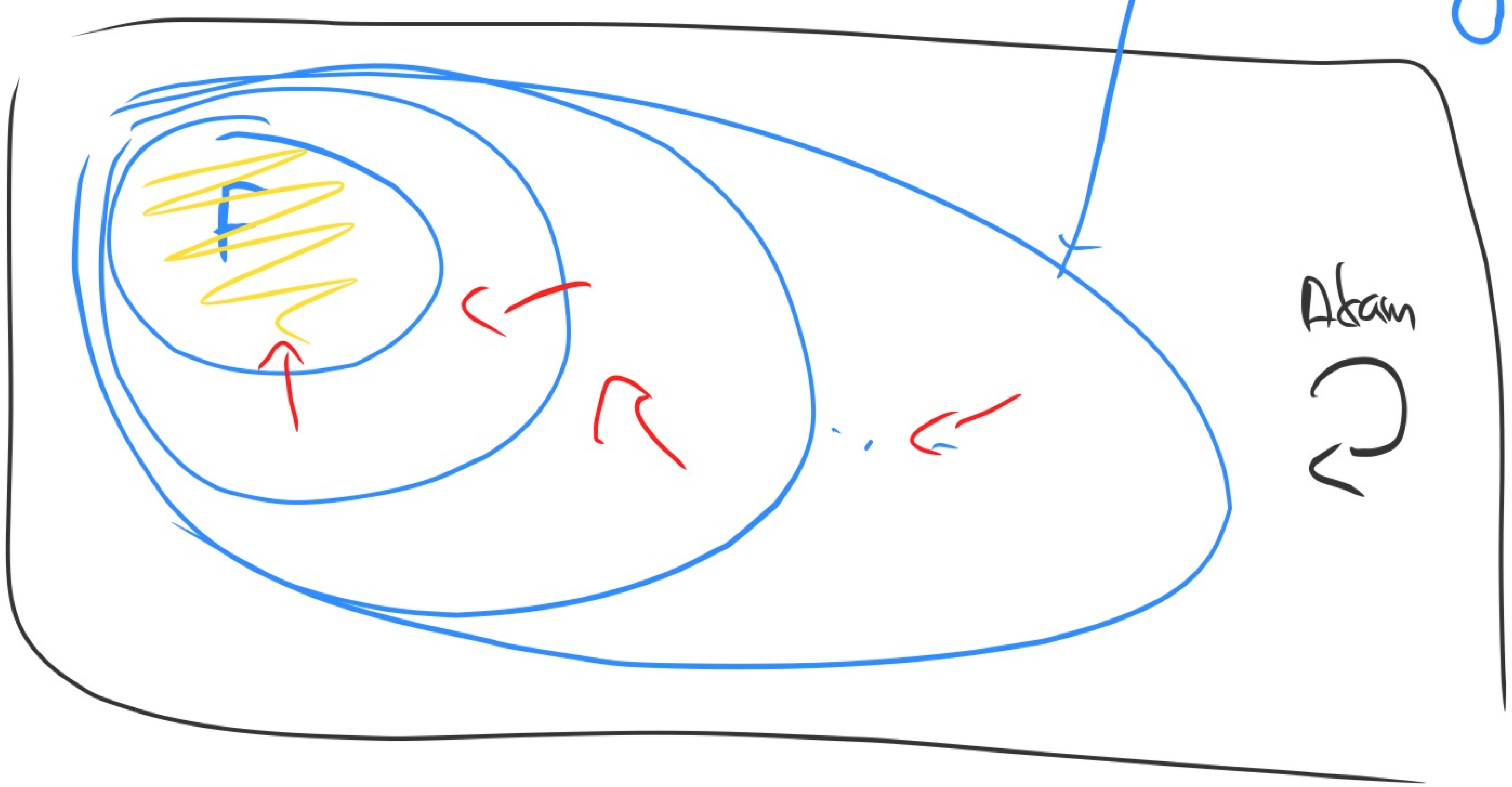
$F \subseteq V$ final vertices

$$\Omega = V^* F V^\omega$$

$Attr_0(F) = F$

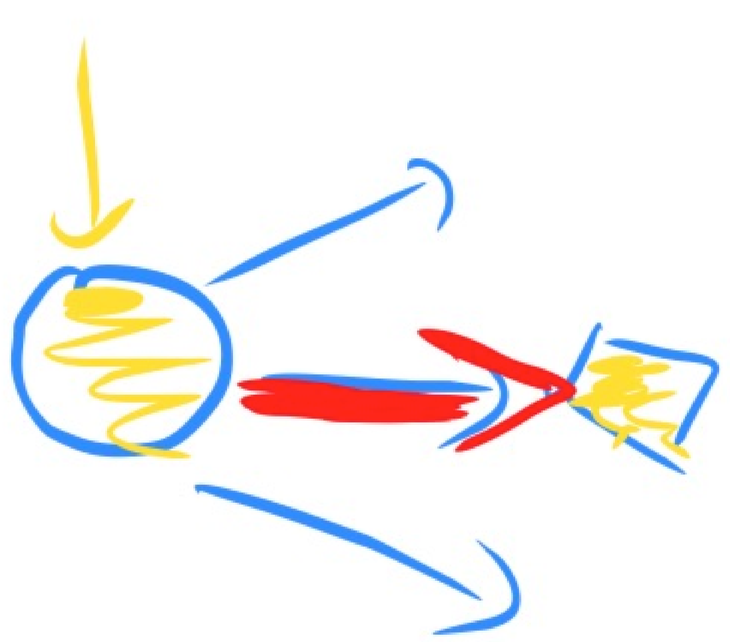
$Attr_{i+1}(F) = Attr_i(F)$
$\cup \{ v \in V_E \mid \exists v' \in Attr_i(F) \text{ st } v \to v' \}$
$\cup \{ v \in V_A \mid \forall v' \text{ if } v \to v' \text{ then } v' \in Attr_i(F) \}$

