1^{k_1}}, \frac{|m_2|}{q_2^{k_2}} \right) e^{-C \log q_1 \log q_2 \log(\max(k_1,k_2)) \log(\max(|m_1|,|m_2|))}.$$

The next lemma is an adapted version of Lemmata 1 and 2 of [1] which are due to Hua [5] and Vinogradov.

Lemma 5 (Lemmata 10 and 11 in [4]). *Let $P(n)$ be a polynomial of degree r with leading coefficient β. For every $\tau_0 > 0$, we have a $\tau > 0$ such that*

$$N^{-r} (\log N)^{\tau} < \beta < (\log N)^{-\tau}$$

implies

$$\frac{1}{N} \sum_{n<N} e(P(n)) = \mathcal{O}\left((\log N)^{-\tau_0} \right)$$

and

$$\frac{1}{\pi(N)} \sum_{p<N} e(P(p)) = \mathcal{O}\left((\log N)^{-\tau_0} \right)$$

as $N \to \infty$.

Proof of Proposition 4. As for Proposition 2, we have to estimate the sums

$$\Sigma = \sum_{(\mathbf{m}_1,\mathbf{m}_2)\in\mathcal{M}} T_{\mathbf{m}_1,\mathbf{m}_2} \sum_{n<N} e\left(\mathbf{m}_1 \cdot \mathbf{v}_1 P_1(n) + \mathbf{m}_2 \cdot \mathbf{v}_2 P_2(n) \right).$$

The case of different degrees of the polynomials is treated by Proposition 1. So we can assume that they have the same degree $r_1 = r_2 = r$.

As in the proof of Proposition 2, we fix $\Delta = (\log N)^{-\delta}$ and restrict to those $(\mathbf{m}_1, \mathbf{m}_2)$ for which $|m_j^{(\ell)}| < (\log N)^{2\delta}$ and $m_j^{(\ell)} \not\equiv 0 \, (q_\ell)$ for all j, ℓ.

Suppose now $g_r^{(1)} \mathbf{m}_1 \cdot \mathbf{v}_1 + g_r^{(2)} \mathbf{m}_2 \cdot \mathbf{v}_2 \neq 0$ and set $\varepsilon = \eta/(h_1 + h_2 - 1)$. Then there exists an integer K with $0 \leq K \leq h_1 + h_2 - 2$ such that for all j and $\ell = 1, 2$

$$k_{j+1}^{(\ell)} - k_j^{(\ell)} \notin \left[(\log N)^{K\varepsilon}, (\log N)^{(K+1)\varepsilon} \right).$$

So fix K with this property.

First suppose $k_{j+1}^{(\ell)} - k_j^{(\ell)} < (\log N)^{K\varepsilon}$ for all j, ℓ. Then we set

$$\overline{m}_\ell = g_r^{(\ell)} \sum_{j=1}^{h_\ell} m_j^{(\ell)} q_\ell^{k_{h_\ell}^{(\ell)} - k_j^{(\ell)}} \qquad (\ell = 1, 2)$$

and have $\log |\overline{m}_\ell| \ll (\log N)^{K\varepsilon}$. We can apply Lemma 4 to

$$g_r^{(1)} \mathbf{m}_1 \cdot \mathbf{v}_1 + g_r^{(2)} \mathbf{m}_2 \cdot \mathbf{v}_2 = \frac{\overline{m}_1}{q_1^{k_{h_1}^{(1)}+1}} + \frac{\overline{m}_2}{q_2^{k_{h_2}^{(2)}+1}}$$

