Symmetry of information and nonuniform lower bounds

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Outline

Two complexity classes

- EXP: set of languages recognized in exponential time by a deterministic Turing machine

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- Open question: $\mathrm{EXP} \subset \mathrm{P/poly}$?

- Main result: polynomial-time symmetry of information implies $\mathrm{EXP} \not\subset \mathrm{P/poly}$.

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- $\mathrm{EXP} \neq \mathrm{P}$ (time hierarchy theorem).
- Space complexity version:

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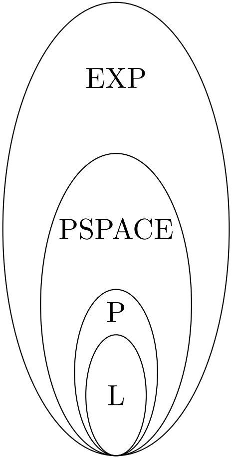
- Even the question "$\mathrm{EXP} \subset \mathrm{L/poly?}$" is open.

Some classes



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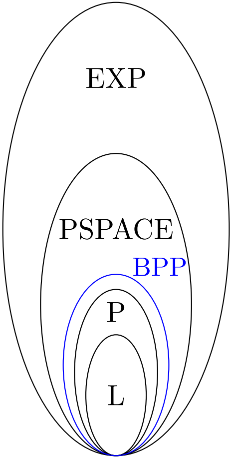

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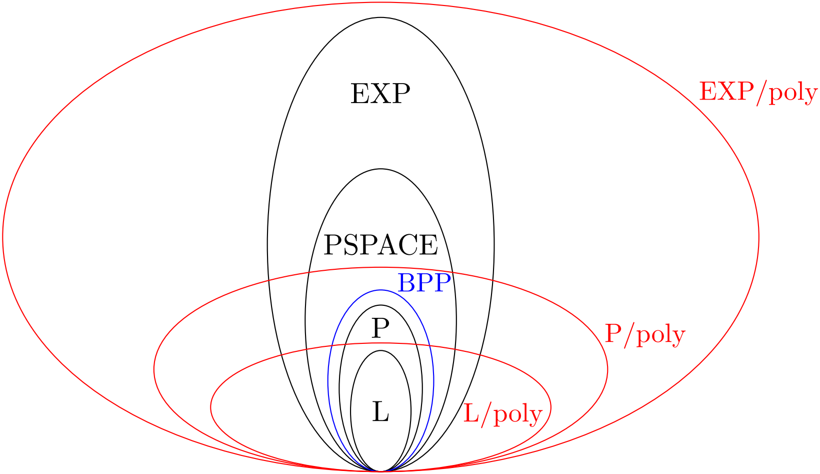

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- If $\mathcal{C}$ is a complexity class and $a : \mathbb{N} \to \mathbb{N}$ a function, then $\mathcal{C}/a(n)$ is the set of languages A such that there exists $B \in \mathcal{C}$ and a function $c : \mathbb{N} \to \{0,1\}^*$ satisfying:
  - $\forall n, |c(n)| \leq a(n)$;
  - $\forall x \in \{0,1\}^*$, $x \in A \iff (x, c(|x|)) \in B$.
- "The class $\mathcal{C}$ is helped by the advice $c(|x|)$" (the same for all words of each length).

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  $\mathrm{P/poly}$: conversion advice $\longleftrightarrow$ boolean circuit.
- $\mathrm{EXP} \subset \mathrm{P/poly} \iff \mathrm{EXP/poly} = \mathrm{P/poly}$.

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- Babai, Fortnow, Nisan & Wigderson 1993: if $\mathrm{EXP} \not\subset \mathrm{P/poly}$ then BPP has subexponential-time deterministic algorithms.

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- Babai, Fortnow, Nisan & Wigderson 1993: if $\mathrm{EXP} \not\subset \mathrm{P/poly}$ then BPP has subexponential-time deterministic algorithms.
- For the other direction, Kabanets & Impagliazzo 2002: if $\mathrm{P} = \mathrm{BPP}$ then NEXP does not have polynomial-size circuits.