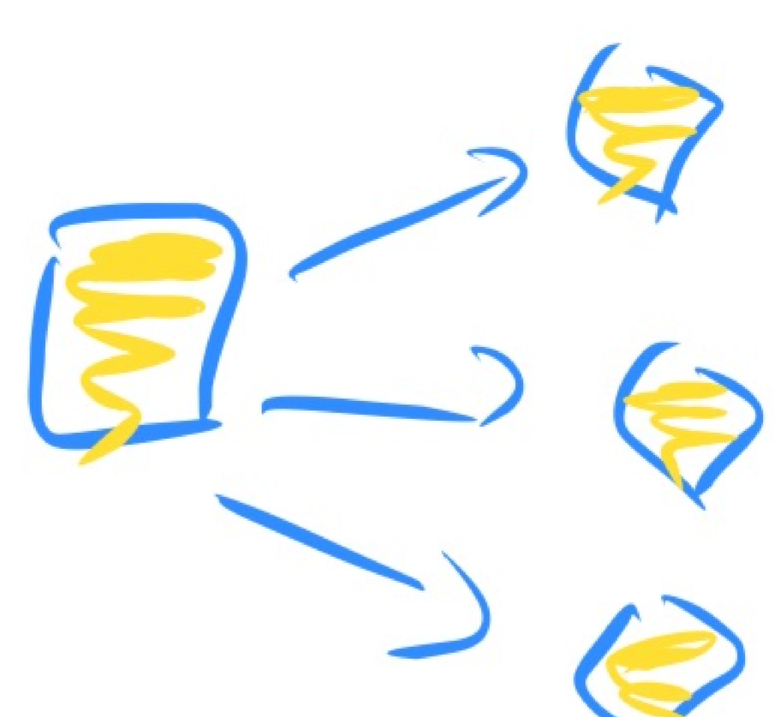
$Attr(F)$



$(Attr_i) \nearrow$

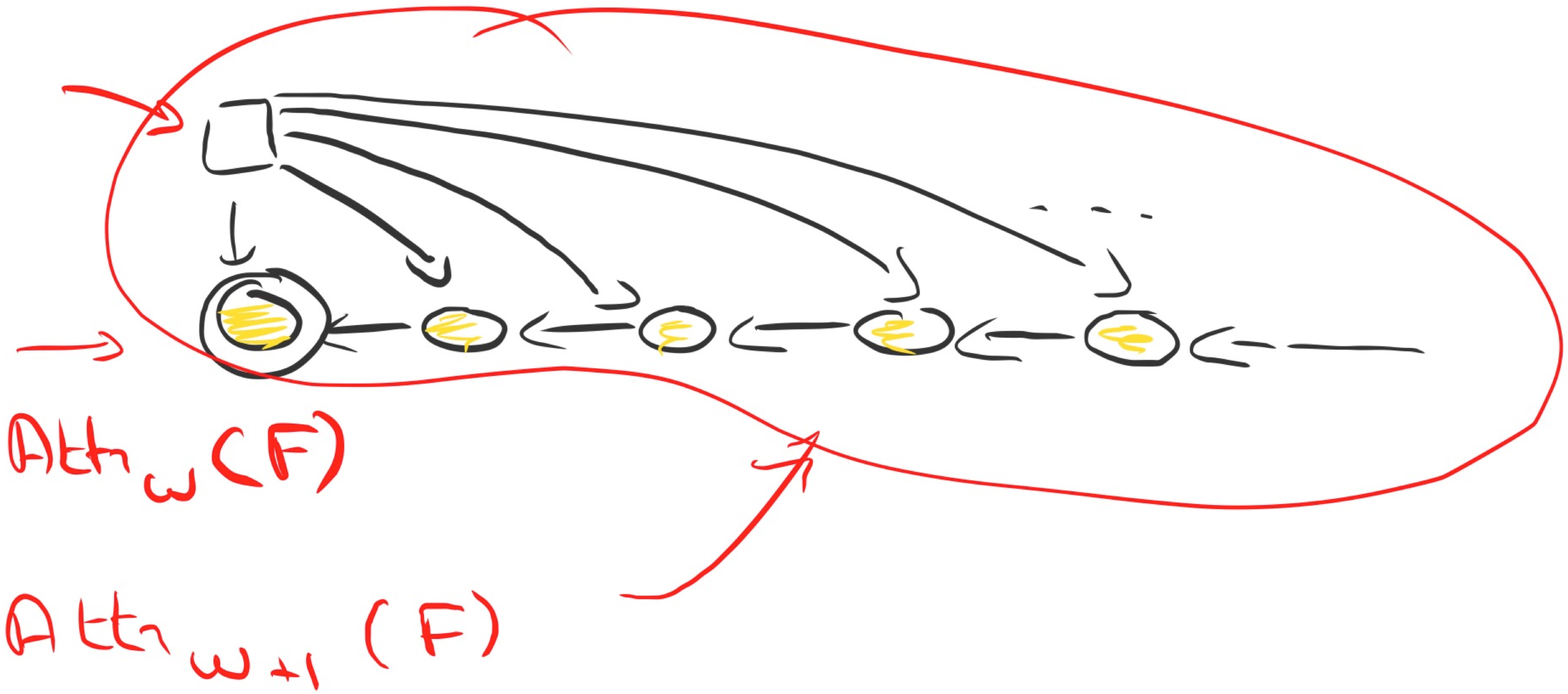
↳ $Attr(F)$ limit.





Th : $Attr(F) \overset{\subseteq}{=} W_E$

+ positional strat


$Attr_{\omega}(F)$
$Attr_{\omega+1}(F)$

Buchi condition:

$F \subseteq V$ final vertices.