and obtain

$$\begin{aligned}\left|g_r^{(1)}\mathbf{m}_1\cdot\mathbf{v}_1+g_r^{(2)}\mathbf{m}_2\cdot\mathbf{v}_2\right| &\ge \max\left(q_1^{-k_{h_1}^{(1)}-1},q_2^{-k_{h_2}^{(1)}-1}\right)e^{-c\log\log N\,(\log N)^{K\varepsilon}}\\ &\ge \frac{\max(q_1,q_2)^{(\log N)^\eta}e^{-c\log\log N\,(\log N)^{K\varepsilon}}}{N^r}\\ &\ge \frac{e^{\log(\max(q_1,q_2))(\log N)^\eta-c\log\log N\,(\log N)^{\eta\frac{h_1+h_2-2}{h_1+h_2-1}}}}{N^r}\ge\frac{(\log N)^\tau}{N^r}\end{aligned}$$

for some constant $c>0$ and all $\tau>0$. Because of

$$\left|g_r^{(1)}\mathbf{m}_1\cdot\mathbf{v}_1+g_r^{(2)}\mathbf{m}_2\cdot\mathbf{v}_2\right|\le\frac{(h_1+h_2)(\log N)^{2\delta}}{\min(q_1,q_2)^{-(\log N)^\eta}},$$

Lemma 5 can be applied.

Otherwise we have some s_ℓ, $\ell=1,2$, such that $k_{j+1}^{(\ell)}-k_j^{(\ell)}<(\log N)^{K\varepsilon}$ for all $j<s_\ell$ and $k_{s_\ell+1}^{(\ell)}-k_{s_\ell}^{(\ell)}\ge(\log N)^{(K+1)\varepsilon}$. Here we set

$$\overline{m}_\ell=g_r^{(\ell)}\sum_{j=1}^{s_\ell}m_j^{(\ell)}q_\ell^{k_{s_\ell}^{(\ell)}-k_j^{(\ell)}}\qquad(\ell=1,2).$$

and have again $\log|\overline{m}_\ell|\ll(\log N)^{K\varepsilon}$. Furthermore, we can estimate the sums

$$\sum_{j=s_\ell+1}^{h_\ell}\frac{m_j^{(\ell)}}{q_\ell^{k_j^{(\ell)}+1}}=\mathcal{O}\left((\log N)^{2\delta}q_\ell^{-k_{s_\ell}-(\log N)^{(K+1)\varepsilon}}\right).$$

Thus we get

$$\begin{aligned}\left|g_r^{(1)}\mathbf{m}_1\cdot\mathbf{v}_1+g_r^{(2)}\mathbf{m}_2\cdot\mathbf{v}_2\right| &\ge\left|\frac{\overline{m}_1}{q_1^{k_{s_1}^{(1)}+1}}+\frac{\overline{m}_2}{q_2^{k_{s_2}^{(2)}+1}}\right|-\left|\sum_{j_1=s_1+1}^{h_1}\frac{m_{j_1}^{(1)}}{q_1^{k_{j_1}^{(1)}+1}}\right|-\left|\sum_{j_2=s_2+1}^{h_2}\frac{m_{j_2}^{(2)}}{q_2^{k_{j_2}^{(2)}+1}}\right|\\ &\ge\max\left(q_1^{-k_{s_1}^{(1)}-1},q_2^{-k_{s_2}^{(2)}-1}\right)e^{-c\log\log N\,(\log N)^{K\varepsilon}}\\ &\quad-\mathcal{O}\left((\log N)^{2\delta}\max\left(q_1^{-k_{s_1}^{(1)}-1},q_2^{-k_{s_2}^{(2)}-1}\right)e^{-(\log N)^{(K+1)\varepsilon}}\right)\\ &\gg\max\left(q_1^{-k_{s_1}^{(1)}-1},q_2^{-k_{s_2}^{(2)}-1}\right)e^{-c\log\log N\,(\log N)^{K\varepsilon}}\end{aligned}$$

and Lemma 5 can again be applied.

If q_1 and q_2 are coprime, then we have $g_r^{(1)}\mathbf{m}_1\cdot\mathbf{v}_1+g_r^{(2)}\mathbf{m}_2\cdot\mathbf{v}_2=0$ only for $\mathbf{m}_1=\mathbf{m}_2=\mathbf{0}$. Otherwise we may have other choices of $(\mathbf{m}_1,\mathbf{m}_2)$.

