Monadic theory of a linear order versus the theory of its subsets with the lifted min/max operations

Christian Choffrut[1] and Serge Grigorieff[1⋆]

LIAFA, CNRS UMR 7089, Université Paris 7 Denis Diderot, France

Abstract. We compare the monadic second-order theory of an arbitrary linear ordering L with the theory of the family of subsets of L endowed with the operation on subsets obtained by lifting the max operation on L. We show that the two theories define the same relations. The same result holds when lifting the min operation or both max and min operations.

1 Introduction

We initiated a couple of years ago an investigation aiming at comparing the theory of a monadic second-order structure $S = \langle U, \mathcal{P}(U); =_U, \in, \omega_1, \ldots, \rangle$ and

⋆ Partially supported by TARMAC ANR agreement 12 BS02 007 01.

that of the associated first-order structure $T = \langle \mathcal{P}(U); =, \Omega_1, \ldots, \rangle$ where Ω_i is the operation ω_i lifted to subsets: $\Omega_i(X_1, \ldots,) = \{\omega_i(x_1, \ldots,) \mid x_1 \in X_1, \ldots\}$. The structure T can be viewed as follows: lift all operations to subsets and consider the sole formulas about S with no occurrence of an individual variable, whether free or bound. Let us stress that *the inclusion relation and the Boolean operations on sets are not given as primitives in* T. The structure T is clearly definable in S : the unique sort of T (namely $\mathcal{P}(U)$) is among the two sorts of S (which are U and $\mathcal{P}(U)$) and the lifted operations Ω_i's are definable in S. The general issue is: what can be known of S within T ? More precisely,

(Q_1) Concerning relations on $\mathcal{P}(U)$, does definability in S implies definability in T ? This question reduces to the following one: is it possible to define in T the class of singleton sets and the set-inclusion relation hence to define in T the most natural isomorphic copy of S.

(Q_2) In case question (Q_1) receives negative answer then
(*) which S-definable families of subsets of U are also T-definable?
(**) is it still possible to define in T an isomorphic copy of S ?

In previous works we studied two particular cases: $S_1 = \langle \mathbb{N}; =_\mathbb{N}, \in, + \rangle$ and $S_2 = \langle \Sigma^*; =_{\Sigma^*}, \in, \cdot \rangle$, cf. [4,5]. We showed that in these two cases question (Q_1) has a negative answer but question ($*$) gets a positive solution: an isomorphic copy of S is definable in T. Let us give a brief account for S_1. Consider the maps $\sigma : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ and $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ such that $\sigma(X) = \{0\} \cup (1 + X)$ and $f(n) = \{0\} \cup (1 + n + \mathbb{N})$. We proved that the ranges of σ and f and the images under σ and f of membership and addition, namely, the four predicates

$$\sigma(\mathcal{P}(\mathbb{N})) \subseteq \mathcal{P}(\mathbb{N}) \qquad f(\mathbb{N}) \subseteq \mathcal{P}(\mathbb{N}) \qquad \{(f(x), \sigma(X)) \mid x \in X\} \subseteq \mathcal{P}N) \times \mathcal{P}(\mathbb{N})$$
$$\{(f(x), f(y), f(z)) \mid x, y, z \in \mathbb{N}, z = x + y\} \subseteq \mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N})$$

are all definable in T with respective complexities Σ_1, Π_3, Δ_4 and Δ_5.

In this paper we consider an arbitrary linear order L with possibly minimum and maximum elements and show that its monadic second-order theory is equivalent to the first-order theory of its power set when the order relation is lifted by defining the predicate $\mathtt{Max}(X, Y, Z)$ where $X = \{\max\{y, z\} \mid y \in Y, z \in Z\}$. The situation is much simpler than above since question (Q_1) gets a positive answer: we can express in T the predicates "X is a singleton" and "X is a subset of Y".

