METRICAL THEORY FOR α-ROSEN FRACTIONS

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Abstract. The Rosen fractions form an infinite family which generalizes the nearest-integer continued fractions. In this paper we introduce a new class of continued fractions related to the Rosen fractions, the α-Rosen fractions. The metrical properties of these α-Rosen fractions are studied.

We find planar natural extensions for the associated interval maps, and show that their domains of definition are closely related to the domains of the 'classical' Rosen fractions. This unifies and generalizes results of diophantine approximation from the literature.

1. Introduction

Although David Rosen [Ros] introduced as early as 1954 an infinite family of continued fractions which generalize the nearest-integer continued fraction, it is only very recently that the metrical properties of these so-called Rosen fractions haven been investigated; see e.g. [Schm], [N2], [GH] and [BKS]. In this paper we will introduce α-Rosen fractions, and study their metrical properties for special choices of α. These choices resemble Nakada's α-expansions, in fact for $q=3$ these are Nakada's α-expansions; see also [N1]. To be more precise, let $q \in \mathbb{Z}$, $q \geq 3$, and $\lambda = \lambda_q = 2\cos\frac{\pi}{q}$. Then we define for $\alpha \in [\frac{1}{2}, \frac{1}{\lambda}]$ the map $T_\alpha : [\lambda(\alpha-1), \lambda\alpha] \to [\lambda(\alpha-1), \lambda\alpha)$ by

$$T_\alpha(x) := \left|\frac{1}{x}\right| - \lambda \left\lfloor \left|\frac{1}{x\lambda}\right| + 1 - \alpha \right\rfloor, \ x \neq 0, \tag{1}$$

and $T_\alpha(0) := 0$. Here, $\lfloor \xi \rfloor$ denotes the *floor* (or *entier*) of ξ, i.e., the greatest integer smaller than or equal to ξ. In order to have positive digits, we demand that $\alpha \leq 1/\lambda$. Setting $d(x) = \lfloor |\frac{1}{x\lambda}| + 1 - \alpha \rfloor$ (with $d(0) = \infty$), $\varepsilon(x) = \operatorname{sgn}(x)$, and more generally

$$\varepsilon_n(x) = \varepsilon_n = \varepsilon\left(T_\alpha^{n-1}(x)\right) \quad \text{and} \quad d_n(x) = d_n = d\left(T_\alpha^{n-1}(x)\right) \tag{2}$$

for $n \geq 1$, one obtains for $x \in I_{q,\alpha} := [\lambda(\alpha-1), \alpha\lambda]$ an expression of the form

$$x = \cfrac{\varepsilon_1}{d_1\lambda + \cfrac{\varepsilon_2}{d_2\lambda + \cdots + \cfrac{\varepsilon_n}{d_n\lambda + T_\alpha^n(x)}}},$$

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where $\varepsilon_i \in \{\pm 1, 0\}$ and $d_i \in \mathbb{N} \cup \{\infty\}$. Setting

$$\frac{R_n}{S_n} = \cfrac{\varepsilon_1}{d_1\lambda + \cfrac{\varepsilon_2}{d_2\lambda + \cdots + \cfrac{\varepsilon_n}{d_n\lambda}}} =: [\, \varepsilon_1 : d_1,\, \varepsilon_2 : d_2, \ldots, \varepsilon_n : d_n \,], \tag{3}$$

we will show in Section 2.3 that

$$\lim_{n\to\infty} \frac{R_n}{S_n} = x,$$

and for convenience we will write

$$x = \cfrac{\varepsilon_1}{d_1\lambda + \cfrac{\varepsilon_2}{d_2\lambda + \cdots}} =: [\, \varepsilon_1 : d_1,\, \varepsilon_2 : d_2, \ldots \,]. \tag{4}$$

We call R_n/S_n the nth α-Rosen convergent of x, and (4) the α-Rosen fraction of x.

The case $\alpha = 1/2$ yields the Rosen fractions, while the case $\alpha = 1/\lambda$ is the Rosen fraction equivalent of the classical regular continued fraction expansion (RCF). In case $q = 3$ (and $1/2 \le \alpha \le 1/\lambda$), the above defined α-Rosen fractions are in fact Nakada's α-expansions (and the case $\alpha = 1/\lambda = 1$ is the RCF). Already from [BKS] it is clear that in order to construct the underlying ergodic system for any α-Rosen fraction and the planar natural extension for the associated interval map T_α, it is fundamental to understand the orbit under T_α of the two endpoints $\lambda(\alpha-1)$ and $\lambda\alpha$ of $X = X_\alpha := [\lambda(\alpha-1), \lambda\alpha]$. Although the situation is in general more complicated than the 'classical case' from [BKS], the natural extension together with the invariant measure can be given, and it is shown that this dynamical system is weakly Bernoulli.

Using the natural extension, metrical properties of the α-Rosen fractions will be given in Section 4.

2. Natural extensions

In this section we find the "smallest" domain $\Omega_\alpha \subset \mathbb{R}^2$ on which the map

$$\mathcal{T}_\alpha(x,y) = \left(T_\alpha(x), \frac{1}{d(x)\lambda + \varepsilon(x) y} \right), \quad (x,y) \in \Omega_\alpha, \tag{5}$$

is bijective a.e.. We will deal with the general case, resembling Nakada's α-expansions, i.e., $1/2 \le \alpha \le 1/\lambda$ and $\lambda = \lambda_q = 2\cos\pi/q$ for some fixed $q \in \mathbb{Z}$, $q \ge 4$ (the case $q = 3$ is in fact the case of Nakada's α-expansions; see also [N1]). As in [BKS], we need to discern between odd and even q's, but some properties are shared by both cases, and these are collected here first.

For $x \in [\lambda(\alpha-1), \lambda\alpha]$, setting

$$A_i = \begin{pmatrix} 0 & \varepsilon_i \\ 1 & d_i\lambda \end{pmatrix}, \quad \text{and} \quad M_n = A_1 \cdots A_n = \begin{pmatrix} K_n & R_n \\ L_n & R_n \end{pmatrix},$$

it immediately follows from $M_n = M_{n-1}A_n$ that $K_n = R_{n-1}$, $L_n = S_{n-1}$, and

$$\begin{aligned} &R_{-1} := 1, \quad R_0 := 0, \quad R_n = d_n\lambda R_{n-1} + \varepsilon_n R_{n-2}, \quad \text{for } n = 1, 2, \dots, \\ &S_{-1} := 0, \quad S_0 := 1, \quad S_n = d_n\lambda S_{n-1} + \varepsilon_n S_{n-2}, \quad \text{for } n = 1, 2, \dots, \end{aligned} \tag{6}$$

if $d_n < \infty$. For a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $\det(A) \neq 0$, we define the corresponding Möbius (or fractional linear) transformation by

$$A(x) = \frac{ax + b}{cx + d}.$$

Consequently, considering M_n as a Möbius transformation, we find that

$$M_n(0) = \frac{R_n}{S_n}, \quad \text{and } M_n(0) = A_1 \cdots A_n(0) = \cdots = [\,\varepsilon_1 : d_1,\, \varepsilon_2 : d_2, \dots, \varepsilon_n : d_n\,].$$

It follows that the numerators and denominators of the α-Rosen convergents of x from (3) satisfy the usual recurrence relations (6); see also [BKS], p. 1279.

Furthermore, since

$$x = M_{n-1}\begin{pmatrix} 0 & \varepsilon_n \\ 1 & d_n\lambda + T_\alpha^n(x) \end{pmatrix}(0),$$

we have that

$$x = \frac{R_n + T_\alpha^n(x)R_{n-1}}{S_n + T_\alpha^n(x)S_{n-1}} \quad \text{and} \quad T_\alpha^n(x) = \frac{R_n - S_n x}{S_{n-1}x - R_{n-1}}. \tag{7}$$

Let $\ell_0 = (\alpha - 1)\lambda$ be the left-endpoint of the interval on which the continued fraction map T_α was defined in (1), $r_0 = \alpha\lambda$ its right-endpoint and let

$$\Delta(\varepsilon : d) = \{\, x \in [(\alpha - 1)\lambda, \alpha\lambda] \mid \varepsilon_1(x) = \varepsilon,\; d_1(x) = d\,\},$$

be the cylinders of order 1 of numbers with same first digits given by (4). If we set

$$\delta_d = \frac{1}{(\alpha + d)\lambda}$$

for all $d \geq 1$, then the cylinders are given by the following table

$\Delta(-1 : 1)$	$\Delta(-1 : d)$, $d \geq 2$	$\Delta(0 : \infty)$	$\Delta(+1 : d)$, $d \geq 2$	$\Delta(+1 : 1)$
$[\ell_0, \delta_1)$	$[-\delta_{d-1}, -\delta_d)$	$\{0\}$	$(\delta_d, \delta_{d-1}]$	$(\delta_1, r_0]$

where we have used that $r_0 > \delta_1$ since $\lambda \geq \sqrt{2}$ for $q \geq 4$. Note that we have by definition that

$$T_\alpha(x) = \varepsilon/x - \lambda d,$$

for all $x \in \Delta(\varepsilon, d)$, $x \neq 0$.

Setting $\ell_n = T_\alpha^n(\ell_0)$, $r_n = T_\alpha^n(r_0)$, $n \geq 0$, we have that

$$r_1 = \frac{1}{\alpha\lambda} - \lambda = -\frac{\alpha\lambda^2 - 1}{\alpha\lambda} < 0.$$

In case $\alpha = 1/2$, we write ϕ_n instead of ℓ_n, for $n \geq 0$. In [BKS], it was shown that

$$-\lambda/2 = \begin{cases} [\,(-1\,:\,1)^{p-1}\,], & \text{if } q = 2p \\ [\,(-1\,:\,1)^h,\, -1\,:\,2,\, (-1\,:\,1)^h\,], & \text{if } q = 2h+3, \end{cases}$$

from which it immediately follows that

$$\phi_0 = -\frac{\lambda}{2} < \phi_1 < \cdots < \phi_{p-2} = -\frac{1}{\lambda} < \phi_{p-1} = 0, \quad \text{if } q = 2p, \tag{8}$$

and that for $q = 2h+3$,

$$\phi_0 = -\frac{\lambda}{2} < \phi_1 < \cdots < \phi_{h-2} < \phi_{h-1} < -\frac{2}{3\lambda} < \phi_h < -\frac{2}{5\lambda},$$

$\phi_0 < \phi_{h+1} < \phi_1$, $\phi_{h+1} = 1 - \lambda$, and

$$\phi_{h+1} < \phi_{h+2} < \cdots < \phi_{2h} = -\frac{1}{\lambda} < -\frac{2}{3\lambda} < \phi_{2h+1} = 0;$$

see also Figure 1.

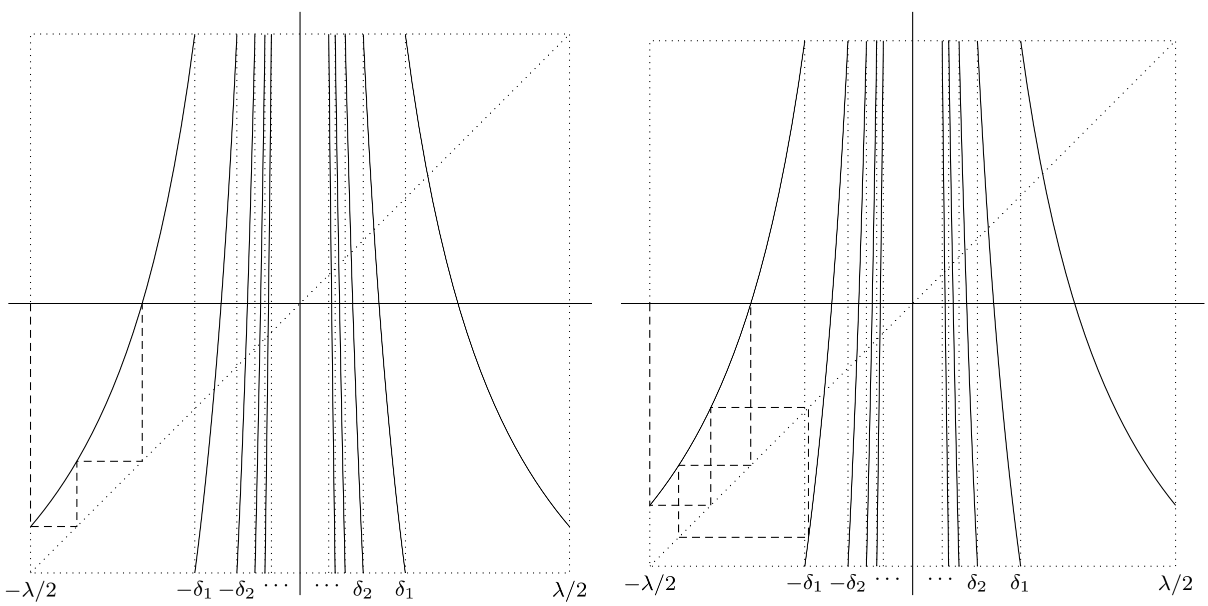

FIGURE 1. The map $T_{1/2}$ and the orbit of $-\lambda/2$ (dashed broken line) for $q = 8$ (left) and $q = 7$ (right).