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- Here: symmetry of information $(\mathrm{SI_p}) \Rightarrow \mathrm{EXP} \not\subset \mathrm{P/poly}$;
- Lee & Romashchenko 2004: $(\mathrm{SI_p}) \Rightarrow \mathrm{EXP} \neq \mathrm{BPP}$
  (remark: $\mathrm{BPP} \subset \mathrm{P/poly}$, Adleman 1978).

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Lemma

If $A \in \mathrm{P}/n^c$ then there exists a constant k and a family (p_n) of programs of size $k + n^c$ such that

- $\mathcal{U}(p_n, x) = 1$ *iff* $x \in A$*;*
- $\mathcal{U}(p_n, x)$ *works in polynomial time.*

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Proof

By definition, $x \in A \Longleftrightarrow (x, c(|x|)) \in B$. Then p_n is merely the concatenation of the program for B and of $c(n)$. □

Advices of size n^c (continued)

Proposition (warm-up)

For all constants $c_1, c_2 \geq 1$, there exists a sparse language A in $\mathrm{DTIME}(2^{n^{1+c_1 c_2}})$ but not in $\mathrm{DTIME}(2^{n^{c_1}})/n^{c_2}$.

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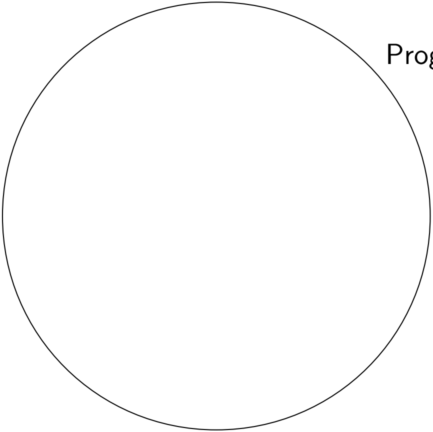

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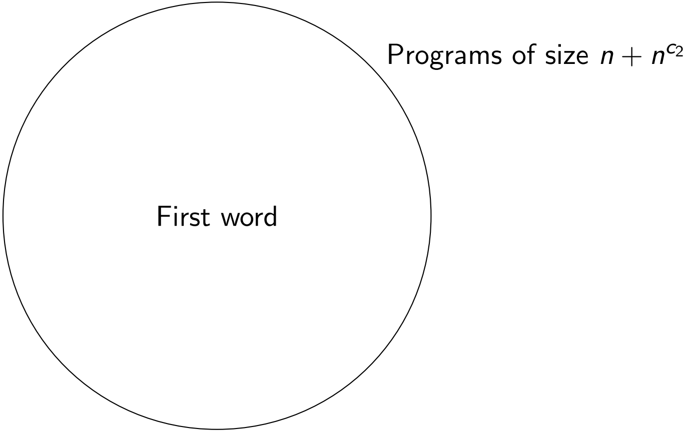

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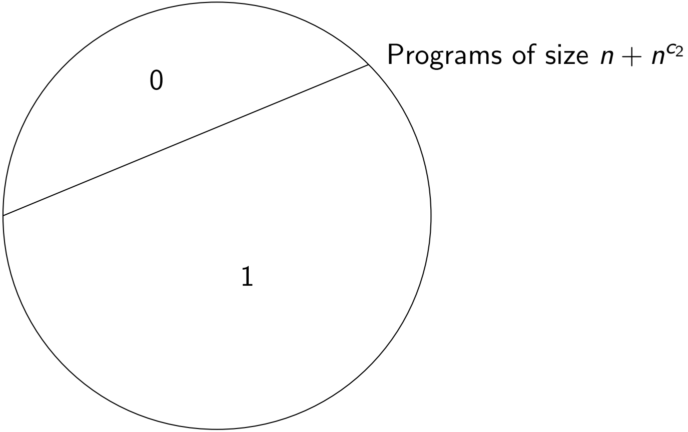