Eve wins a play iff it visits F ∞ often.

$$\Omega = \bigcap_{i \geq 0} V^i V^* F V^\omega = \{ v_0 v_1 v_2 \ldots \mid \exists^\infty i \; v_i \in F \}.$$

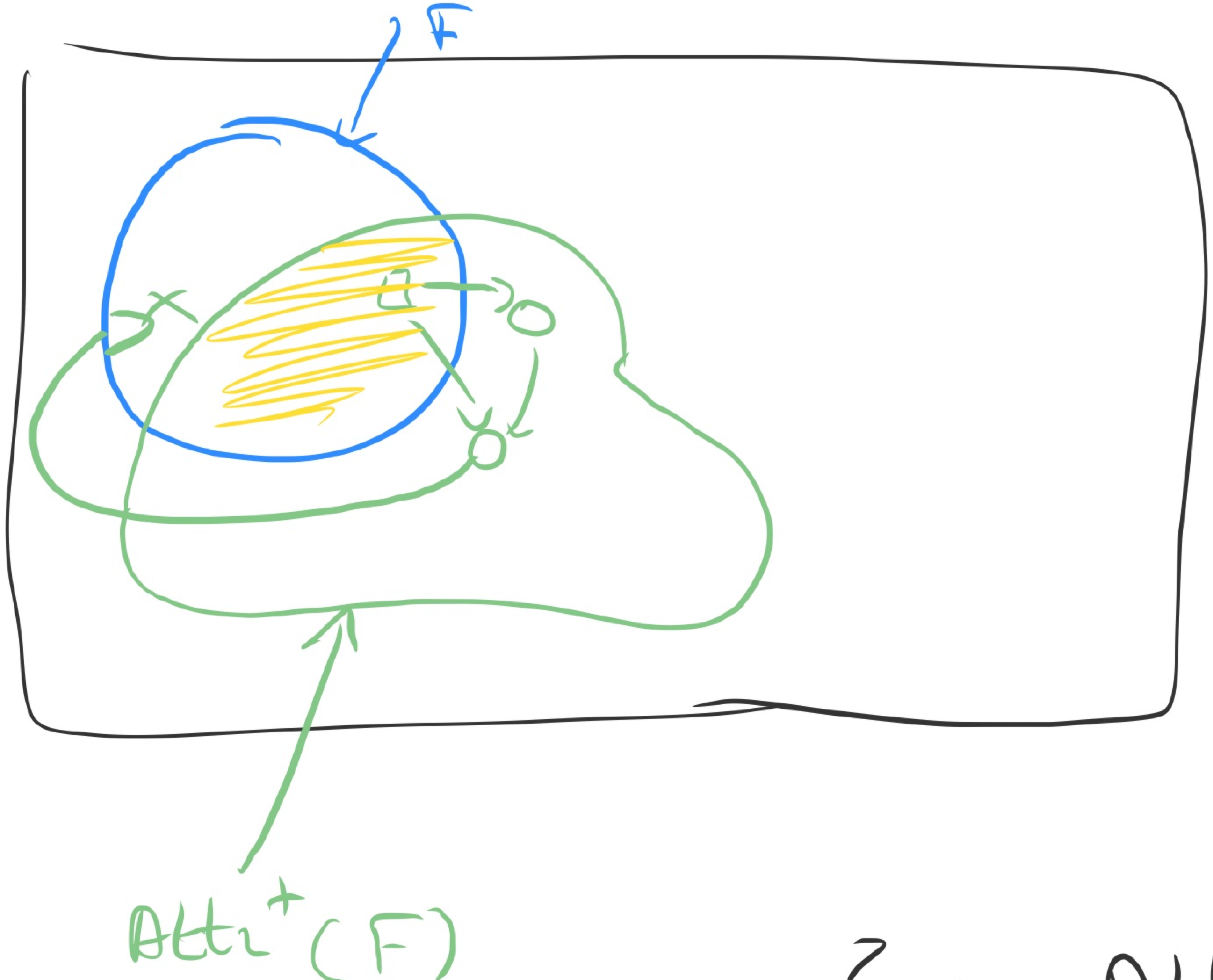
$$\mathrm{Attr}^+_0(S) = \emptyset$$

$$\mathrm{Attr}^+_{i+1}(S) = \mathrm{Attr}^+_i(S) \cup \{ v \in V_E \mid \exists v' \text{ st } v \to v' \wedge v' \in \mathrm{Attr}^+_i(S) \cup S \}$$
$$\cup \{ v \in V_A \mid \forall v' \text{ st } v \to v' \text{ then } v' \in \mathrm{Attr}^+_i(S) \cup S \}.$$

$$\mathrm{Attr}^+(S) = \lim (\mathrm{Attr}^+_i(S))_{i \geq 0}$$

↳ vertices from which Eve can force to visit S in at least one step

Proof: like reachability games.