Thus we see that the behavior of the orbit of $-\lambda/2$ is very different in the even case compared to the odd case; see also Figure 2, where the relevant terms of $(\phi_n)_{n\geq 0}$, $(\ell_n)_{n\geq 0}$, and $(r_n)_{n\geq 0}$ are displayed for even q.

Direct verification yields the following lemma.

LEMMA 2.1. *For $q \geq 4$, $1/2 \leq \alpha \leq 1/\lambda$, we have that*

$$\phi_0 = -\frac{\lambda}{2} \leq \ell_0 \leq r_1 < \phi_1 = -\frac{\lambda^2 - 2}{\lambda},$$

with $\phi_0 = \ell_0$ if and only if $\alpha = 1/2$, and $\ell_0 = r_1$ if and only if $\alpha = 1/\lambda$.

In [BKS], the sequence $(\phi_n)_{n\geq 0}$ plays a crucial role in the construction of the natural extension of the Rosen fractions. Due to the fact that the orbits of both $-\lambda/2$ and $\lambda/2$ would become constant 0 after a finite number of steps (depending on q), the natural extension of the Rosen fraction could be easily constructed. In this paper, the $(\ell_n)_{n\geq 0}$ and $(r_n)_{n\geq 0}$ play a role comparable to that of the sequence $(\phi_n)_{n\geq 0}$ (even though the ϕ_n's are frequently used as well).

Let $x \in [\lambda(\alpha-1), \alpha\lambda]$ be such, that $(\varepsilon_n(x) : d_n(x)) = (-1 : 1)$ for $n = 1, 2, \ldots, m$. Then it follows from (6) that the α-Rosen convergents of x satisfy

$$R_{-1} = 1, \quad R_0 = 0, \quad R_n = \lambda R_{n-1} - R_{n-2}, \quad \text{for } n = 1, 2, \ldots, m\,,$$
$$S_{-1} = 0, \quad S_0 = 1, \quad S_n = \lambda S_{n-1} - S_{n-2}, \quad \text{for } n = 1, 2, \ldots, m\,.$$

As in [BKS], we define the auxiliary sequence $(B_n)_{n\geq 0}$ by

$$B_0 = 0,\ B_1 = 1, \quad B_n = \lambda B_{n-1} - B_{n-2}, \quad \text{for } n = 2, 3, \ldots\,. \tag{9}$$

This yields for $n = 1, \ldots, m$ that $R_n = -B_n$, $S_n = B_{n+1}$, and $T_\alpha^n(x) = -\frac{B_n + B_{n+1}x}{B_{n-1}+B_n x}$ by (7). It follows that

$$\ell_n = -\frac{B_n + B_{n+1}(\alpha-1)\lambda}{B_{n-1} + B_n(\alpha-1)\lambda} = -\frac{B_{n+1}\alpha\lambda - B_{n+2}}{B_n\alpha\lambda - B_{n+1}} \quad \text{if } \ell_0 = [(-1:1)^n, \ldots]. \tag{10}$$

For $x = [+1 : 1, (-1 : 1)^{n-1}, \ldots] = -[(-1 : 1)^n, \ldots]$, we obtain similarly $R_n = B_n$, $S_n = B_{n+1}$, thus $T_\alpha^n(x) = \frac{B_n - B_{n+1}x}{B_n x - B_{n-1}}$ and

$$r_n = -\frac{B_{n+1}\alpha\lambda - B_n}{B_n\alpha\lambda - B_{n-1}} \qquad \text{if } r_0 = [+1 : 1, (-1 : 1)^{n-1}, \ldots]\,. \tag{11}$$

It is easy to see that $B_n = \sin\frac{n\pi}{q}/\sin\frac{\pi}{q}$ (see also [W], Equation 15 in Section 144). Clearly B_n, $n \geq 0$, is a periodic sequence, with period length q.

2.1. **Even indices.** Let $q = 2p$, $p \in \mathbb{N}$, $p \geq 2$. Essential in the construction of the natural extension is the following theorem.

THEOREM 2.2. *Let $q = 2p$, $p \in \mathbb{N}$, $p \geq 2$, and let the sequences $(\ell_n)_{n\geq 0}$ and $(r_n)_{n\geq 0}$ be defined as before. If $1/2 < \alpha < 1/\lambda$, then we have that*

$$\ell_0 < r_1 < \ell_1 < \cdots < r_{p-2} < \ell_{p-2} < -\delta_1 < r_{p-1} < 0 < \ell_{p-1} < r_0, \tag{12}$$

$d_p(r_0) = d_p(\ell_0) + 1$ and $\ell_p = r_p$. If $\alpha = 1/2$, then we have that

$$\ell_0 < r_1 = \ell_1 < \cdots < r_{p-2} = \ell_{p-2} < -\delta_1 < r_{p-1} = \ell_{p-1} = 0 < r_0. \tag{13}$$

If $\alpha = 1/\lambda$, then we have that

$$\ell_0 = r_1 < \ell_1 = r_2 < \cdots < \ell_{p-2} = r_{p-1} = -\delta_1 < 0 < r_0. \tag{14}$$

PROOF. If $\alpha = 1/2$, then $\ell_0 = \phi_0$ and $r_0 = -\phi_0$, hence (13) is an immediate consequence of (8).

In general, in view of Lemma 2.1 and the fact that $\phi_0 = [(-1:1)^{p-1}]$, we have the following situation: $T_\alpha([\ell_0, \phi_1)) = [\ell_1, \phi_2)$ and $T_\alpha([\phi_{j-1}, \phi_j)) = [\phi_j, \phi_{j+1})$ for $j = 2, 3, \ldots, p-2$, cf. Figure 2. This yields that $\ell_0 = [(-1:1)^{p-2}, \ldots]$.

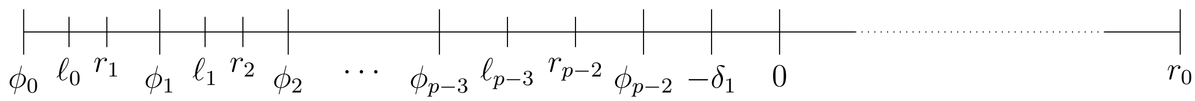

FIGURE 2. The relevant terms of $(\phi_n)_{n\ge 0}$, $(\ell_n)_{n\ge 0}$, and $(r_n)_{n\ge 0}$ for even q.

Since $\sin\frac{(p-1)\pi}{2p} = \sin\frac{(p+1)\pi}{2p}$, we obtain

$$B_{p-1} = B_{p+1}, \quad B_{p-1} = \frac{\lambda}{2} B_p, \quad B_{p-2} = \left(\frac{\lambda^2}{2} - 1\right) B_p.$$

By (10), we have therefore that

$$\ell_{p-2} = -\frac{B_{p-1}\alpha\lambda - B_p}{B_{p-2}\alpha\lambda - B_{p-1}} = -\frac{2-\alpha\lambda^2}{\lambda(1-\alpha\lambda^2+2\alpha)} \le -\frac{1}{(\alpha+1)\lambda} = -\delta_1,$$

with $\ell_{p-2} = -\delta_1$ if and only if $\alpha = 1/\lambda$. For $\alpha = 1/\lambda$, we clearly have that $r_0 = 1$, thus $r_1 = 1 - \lambda = \ell_0$ and (14) is proved.

If $1/2 < \alpha < 1/\lambda$, then we have that $\ell_{p-2} < -\delta_1$, hence $\ell_0 = [(-1:1)^{p-1}, \ldots, (1:d_p), \ldots]$, with $d_p \ge 1$, and again due to (10) we obtain

$$\ell_{p-1} = -\frac{B_p\alpha\lambda - B_{p-1}}{B_{p-1}\alpha\lambda - B_p} = \frac{(2\alpha-1)\lambda}{2-\alpha\lambda^2} > 0.$$

Similarly, we have that $r_0 = [+1:1, (-1:1)^{p-2}, \ldots]$ and, by (11),

$$r_{p-1} = -\frac{B_p\alpha\lambda - B_{p-1}}{B_{p-1}\alpha\lambda - B_{p-2}} = -\frac{(2\alpha-1)\lambda}{2-(1-\alpha)\lambda^2} \in (-\delta_1, 0),$$

thus (12) is proved. Since

$$\left|\frac{1}{r_{p-1}}\right| - \left|\frac{1}{\ell_{p-1}}\right| = \lambda,$$

it follows from the definition of T_α that $\ell_p = r_p$ and $d_1(r_{p-1}) = d_1(\ell_{p-1}) + 1$. With $d_p(x) = d_1(T_\alpha^{p-1}(x))$, the theorem is proved. $\square$

REMARK. The structure of the ℓ_n's and r_n's allows us to determine all possible sequences of "digits". For example, the longest consecutive sequence of digits $(-1:1)$ contains $p-1$ terms if $\alpha < 1/\lambda$ since $\ell_{p-2} < -\delta_1$ and $\ell_{p-1} \ge -\delta_1$. In case $\alpha = 1/\lambda$, we only have $(-1:1)^{p-2}$. In particular in case $q = 4$, $\alpha = 1/\lambda$, the cylinder $\Delta(-1:1)$ is empty.

On the other hand, $(+1:1)$ is always followed by $(-1:1)^{p-2}$ since $r_{p-2} < -\delta_1$, with $\Delta(+1:1) = \{r_0\}$ in case $\alpha = 1/\lambda$.

Now we construct the domain Ω_α upon which $\mathcal{T}_\alpha$ is bijective.

THEOREM 2.3. *Let $q = 2p$ with $p \geq 2$. Then the system of relations*

$$\begin{cases} (\mathcal{R}_1): & H_1 = 1/(\lambda + H_{2p-1}) \\ (\mathcal{R}_2): & H_2 = 1/\lambda \\ (\mathcal{R}_n): & H_n = 1/(\lambda - H_{n-2}) \qquad \text{for } n = 3,4,\ldots,2p-1 \\ (\mathcal{R}_{2p}): & H_{2p-2} = \lambda/2 \\ (\mathcal{R}_{2p+1}): & H_{2p-3} + H_{2p-1} = \lambda \end{cases}$$

admits the (unique) solution

$$H_{2n} = -\phi_{p-n} = \frac{B_n}{B_{n+1}} = \frac{\sin\frac{n\pi}{2p}}{\sin\frac{(n+1)\pi}{2p}} \quad \text{for } n = 1,2,\ldots,p-1,$$

$$H_{2n-1} = \frac{B_{p-n} - B_{p+1-n}}{B_{p-1-n} - B_{p-n}} = \frac{\cos\frac{n\pi}{2p} - \cos\frac{(n-1)\pi}{2p}}{\cos\frac{(n+1)\pi}{2p} - \cos\frac{n\pi}{2p}} \quad \text{for } n = 1,2,\ldots,p,$$

in particular $H_{2p-1} = 1$.

Let $1/2 < \alpha < 1/\lambda$ and $\Omega_\alpha = \bigcup_{n=1}^{2p-1} J_n \times [0, H_n]$ with $J_{2n-1} = [\ell_{n-1}, r_n)$, $J_{2n} = [r_n, \ell_n)$ for $n = 1,2,\ldots,p-1$, and $J_{2p-1} = [\ell_{p-1}, r_0)$. Then the map $\mathcal{T}_\alpha : \Omega_\alpha \to \Omega_\alpha$ given by (5) is bijective off of a set of Lebesgue measure zero.

PROOF. It is easily seen that the solution of this system of relations is unique and valid, and that $\mathcal{T}_\alpha$ is injective. We thus concern ourselves with the surjectivity of $\mathcal{T}_\alpha$; see also Figure 3.