Set $q=(q_1,q_2)$ and $\tilde{q}_1=q_1/q$, $\tilde{q}_2=q_2/q$. Assume, w.l.o.g., $k_{h_1}^{(1)}\ge k_{h_2}^{(2)}$. Then we have

$$\sum_{j_1=1}^{h_1} \frac{g_r^{(1)} m_{j_1}^{(1)}}{q_1^{k_{j_1}^{(1)}}} + \sum_{j_2=1}^{h_2} \frac{g_r^{(2)} m_{j_2}^{(2)}}{q_2^{k_{j_2}^{(2)}}}$$

$$= g_r^{(1)} \frac{m_1^{(1)} \tilde{q}_1^{k_{h_1}^{(1)}-k_1^{(1)}} \tilde{q}_2^{k_{h_2}^{(2)}} q^{k_{h_1}^{(1)}-k_1^{(1)}} + \cdots + m_{h_1-1}^{(1)} \tilde{q}_1^{k_{h_1}^{(1)}-k_{h_1-1}^{(1)}} \tilde{q}_2^{k_{h_2}^{(2)}} q^{k_{h_1}^{(1)}-k_{h_1-1}^{(1)}} + m_{h_1}^{(1)} \tilde{q}_2^{k_{h_2}^{(2)}}}{\tilde{q}_1^{k_{h_1}^{(1)}} \tilde{q}_2^{k_{h_2}^{(2)}} q^{k_{h_1}^{(1)}}}$$

$$+ g_r^{(2)} \frac{m_1^{(2)} \tilde{q}_1^{k_{h_1}^{(1)}} \tilde{q}_2^{k_{h_2}^{(2)}-k_1^{(2)}} q^{k_{h_1}^{(1)}-k_1^{(2)}} + \cdots + m_{h_2}^{(2)} \tilde{q}_1^{k_{h_1}^{(1)}} q^{k_{h_1}^{(1)}-k_{h_2}^{(2)}}}{\tilde{q}_1^{k_{h_1}^{(1)}} \tilde{q}_2^{k_{h_2}^{(2)}} q^{k_{h_1}^{(1)}}},$$

where we have omit the "+1" in the denominator for simplicity. (Just consider $k_j^{(\ell)} - 1$ instead of $k_j^{(\ell)}$.) Hence we must have

$$g_r^{(1)} \left(m_1^{(1)} \tilde{q}_1^{k_{h_1}^{(1)}-k_1^{(1)}} q^{k_{h_1}^{(1)}-k_1^{(1)}} + \cdots + m_{h_1-1}^{(1)} \tilde{q}_1^{k_{h_1}^{(1)}-k_{h_1-1}^{(1)}} q^{k_{h_1}^{(1)}-k_{h_1-1}^{(1)}} + m_{h_1}^{(1)} \right) \equiv 0 \left(\tilde{q}_1^{k_{h_1}^{(1)}} \right). \tag{9}$$

Of course this is useful only if $\tilde{q}_1 > 1$, which we assume first. We have to distinguish several cases. (9) implies

$$m_{j+1}^{(1)} q_1^{k_{h_1}^{(1)}-k_{j+1}^{(1)}} + \cdots + m_{h_1-1}^{(1)} q_1^{k_{h_1}^{(1)}-k_{h_1-1}^{(1)}} + \cdots + m_{h_1}^{(1)} \equiv 0 \left(\tilde{q}_1^{k_{h_1}^{(1)}-k_j^{(1)}} \right) \tag{10}$$

for all j, $1 \le j \le h_1 - 1$. If $k_{j+1}^{(1)} - k_j^{(1)} \ge (\log N)^\varepsilon$ for some j, then $|m_j^{(\ell)}| < (\log N)^{2\delta}$ implies that the left hand side of (10) must be zero. Hence $m_{h_1}^{(1)} \equiv 0\,(q_1)$ which implies $T_{\mathbf{m}_1,\mathbf{m}_2} = 0$ since we have excluded $m_{h_1}^{(1)} = 0$. If $k_{j+1}^{(1)} - k_j^{(1)} \le (\log N)^\varepsilon$ for all j, then the left hand side of (9) must be zero and $m_{h_1}^{(1)} \equiv 0\,(q_1)$.