$(Z_i)_i \downarrow$

$Z_0 = F$

$Z_{i+1} = Attr^+(Z_i) \cap F$

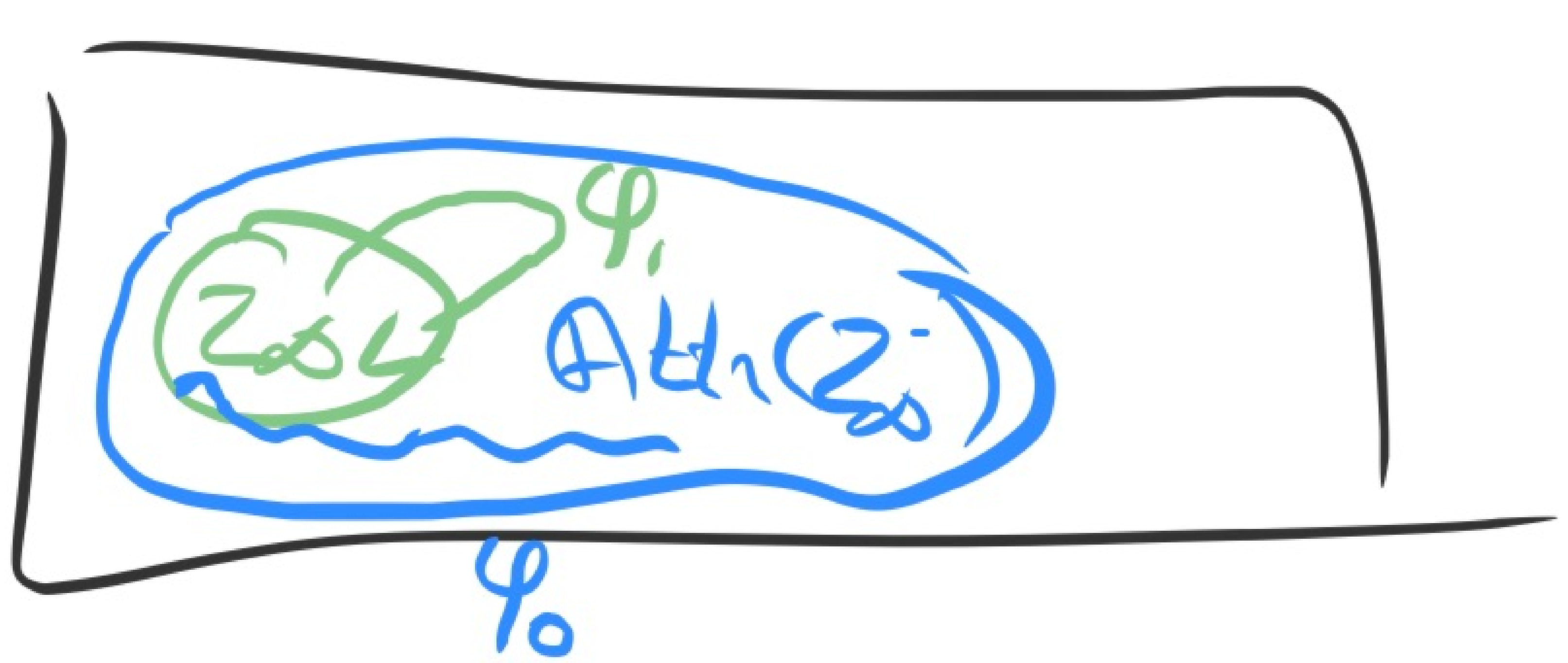
$Z_\infty = \lim_{i \to \infty} Z_i$

↳ greatest fixpoint

$Z_\infty = Attr^+(Z_\infty) \cap F$

↳ Eve has a positional strat to go back to Z_∞ from Z_∞ in at least one step.

↳ φ_1



<u>Th</u> : $W_E = Attr(Z_\infty)$

Proof $Attr(Z^\infty) \subseteq W_E$ $\Leftarrow$

Define φ positional on $Attr(Z^\infty)$ by:

$$\varphi(v) = \varphi_1(v) \quad \text{if } v \in Z_\infty$$
$$= \varphi_0(v) \quad \text{if } v \notin Z_\infty$$

Let us prove that any play starting in $Attr(Z^\infty)$ where Eve respects φ is winning for Eve.

$$\lambda = v_0 v_1 v_2 \dots$$

$\forall i \quad v_i \in Attr(Z^\infty)$ because 1) v_0 does
2) if v_i does then v_{i+1} does too

$$\lambda = \overbrace{v_0 \dots v_{k_0}}^{\varphi_0}\overbrace{\,}^{\varphi_1}\overbrace{v_{k_0+1} \dots v_{k_1}}^{\varphi_0} \dots v_{k_2} \dots v_{k_i} \Rightarrow \text{winning}$$

$\hookrightarrow \in Z_\infty$ $\qquad$ $Z_\infty \subseteq F$

$V \setminus Attr(Z_\infty) \subseteq W_A$.

$v \notin Attr(Z_\infty) \Rightarrow \exists i$ st $v \notin Attr(Z_i)$

Call $rk(v)$ = smallest i st $v \in Attr(Z_i) \setminus Attr(Z_{i+1})$

$[Z_{-1} = V]$

Property: $\forall v \notin Attr(Z_\infty)$ then we have:

- $v \in V_A$, v has a successor v' st $rk(v') \leq rk(v)$ and if $v \in F$ then the inequality is strict

- $v \in V_E$, $\forall v'$ st $v \to v'$ one has $rk(v') \leq rk(v)$ and if $v \in F$ the inequality is strict.

Proof: v with $rk(v) = i$ and $v \notin Attr(Z_\infty)$

$\hookrightarrow$ $v \notin Attr(Z_{i+1}) \to$ Adam can force to stay outside of $Attr(Z_{i+1}) \Leftrightarrow$ to stay in vertices of $rk \leq i$. $v \in Attr^+(Z_i)$

$v \in F$, if for all vertices v' st $v \to v'$ one has $rk(v') = i \Rightarrow \cap F = Z_{i+1} \Rightarrow rk(v) \geq i+1$

Define ψ positional strat for Adam by letting

$$\psi(v) = v' \text{ for some } v' \text{ st } rk(v') \leq rk(v)$$

strict if $v \in F$.

$\lambda = v_0 v_1 v_2 \cdots$ where Adam respects ψ and starting outside of $Attr(Z_\infty)$

Then: $rk(v_i) \searrow$ and decreases strictly when $v_i \in F$

$\Rightarrow$ this can only happen finitely often

$\Rightarrow$ $|\{i \mid v_i \in F\}| < \infty$ $\Rightarrow$ Adam wins in λ.

$V \setminus Attr(Z_\infty) \subseteq W_A$

$\blacksquare$

Parity

$G = (V, E)$

$C = \{0, \dots, d\} \subseteq \mathbb{N}$ colours.

$c : V \to C$: colouring fct.