By (12), we have $J_{n-2} \subset \Delta(-1:1)$ for $n = 3,4,\ldots,2p-2$, thus

$$\mathcal{T}_\alpha(J_{n-2} \times [0, H_{n-2}]) = J_n \times \left[\frac{1}{\lambda}, \frac{1}{\lambda - H_{n-2}}\right] = J_n \times [H_2, H_n],$$

where we have used $(\mathcal{R}_2)$ and $(\mathcal{R}_n)$. Furthermore, $(\mathcal{R}_{2p-1})$ gives

$$\mathcal{T}_\alpha([\ell_{p-2}, -\delta_1) \times [0, H_{2p-3}]) = [\ell_{p-1}, r_0) \times \left[\frac{1}{\lambda}, \frac{1}{\lambda - H_{2p-3}}\right] = J_{2p-1} \times [H_2, H_{2p-1}].$$

For $n = 2,3,\ldots,d_1(r_{p-1}) - 1 = d_p(r_0) - 1$, we have that

$$\mathcal{T}_\alpha([-\delta_{n-1}, -\delta_n) \times [0, H_{2p-3}]) = [\ell_0, r_0) \times \left[\frac{1}{n\lambda}, \frac{1}{n\lambda - H_{2p-3}}\right].$$

The remaining part of the rectangle $J_{2p-3} \times [0, H_{2p-3}]$ is mapped to

$$\mathcal{T}_\alpha([-\delta_{d_p(r_0)-1}, r_{p-1}) \times [0, H_{2p-3}]) = [\ell_0, r_p) \times \left[\frac{1}{d_p(r_0)\lambda}, \frac{1}{d_p(r_0)\lambda - H_{2p-3}}\right].$$

Now consider the image of $J_{2p-1} \times [0, H_{2p-1}]$. If $d_p(\ell_0) \geq 2$, then it is split into

$$\mathcal{T}_\alpha((\ell_{p-1}, \delta_{d_p(\ell_0)-1}] \times [0, H_{2p-1}] = [\ell_0, r_p) \times \left[\frac{1}{d_p(\ell_0)\lambda + H_{2p-1}}, \frac{1}{d_p(\ell_0)\lambda}\right],$$

$$\mathcal{T}_\alpha((-\delta_n, -\delta_{n-1}] \times [0, H_{2p-1}]) = [\ell_0, r_0) \times \left[\frac{1}{n\lambda + H_{2p-1}}, \frac{1}{n\lambda}\right] \quad \text{for } n = 2, 3, \ldots, d_p(\ell_0) - 1,$$

$$\mathcal{T}_\alpha((\delta_1, r_0) \times [0, H_{2p-1}]) = (r_1, r_0) \times \left[\frac{1}{\lambda + H_{2p-1}}, \frac{1}{\lambda}\right] = (r_1, r_0) \times [H_1, H_2],$$

where we have used $(\mathcal{R}_1)$. Since $H_{2p-3} + H_{2p-1} = \lambda$ and $d_p(r_0) = d_p(\ell_0) + 1$, the different parts of $\mathcal{T}_\alpha([-\delta_1, r_{p-1}) \times [0, H_{2p-3}])$ and $\mathcal{T}_\alpha((\ell_{p-1}, \delta_1] \times [0, H_{2p-1}])$ "layer one under the other" and "fill up like a jig-saw puzzle"

$$\left([\ell_0, r_p) \times \left[\frac{1}{d_p(r_0)\lambda}, \frac{1}{d_p(\ell_0)\lambda}\right]\right) \cup \left([\ell_0, r_0) \times \left[\frac{1}{d_p(r_0)\lambda}, H_1\right]\right).$$

In case $d_p(\ell_0) = 1$, we simply have

$$\mathcal{T}_\alpha([-\delta_1, r_{p-1}) \times [0, H_{2p-3}]) = [\ell_0, r_p) \times [1/(2\lambda), H_1]\,,$$
$$\mathcal{T}_\alpha((\ell_{p-1}, r_0) \times [0, H_{2p-1}] = (r_1, r_p) \times [H_1, H_2]\,.$$

Finally, the image of the central rectangle $J_{2p-2} \times [0, H_{2p-2}]$ is split into

$$\mathcal{T}_\alpha([r_{p-1}, -\delta_{d_p(r_0)}) \times [0, H_{2p-2}]) = [r_p, r_0) \times \left[\frac{1}{d_p(r_0)\lambda}, \frac{1}{d_p(r_0)\lambda - H_{2p-2}}\right],$$

$$\mathcal{T}_\alpha([-\delta_{n-1}, -\delta_n) \times [0, H_{2p-2}]) = [\ell_0, r_0) \times \left[\frac{1}{n\lambda}, \frac{1}{n\lambda - H_{2p-2}}\right] \quad \text{for } n > d_p(r_0),$$

$$\mathcal{T}_\alpha((\delta_n, \delta_{n-1}] \times [0, H_{2p-2}]) = [\ell_0, r_0) \times \left[\frac{1}{n\lambda + H_{2p-2}}, \frac{1}{n\lambda}\right] \quad \text{for } n > d_p(\ell_0),$$

$$\mathcal{T}_\alpha((\delta_{d_p(\ell_0)}, \ell_{p-1}] \times [0, H_{2p-2}]) = [r_p, r_0) \times \left[\frac{1}{d_p(\ell_0)\lambda + H_{2p-2}}, \frac{1}{d_p(\ell_0)\lambda}\right].$$

Since $H_{2p-2} = \lambda/2$ and $d_p(r_0) = d_p(\ell_0) + 1$, the union of these images is

$$\left([\ell_0, r_0) \times \left(0, \frac{1}{d_p(r_0)\lambda}\right]\right) \cup \left([r_p, r_0) \times \left[\frac{1}{d_p(r_0)\lambda}, \frac{1}{d_p(\ell_0)\lambda}\right]\right).$$

Therefore $\mathcal{T}_\alpha(\Omega_\alpha)$ and Ω_α differ only by a set of Lebesgue measure zero. □

REMARK. If $\alpha = 1/2$, then the intervals J_{2n} are empty and $r_{p-1} = \ell_{p-1} = 0$. The proof of Theorem 2.3 remains valid, with $d_1(r_{p-1}) = d_1(\ell_{p-1}) = \infty$; see also [BKS]. Since $\ell_n = \phi_n$ for $n = 0, 1, \ldots, p-1$, we have

$$\Omega_{1/2} = \bigcup_{n=1}^{p-1} \left(\left[-\frac{B_n}{B_{n+1}}, -\frac{B_{n-1}}{B_n}\right) \times \left[0, \frac{B_n - B_{n+1}}{B_{n-1} - B_n}\right]\right) \cup \left(\left[0, \frac{\lambda}{2}\right) \times [0, 1]\right).$$

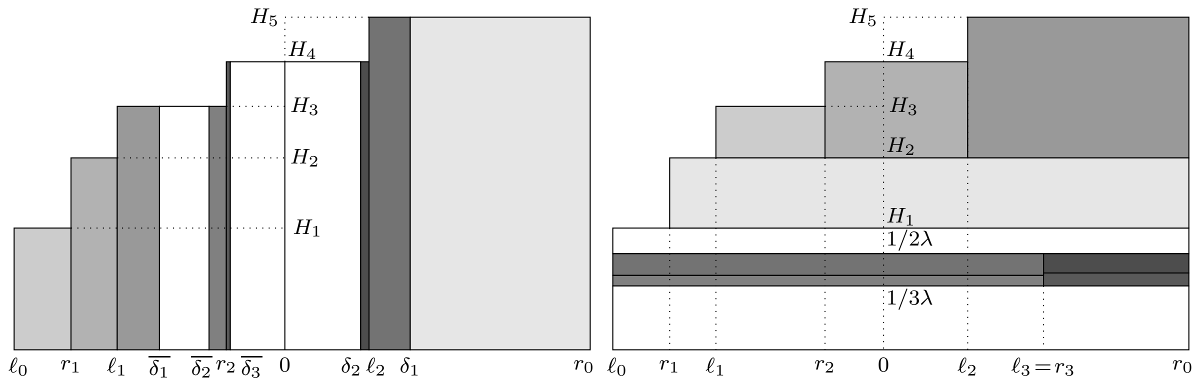

FIGURE 3. The natural extension domain Ω_α (left) and its image under $\mathcal{T}_\alpha$ (right) of the α-Rosen continued fraction ($\overline{\delta_n} = -\delta_n$); here $q = 6$, $\alpha = 0.53$, $d_p(\ell_0) = 2$, $d_p(r_0) = 3$.

For $\alpha = 1/\lambda$, we just have the intervals J_{2n}, $n = 1, 2, \ldots, p-2$ and add $J_{2p-2} = [r_{p-1}, r_0)(= [-\delta_1, 1))$. Furthermore, we have that $r_n = \frac{B_n - B_{n+1}}{B_n - B_{n-1}}$ for $n = 1, 2, \ldots, p-1$ and $r_0 = \frac{B_p - B_{p+1}}{B_p - B_{p-1}} = 1$. This provides the following theorem.

THEOREM 2.4. *Let $q = 2p$ with $p \geq 2$ and*

$$\Omega_{1/\lambda} = \bigcup_{n=1}^{p-1} \left[\frac{B_n - B_{n+1}}{B_n - B_{n-1}}, \frac{B_{n+1} - B_{n+2}}{B_{n+1} - B_n} \right) \times \left[0, \frac{B_n}{B_{n+1}} \right].$$

Then $\mathcal{T}_{1/\lambda} : \Omega_{1/\lambda} \to \Omega_{1/\lambda}$ is bijective off of a set of Lebesgue measure zero.

PROOF. By (14), we have that $\ell_0 = r_1 < \ell_1 = r_2 < \ldots < \ell_{p-2} = r_{p-1} = -\delta_1$, thus

$$\begin{aligned} \mathcal{T}_{1/\lambda} & \left(\left[\frac{B_n - B_{n+1}}{B_n - B_{n-1}}, \frac{B_{n+1} - B_{n+2}}{B_{n+1} - B_n} \right) \times \left[0, \frac{B_n}{B_{n+1}} \right] \right) \\ & = \left[\frac{B_{n+1} - B_{n+2}}{B_{n+1} - B_n}, \frac{B_{n+2} - B_{n+3}}{B_{n+2} - B_{n+1}} \right) \times \left[\frac{1}{\lambda}, \frac{B_{n+1}}{B_{n+2}} \right] \quad \text{for } n = 1, 2, \ldots, p-2. \end{aligned}$$

The different parts of $[-r_{p-1}, r_0)$ are mapped to

$$\begin{aligned} \mathcal{T}_{1/\lambda} \left([-\delta_{n-1}, -\delta_n) \times \left[0, \frac{\lambda}{2} \right] \right) &= [1-\lambda, 1) \times \left[\frac{1}{n\lambda}, \frac{2}{(2n-1)\lambda} \right] \quad \text{for } n = 2, 3, \ldots \\ \mathcal{T}_{1/\lambda} \left((\delta_n, \delta_{n-1}] \times \left[0, \frac{\lambda}{2} \right] \right) &= [1-\lambda, 1) \times \left[\frac{2}{(2n+1)\lambda}, \frac{1}{n\lambda} \right] \quad \text{for } n = 2, 3, \ldots \\ \mathcal{T}_{1/\lambda} \left((\delta_1, 1) \times \left[0, \frac{\lambda}{2} \right] \right) &= [1-\lambda, 1) \times \left[\frac{2}{3\lambda}, \frac{1}{\lambda} \right] \end{aligned}$$

and the union of these images is $[1-\lambda, 1) \times (0, 1/\lambda]$. $\square$

REMARK. Note that there is a simple relation between $\Omega_{1/2}$ and $\Omega_{1/\lambda}$, which will be useful in Section 3; reflect $\Omega_{1/2}$ in the line $y = x$ in case $x \geq 0$, and reflect $\Omega_{1/2}$ in the line $y = -x$ in case $x \leq 0$, to find $\Omega_{1/\lambda}$; see also Figure 4.

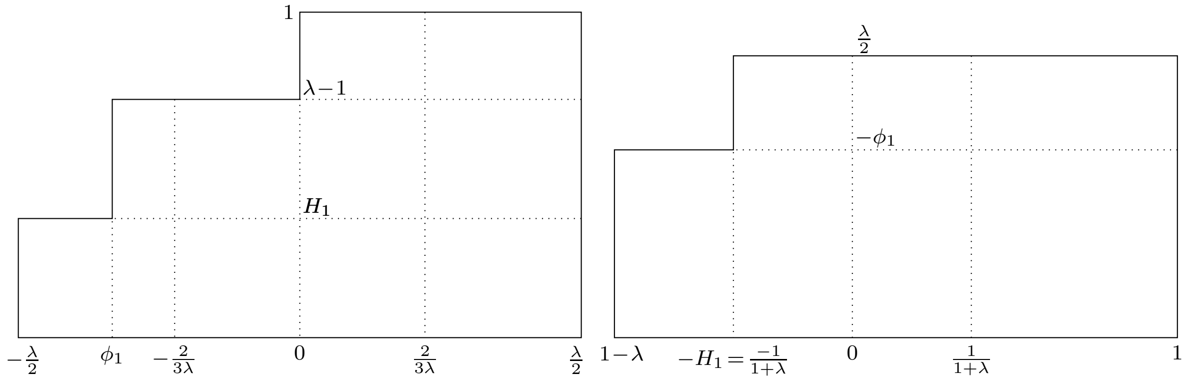

FIGURE 4. $\Omega_{1/2}$ (left) and $\Omega_{1/\lambda}$ (right); here $q=6$.