Let us recall that the monadic theory of a linear order has been intensively studied. As a prelude to the general monadic theory of linear orders, Gurevich 1964 [6] proved the decidability of the theory of linear orders with one-place predicates. Büchi 1960 [1] proved the decidability of the monadic theory of the order on $\mathbb{N}$. The result has been extended to all countable ordinals, Büchi 1973 [2], and then to all ordinals $< \omega_2$, Büchi & Zaiontz 1983 [3]. The decision problem for the monadic theory of the ordinal ω_2 happens to depend on axioms of set theory, Gurevich & Magidor & Shelah 1983 [8], Lifsches & Shelah 1992 [13]. The monadic theory of the order on $\mathbb{R}$ is undecidable, Shelah 1975 [15], and in fact very complex, Gurevich 1979 [9], Gurevich & Shelah 1982-84 [10,11,12].

We now give a brief outline of the paper. Section 2 recalls the basics on linear orderings. It also introduces the two structures to be compared. In section

3 we study for its own sake the structure obtained by lifting a linear ordering to subsets. We consider it both as a monoid and as a partial ordering of which we give a couple of alternative characterizations.

The expressibility of singletons is obtained in section 4 by a careful study of the set of immediate predecessors of a given subset in the lifted ordering since the cardinal of this set discriminates the singletons among all subsets. The same is done for pairs.

Membership of an element to an arbitrary subset (more exactly, inclusion of a singleton set in a set) is the second ingredient to prove the equivalence of the two structures and it is considered in Section 5. Two different expressions are given according to whether or not the ordering possesses a zero. The two expressions have the same complexity Δ_4 but they are based on different approaches which we found interesting to keep.

The equivalence of different structures, mainly those introduced in paragraph 2.2 along with their natural variants is established in section 6.

2 Preliminaries

2.1 Linear orders

This section is meant to keep this paper self-contained. We recall the basic definitions on orderings, see e.g., [14]

Definition 1. *An element $a \in L$ is an* upper bound *of $X \subseteq L$ if $x \leq a$ for all $x \in X$. It is a* least upper bound *if it is an upper bound and for all upper bounds b it holds $a \leq b$. It is the* maximum element *of X and denoted* $\max(X)$ *if furthermore it belongs to X. If L has a maximum element we denote it by* 1.

A final segment *is a subset which is* upward saturated, *i.e., $x \in X$ and $y \geq x$ implies $y \in X$. It is a* closed final segment *if it is of the form $\{x \mid x \geq a\}$ for some $a \in L$, else it is an* open final segment.

The notions of lower bound, greatest lower bound $\mathrm{glb}(X)$ *and minimum element* $\min(X)$ *of a set X are defined in the obvious similar way. So is the notion of minimum element* 0.

Definition 2. *Two elements a, b are* successive *if $a < b$ and the condition $a \leq c \leq b$ implies $c = a$ or $c = b$. We then say that a is an* immediate predecessor *of b and that b is an* immediate successor *of a.*

An element is a successor *if it admits an immediate predecessor, it is a* predecessor *if it admits an immediate successor.*

If $a \in L$ is not a successor and if it is not 0, it is a limit.

Notation 3 *The final segments canonically associated to a subset $X \subseteq L$ are denoted by*

$$X^{\geq} = \{y \mid \exists x \in X \quad x \leq y\} \text{ (the smallest final segment containing } X\text{)}$$
$$X^{>} = \{y \mid \exists x \in X \quad x < y\}$$

and the set of strict lower bounds by

$$X^{<} = \{y \mid \forall x \in X \quad y < x\}$$

Lemma 4. *Given two final segments $F, G \subseteq L$ we have*

$$F \subsetneq G \quad \text{or} \quad F = G \quad \text{or} \quad G \subsetneq F$$

Proof. Assume $F \neq G$, i.e., without loss of generality assume there exists $x \in F \setminus G$. Then for all $y \in G$ we have $x \ngeq y$ or equivalently $x < y$. But then $G \subseteq \{z \in L \mid x < z\} \subsetneq F$. □

Remark 5. The following elementary observation underlies many proofs of this paper. It helps having it in mind.