$\Omega = \{ \lambda = v_0 v_1 \dots \mid \limsup (c(v_i))_{i \geq 0} \text{ even} \}$

↳ largest ∞ often visited colour is even

Th: One can compute winning region + positional strat that traps in the winning region

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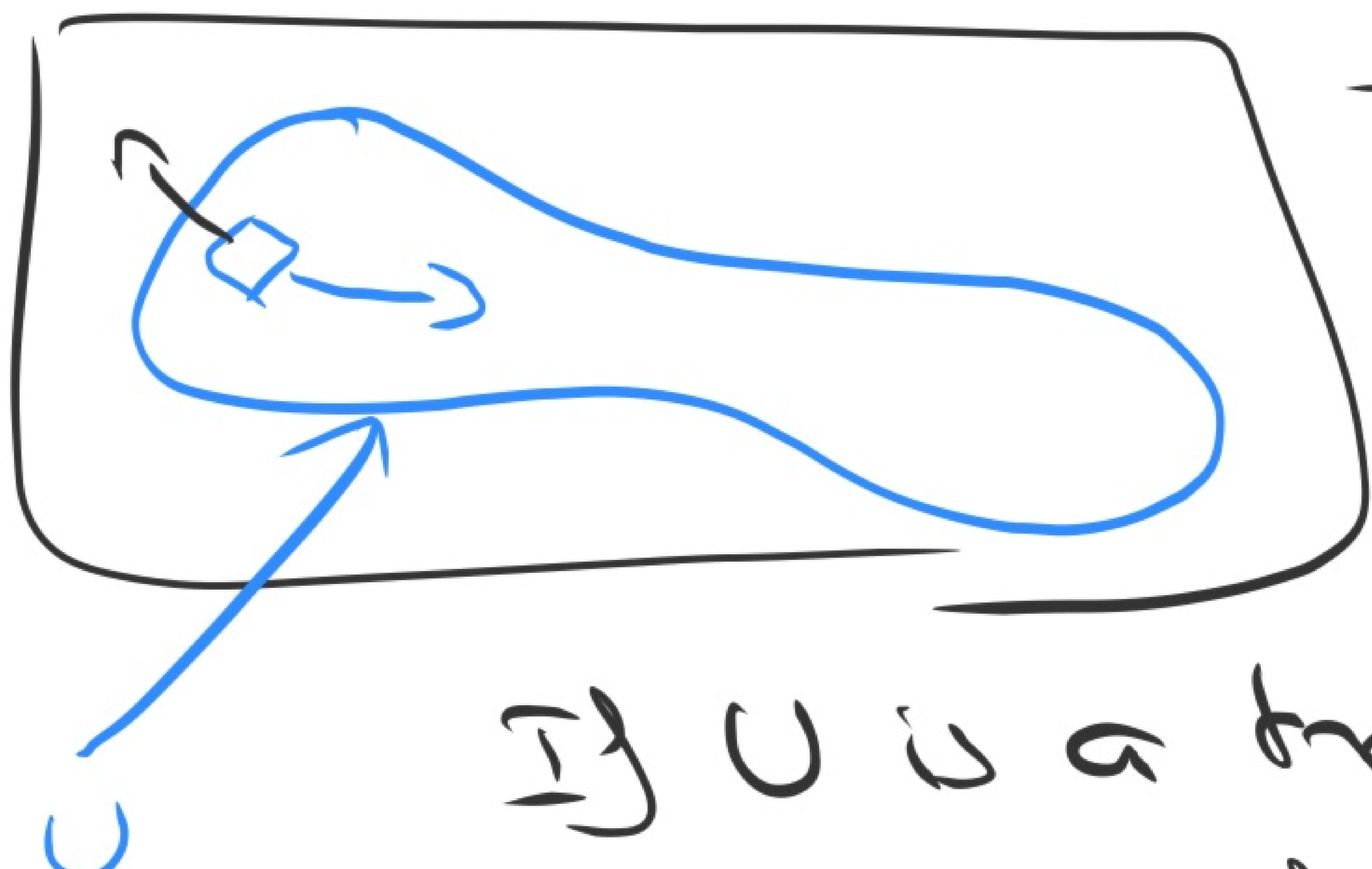
Subarena. U is a subarena if $G[U]$ has no dead-end

↳ $(U, E \cap U \times U)$



Trap: $U \subseteq V$ is a trap for player σ if $\bar{\sigma}$ has a strat to trap the play in U if it starts from

If U is a trap then U is a subarena

→ The complement of an attractor for σ is a trap for σ

Proof is by induction on the number n of colours.