As in [BKS], a Jacobian calculation shows that $\mathcal{T}_\alpha$ preserves the probability measure ν_α with density

$$\frac{C_{q,\alpha}}{(1+xy)^2},$$

where $C_{q,\alpha}$ is a normalizing constant. For the calculation of this constant, we need the following lemma.

LEMMA 2.5. *If $m_1-m_2=m_3-m_4$, then we have that*

$$B_{n+m_1}B_{-n+m_2}-B_{n+m_3}B_{-n+m_4}=B_{m_1-m_3}B_{m_2+m_3} \quad \textit{for all } n\in\mathbb{Z}.$$

PROOF. With $\zeta=\exp(\pi i/q)$, we have that

$$\begin{aligned} & B_{n+m_1}B_{-n+m_2}-B_{n+m_3}B_{-n+m_4} \\ & \quad = \frac{(\zeta^{n+m_1}-\zeta^{-n-m_1})(\zeta^{-n+m_2}-\zeta^{n-m_2})-(\zeta^{n+m_3}-\zeta^{-n-m_3})(\zeta^{-n+m_4}-\zeta^{n-m_4})}{(\zeta-\zeta^{-1})^2} \\ & \quad = \frac{\zeta^{m_1+m_2}-\zeta^{-m_1-m_2}-\zeta^{m_3+m_4}-\zeta^{-m_3-m_4}}{(\zeta-\zeta^{-1})^2}=B_{m_1-m_3}B_{m_2+m_3}. \end{aligned}$$

□

PROPOSITION 2.6. *For $1/2\le\alpha\le1/\lambda$, the normalizing constant is*

$$C_{q,\alpha}=1/\log\frac{1+\cos\frac{\pi}{q}}{\sin\frac{\pi}{q}}.$$

PROOF. Similarly to [BKS], integration of the density over Ω_α gives

$$C_{q,\alpha}=1/\log\left(\frac{1+r_0}{1+\ell_{p-1}}\prod_{n=1}^{p-1}\frac{1+r_nH_{2n-1}}{1+\ell_{n-1}H_{2n-1}}\frac{1+\ell_nH_{2n}}{1+r_nH_{2n}}\right)$$

for $1/2<\alpha<1/\lambda$, by Theorem 2.3. Using (10), (11) and Lemma 2.5, we find

$$\frac{1+r_nH_{2n-1}}{1+\ell_{n-1}H_{2n-1}}=\frac{B_n-B_{n-1}\alpha\lambda}{B_n\alpha\lambda-B_{n-1}}, \qquad \frac{1+\ell_nH_{2n}}{1+r_nH_{2n}}=\frac{B_n\alpha\lambda-B_{n-1}}{B_{n+1}-B_n\alpha\lambda}$$

for $n = 1, 2, \ldots, p-1$, and
$$\frac{1+r_0}{1+\ell_{p-1}} = \frac{B_p - B_{p-1}\alpha\lambda}{B_p - B_{p-1}}.$$
Putting everything together, we obtain
$$C_{q,\alpha} = 1/\log\frac{1}{B_p - B_{p-1}} = 1/\log\frac{\sin\frac{\pi}{q}}{1-\cos\frac{\pi}{q}} = 1/\log\frac{1+\cos\frac{\pi}{q}}{\sin\frac{\pi}{q}}.$$
For $\alpha = 1/2$, we have the same constant by the remark following Theorem 2.3 and by [BKS]. Finally, the remark following Theorem 2.4 shows that $C_{q,1/\lambda}$ is the same constant as well. □

Let μ_α be the projection of ν_α on the first coordinate, let $\bar{\mathcal{B}}$ be the restriction of the two-dimensional σ-algebra on Ω_α, and $\mathcal{B}$ be the Lebesgue σ-algebra on $I_{q,\alpha} = [\lambda(\alpha-1), \alpha\lambda]$. In [Roh], Rohlin introduced and studied the concept of *natural extension* of a dynamical system. In our setting, a natural extension of $(I_{q,\alpha}, \mathcal{B}, \mu_\alpha, T_\alpha)$ is an invertible dynamical system $(X_\alpha, \mathcal{B}_{X_\alpha}, \rho_\alpha, \mathcal{S}_\alpha)$, which contains $(I_{q,\alpha}, \mathcal{B}, \mu_\alpha, T_\alpha)$ as a factor, such that $\mathcal{B}_{X_\alpha} = \bigvee_{n\geq 0} \mathcal{S}_\alpha^n \pi^{-1}\mathcal{B}$, where π is the factor map. A natural extension is unique up to isomorphism.

We have the following theorem.

THEOREM 2.7. *Let $q \geq 4$, $q = 2p$, and let $\frac{1}{2} \leq \alpha \leq \frac{1}{\lambda}$. Then the dynamical system $(\Omega_\alpha, \bar{\mathcal{B}}, \nu_\alpha, \mathcal{T}_\alpha)$ is the natural extension of the dynamical system $(I_{q,\alpha}, \mathcal{B}, \mu_\alpha, T_\alpha)$.*

PROOF. Let $\pi_1 : \Omega_\alpha \to I_{q,\alpha}$ be the projection onto the first coordinate. An easy calculation shows that $\pi_1 \circ T_\alpha = \mathcal{T}_\alpha \circ \pi_1$, $\mu_\alpha = \nu_\alpha \circ \pi_1^{-1}$, and $\pi_1^{-1}\mathcal{B} \subset \bar{\mathcal{B}}$ so that π_1 is a factor map. It remains to show that
$$\bar{\mathcal{B}} = \bigvee_{n\geq 0} \mathcal{T}_\alpha^n \pi_1^{-1}\mathcal{B}. \tag{15}$$
For each admissible block $(\varepsilon_1, d_1), (\varepsilon_2, d_2), \ldots, (\varepsilon_n, d_n)$, define
$$\Delta_n((\varepsilon_1, d_1), \ldots, (\varepsilon_n, d_n)) = \Delta(\varepsilon_1, d_1) \cap T_\alpha^{-1}\Delta(\varepsilon_2, d_2) \cap T_\alpha^{-(n-1)}\Delta(\varepsilon_n, d_n).$$
The intervals Δ_n defined above are called *fundamental intervals of order n*. Since T_α is expanding, the Lebesgue measure of $\Delta_n((\varepsilon_1, d_1), \ldots, (\varepsilon_n, d_n))$ tends to 0 as $n \to \infty$ for any admissible sequence $(\varepsilon_1, d_1), (\varepsilon_2, d_2), \ldots$. Thus, the collection
$$P = \{\Delta_n((\varepsilon_1, d_1), \ldots, (\varepsilon_n, d_n)) : n \geq 1, \text{ with } (\varepsilon_1, d_1), \ldots, (\varepsilon_n, d_n) \text{ admissible}\}$$
generates $\mathcal{B}$, i.e. $\sigma(\bigvee_{n\geq 0} T_\alpha^{-n}P) = \mathcal{B}$. Let $\mathcal{P}_\alpha = \pi_1^{-1}P$, to prove (15) it is enough to show that $\bigvee_{n\geq 0} \mathcal{T}_\alpha^n \mathcal{P}_\alpha$ generates $\bar{\mathcal{B}}$ which is equivalent to showing that $\bigvee_{n\geq 0} \mathcal{T}_\alpha^n \mathcal{P}_\alpha$ separates points of Ω_α. To do this, we first study the action of $\mathcal{T}_\alpha^{-1}$ on Ω_α.

From Theorem 2.3, one sees that $\mathcal{T}_\alpha^{-1}$ must take horizontal stripes to vertical stripes, so we need a partition in the vertical direction. Unfortunately, it is not always possible to find a uniform partition on the y-axis that works for all x; see Figure 5. Instead, we

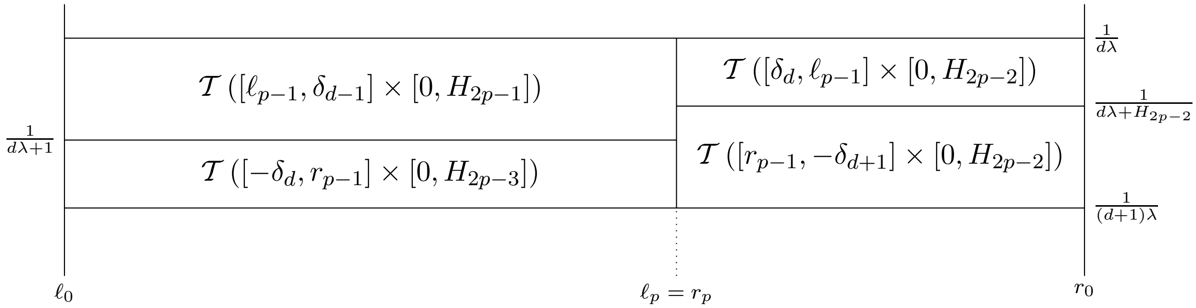

FIGURE 5. "Blow-up" of the relevant part of Ω_α for even q, $1/2 < \alpha < 1/\lambda$.

partition per fiber. To be more specific, for each $x \in I_{q,\alpha}$, let $D(x) = \{y;\, (x,y) \in \Omega_\alpha\}$, so $D(x)$ is the fiber over x. Consider the following partition of $D(x)$,

$$\Delta^{\#}(-1,1,x) = \begin{cases} [H_2, H_{2p-2}] & \text{if } x \geq \ell_{p-1} \\ [H_2, H_{2p-3}] & \text{if } r_{p-1} \leq x < \ell_{p-1} \\ [H_2, H_{2p-4}] & \text{if } \ell_{p-2} \leq x < r_{p-2} \\ \quad\vdots & \qquad\vdots \\ [H_2, H_3] & \text{if } \ell_1 \leq x < r_2 \\ \quad\emptyset & \text{if } x < \ell_1, \end{cases}$$

and

$$\Delta^{\#}(1,1,x) = \begin{cases} [H_1, H_2] & \text{if } x \geq r_1 \\ \quad\emptyset & \text{if } x < r_1. \end{cases}$$

For $(\varepsilon, d) \notin \{(-1, d_p(r_0)), (1, d_p(\ell_0))\}$, set

$$\Delta^{\#}(-1,d,x) = \left[\frac{1}{d\lambda}, \frac{1}{d\lambda - H_{2p-2}}\right], \quad \text{and} \quad \Delta^{\#}(1,d,x) = \left[\frac{1}{d\lambda + H_{2p-1}}, \frac{1}{d\lambda}\right].$$

Finally,

$$\Delta^{\#}(-1,d_p(r_0),x) = \begin{cases} \left[\dfrac{1}{d_p(r_0)\lambda}, \dfrac{1}{(d_p(r_0)-1)\lambda + 1}\right] & \text{if } x < \ell_p \\[2ex] \left[\dfrac{1}{d_p(r_0)\lambda}, \dfrac{1}{(d_p(r_0)-1)\lambda + H_{2p-2}}\right] & \text{if } x \geq \ell_p, \end{cases}$$

and

$$\Delta^{\#}(1,d_p(\ell_0),x) = \begin{cases} \left[\dfrac{1}{d_p(\ell_0)\lambda + 1}, \dfrac{1}{d_p(\ell_0)\lambda}\right] & \text{if } x < \ell_p \\[2ex] \left[\dfrac{1}{d_p(\ell_0)\lambda + H_{2p-2}}, \dfrac{1}{d_p(\ell_0)\lambda}\right] & \text{if } x \geq \ell_p. \end{cases}$$

One can give $\mathcal{T}_\alpha^{-1}$ explicitly,

$$\mathcal{T}_\alpha^{-1}(x,y) = \left(\frac{\varepsilon}{d\lambda + x}, \varepsilon\left(\frac{1}{y} - d\lambda\right)\right), \qquad \text{if } y \in \Delta^{\#}(\varepsilon, d, x).$$

From the definitions of $\mathcal{T}_\alpha$ and $\mathcal{T}_\alpha^{-1}$ one sees that $\mathcal{T}_\alpha$ is expanding in the x-direction, while $\mathcal{T}_\alpha^{-1}$ is expanding in the y-direction.

