Lower bounds
in Boolean and algebraic complexity

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Lower bounds

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- What problems are not efficiently computable?

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- Foundation of computational complexity theory: hierarchy theorems (Hartmanis & Stearns 1965)

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  = lower bound on certain ad-hoc problems
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- Links with derandomization, cryptography, etc.

Algebraic vs Boolean

- $\{0, 1\}$: poor structure $\rightarrow$ extension to other finite fields, or to $\mathbb{R}$, $\mathbb{C}$, etc. (More mathematical tools)
- "Arithmetization": e.g. $\mathsf{IP} = \mathsf{PSPACE}$ (Shamir 1992)

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- Algebraic models: natural models to deal with algebraic problems (polynomials), operations $+$ and $\times$.
  Rich structure.
- Same questions as in the Boolean world (P vs NP, etc.).
  Many links (transfer theorems): e.g. on some structures, equivalent "P vs NP" questions

Outline

1. Boolean circuits

2. Derandomization

3. Lower bounds on the computation of polynomials

4. Permanent

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Why circuits?

- Easier model than Turing machines
  (→ easier to study? "Look inside circuits")
- If NP does not have polynomial-size circuits
  then $\mathsf{NP} \neq \mathsf{P}$

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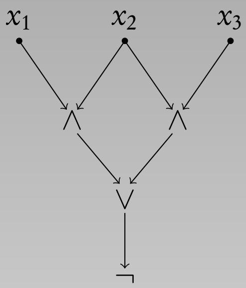
- Easier model than Turing machines
  (→ easier to study? "Look inside circuits")
- If NP does not have polynomial-size circuits
  then NP $\neq$ P

- Idea behind circuits: one algorithm for one input size
  → possibly different algorithms for different input sizes
- Example
  adder for 32 bit integers vs adder for 33 bit integers:
  why use the same algorithm?

Boolean circuits: definition

- Directed acyclic graph with nodes labeled by $\neg$ (fan-in 1), $\vee, \wedge$ (fan-in 2). Inputs $x_1, \ldots, x_n$.



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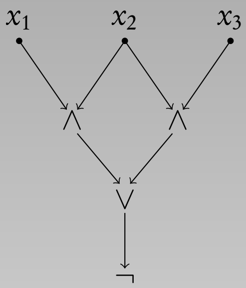
  $\rightarrow$ words of same length



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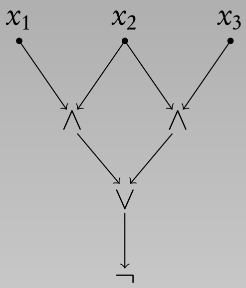
- Size = number of nodes
- Fixed number of inputs $\rightarrow$ words of same length
- A word x is accepted iff $C(x) = 1$
- C accepts a subset of $\{0, 1\}^n$
  $\rightarrow$ family of circuits (C_n) to recognize a language
  (C_n has n inputs)



Circuits vs TM

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- If a family of circuits (C_n) can be described efficiently (uniformity), then it can be simulated by a Turing machine running in time $t(n) \simeq |C_n|$
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- Measure of complexity: size of the smallest circuits

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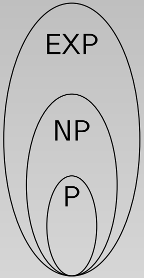

$$\mathsf{EXP} = \mathsf{DTIME}(2^{n^{O(1)}})$$
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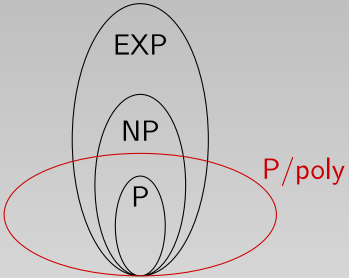

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$$\mathsf{P/poly} = \mathsf{SIZE}(n^{O(1)})$$

Diagonalization

- Rooted in Cantor's diagonal argument
- Diagonalization against Turing machines:
  definition of a language L not recognized
  by any TM working in time $t(n)$:
  - (M_i) = enumeration of TMs working in time $t(n)$
  - $\mathtt{0}^n \in L \iff M_n(\mathtt{0}^n)$ does not accept

  $\rightarrow$ every M_n makes a mistake.

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  $\rightarrow$ every M_n makes a mistake.