__Base case__: $n = 1 \to$ trivial.

Induction $n \to$ ✓

largest colour in C

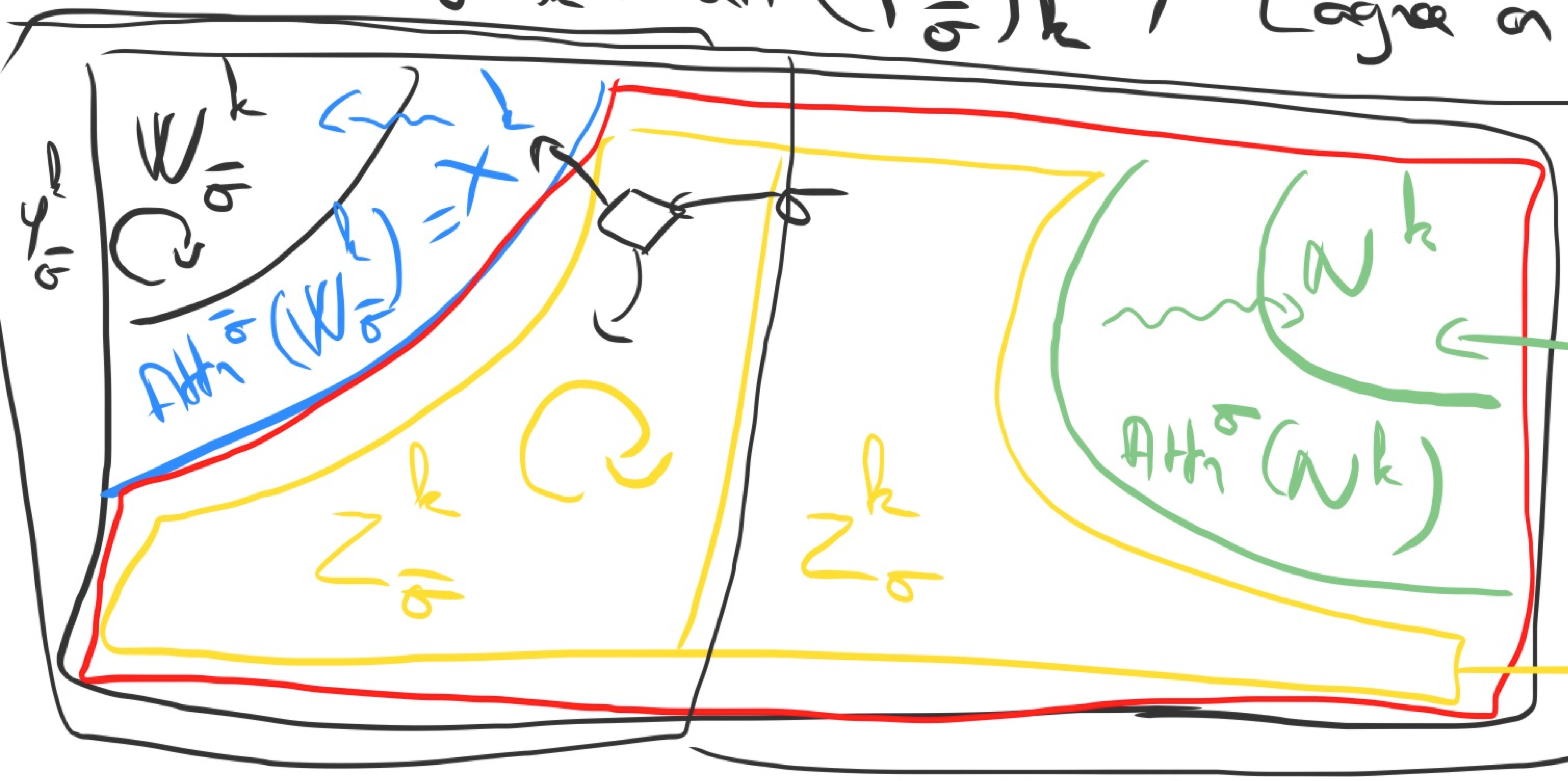
Call σ the play that wins if d is ∞ often repeated

We will construct an ↑ sequence $(W^k_{\bar\sigma})_k$, $(\varphi^k_{\bar\sigma})_k$ st:

positional strat

1) $W^k_{\bar\sigma}$ is a trap for σ $\subseteq V$ and $\varphi^k_{\bar\sigma}$ is winning on it and traps the play in $W^k_{\bar\sigma}$

2) $(W^k_{\bar\sigma})_k$ ↑ an $(\varphi^k_{\bar\sigma})_k$ ↑ [agree on th common domain + domain ↑].



$W^0_{\bar\sigma} = \emptyset$ $\quad$ $\varphi^0_{\bar\sigma}$ def nowhere

vertices with colour d

$W^{k+1}_{\bar\sigma} = W^k_{\bar\sigma} \cup X^k \cup Z^k_{\bar\sigma}$

Subarena with $n-1$ colours

$W^k_{\bar\sigma}$ | $\varphi^k_{\bar\sigma}$

$X_k = Attr^{\bar\sigma}(W^k_{\bar\sigma})$ | φ_{attr}

$Y_k = V \setminus X_k$

$Z_k = Y_k \setminus Attr^{\sigma}(N_k)$ where $N_k = \{v \in Y_k \mid \rho(v) = d\}$

↳ subarena with n-1 colours

$\Rightarrow$ $\left.\begin{array}{ll} Z^k_{\bar\sigma} & / \; Z^k_{\sigma} \\ \varphi^k_{Z\bar\sigma} & / \; \varphi^k_{Z\sigma} \end{array}\right\}$ by induction hyp on the # colours.

$$W^{k+1}_{\bar\sigma} = X_k \cup Z^k_{\bar\sigma}$$

$$\varphi^{k+1}_{\bar\sigma}(v) = \varphi^k_{Z,\bar\sigma}(v) \quad \text{if } v \in Z^k_{\bar\sigma}$$
$$= \varphi^k_{\bar\sigma}(v) \quad \text{if } v \in W^k_{\bar\sigma}$$
$$= \varphi_{attr}(v) \quad \text{if } X_k \setminus W^k_{\bar\sigma}$$

ρ starting from $W^{k+1}_{\bar\sigma}$ where $\bar\sigma$ follows $\varphi^{k+1}_{\bar\sigma}$.

→ either ρ stays in $Z^k_{\bar\sigma}$ forever $\Rightarrow$ respects $\varphi^k_{Z,\bar\sigma}$