Now, let $(x,y),(x',y')$ be two distict elements of Ω_α. If $x\neq x'$, then there exist two distinct fundamental intervals $\Delta_n((\varepsilon_1,d_1),\dots,(\varepsilon_n,d_n))$ and $\Delta_n((\varepsilon'_1,d'_1),\dots,(\varepsilon'_n,d'_n))$ such that $(x,y)\in\pi_1^{-1}\Delta_n((\varepsilon_1,d_1),\dots,(\varepsilon_n,d_n))$, and $(x',y')\in\pi_1^{-1}\Delta_n((\varepsilon'_1,d'_1),\dots,(\varepsilon'_n,d'_n))$, i.e. they belong to different elements of $\mathcal{P}_\alpha$. Suppose now that $x=x'$ but $y\neq y'$. Since $\mathcal{T}_\alpha^{-1}$ is expanding in the y-coordinate, then there exist $n\geq 0$, and $(\varepsilon_1,d_1),\dots,(\varepsilon_{n+1},d_{n+1})$, $(\varepsilon'_{n+1},d'_{n+1})$ such that

(i) $(\varepsilon_{-n},d_{-n})\neq(\varepsilon'_{-n},d'_{-n})$,
(ii) $\mathcal{T}_\alpha^{-j}(x,y),\mathcal{T}_\alpha^{-j}(x,y')\in\mathcal{T}_\alpha\pi_1^{-1}\Delta(\varepsilon_{-j},d_{-j})$ for $j=0,\dots,n-1$ (this is void if $n=0$),
(iii) $\mathcal{T}_\alpha^{-n}(x,y)\in\mathcal{T}_\alpha\pi_1^{-1}\Delta(\varepsilon_{-n},d_{-n})$ and $\mathcal{T}_\alpha^{-n}(x,y')\in\mathcal{T}_\alpha\pi_1^{-1}\Delta(\varepsilon'_{-n},d'_{-n})$.

Then,

$$(x,y)\in\mathcal{T}_\alpha^{n+1}\pi_1^{-1}\Delta(\varepsilon_{-n},d_{-n})\cap\mathcal{T}_\alpha^{n}\pi_1^{-1}\Delta(\varepsilon_{-n+1},d_{-n+1})\cap\cdots\cap\mathcal{T}_\alpha\pi_1^{-1}\Delta(\varepsilon_0,d_0),$$
$$(x,y')\in\mathcal{T}_\alpha^{n+1}\pi_1^{-1}\Delta(\varepsilon'_{-n},d'_{-n})\cap\mathcal{T}_\alpha^{n}\pi_1^{-1}\Delta(\varepsilon_{-n+1},d_{-n+1})\cap\cdots\cap\mathcal{T}_\alpha\pi_1^{-1}\Delta(\varepsilon_0,d_0).$$

Thus, (x,y) and (x,y') belong to different elements of $\bigvee_{n\in\mathbb{Z}}\mathcal{T}_\alpha^n\mathcal{P}_\alpha$. In all cases, we see that $\bigvee_{n\in\mathbb{Z}}\mathcal{T}_\alpha^n\mathcal{P}_\alpha$ separates points of Ω_α. Therefore, $(\Omega_\alpha,\bar{\mathcal{B}},\nu_\alpha,\mathcal{T}_\alpha)$ is the natural extension of $(I_{q,\alpha},\mathcal{B},\mu_\alpha,T_\alpha)$. □

Remark. In case $\alpha=1/2$ and $\alpha=1/\lambda$ the proof of Theorem 2.7 is a straightforward application of Theorem 21.2.2 from [Schw]; see also Examples 21.3.1 (the case of the RCF) and 21.3.2 (the NICF) in [Schw]. However, for $1/2<\alpha<1/\lambda$, a lot of extra work is needed, due to the problem mentioned in the above proof, and illustrated in Figure 5.

2.2. **Odd indices.** Let $q=2h+3$, $h\in\mathbb{N}$. The ℓ_n's and r_n's are ordered in the following way.

Theorem 2.8. *Let $q=2h+3$, $h\in\mathbb{N}$, let the sequences $(\ell_n)_{n\geq 0}$ and $(r_n)_{n\geq0}$ be defined as above, and let $\rho=\frac{\lambda-2+\sqrt{\lambda^2-4\lambda+8}}{2}$. Then we have the following cases:*

$\alpha=1/2$: $\ell_0<r_{h+1}=\ell_{h+1}<r_1=\ell_1<\cdots<r_{2h-1}=\ell_{2h-1}<r_{h-1}=\ell_{h-1}<r_{2h}=\ell_{2h}<-\delta_1<r_h=\ell_h<-\delta_2<r_{2h+1}=\ell_{2h+1}=0<r_0$

$1/2<\alpha<\rho/\lambda$: $\ell_0<r_{h+1}<\ell_{h+1}<r_1<\cdots<\ell_{h-2}<r_{2h-1}<\ell_{2h-1}<r_{h-1}<\ell_{h-1}<r_{2h}<\ell_{2h}<-\delta_1<r_h<\ell_h<-\delta_2<r_{2h+1}<0<\ell_{2h+1}<r_0$
Furthermore, we have $\ell_{2h+2}=r_{2h+2}$ and $d_{2h+2}(r_0)=d_{2h+2}(\ell_0)+1$.

$\alpha=\rho/\lambda$: $\ell_0=r_{h+1}<\ell_{h+1}=r_1<\cdots<\ell_{h-1}<r_h=-\delta_1<\ell_h<-\delta_2<0<r_0$

$\rho/\lambda<\alpha<1/\lambda$: $\ell_0<r_1<\cdots<\ell_{h-1}<r_h<-\delta_1<\ell_h<0<r_{h+1}<r_0$
Furthermore, we have $\ell_{h+1}=r_{h+2}$ and $d_{h+1}(\ell_0)=d_{h+2}(r_0)+1$.

$\alpha=1/\lambda$: $\ell_0=r_1<\cdots<\ell_{h-1}=r_h<-\delta_1<\ell_h=r_{h+1}=0<r_0$

Proof. In [BKS], Section 3.2 (see also the introduction of this section), it was shown that

$$\phi_0 = -\frac{\lambda}{2} < \phi_1 < \cdots < \phi_{h-2} < \phi_{h-1} < -\frac{2}{3\lambda} < \phi_h < -\frac{2}{5\lambda},$$

$\phi_0 < \phi_{h+1} < \phi_1$, and

$$\phi_{h+1} < \phi_{h+2} < \cdots < \phi_{2h} = -\frac{1}{\lambda} < -\frac{2}{3\lambda} < \phi_{2h+1} = 0.$$

In view of this and Lemma 2.1, we therefore have that $\phi_{h-1} \le \ell_{h-1} \le r_h < \phi_h$. An important question is to know, where $-\delta_1$ is located. Since $3/2 \le 1+\alpha$, we have $\phi_{h-1} < -\delta_1$. For $q = 2h+3$, we have $\sin\frac{(h+1)\pi}{q} = \sin\frac{(h+2)\pi}{q}$, thus

$$B_{h+1} = B_{h+2}, \qquad B_h = (\lambda-1)B_{h+1}, \qquad B_{h-1} = (\lambda^2-\lambda-1)B_{h+1}.$$

Hence we obtain, by (10),

$$\ell_{h-1} = -\frac{\alpha\lambda B_h - B_{h+1}}{\alpha\lambda B_{h-1} - B_h} = -\frac{1-\alpha\lambda(\lambda-1)}{\lambda-1-\alpha\lambda(\lambda^2-\lambda-1)} < -\delta_1.$$

The position of r_h with respect to $-\delta_1$ leads us to distinguish between the possible cases. We have that

$$r_h = -\frac{B_{h+1}\alpha\lambda - B_h}{B_h\alpha\lambda - B_{h-1}} = -\frac{1-(1-\alpha)\lambda}{1-(1-\alpha)\lambda(\lambda-1)} < -\delta_1$$

if and only if $\alpha^2\lambda^2 + \alpha\lambda(2-\lambda) - 1 > 0$, i.e., $\alpha\lambda > \frac{\lambda-2+\sqrt{\lambda^2-4\lambda+8}}{2} = \rho$. Note that

$$\frac{1}{2} < \frac{\lambda-2+\sqrt{\lambda^2-4\lambda+8}}{2\lambda} < \frac{1}{\lambda} \qquad \text{for } 0<\lambda<2.$$

Assume first that $\alpha > \rho/\lambda$. Then we have that $r_h < -\delta_1$, from which it immediately follows that

$$r_{h+1} = \frac{B_{h+1} - B_{h+2}\alpha\lambda}{B_{h+1}\alpha\lambda - B_h} = \frac{1-\alpha\lambda}{1-(1-\alpha)\lambda} \ge 0,$$
$$\ell_h = \frac{B_{h+1}\alpha\lambda - B_{h+2}}{B_{h+1} - B_h\alpha\lambda} = -\frac{1-\alpha\lambda}{1-\alpha\lambda(\lambda-1)} \le 0.$$

If $\alpha < 1/\lambda$, then $|1/\ell_h| - |1/r_{h+1}| = \lambda$, hence $\ell_{h+1} = r_{h+2}$ and $d_1(r_{h+1}) = d_1(\ell_h)+1$. In case $\alpha = 1/\lambda$, then $r_{h+1} = \ell_h = 0$. Hence the last two cases are proved.

It remains to consider $1/2 < \alpha \le \rho/\lambda$. Now we have that $-\delta_1 \le r_h (< \phi_h)$, with $-\delta_1 = r_h$ if and only if $\alpha = \rho/\lambda$. Consequently, we immediately find that

$$r_0 = [\,+1:1, (-1:1)^{h-1}, -1:2, (-1:1)^h, \dots\,].$$

To see that $\ell_h < -\delta_2$, note that this is equivalent to

$$\alpha^2\lambda^2 - \alpha\lambda^2 + 2\alpha\lambda - 1 < 2(\lambda-1)(1-\alpha\lambda),$$

which holds because of the assumption $\alpha^2\lambda^2 + \alpha\lambda(2-\lambda) - 1 \le 0$. This assumption also implies that $\ell_{h+1} \le r_1$, where again equality holds if and only if $\alpha = \rho/\lambda$. This proves the case $\alpha = \rho/\lambda$.

For $\alpha < \rho/\lambda$, we have

$$\ell_0 = [(-1:1)^h, -1:2, (-1:1)^h, \dots],$$

hence the convergents of ℓ_0 satisfy $-R_{h-1} = B_{h-1} = (\lambda^2-\lambda-1)B_{h+1}$, $-R_h = (\lambda-1)B_{h+1}$ and

$$R_{h+1} = 2\lambda R_h - R_{h-1} = -(\lambda^2-\lambda+1)B_{h+1}.$$

The recurrence $R_{h+n+1} = \lambda R_{h+n} - R_{h+n-1}$ for $n = 1, 2, \dots, h$, yields

$$-R_{h+n} = (B_{n+2} - B_{n+1} + 2B_n)B_{h+1} \quad \text{for } n = 0, 1, \dots, h+1.$$

For the S_n's, we have similarly that $S_{h-1} = (\lambda-1)B_{h+1}$, $S_h = B_{h+1}$, thus $S_{h+1} = 2\lambda S_h - S_{h-1} = (\lambda+1)B_{h+1}$ and

$$S_{h+n} = (B_{n+1} + B_n)B_{h+1} \qquad \text{for } n = 0, 1, \dots, h+1.$$

By (7), we obtain, for $n = 0, 1, \dots, h+1$,

$$\ell_{h+n} = -\frac{(B_{n+1}+B_n)(\alpha-1)\lambda + B_{n+2} - B_{n+1} + 2B_n}{(B_n + B_{n-1})(\alpha-1)\lambda + B_{n+1} - B_n + 2B_{n-1}}. \tag{16}$$

For the convergents of r_0, only the sign of the R_n is different and we get

$$r_{h+n} = -\frac{(B_{n+1}+B_n)\alpha\lambda - (B_{n+2} - B_{n+1} + 2B_n)}{(B_n + B_{n-1})\alpha\lambda - (B_{n+1} - B_n + 2B_{n-1})}. \tag{17}$$

This yields that

$$r_{2h+1} = -\frac{(2\alpha-1)\lambda}{\alpha\lambda^2 - 2\lambda + 2}\ (<0), \qquad \ell_{2h+1} = \frac{(2\alpha-1)\lambda}{(1-\alpha)\lambda^2 - 2\lambda + 2}\ (>0),$$

hence $|1/r_{2h+1}| - |1/\ell_{2h+1}| = \lambda$, $d_1(r_{2h+1}) = d_1(\ell_{2h+1}) + 1$, and the theorem is proved. □

For the construction of the natural extension, we have to distinguish between the different cases of the previous theorem. Consider first $\alpha > \rho/\lambda$.

Theorem 2.9. *Let $q = 2h+3$ with $h \geq 1$. Then the system of relations*

$$\begin{cases} (\mathcal{R}_1): & H_1 = 1/(\lambda + H_{2h+2}) \\ (\mathcal{R}_2): & H_2 = 1/\lambda \\ (\mathcal{R}_n): & H_n = 1/(\lambda - H_{n-2}) \quad \textit{for } n = 3, 4, \dots, 2h+2 \\ (\mathcal{R}_{2h+3}): & H_{2h+1} = \lambda/2 \\ (\mathcal{R}_{2h+4}): & H_{2h} + H_{2h+2} = \lambda \end{cases}$$

admits the (unique) solution

$$H_{2n} = -\phi_{2h+1-n} = \frac{B_n}{B_{n+1}} = \frac{\sin\frac{n\pi}{q}}{\sin\frac{(n+1)\pi}{q}} \quad \textit{for } n = 1, 2, \dots, h+1,$$

$$H_{2n-1} = \frac{B_{n-1} + B_n}{B_n + B_{n+1}} = \frac{\sin\frac{(n-1)\pi}{q} + \sin\frac{n\pi}{q}}{\sin\frac{n\pi}{q} + \sin\frac{(n+1)\pi}{q}} \quad \textit{for } n = 1, 2, \dots, h+1,$$

in particular $H_{2h+2} = 1$.

