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Algebraic complexity – Exercise session 1
NP-completeness

The goal of the following exercises is to study two usual NP-complete problems over the structures $(\mathbf{C}, +, -, \times, =)$ and $(\mathbf{R}, +, -, \times, =, \leq)$.

Let $HN_{\mathbf{C}}$ (sometimes called $FEAS_{\mathbf{C}}$) be the following problem over the structure $(\mathbf{C}, +, -, \times, =)$.

- Input: k multivariate polynomials $f_1, \ldots, f_k \in \mathbf{C}[X_1, \ldots, X_n]$.
- Problem: decide whether the polynomials f_i share a common complex root.

Exercise 1

1. The name $HN_{\mathbf{C}}$ comes from Hilbert's Nullstellensatz. Why?
2. Show that the problem $HN_{\mathbf{C}}$ is decidable over $(\mathbf{C}, +, -, \times, =)$ (i.e. can be decided by a uniform family of algebraic circuits—no matter their size).
3. Show that $HN_{\mathbf{C}}$ is in $NP_{\mathbf{C}}$.
4. Let x be a variable. Define via a conjunction of polynomial equalities (with possibly existential quantifiers) a variable z whose value is 1 if $x = 0$ and 0 otherwise.
5. Show that $HN_{\mathbf{C}}$ is $NP_{\mathbf{C}}$-hard.

Let 4FEAS be the following problem over the structure $(\mathbf{R}, +, -, \times, =, \leq)$.

- Input: a multivariate polynomial $f \in \mathbf{R}[X_1, \ldots, X_n]$ of degree ≤ 4.
- Problem: decide whether the polynomial f has a real root.

Exercise 2

1. Show that the problem 4FEAS is decidable over $(\mathbf{R}, +, -, \times, =, \leq)$.
2. Show that 4FEAS is in $NP_{\mathbf{R}}$.
3. Let x be a variable. Define via a conjunction of polynomial identities (with possibly existential quantifiers) a variable z whose value is 1 if $x \geq 0$ and 0 otherwise.
4. Show that 4FEAS is $NP_{\mathbf{R}}$-hard. Hints: compared to the problem $HN_{\mathbf{C}}$, why is only one polynomial enough? How to reduce the degree to 4?