$\Rightarrow$ winning by IH

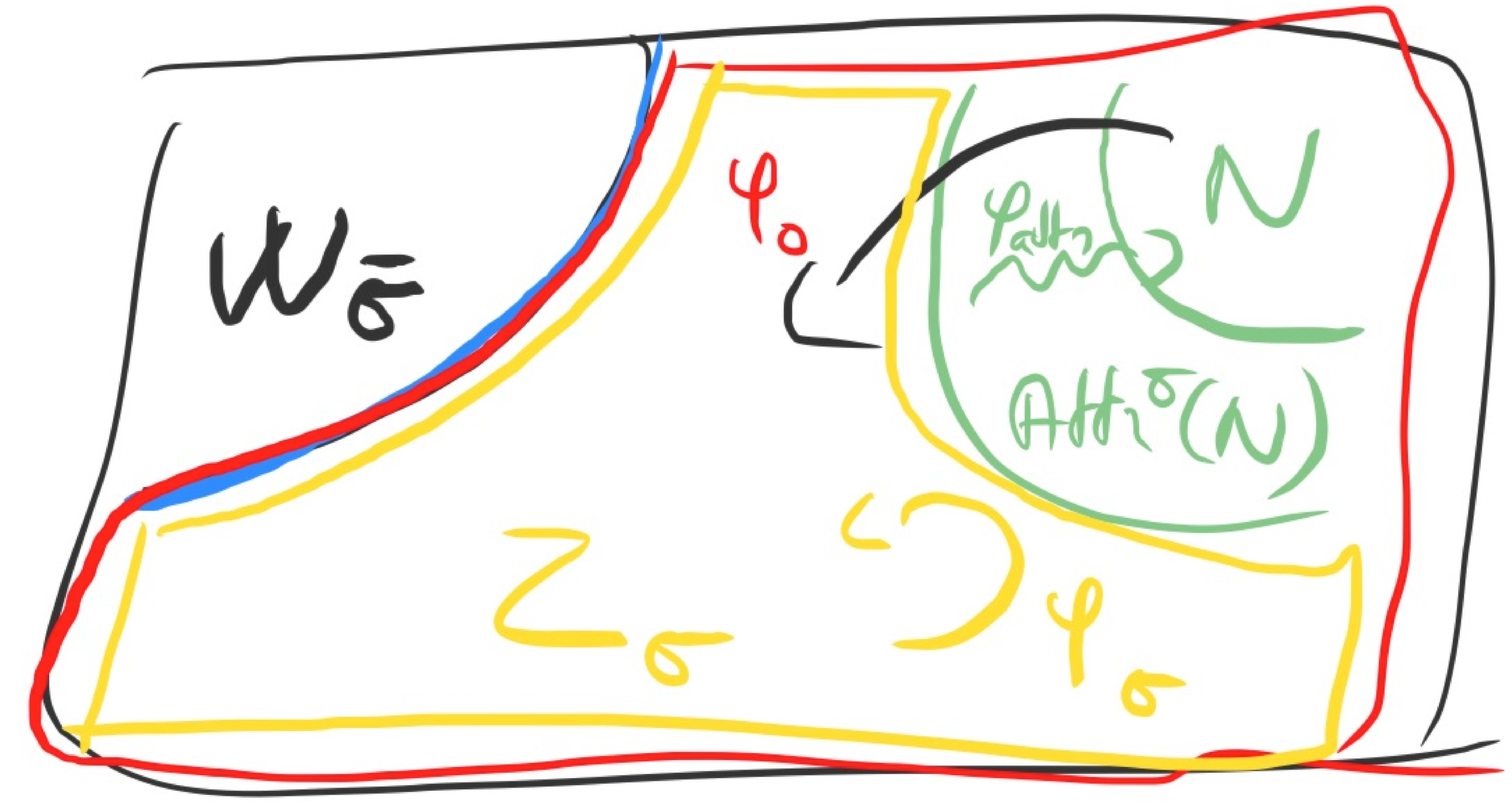
→ or it goes eventually to X_k

$\Rightarrow$ hence eventually stay in $W^k_{\bar\sigma}$ forever $\Rightarrow$ winning by IH on k.

$W_{\bar\sigma} = \text{Limit}(W^k_{\bar\sigma})$; winning for $\bar\sigma$

Consider $\underbrace{V \setminus W_{\bar{\sigma}}}$

situation when fixpoint is reached $\rightarrow$



Define φ strat for σ

$$\varphi(v) = \begin{cases} \varphi_0(v) & \text{if } v \in N \\ \varphi_{attr}(v) & \text{if } v \in Attr^{\sigma}(N) \setminus N \\ \varphi_{\sigma}(v) & \text{if } v \in Z_{\sigma} \end{cases}$$

$\rightarrow$ positional.

λ play starting in $V \setminus W_{\bar{\sigma}}$ when σ respects φ.

1) either λ stays in Z_{σ} eventually forever $\Rightarrow$ winning by IH on # colours

2) ∞ often gets in $Attr^{\sigma}(N)$ $\Rightarrow$ ∞ often visits N $\Rightarrow$ max ∞ often visited colour is d $\Rightarrow$ σ wins.

Corollary: Solving parity game is in $NP \cap co\text{-}NP$.