Let $\rho/\lambda < \alpha \le 1/\lambda$ and $\Omega_\alpha = \bigcup_{n=1}^{2h+2} J_n \times [0, H_n]$ with $J_{2n-1} = [\ell_{n-1}, r_n)$, $J_{2n} = [r_n, \ell_n)$ for $n = 1, 2, \ldots, h$, $J_{2h+1} = [\ell_h, r_{h+1})$ and $J_{2h+2} = [r_{h+1}, r_0)$. Then the map $\mathcal{T}_\alpha : \Omega_\alpha \to \Omega_\alpha$ given by (5) is bijective off of a set of Lebesgue measure zero.

Remark. The case $q = 3$, $\rho/\lambda \le \alpha \le 1/\lambda$, which is the case of Nakada's α-expansions for $(\sqrt{5}-1)/2 \le \alpha \le 1$, has been dealt with in [N1]; see also [NIT], [TI], and [K1, K2].

The proof of Theorem 2.9 is very similar to that of Theorem 2.3 and therefore omitted, see also Figure 6. In case $\alpha = 1/\lambda$, the intervals J_{2n-1} are empty.

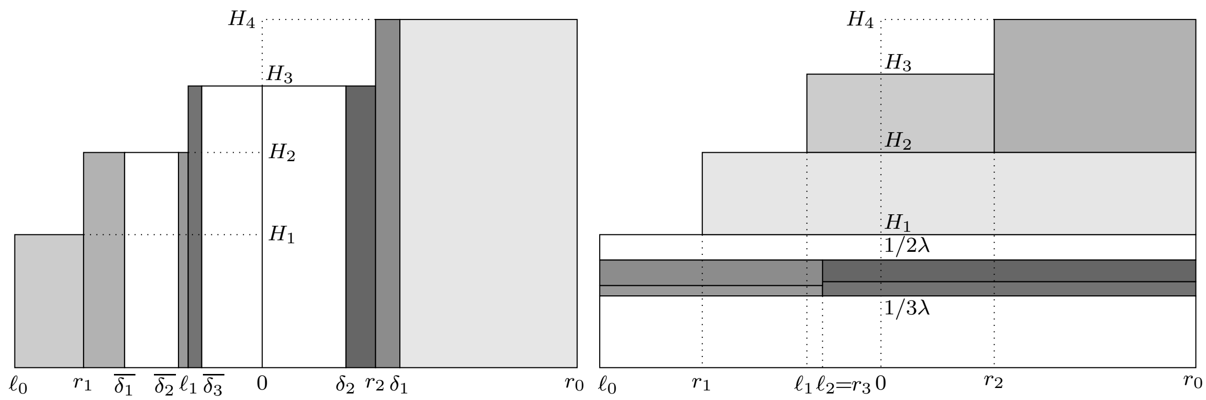

Figure 6. The natural extension domain Ω_α (left) and its image under $\mathcal{T}_\alpha$ (right) of the α-Rosen continued fraction ($\overline{\delta_n} = -\delta_n$); here $q = 5$, $\alpha = 0.56$, $d_{h+1}(\ell_0) = 3$, $d_{h+2}(r_0) = 2$.

Once more, a Jacobian calculation shows that $\mathcal{T}_\alpha$ preserves the probability measure ν_α with density

$$\frac{C_{q,\alpha}}{(1+xy)^2},$$

where $C_{q,\alpha}$ is a normalizing constant given by the following proposition.

Proposition 2.10. *If $q = 2h+3$ and $\rho/\lambda < \alpha \le 1/\lambda$, then the normalizing constant is*

$$C_{q,\alpha} = 1/\log \frac{1+\alpha\lambda}{\sqrt{2-\lambda}} = 1/\log \frac{1 + 2\alpha \cos\frac{\pi}{q}}{2 \sin\frac{\pi}{2q}}.$$

Proof. Integration gives

$$C_{q,\alpha} = 1/\log \left(\frac{1+r_0}{1+r_{h+1}} \prod_{n=1}^{h+1} \frac{1+r_n H_{2n-1}}{1+\ell_{n-1} H_{2n-1}} \prod_{n=1}^{h} \frac{1+\ell_n H_{2n}}{1+r_n H_{2n}} \right).$$

Using (10) and (11), we find

$$\frac{1+r_n H_{2n-1}}{1+\ell_{n-1} H_{2n-1}} = \frac{B_n - B_{n-1}\alpha\lambda}{B_n\alpha\lambda - B_{n-1}}, \qquad \frac{1+\ell_n H_{2n}}{1+r_n H_{2n}} = \frac{B_n\alpha\lambda - B_{n-1}}{B_{n+1} - B_n\alpha\lambda},$$

and

$$\frac{1+r_0}{1+r_{h+1}} = \frac{(1+\alpha\lambda)(B_{h+1}\alpha\lambda - B_h)}{B_{h+1}-B_h} = (1+\alpha\lambda)(B_{h+1}\alpha\lambda - B_h)B_{h+1}\,,$$

where we have used

$$(2-\lambda)B_{h+1}^2 = \frac{2(1-\cos\frac{\pi}{q})\sin^2\frac{(h+1)\pi}{q}}{\sin^2\frac{\pi}{q}} = \frac{4\sin^2\frac{\pi}{2q}\cos^2\frac{\pi}{2q}}{4\sin^2\frac{\pi}{2q}\cos^2\frac{\pi}{2q}} = 1.$$

Putting everything together, we obtain

$$C_{q,\alpha} = 1/\log((1+\alpha\lambda)B_{h+1}) = 1/\log\frac{1+\alpha\lambda}{\sqrt{2-\lambda}} = 1/\log\frac{1+\alpha\lambda}{2\sin\frac{\pi}{2q}}\,.$$

□

REMARK. Note that for $q=3$ this result confirms Nakada's result from [N1] for α between $(\sqrt{5}-1)/2$ and 1; in this case, the normalizing constant is indeed $1/\log(1+\alpha)$.

Now consider $\alpha < \rho/\lambda$.

THEOREM 2.11. *Let $q = 2h+3$ with $h \geq 1$. Then the system of relations*

$$\begin{cases} (\mathcal{R}_1): & H_1 = 1/(2\lambda - H_{4h-1}) \\ (\mathcal{R}_2): & H_2 = 1/(2\lambda - H_{4h}) \\ (\mathcal{R}_3): & H_3 = 1/(\lambda + H_{4h+3}) \\ (\mathcal{R}_4): & H_4 = 1/\lambda \\ (\mathcal{R}_n): & H_n = 1/(\lambda - H_{n-4}) \qquad \textit{for } n = 5,6,\ldots,4h+3 \\ (\mathcal{R}_{4h+4}): & H_{4h+2} = \lambda/2 \\ (\mathcal{R}_{4h+5}): & H_{4h+1} + H_{4h+3} = \lambda \end{cases}$$

admits the (unique) solution

$$H_{4n} = \frac{B_n}{B_{n+1}}, \qquad H_{4n-2} = \frac{B_{n-1}+B_n}{B_n+B_{n+1}},$$
$$H_{4h+3-4n} = \frac{B_{n+1}\rho - B_n}{B_n\rho - B_{n-1}}, \qquad H_{4h+1-4n} = \frac{B_{n+1}\rho - B_{n+2}}{B_n\rho - B_{n+1}},$$

in particular $H_{4h+3} = \rho$.

Let $1/2 \leq \alpha < \rho/\lambda$, and $\Omega_\alpha = \bigcup_{n=1}^{4h+3} J_n \times [0, H_n]$ with $J_{4n-3} = [\ell_{n-1}, r_{h+n})$, $J_{4n-2} = [r_{h+n}, \ell_{h+n})$, $J_{4n-1} = [\ell_{h+n}, r_n)$, $J_{4n} = [r_n, \ell_n)$ for $n = 1,2,\ldots,h$, $J_{4h+1} = [\ell_h, r_{2h+1})$, $J_{4h+2} = [r_{2h+1}, \ell_{2h+1})$ and $J_{4h+3} = [\ell_{2h+1}, r_0)$. Then the map $\mathcal{T}_\alpha : \Omega_\alpha \to \Omega_\alpha$ given by (5) is bijective off of a set of Lebesgue measure zero.

PROOF. The proof of the bijectivity runs along the same lines as the proof of Theorem 2.3 and is therefore omitted, see also Figure 7.

The H_{4n}'s are determined by $(\mathcal{R}_4), (\mathcal{R}_8), \ldots, (\mathcal{R}_{4h})$. The H_{4n-2}'s are determined by $(\mathcal{R}_2), (\mathcal{R}_6), \ldots, (\mathcal{R}_{4h+2})$ and $(\mathcal{R}_{4h+4})$. By $(\mathcal{R}_3), (\mathcal{R}_7), \ldots, (\mathcal{R}_{4h+3})$, we obtain $H_{4h+3-4n} =$

$\frac{B_{n+1}H_{4h+3}-B_n}{B_nH_{4h+3}-B_{n-1}}$ and

$$\frac{1}{\lambda+H_{4h+3}} = H_3 = \frac{B_{h+1}H_{4h+3}-B_h}{B_hH_{4h+3}-B_{h-1}} = \frac{H_{4h+3}-(\lambda-1)}{(\lambda-1)H_{4h+3}-(\lambda^2-\lambda-1)},$$

thus $H_{4h+3}^2+(2-\lambda)H_{4h+3}-1=0$, i.e. $H_{4h+3}=\rho$. Finally, the $H_{4h+1-4n}$'s are determined by $(\mathcal{R}_1),(\mathcal{R}_5),\ldots,(\mathcal{R}_{4h+5})$. For $\alpha=1/2$, the intervals J_{4n} and J_{4n-2} are empty. $\square$

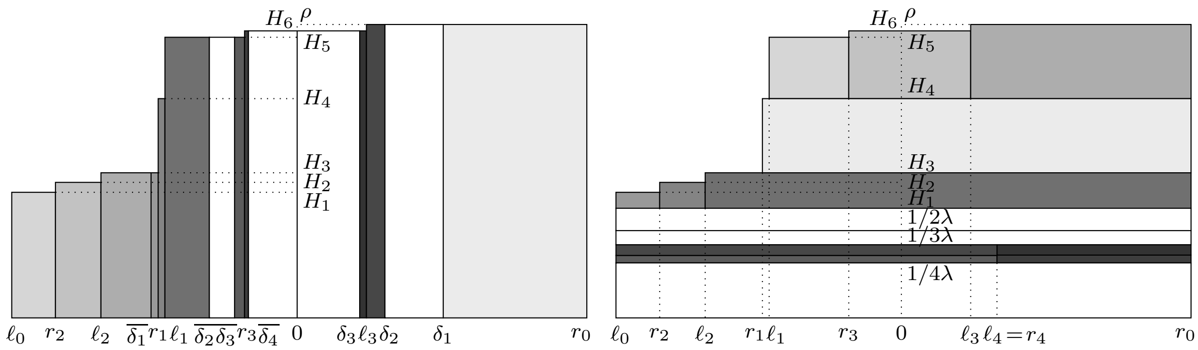

FIGURE 7. The natural extension domain Ω_α (left) and its image under $\mathcal{T}_\alpha$ (right) of the α-Rosen continued fraction ($\overline{\delta_n}=-\delta_n$); here $q=5$, $\alpha=0.5038$, $d_{2h+2}(\ell_0)=2$, $d_{2h+2}(r_0)=3$.

Again, $\mathcal{T}_\alpha$ preserves the probability measure ν_α with density $C_{q,\alpha}/(1+xy)^2$, where $C_{q,\alpha}$ is a normalizing constant given by the following proposition.