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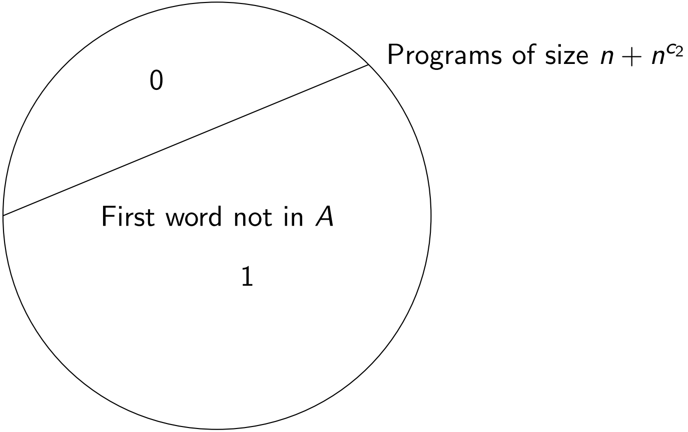

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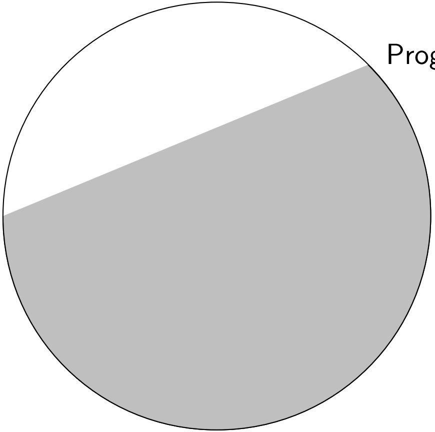

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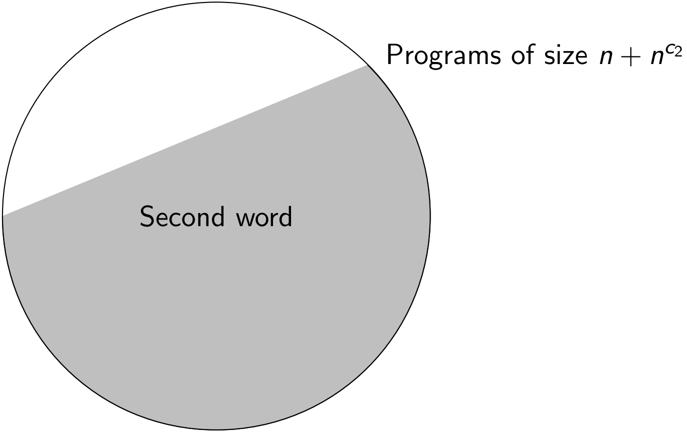

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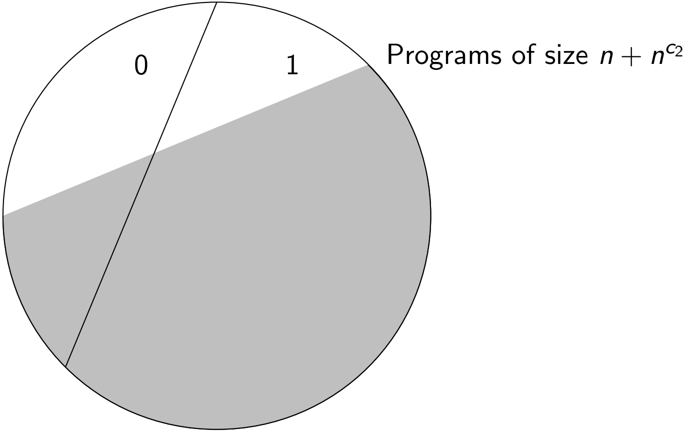

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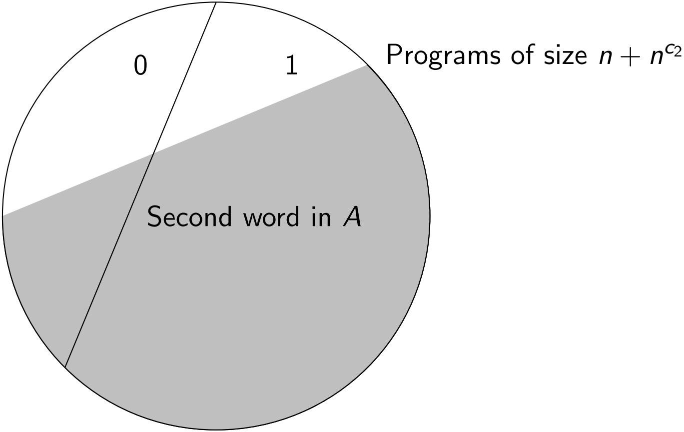