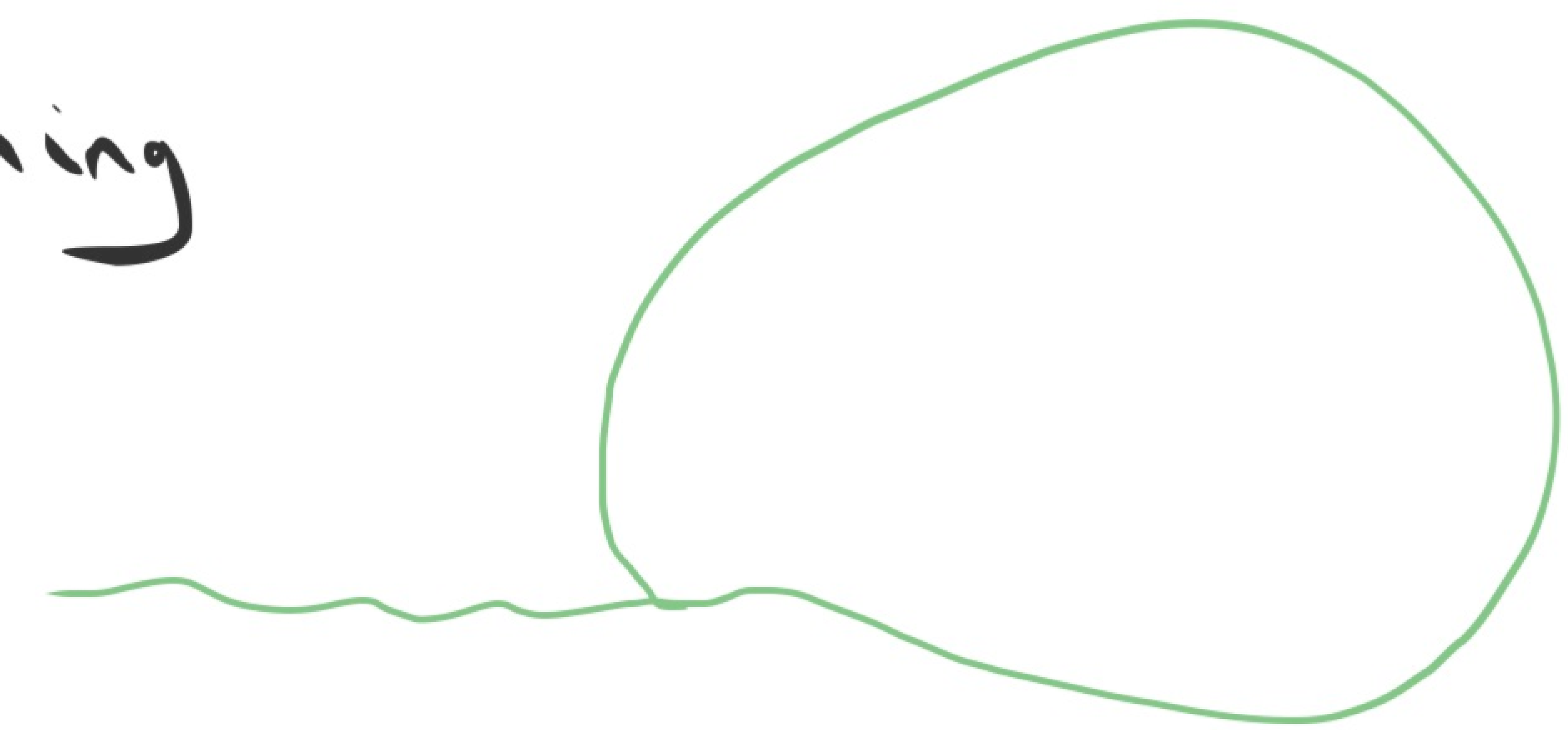
Proof **NP:**

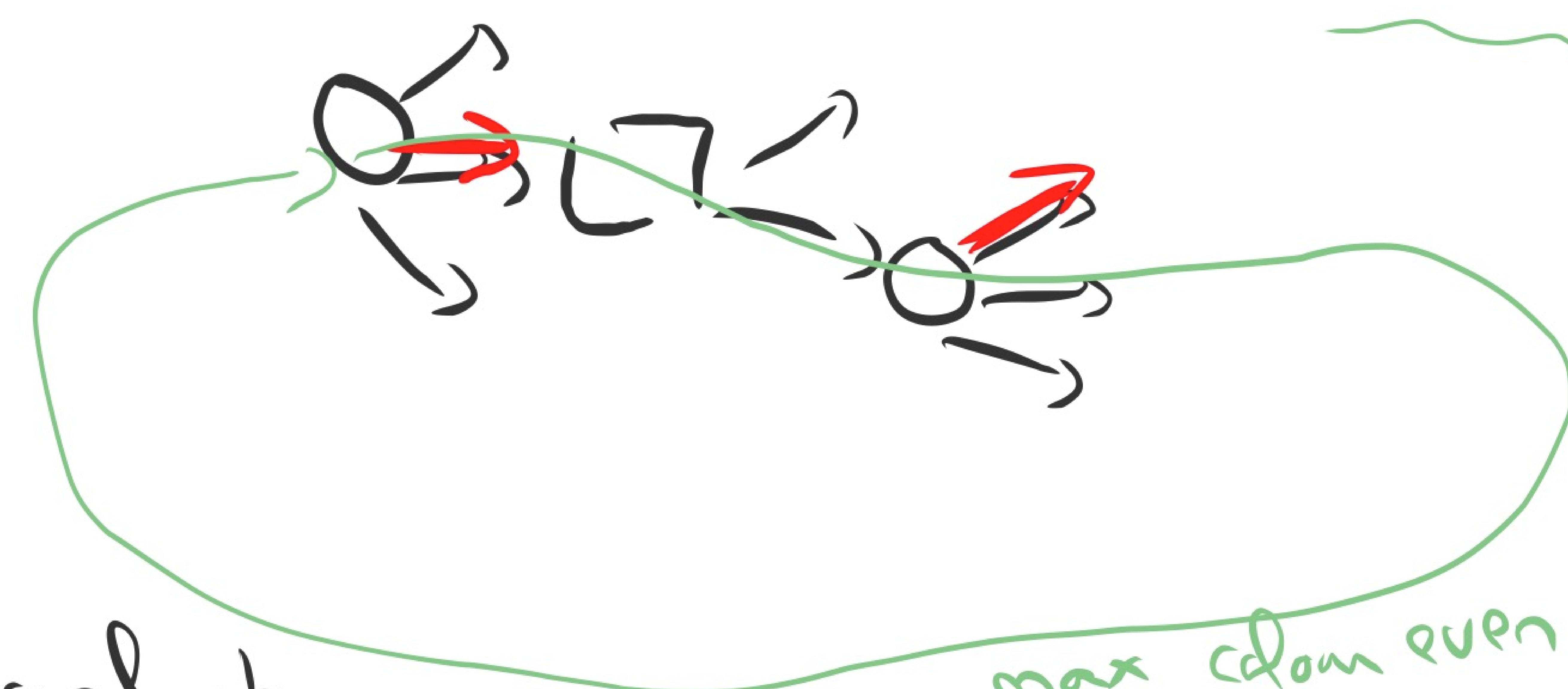
Eve wins $\iff$ she has a **positional** winning strat.

Algo: 1) Guess φ positional for Eve

2) Check that φ is winning

$\hookrightarrow$ PTIME





co-NP: dual pb.

Big question: is it in P?