PROPOSITION 2.12. *If $q=2h+3$ and $1/2\le\alpha<\rho/\lambda$, then the normalizing constant is*

$$C_{q,\alpha} = 1/\log\frac{1+\rho}{\sqrt{2-\lambda}} = 1/\log\frac{1+\rho}{2\sin\frac{\pi}{2q}}.$$

PROOF. Integration yields that $C_{q,\alpha}$ is equal to

$$1/\log\left(\frac{1+r_0\rho}{1+\ell_{2h+1}\rho}\prod_{j=1}^{h+1}\frac{1+r_{h+j}H_{4j-3}}{1+\ell_{j-1}H_{4j-3}}\frac{1+\ell_{h+j}H_{4j-2}}{1+r_{h+j}H_{4j-2}}\prod_{j=1}^{h}\frac{1+r_jH_{4j-1}}{1+\ell_{h+j}H_{4j-1}}\frac{1+\ell_jH_{4j}}{1+r_jH_{4j}}\right).$$

Using (10), (11), (16), (17) and Lemma 2.5, we find

$$\frac{1+r_{h+j}H_{4j-3}}{1+\ell_{j-1}H_{4j-3}} = \frac{(\lambda(1-2\alpha)\rho+\alpha\lambda^2-\lambda^2+2\lambda-2)(B_j-B_{j-1}\alpha\lambda)}{((\alpha\lambda-1)\rho-\alpha\lambda+\lambda-1)((B_j+B_{j-1})\alpha\lambda-B_{j+1}+B_j-2B_{j-1})}$$

$$\frac{1+\ell_{h+j}H_{4j-2}}{1+r_{h+j}H_{4j-2}} = \frac{(B_j+B_{j-1})\alpha\lambda-B_{j+1}+B_j-2B_{j-1}}{2B_j-B_{j-1}+B_{j-2}-(B_j+B_{j-1})\alpha\lambda}$$

$$\frac{1+r_jH_{4j-1}}{1+\ell_{h+j}H_{4j-1}} = \frac{((\alpha\lambda-1)\rho-\alpha\lambda+\lambda-1)(2B_j-B_{j-1}+B_{j-2}-(B_j+B_{j-1})\alpha\lambda)}{(\lambda(1-2\alpha)\rho+\alpha\lambda^2-\lambda^2+2\lambda-1)(B_j\alpha\lambda-B_{j-1})}$$

$$\frac{1+\ell_jH_{4j}}{1+r_jH_{4j}} = \frac{B_j\alpha\lambda-B_{j-1}}{B_{j+1}-B_j\alpha\lambda}$$

and

$$\frac{1+r_0\rho}{1+\ell_{2h+1}\rho} = \frac{(1+\alpha\lambda\rho)((2B_{h+1}-B_h+B_{h-1}-(B_{h+1}+B_h)\alpha\lambda)}{((2\alpha-1)\lambda\rho-\alpha\lambda^2+\lambda^2+2\lambda-2)B_{h+1}}.$$

Putting everything together, we obtain that

$$C_{q,\alpha} = 1/\log\frac{(1+\alpha\lambda\rho)\sqrt{2-\lambda}}{1+\alpha\lambda-\lambda+(1-\alpha\lambda)\rho} = 1/\log\frac{1+\rho}{\sqrt{2-\lambda}}.$$

□

This confirms again Nakada's result for $q=3$, i.e., $C_{3,\alpha}=1/\log\frac{\sqrt{5}+1}{2}$ for $\frac{1}{2}\le\alpha<\frac{\sqrt{5}-1}{2}$.

The case $\alpha=\rho/\lambda$ is slightly different from both other cases (similarly to $\alpha=1/\lambda$ for even q).

Theorem 2.13. *Let $q=2h+3$ with $h\ge 1$, $\alpha=\rho/\lambda$ and*

$$\Omega_\rho = \bigcup_{j=1}^{h}\left([\ell_{j-1},r_j)\times\left[0,\frac{B_{j-1}+B_j}{B_j+B_{j+1}}\right]\right)\cup\left([r_j,\ell_j)\times\left[0,\frac{B_j}{B_{j+1}}\right]\right)\cup\left([\ell_h,\rho)\times\left[0,\frac{\lambda}{2}\right]\right).$$

Then $\mathcal{T}_\rho:\Omega_\rho\to\Omega_\rho$ is bijective off of a set of Lebesgue measure zero.

The normalizing constant in this case is $C_{q,\rho/\lambda}=1/\log\frac{1+\rho}{\sqrt{2-\lambda}}$ as above. As in the even case, we set μ_α the projection of ν_α on the first coordinate, $\bar{\mathcal{B}}$ be the restriction of the two-dimensional σ-algebra on Ω_α, and $\mathcal{B}$ be the Lebesgue σ-algebra on $I_{q,\alpha}=[\lambda(\alpha-1),\alpha\lambda]$. We have the following theorem, whose proof is similar to the proof of Theorem refthm:natural-extension-even-case in the even case.

Theorem 2.14. *Let $q\ge 3$, $q=2h+1$, and let $\frac{1}{2}\le\alpha\le\frac{1}{\lambda}$. Then the dynamical system $(\Omega_\alpha,\bar{\mathcal{B}},\nu_\alpha,\mathcal{T}_\alpha)$ is the natural extension of the dynamical system $(I_{q,\alpha},\mathcal{B},\mu_\alpha,T_\alpha)$.*

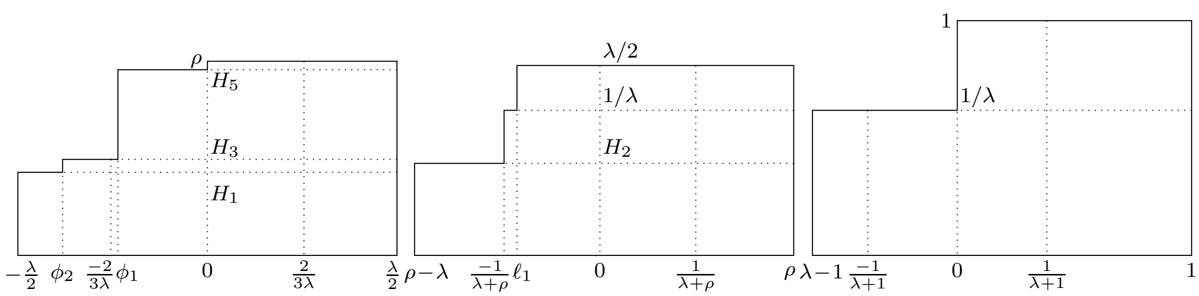

Figure 8. $\Omega_{1/2}$ (left), $\Omega_{\rho/\lambda}$ (middle) and $\Omega_{1/\lambda}$ (right); here $q=5$.

2.3. **Convergence of the continued fractions.** Now we can prove easily that the α-Rosen continued fractions converge. If $T_\alpha^n(x) = 0$ for some $n \geq 0$, then this is clear. Therefore assume that $T_\alpha^n(x) \neq 0$ for all $n \geq 0$. Setting $(t_n, v_n) = \mathcal{T}_\alpha^n(x, 0)$, it follows directly from the definition (5) of $\mathcal{T}_\alpha$ that $v_n = [\,1 : d_n,\, \varepsilon_n : d_{n-1}, \ldots, \varepsilon_2 : d_1\,]$. Furthermore, an immediate consequence of (6) is that $S_{n-1}/S_n = [\,1 : d_n,\, \varepsilon_n : d_{n-1}, \ldots, \varepsilon_2 : d_1\,]$, i.e., $v_n = S_{n-1}/S_n$.

Theorems 2.3, 2.9 and 2.11 (see also Figures 3, 6 and 7) show that $v_n \leq 1$, i.e., $S_n \geq S_{n-1}$, and that $S_n = S_{n-1}$ if and only if $q = 2h+3$, $n = h+1$, $d_1 = 1$, $(\varepsilon_i : d_i) = (-1 : 1)$ for $i = 2, 3, \ldots, h+1$, which is possible only if $\alpha > \rho/\lambda$. Furthermore we have that $v_{n-1} v_n \leq 1/c$ for some constant $c > 1$, i.e., $S_n \geq c S_{n-2}$. It follows from (7) that

$$\left| x - \frac{R_n}{S_n} \right| = \left| \frac{T_\alpha^n(x)(R_{n-1}S_n - R_n S_{n-1})}{S_n(S_n + T_\alpha^n(x) S_{n-1})} \right| = \frac{|t_n|}{S_n^2(1 + t_n v_n)} \leq \frac{\alpha\lambda}{(1 + \alpha\lambda - \lambda) S_n^2}\,, \tag{18}$$

hence the α-Rosen convergents R_n/S_n converge to x as $n \to \infty$.

3. Mixing properties of α-Rosen fractions

In case q is even, and $\alpha = 1/\lambda$, we saw in the previous section that there is a simple relation between $\Omega_{1/2}$ and $\Omega_{1/\lambda}$; see also Figure 4. Define in this case the map $\mathcal{M} : \Omega_{1/\lambda} \to \Omega_{1/2}$ by

$$\mathcal{M}(x, y) = \begin{cases} (-y, -x) & \text{if } (x, y) \in \Omega_{1/\lambda},\ x < 0, \\ (y, x) & \text{if } (x, y) \in \Omega_{1/\lambda},\ x \geq 0. \end{cases}$$

Clearly, $\mathcal{M} : \Omega_{1/\lambda} \to \Omega_{1/2}$ is bijective and bi-measurable transformation, and $\nu_{1/\lambda}(\mathcal{M}^{-1}(A)) = \nu_{1/2}(A)$, for every Borel set $A \subset \Omega_{1/2}$. By comparing the partitions of $\mathcal{T}_{1/\lambda}$ (on $\Omega_{1/\lambda}$) and that of $\mathcal{T}_{1/2}^{-1}$ (on $\Omega_{1/2}$), we find that

$$\mathcal{T}_{1/\lambda}(x, y) = \mathcal{M}^{-1}\left(\mathcal{T}_{1/2}^{-1}\left(\mathcal{M}(x, y) \right) \right), \qquad (x, y) \in \Omega_{1/\lambda},\ x \neq 0.$$

This implies that the dynamical systems $(\Omega_{1/2}, \bar{\mathcal{B}}, \nu_{1/2}, \mathcal{T}_{1/2}^{-1})$ and $(\Omega_{1/\lambda}, \bar{\mathcal{B}}, \nu_{1/\lambda}, \mathcal{T}_{1/\lambda})$ are isomorphic. In [BKS] it was shown that the dynamical system system $(\Omega_{1/2}, \bar{\mathcal{B}}, \nu_{1/2}, \mathcal{T}_{1/2})$ is weakly Bernoulli with respect to the natural partition, hence $(\Omega_{1/2}, \bar{\mathcal{B}}, \nu_{1/2}, \mathcal{T}_{1/2})$ and $(\Omega_{1/2}, \bar{\mathcal{B}}, \nu_{1/2}, \mathcal{T}_{1/2}^{-1})$ are isomorphic. As a consequence we find that the dynamical systems $(\Omega_{1/2}, \bar{\mathcal{B}}, \nu_{1/2}, \mathcal{T}_{1/2})$ and $(\Omega_{1/\lambda}, \bar{\mathcal{B}}, \nu_{1/\lambda}, \mathcal{T}_{1/\lambda})$ are isomorphic.

In this section, we will show that this result also holds for all q and all α strictly between $1/2$ and $1/\lambda$, using a result by M. Rychlik [Ry]. For completeness, we state explicitly the hypothesis needed for Rychlik's result (the reader is referred to [Ry] for more details).

Let X be a totally ordered order-complete set. Open intervals constitute a base of a complete topology in X, making X into a topological space. If X is separable, then X is homeomorphic with a closed subset of an interval. Let $\mathcal{B}$ be the Borel σ-algebra on X, and m a fixed regular, Borel probability measure on X (in our case m will be the normalized Lebesgue measure restricted to X). Let $U \subset X$ be an open dense subset of X, such that $m(U) = 1$. Let $S = X \setminus U$, clearly $m(S) = 0$.

Let $T : U \to X$ be a continuous map, and β a countable family of closed intervals with disjoint interiors, such that $U \subset \bigcup \beta$. Furthermore, suppose that for any $B \in \beta$ one has that $B \cap S$ consists only of endpoints of B, and that T restricted to $B \cap U$ admits an extension to a homeomorphism of B with some interval in X. Suppose that $T'(x) \neq 0$ for $x \in U$, and let $g(x) = 1/|T'(x)|$ for $x \in U$, $g|_S = 0$. Let $P : L^1(X, m) \to L^1(X, m)$ be the Perron-Frobenius operator of T,

$$Pf(x) = \sum_{y \in T^{-1}x} g(y)f(y).$$

In [Ry], it was proved (among many other things) that, if $\|g\|_\infty < 1$ and $\operatorname{Var} g < \infty$, then there exist functions $\varphi_1, \varphi_2, \ldots, \varphi_s$ of bounded variation, such that

(i) $P\varphi_i = \varphi_i$;
(ii) $\int \varphi_i \, \mathrm{d}m = 1$;
(iii) There exists a measurable partition $C_1, C_2, \ldots, C_s$ of X with $T^{-1}C_i = C_i$ for $i = 1, 2, \ldots, s$;
(iv) The dynamical system (C_i, T_i, ν_i), where $T_i = T|_{C_i}$ and $\nu_i(B) = \int_B \varphi_i \, \mathrm{d}m$ are exact, and ν_i is the unique invariant measure for T_i, absolutely continuous with respect to $m|_{C_i}$.