- If M_n makes a mistake on a single input (0^n among an infinite number of words), then it does not recognize the language
  $\rightarrow$ an infinite number of words can be used to diagonalize

Diagonalization against circuits

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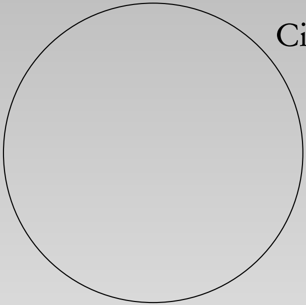

Circuits of size n^k

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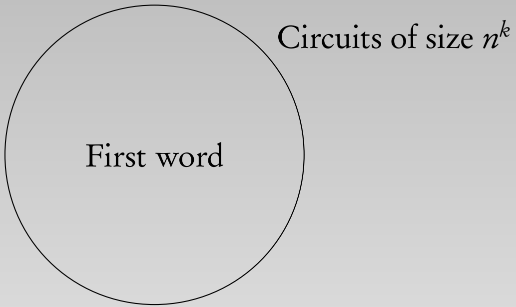

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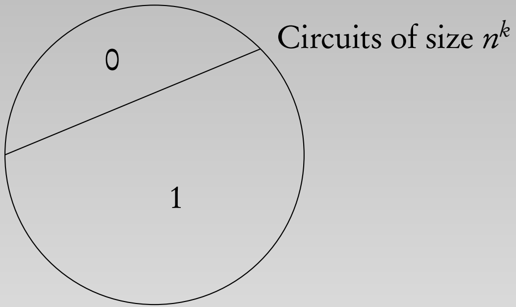

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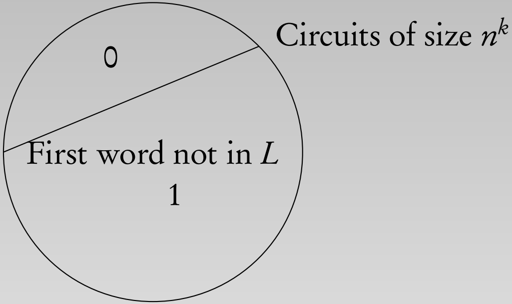

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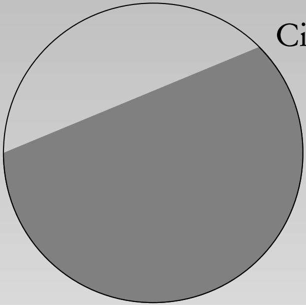

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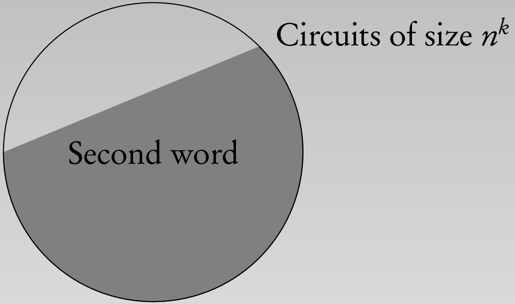

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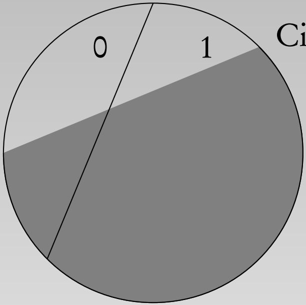

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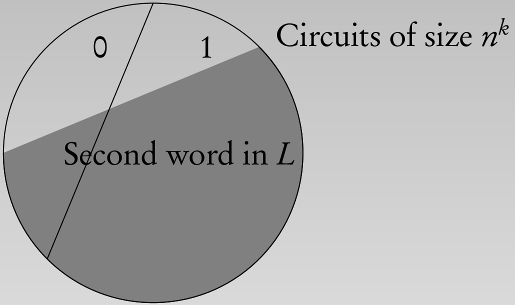

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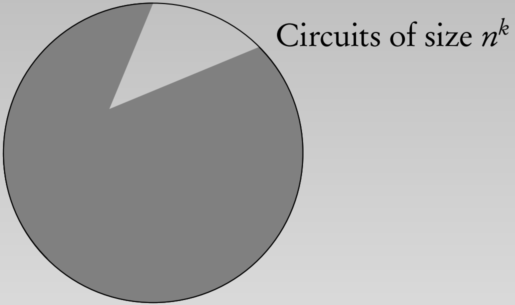