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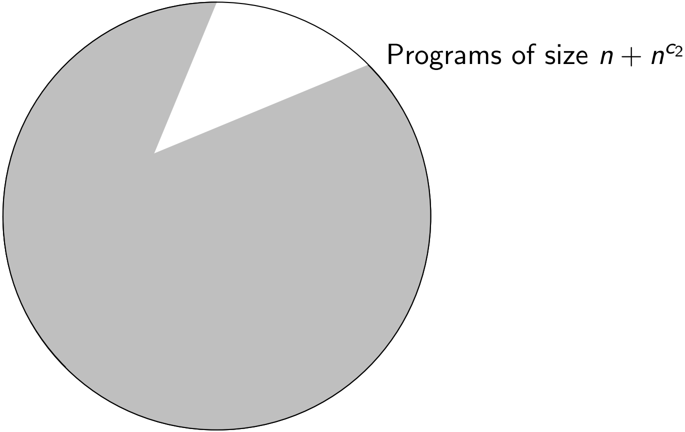

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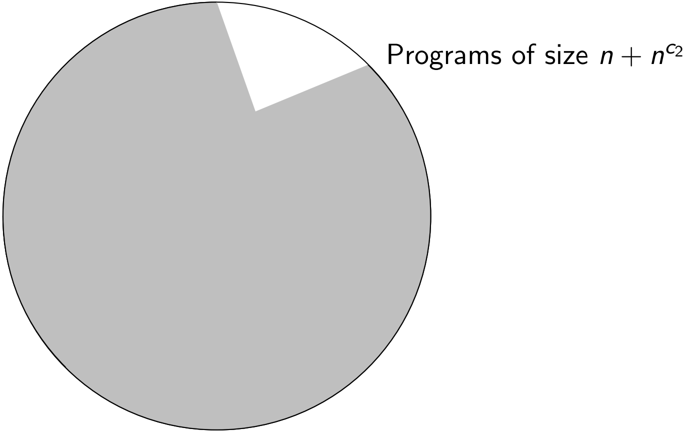

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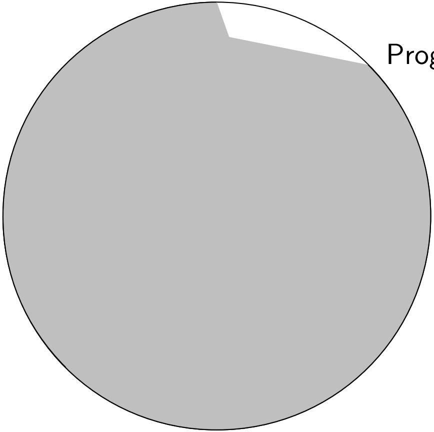

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Proof

We build A by input sizes and word by word. Let $t(n) = 2^{n^{1+c_1 c_2}}$ and $a(n) = n + n^{c_2}$. Let us fix n and define $A^{=n}$:

$$x_1 \in A \iff \begin{array}{l} \text{for at least half of the programs } p \text{ of size } \leq a(n), \\ \mathcal{U}^{t(n)}(p, x_1) = 0. \end{array}$$

(at least half of the programs give the wrong answer for x_1).

Let V_1 be the set of programs giving the right answer for x_1.

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$$x_2 \in A \iff \begin{array}{l} \text{for at least half of the programs } p \in V_1, \\ \mathcal{U}^{t(n)}(p, x_2) = 0. \end{array}$$

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$$x_k \in A \iff \begin{array}{l} \text{for at least half of the programs } p \in V_{k-1}, \\ \mathcal{U}^{t(n)}(p, x_k) = 0. \end{array}$$

The process stops when V_k is empty, that is, for $k = n + n^{c_2}$. We decide that $x_j \notin A$ for $j > k$.