Rychlik also showed that if $s = 1$, i.e., if 1 is the only eigenvalue of P on the unit circle and if there exists only one $\varphi \in L^1(X, M)$ with $P\varphi = \varphi$ and $m(\varphi) = 1$, ($\varphi \geq 0$), then the natural extension of (X, T, ν) is isomorphic to a Bernoulli shift.

Returning to our map T_α, defined on $X = I_{q,\alpha} = [\lambda(\alpha - 1), \alpha\lambda]$, and using the same notation as above, we let m be normalized Lebesgue measure on X,

$$S = \{\lambda(\alpha - 1)\} \cup \left\{ \pm \frac{1}{\lambda(\alpha + d)} \, ; \ d = 1, 2, \ldots \right\}$$

and $U = X \setminus S$. Note that $T_\alpha : U \to X$ is continuous, and that the restriction of T_α to each open interval is homeomorphic to an interval (in fact to X itself, except for the first and last interval).

We have that $g(x) = 1/|T'_\alpha(x)| = x^2$ on U, hence $\|g\|_\infty < 1$ (since $\alpha \neq 1/\lambda$), and $\operatorname{Var} g < \infty$. It is easy, but tedious (cf. [DK] for a proof of the regular case), to see that T is ergodic, hence $s = 1$, and we can apply Rychlik's result to obtain the following theorem.

Theorem 3.1. *The natural extension $(\Omega_\alpha, \nu_\alpha, \mathcal{T}_\alpha)$ of $(X_\alpha, \mu_\alpha, T_\alpha)$ is weakly Bernoulli. Hence, the natural extension is isomorphic to any Bernoulli shift with the same entropy.*

4. Metrical properties of 'regular' Rosen fractions

An important reason to introduce and study the natural extension of the ergodic system underlying any continued fraction expansion, is that such a natural extension facilitates the study of the continued fraction expansion at hand; see e.g. [DK] and [IK], Chapter 4. The following theorem is a consequence of this; see [BJW], [DK], or [IK], Chapter 4.

THEOREM 4.1. *Let $q \geq 3$, and let $1/2 \leq \alpha \leq 1/\lambda$. For almost all G_q-irrational numbers x, the two-dimensional sequence $\mathcal{T}_\alpha(x,0) = (\, T_\alpha^n(x),\, S_{n-1}/S_n\,)$, $n \geq 1$, is distributed over Ω_α according to the density function g_α, given by*

$$g_\alpha(t,v) = \frac{C_{q,\alpha}}{(1+tv)^2}$$

for $(t,v) \in \Omega_\alpha$, and $g_\alpha(t,v) = 0$ otherwise. Here $C_{q,\alpha}$ is the normalizing constant of the $\mathcal{T}_\alpha$-invariant measure ν_α.

Due to Proposition 4.1, it is possible to study the distribution of various sequences related to the α-Rosen expansion of almost every $x \in X_\alpha$. Classical examples of these are the frequency of digits, or the analogs of various classical results by Lévy and Khintchine. However, these results can already be obtained from the projection $(X_\alpha, \mathcal{B}_\alpha, \mu_\alpha, T_\alpha)$ of $(\Omega_\alpha, \bar{\mathcal{B}}_\alpha, \nu_\alpha, \mathcal{T}_\alpha)$ — which is also ergodic — and the Ergodic Theorem. For the distribution of the so-called *approximation coefficients*, the natural extension $(\Omega_\alpha, \bar{\mathcal{B}}_\alpha, \nu_\alpha, \mathcal{T}_\alpha)$ is necessary. These approximation coefficients $\Theta_n = \Theta_n(x)$, are defined by

$$\Theta_n = \Theta_n(x) := S_n^2 \left| x - \frac{R_n}{S_n} \right|, \quad n \geq 0, \tag{19}$$

where R_n/S_n is the nth α-Rosen convergent, which is obtained by truncating the α-Rosen expansion.

With $(t_n, v_n) = \mathcal{T}_\alpha^n(x,0)$, it follows from (18), that

$$\Theta_n = \frac{\varepsilon_{n+1} t_n}{1 + t_n v_n}, \quad \text{for } n \geq 1. \tag{20}$$

Similarly, since $t_n = \varepsilon_n / t_{n-1} - d_n \lambda$ and $v_n = S_{n-1}/S_n$, it follows from (6) that

$$\Theta_{n-1} = \frac{v_n}{1 + t_n v_n}, \quad \text{for } n \geq 1. \tag{21}$$

In view of (20) and (21), we define the map

$$F(t,v) = \left(\frac{v}{1+tv}, \frac{t}{1+tv} \right) =: (\xi, \eta), \qquad \text{for } tv \neq -1.$$

It is now easy calculation, see e.g. [BKS], p. 1293, that due to Proposition 4.1 one has for almost all $x \in X_\alpha$ that the sequence $(\Theta_{n-1}(x), \varepsilon_{n+1}\Theta_n(x))_{n\geq 0}$ is distributed on $F(\Omega_\alpha)$ according to the density function $C_{q,\alpha}/\sqrt{1-4\xi\eta}$. Setting

$$\Gamma_\alpha^+ = F(\{(t,v) \in \Omega_\alpha \mid t \geq 0\}) \quad \text{and} \quad \Gamma_\alpha^- = F(\{(t,v) \in \Omega_\alpha \mid t \leq 0\}),$$

we have found the following theorem.

THEOREM 4.2. *Let $q \geq 3$, $1/2 \leq \alpha \leq 1/\lambda$, and define the functions d_α^+ and d_α^- by*

$$d_\alpha^{\pm}(\xi,\eta) = \frac{C_{q,\alpha}}{\sqrt{1 \mp 4\xi\eta}} \quad \text{for} \quad (\xi,\eta) \in \Gamma_\alpha^{\pm}, \tag{22}$$

and $d_\alpha^{\pm}(\xi,\eta) = 0$ otherwise. Then the sequence $(\Theta_{n-1}(x), \Theta_n(x))_{n\geq 1}$ lies in the interior of $\Gamma = \Gamma_\alpha^+ \cup \Gamma_\alpha^-$ for all G_q-irrational numbers x, and for almost all x this sequence is distributed according to the density function d_α, where

$$d_\alpha(\xi,\eta) = d_\alpha^+(\xi,\eta) + d_\alpha^-(\xi,\eta)\,.$$

By this last statement we mean that, for almost all x and for all $a, b \geq 0$, the limit

$$\lim_{N\to\infty} \frac{1}{N} \#\{j \,:\, 1 \leq j \leq N, \Theta_{j-1}(x) < a,\, \Theta_j(x) < b\,\}$$

exists, and equals

$$\int_0^a \int_0^b d_\alpha(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

Several corollaries can be drawn from Theorem 4.2, see e.g. [K1], where (for $q = 3$) for almost all x the distributions of the sequences $(\Theta_n)_{n\geq 1}$, $(\Theta_{n-1} + \Theta_n)_{n\geq 1}$, $(\Theta_{n-1} - \Theta_n)_{n\geq 1}$ were determined.

Here we only mention the following result for even values of q, a result which was previously obtained in [BKS] for both even and odd values of q, and $\alpha = 1/2$.

PROPOSITION 4.3. *Let $q \geq 4$ be an even integer, $1/2 \leq \alpha \leq 1/\lambda$, and let*

$$\mathcal{L}_\alpha := \min\left\{ \frac{\lambda}{\lambda+2}, \frac{\lambda(2-\alpha\lambda^2)}{4-\lambda^2} \right\}.$$

Then for almost all G_q-irrational numbers x and all $c \geq 1/\mathcal{L}_\alpha$, we have that

$$\lim_{N\to\infty} \frac{1}{N} \# \left\{ n : 1 \leq n \leq N,\, \Theta_n(x) < \frac{1}{c} \right\} = \frac{\lambda C_{q,\alpha}}{c}\,.$$

PROOF. In view of the expression of $\Theta_{n-1}(x)$ in (20), we consider curves given by

$$c = \frac{v}{1+tv},$$

where $c > 0$ is a constant, and $t \in [\ell_0, r_0]$. Note that these curves are monotonically increasing on $[\ell_0, r_0]$, and that the curve given by $v = \frac{c_1}{1-c_1 t}$ lies "above" the curve given by $v = \frac{c_2}{1-c_2 t}$, if and only if $c_1 > c_2$.

Now let $\mathcal{L}_\alpha$ be defined as the positive largest c for which the curve $c = \frac{v}{1+tv}$ lies in Ω_α for $t \in [\ell_0, r_0]$, i.e.,

$$\mathcal{L}_\alpha = \max\left\{ c > 0 :\ \left(t, \frac{c}{1-ct}\right) \in \Omega_\alpha,\ \text{for all } t \in [\ell_0, r_0] \right\}.$$

It follows from Theorem 4.1 that for all $z \leq \mathcal{L}_\alpha$, and for almost all G_q-irrationals x one has that

$$\lim_{N\to\infty} \frac{1}{N} \# \{n : 1 \leq n \leq N,\, \Theta_n(x) < z\} = \int_{\ell_0}^{r_0} \left(\int_0^{\frac{z}{1-zt}} g_\alpha(t,v)\,\mathrm{d}v \right) \mathrm{d}t = \lambda C_{q,\alpha} z\,,$$

where $C_{q,\alpha}$ is the normalizing constant of the invariant measure (which has density g_α). So we are left to show that $\mathcal{L}_\alpha = \min\left\{\frac{\lambda}{\lambda+2}, \frac{\lambda(2-\alpha\lambda^2)}{4-\lambda^2}\right\}$.

In the even case we can discern three cases: $\alpha = 1/2$, $1/2 < \alpha < 1/\lambda$, and $\alpha = 1/\lambda$. Note that the first case has been dealt with in [BKS]; in case $\alpha = 1/2$ one has that $\mathcal{L}_\alpha = \frac{\lambda}{\lambda+2}$.

In case $1/2 < \alpha < 1/\lambda$, first note that the curve $c_1 = \frac{v}{1+tv}$ goes through $(r_1, H_1) = (\frac{1}{\alpha\lambda} - \lambda, \frac{1}{\lambda+1})$ if and only if $c_1 = \frac{\alpha\lambda}{\alpha\lambda+1}$. Since in this case

$$\frac{\alpha\lambda}{\alpha\lambda+1} < \frac{1}{2} < \frac{1}{\lambda} = H_2,$$

and the curve $c_1 = \frac{v}{1+tv}$ is monotonically increasing on $[\ell_0, r_0]$, we immediately find that this curve is in Ω_α for $t \in [\ell_0, 0]$, yielding that $\mathcal{L}_\alpha \leq \frac{\alpha\lambda}{\alpha\lambda+1}$.

From Theorem 2.2 we see that $\ell_{p-2} < 0 < \ell_{p-1}$. Setting

$$c_2 = \frac{H_{2p-2}}{1+\ell_{p-1}H_{2p-2}} = \frac{\lambda(2-\alpha\lambda^2)}{4-\lambda^2}, \quad c_3 = \frac{H_{2p-1}}{1+r_0H_{p-1}} = \frac{1}{1+\alpha\lambda},$$

it follows from Theorem 2.3 (see also Figure 3 for $q = 6$) that $\mathcal{L}_\alpha = \min\{c_1, c_2\}$, since $c_1 < c_3$. For q (and therefore λ) fixed, and for $\alpha \in [1/2, 1/\lambda]$, one easily shows that $c_1 = c_1(\alpha)$ is a monotonically increasing function of α, with $c_1(1/2) = \frac{\lambda}{\lambda+2}$, and $c_1(1/\lambda) = 1/2$, while $c_2 = c_2(\alpha)$ is a line with slope $-\lambda^3/(4-\lambda^2)$. Since $c_2(1/2) = \lambda/2 > 1/2 > \lambda/(\lambda+2)$, and $c_2(1/\lambda) = \lambda/(\lambda+2) = c_1(1/2) < c_1(1/\lambda) = 1/2$, we find for $1/2 < \alpha < 1/\lambda$ that

$$\mathcal{L}_\alpha = \min\left\{\frac{\lambda}{\lambda+2}, \frac{\lambda(2-\alpha\lambda^2)}{4-\lambda^2}\right\}.$$

In case $\alpha = 1/\lambda$, the point $(1, \lambda/2)$ yields $c = \lambda/(\lambda+2)$. Since the curve $c = \frac{v}{1+tv}$ is monotonically increasing on $[\ell_0, r_0]$, and from the fact that for $t = 0$ we have that $v = c = \lambda/(\lambda+2) < 1/\lambda$, where $1/\lambda$ is the "smallest height" of Ω_α, we find that

$$\mathcal{L}_{1/\lambda} = \frac{\lambda}{\lambda+2}.$$

This proves the theorem. □

Remark. We only deal with the even case in Proposition 4.3; a result for the odd case is obtained similarly, but has a more involved expression.

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