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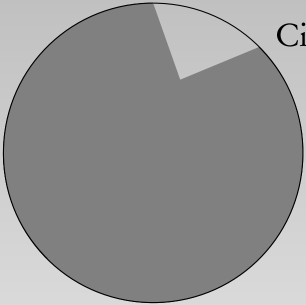

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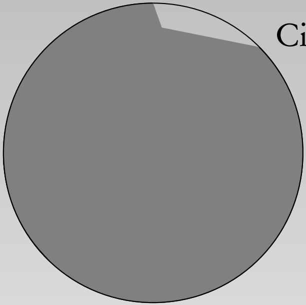

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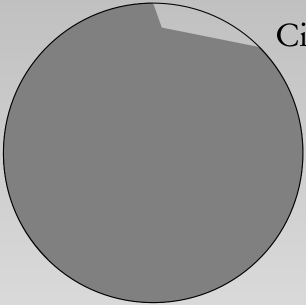

- Not efficient enough for polynomial-size circuits

Diagonalization (2)

- Diagonalization possible on "small pieces" of circuits
  $\rightarrow$ ok if we can glue the pieces together
- Kolmogorov complexity "Symmetry of Information":
  $C(x, y) \simeq C(x) + C(y|x)$

Theorem

"Polynomial-time (SI)" implies $\mathsf{EXP} \not\subset \mathsf{P/poly}$

Proof idea

Diagonalize on pieces of circuits (by eliminating half of the circuits at each step). Glue pieces together thanks to (SI). ∎

Outline

Derandomization

- Can every probabilistic algorithm be turned into a deterministic one without slowdown? (or with "only" polynomial slowdown)
- In other words: does randomness speed up computation more than polynomially?

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Examples:

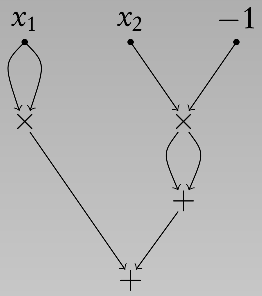
- derandomization of primality testing (AKS 2002) ($O(n^6)$ vs $O(n^2)$ probabilistically)
- PIT: testing if an arithmetic circuit computes the identically 0 polynomial

Arithmetic circuits and PIT

Arithmetic circuits:

- indeterminates and constants as input,
- gates $+$, $\times$ (fan-in 2)

$\rightarrow$ computes a polynomial



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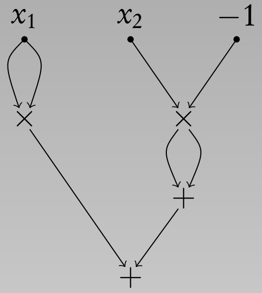

Problem PIT (Polynomial Identity Test):
decide whether an arithmetic circuit computes the identically zero polynomial :

- easy polynomial-time probabilistic algorithm
- no known polynomial-time deterministic algorithm

Randomized algorithms and circuits

Theorem (Adleman 1978)

Probabilistic polynomial-time algorithms have polynomial-size circuits

Proof idea

If the error probability is small ($< 2^{-n}$), then there is a sequence of random choices that works for all inputs of size n
→ these good random choices can be "hard wired" into the circuits. ∎

Derandomization and lower bounds

1. Lower bounds imply derandomization.

Impagliazzo, Nisan, Wigderson, … (1993–1997):

Theorem

If exponential-time problems do not have polynomial-size circuits, then derandomization is possible

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THEOREM

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Proof idea

use a hard problem in EXP to produce a pseudo-random generator that fools polynomial-time algorithms

Pseudo-random generator

Pseudo-random generator:

function $f : \{0, 1\}^{\log n} \rightarrow \{0, 1\}^n$ such that

for any circuit C of size $\leq s$:

$$|\Pr_x(C(x) = 1) - \Pr_a(C(f(a)) = 1)| < 1/10.$$

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f needs only be computed in exponential time

since its input is of size $\log n$.

If $A \in \mathsf{EXP}$ does not have polynomial-size circuits,

it can be turned into a PRG.

Derandomization and lower bounds (2)

2. Derandomization implies lower bounds.

THEOREM (Kabanets & Impagliazzo 2004)

If derandomization is possible, then NEXP does not have polynomial-size circuits (boolean or arithmetic)

For the idea

We shall only look at a special case of derandomization:

"black-box derandomization"

Black-box derandomization

PIT: decide whether an arithmetic circuit computes the polynomial 0.