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- $A \in \mathrm{DTIME}(2^{n^{1+c_1 c_2}})$. □

Some consequences

Corollary

For all constant $c > 0$, $\mathrm{EXP} \not\subset \mathrm{P}/n^c$ *and*
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Corollary

For all $c > 0$, $\mathrm{PP} \not\subset \mathrm{DTIME}(n^c)/(n - \log n)$.

Proof idea

It $t(n) = n^c$, deciding whether "for at least half of the programs $p \in V_{k-1}$, $\mathcal{U}^{t(n)}(p, x_k) = 0$" is a PP problem.

Therefore we can decide A with a PP oracle. □

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Theorem (Vinodchandran 2004)

For any fixed $c > 0$, PP *does not have circuits of size* n^c.

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- Typical time bound: polynomial or exponential. There could also be a space bound.

Links Kolmogorov/nonuniform complexity

Characteristic string $\chi^n \in \{0,1\}^{2^n}$ of $A^{=n}$:

$$\chi^n_i = 1 \iff x_i \in A^{=n}.$$

Lemma

Suppose that for all n and some $1 \le i \le 2^n$ we have

$$C^{ir(n)}(\chi^n[1..i]) > n + a(n).$$

Then $A \notin \mathrm{DTIME}(r(n))/a(n)$.

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Proof

If $A \in \mathrm{DTIME}(r(n))/a(n)$ then $\chi^n[1..i]$ is computed in time $ir(n)$ with a program of size $a(n) + O(1)$. □

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- The (equivalent) version we will use:

$$C(x, y) \simeq C(x) + C(y|x).$$

$\leq$: easy direction $\geq$: hard direction.

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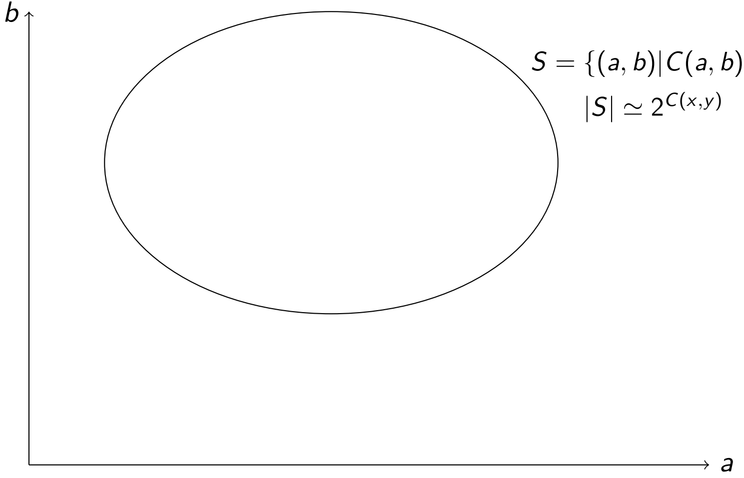

$S = \{(a, b) | C(a, b) \leq C(x, y)\}$

$|S| \simeq 2^{C(x,y)}$

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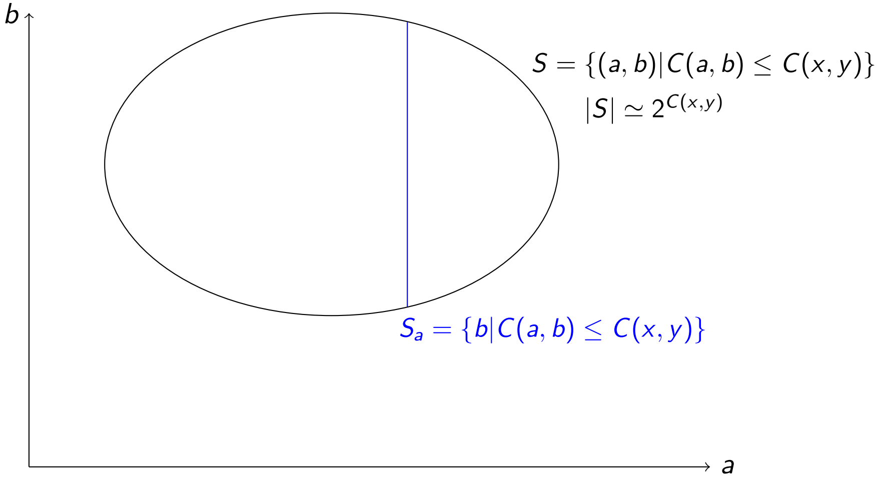