Now consider the case $\tilde{q}_1 = 1$, i.e. $q_1 | q_2$. Then we have to check

$$g_r^{(1)} \left(m_1^{(1)} \tilde{q}_2^{k_{h_2}^{(2)}} q^{k_{h_1}^{(1)}-k_1^{(1)}} + \cdots + m_{h_1-1}^{(1)} \tilde{q}_2^{k_{h_2}^{(2)}} q^{k_{h_1}^{(1)}-k_{h_1-1}^{(1)}} + m_{h_1}^{(1)} \tilde{q}_2^{k_{h_2}^{(2)}} \right) + \tag{11}$$
$$g_r^{(2)} \left(m_1^{(2)} \tilde{q}_2^{k_{h_2}^{(2)}-k_1^{(2)}} q^{k_{h_1}^{(1)}-k_1^{(2)}} + \cdots + m_{h_2-1}^{(2)} \tilde{q}_2^{k_{h_2}^{(2)}-k_{h_2-1}^{(2)}} q^{k_{h_1}^{(1)}-k_{h_2-1}^{(2)}} + m_{h_2}^{(2)} q^{k_{h_1}^{(1)}-k_{h_2}^{(2)}} \right) = 0.$$

This implies

$$g_r^{(2)} q^{k_{h_1}^{(1)}-k_{h_2}^{(2)}} \left(m_{j+1}^{(2)} q_2^{k_{h_2}^{(2)}-k_{j+1}^{(2)}} + \cdots + m_{h_2-1}^{(2)} q_2^{k_{h_2}^{(2)}-k_{h_2-1}^{(2)}} + m_{h_2}^{(2)} \right) \equiv 0 \left(\tilde{q}_2^{k_{h_2}^{(2)}-k_j^{(2)}} \right) \tag{12}$$

for $1 \le j \le h_2 - 1$ and for $j = 0$, if we set $k_0^{(2)} = 0$.

Assume first $k_{h_1}^{(1)} - k_{h_2}^{(2)} \le (\log N)^{\varepsilon/2}$. Then we can do the same reasonings as above and obtain $m_{h_2}^{(2)} \equiv 0\,(q_2)$.

The last (and most difficult) case is $k_{h_1}^{(1)} - k_{h_2}^{(2)} \ge (\log N)^{\varepsilon/2}$. First suppose that $\tilde{q}_2$ has some prime divisor $\tilde{p}_2 \not| q$. Then we get from (12)

$$g_r^{(2)} \left(m_{j+1}^{(2)} q_2^{k_{h_2}^{(2)}-k_{j+1}^{(2)}} + \cdots + m_{j+1}^{(2)} q_2^{k_{h_2}^{(2)}-k_{h_2-1}^{(2)}} + m_{h_2}^{(2)} \right) \equiv 0 \left(\tilde{p}_2^{k_{h_2}^{(2)}-k_j^{(2)}} \right)$$

for $0 \le j \le h_2 - 1$ and again $m_{h_2}^{(2)} \equiv 0\,(q_2)$. Suppose next that q has some prime divisor $p \nmid \tilde{q}_2$. Then we have

$$g_r^{(1)} \left(m_1^{(1)} q^{k_{h_1}^{(1)} - k_1^{(1)}} + \cdots + m_{h_1-1}^{(1)} q^{k_{h_1}^{(1)} - k_{h_1-1}^{(1)}} + m_{h_1}^{(1)} \right) \equiv 0 \left(p^{k_{h_1}^{(1)} - k_{h_2}^{(2)}} \right)$$

and we can do the same reasonings with ε/h_1 instead of ε.