- Black-box derandomization of PIT: construct witness points, i.e. s^k points $a_1, \ldots, a_{s^k} \in \mathbb{Z}$ such that for all arithmetic circuit C of size s,

$$C(x) \equiv 0 \iff \forall i, C(a_i) = 0.$$

- Such points exist (e.g., by Schwartz-Zippel lemma).

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- Such points exist (e.g., by Schwartz-Zippel lemma).
- The polynomial $p(x) = \prod_i (X - a_i)$ is nonzero but vanishes on all witness points: thus it does not have circuits of size s.
- If the witness points a_i can be constructed efficiently, this implies that the algebraic versions of P and NP differ.

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Lower bound on ad-hoc polynomials

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find a polynomial that does not have circuits of size s

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Examples:

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Not satisfactory??
Trivial lower bounds: degree, number of variables, etc.

THEOREM (Schnorr, 1978)

There is a univariate polynomial of degree s^2 and coefficients in $\{0, 1\}$ that does not have arithmetic circuits of size s (with arbitrary constants)

Proof of Schnorr's result

Lemma

There exist polynomials $g_i \in \mathbb{Z}[y_1, \ldots, y_{s^2}]$ such that if $p(x) = \sum_{i=1}^{k} a_i x^i$ is computed by a circuit of size s, then there are $\alpha_1, \ldots, \alpha_{s^2} \in \mathbb{C}$ satisfying $a_i = g_i(\alpha_1, \ldots, \alpha_{s^2})$.

Proof idea

The coefficients of the polynomial p can be expressed as a function of the "parameters" of the circuit (the constants and the type of gates). ∎

Proof of Schnorr's result (2)

Corollary

There is a nonzero polynomial $h \in \mathbb{Z}[y_1, \ldots, y_{s^2}]$ of degree $\leq s^3$ such that if $p(x) = \sum_{i=1}^{s^2} a_i x^i$ is computed by a circuit of size s, then $h(a_1, \ldots, a_{s^2}) = 0$.

Proof idea

Take a polynomial h such that $h(g_1, \ldots, g_k) \equiv 0$. $\square$

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Proof idea

Take a polynomial h such that $h(g_1, \ldots, g_k) \equiv 0$. ∎

Corollary

If $a_1, \ldots, a_{s^2} \in \mathbb{Z}$ are not roots of h, then $p(x) = \sum_{i=1}^{s^2} a_i x^i$ does not have circuits of size s.

Question: how efficiently can we construct a_i?

Outline

Permanent

- Deciding the existence of a perfect matching in a bipartite graph: polynomial-time algorithm
- Counting the number of perfect matchings in a bipartite graph: hard!
- Permanent: counts the number of perfect matchings in a bipartite graph

$$\operatorname{per}(x_{1,1}, \ldots, x_{n,n}) = \sum_{\sigma \in S_n} x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$$

- Determinant: same with signature of the permutation – easy to compute! (Gaussian elimination)
- Permanent of 0-1-matrices: #P-complete (Valiant 1979) (#P: counting class — big class: well above NP)

Small depth circuits

Depth: length of the longest path from a leaf to the root
("parallel" time complexity)

Circuits of restricted depth: arbitrary fan-in allowed.

Small depth circuits

THEOREM

The permanent cannot be computed by uniform polynomial-size circuits of depth $o(\log \log n)$ (boolean or arithmetic).

Proof idea: if the permanent had uniform polynomial-size circuits of depth $o(\log \log n)$ then:

1. the value of a circuit of size s and depth d could be computed in time $(\log s)^{2^d}$;
2. every language in EXP would have uniform circuits of size $2^{n^{O(1)}}$ and depth $o(\log n)$.

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2. every language in EXP would have uniform circuits of size $2^{n^{O(1)}}$ and depth $o(\log n)$.

Hence: every language in EXP would have subexponential-time algorithms,
→ a contradiction with the time hierarchy theorem. ■

Conclusion

- Proving lower bounds: central (and challenging!) task
- Importance of nonuniform lower bounds

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- Interconnected Boolean and algebraic worlds
- Many links with other domains (e.g. derandomization)

Conclusion

- Proving lower bounds: central (and challenging!) task
- Importance of nonuniform lower bounds

- Interconnected Boolean and algebraic worlds
- Many links with other domains (e.g. derandomization)

- Study of restricted circuits: small depth, uniform, no negations, multilinear…