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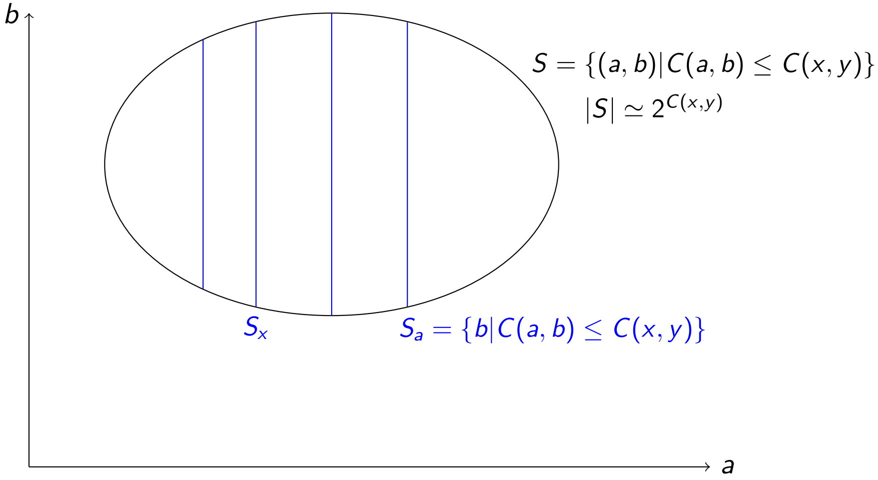

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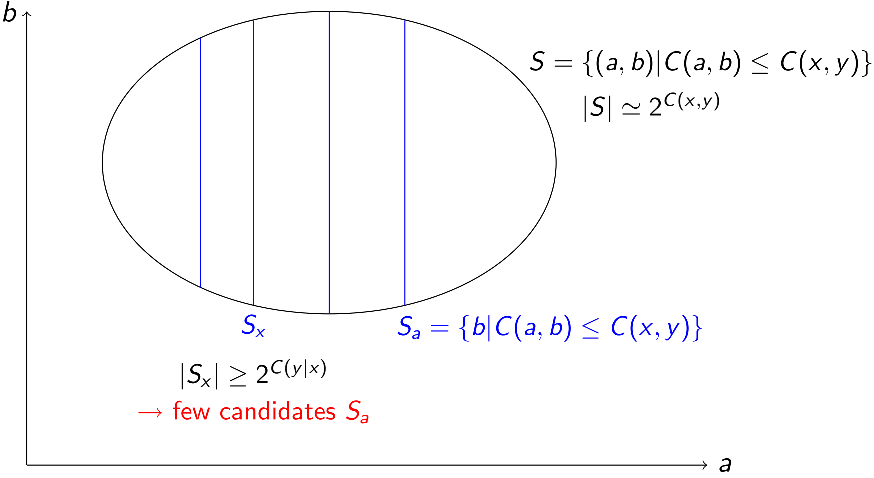

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Polynomial-time symmetry of information: easy direction still holds; hard direction is open!
(true if $\mathrm{P} = \mathrm{NP}$, Longpré & Watanabe 1995).

Symmetry of information

Hypothesis ($\mathrm{SI_p}$)

There exist a polynomial q and a constant $\alpha > 1/2$ such that for all t and all words x, y of size n:

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Suppose $(\mathrm{SI_p})$ *holds.*
Let $u_1, \ldots, u_n$ *be words of size* s. *Let* $m = ns$. *Suppose there exists* k *such that for all* $j \leq n$,

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Proof sketch

$$C^t(u_1, \ldots, u_n) \geq \alpha(C^{tq(m)}(u_1, \ldots, u_{n/2}) +$$
$$C^{tq(m)}(u_{n/2+1}, \ldots, u_n | u_1, \ldots, u_{n/2})). \quad \square$$

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- In EXP, impossible to diagonalize over all advices of polynomial size
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- then we "glue" these blocks together thanks to $(\mathrm{SI_p})$.