It remains to consider q and $\tilde{q}_2$ with prime factorisations $q = p_1^{e_1} \ldots p_s^{e_s}$, $\tilde{q}_2 = p_1^{\tilde{e}_1} \ldots p_s^{\tilde{e}_s}$, where all e_i and $\tilde{e}_i$ are positive integers. Let us rewrite (11):

$$\begin{aligned} g_r^{(1)} &\left(m_1^{(1)} \prod_{i=1}^{s} p_i^{k_{h_2}^{(2)} \tilde{e}_i + (k_{h_1}^{(1)} - k_1^{(1)}) e_i} + \cdots + m_{h_1}^{(1)} \prod_{i=1}^{s} p_i^{k_{h_2}^{(2)} \tilde{e}_i} \right) \\ + g_r^{(2)} &\left(m_1^{(2)} \prod_{i=1}^{s} p_i^{(k_{h_2}^{(2)} - k_1^{(2)}) \tilde{e}_i + (k_{h_1}^{(1)} - k_1^{(2)}) e_i} + \cdots + m_{h_2}^{(2)} \prod_{i=1}^{s} p_i^{(k_{h_1}^{(1)} - k_{h_2}^{(2)}) e_i} \right) = 0. \end{aligned}$$

By assumption, q_1 and q_2 are multiplicatively independent. Thus we have $s \ge 2$ and $e_i/\tilde{e}_i \ne e_j/\tilde{e}_j$ for some i, j. Therefore $k_{h_2}^{(2)} \tilde{e}_i - (k_{h_1}^{(1)} - k_{h_2}^{(2)}) e_i$ cannot be zero for all i and the difference must be at least $\frac{1}{2}(\log N)^{\varepsilon/2}$ for some i. Let

$$(k_{h_1}^{(1)} - k_{h_2}^{(2)}) e_{i_0} - k_{h_2}^{(2)} \tilde{e}_{i_0} \ge \frac{1}{2} (\log N)^{\varepsilon/2}.$$

Then we have

$$g_r^{(1)} \left(m_1^{(1)} \prod_{i=1}^{s} p_i^{(k_{h_1}^{(1)} - k_1^{(1)}) e_i} + \cdots + m_{h_1}^{(1)} \right) \equiv 0 \left(p_{i_0}^{(k_{h_1}^{(1)} - k_{h_2}^{(2)}) e_{i_0} - k_{h_2}^{(2)} \tilde{e}_{i_0}} \right)$$

and we can again do the same reasonings. Similarly

$$k_{h_2}^{(2)} \tilde{e}_{i_0} - (k_{h_1}^{(1)} - k_{h_2}^{(2)}) e_{i_0} \ge \frac{1}{2} (\log N)^{\varepsilon/2}$$

leads to

$$g_r^{(2)} \left(m_1^{(2)} \prod_{i=1}^{s} p_i^{(k_{h_2}^{(2)} - k_1^{(2)})(\tilde{e}_i + e_i)} + \cdots + m_{h_2}^{(2)} \right) \equiv 0 \left(p_{i_0}^{\frac{1}{2} (\log N)^{\varepsilon/2}} \right)$$

and the same result.

Hence, we finally get

$$\begin{aligned} \sum_{(\mathbf{m}_1, \mathbf{m}_2) \ne (\mathbf{0}, \mathbf{0})} |T_{\mathbf{m}_1, \mathbf{m}_2}| \cdot &\left| \frac{1}{N} \sum_{n < N} e \left((g_r^{(1)} \mathbf{m}_1 \cdot \mathbf{v}_1 + g_r^{(2)} \mathbf{m}_2 \cdot \mathbf{v}_2) n \right) \right| \\ &= \mathcal{O} \left((\log N)^{-\delta/2} \right) + \mathcal{O} \left((\log N)^{2(h_1 + h_2)\delta - \lambda} \right), \end{aligned}$$

which completes the proof of Proposition 4.

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