- Other point of view: thanks to $(\mathrm{SI_p})$, build a characteristic string of high Kolmogorov complexity.

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If $(\mathrm{SI_p})$ *holds, then* $\mathrm{EXP} \not\subset \mathrm{P/poly}$.

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Outline of the proof: feedback with previously defined segments.

Proof

We build A by input sizes and word by word. Let $t(n) = n^{\log^3 n}$.
Let us fix n and define $A^{=n}$:

$$x_1 \in A \Longleftrightarrow \begin{array}{l} \text{for at least half of the programs } p \text{ of size } \leq n, \\ \mathcal{U}^{t(n)}(p, x_1) = 0. \end{array}$$

(at least half of the programs give the wrong answer for x_1).

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Let V_1 be the set of programs giving the right answer for x_1.

Proof (continued)

We go on like this, discarding half of the remaining programs at each step, until x_n:

$$x_n \in A \iff \begin{array}{l} \text{for at least half of the programs } p \in V_{n-1}, \\ \mathcal{U}^{t(n)}(p, x_n) = 0. \end{array}$$

We call $u^{(1)}$ the n first bits of the characteristic string of $A^{=n}$ just defined.

Proof (continued)

Then:

$$x_{n+1} \in A \iff \begin{array}{l}\text{for at least half of the programs } p \text{ of size } \leq n,\\ \mathcal{U}^{t(n)}(p, u^{(1)}, x_{n+1}) = 0.\end{array}$$

(at least half of the programs are wrong on x_{n+1}, even with the advice $u^{(1)}$).

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Keep going: call V_1 the set of programs that where right at the preceding step.

$$x_{n+2} \in A \iff \begin{array}{l} \text{for at least half of the programs } p \in V_1, \\ \mathcal{U}^{t(n)}(p, u^{(1)}, x_{n+2}) = 0. \end{array}$$

Proof continued

And so on, until the next segment $u^{(2)}$ of size n is defined. Then:

$$x_{2n+1} \in A \iff \begin{array}{l} \text{for at least half of the programs } p \text{ of size } \leq n, \\ \mathcal{U}^{t(n)}(p, u^{(1)}, u^{(2)}, x_{2n+1}) = 0. \end{array}$$

(at least half of the programs give the wrong answer for x_{2n+1}, even with the advice $u^{(1)}, u^{(2)}$).

We define $n^{\log n}$ segments of size n and decide that $x_j \notin A^{=n}$ for $j > n \times n^{\log n}$.

Proof continued

- $A \notin \mathrm{P/poly}$ because for all j, $C^{t(n)}(u^{(j)} | u^{(1)}, \ldots, u^{(j-1)}) \geq n - 1$. Thus by iteratively applying $(\mathrm{SI_p})$, $C^{t}(\chi^{n}[1..n^{1+\log n}]) \geq n^{\Omega(\log n)}$.
- $A \in \mathrm{EXP}$. $\square$

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- $A \in \mathrm{EXP}$. □

Corollary

If $(\mathrm{SI_p})$ *holds, then there exists a constant* $c > 0$ *such that*

$$\mathrm{BPP} \subseteq \mathrm{DTIME}(2^{\log^c n}).$$

Conclusion

- $(\mathrm{SI_p})$ is a central (and hard) question: if true, then $\mathrm{EXP} \not\subset \mathrm{P/poly}$; if false, then $\mathrm{P} \neq \mathrm{NP}\ldots$
- What about the usual version of $(\mathrm{SI_p})$ (with time bound $q(t)$ instead of $tq(n)$)?
- Can we obtain unconditionnal results by using variants of Kolmogorov complexity ? (for instance CAMD, a version based on the class AM).

Outline

1. Complexity classes
2. Advices of size n^c
3. Symmetry of information
4. Polynomial-size advices