For all nonempty subsets $X \subseteq L$ exactly one of the following conditions holds.

- X has a minimum and $X^{<}$ is empty or has a maximum (e.g., in any finite linear order).
- X has a minimum and $X^{<}$ is nonempty and has no maximum (consider the order $A + B$ with $A = B = \mathbb{N}$ and take $X = B$).
- X has no minimum and $X^{<}$ is empty or has a maximum (consider the order $A + B$ with $A = B = -\mathbb{N}$ and take $X = B$).
- X has no minimum and $X^{<}$ is nonempty and has no maximum (consider the order $A + B$ with $A = \mathbb{N}, B = -\mathbb{N}$ and take $X = B$).

2.2 Logical structures

Given an arbitrary linear order $\leq$ on a nonempty set L, we consider the structure $\langle L; =, \max\rangle$ or $\langle L; =, \max, 0, 1\rangle$ where max has the natural interpretation $\max\{x, y\} = x$ if $x \leq y$ and y otherwise and 0 and 1 are respectively the minimum and maximum elements (in case they exist).

We consider the operation on $\mathcal{P}(L)$ obtained by lifting the max operation on L.

Definition 6. *For $X, Y \subseteq L$, we set*

$$X \uparrow Y = \{\max\{x, y\} \mid x \in X,\ y \in Y\}$$

We compare the two associated structures dealing with sets:

$$S = \begin{cases} \langle L, \mathcal{P}(L); =, \in, \max\rangle \\ \text{or } \langle L, \mathcal{P}(L); =, \in, \max, 0, 1\rangle \end{cases} \quad , \quad T = \langle \mathcal{P}(L); =, \uparrow\rangle \tag{1}$$

Now, we define precisely what question (Q_1) supra means for the two structures S and T. Question (Q_1) (slightly revisited) is as follows: given any second-order formula ϕ for S with m first-order and n second-order variables, does there exist some first-order formula ψ for T with $m+n$ first-order variables such that, for all $a_1,\ldots,a_m \in L$ and $A_1,\ldots,A_n \in \mathcal{P}(L)$ the following equivalence holds

$$\begin{array}{c}\langle L,\mathcal{P}(L);=,\in,\max\rangle \models \phi(a_1,\ldots,a_m,A_1,\ldots,A_n)\\ \Longleftrightarrow \\ \langle \mathcal{P}(L);=,\uparrow\rangle \models \psi(\{a_1\},\ldots,\{a_m\},A_1,\ldots,A_n)\end{array} \tag{2}$$

An easy induction on formulas ϕ shows that it suffices to get such a formula ψ for the two particular formulas ϕ expressing the predicates "$X = \{x\}$" and "$\{x\} \subseteq X$".

Observe that the reverse question "given ψ get ϕ" is straightforward since the lifting of operations from L to $\mathcal{P}(L)$ can be expressed in S.

In all cases, when showing that a predicate is expressible in the language we give an estimate of its syntactic complexity. We recall that a predicate is Σ_n (resp. Π_n) if it is defined by a formula that begins with some existential (resp. universal) quantifiers and alternates $n-1$ times between series of existential and universal quantifiers. It is Δ_n if it is both Σ_n and Π_n. It is $\Sigma_n \wedge \Pi_n$ if it is defined by a conjunction of a Σ_n formula and a Π_n formula.

3 Lifted structure

Every linear ordering L is a lattice which allows one to view it as a universal algebra equipped with binary operations of lower and upper bound of two elements. Here we show that the lifted binary operation of $\mathcal{P}(L)$ allows us to define a partial ordering which makes it a join-semilattice. We investigate $\mathcal{P}(L)$ both as an algebra and as a partially ordered set.

3.1 The semigroup $\langle \mathcal{P}(L),\uparrow\rangle$

Here we are interested in the algebraic structure of the operation $\uparrow$ on the subsets of L.

Lemma 7. *1. The operation $\uparrow$ on $\mathcal{P}(L)$ is idempotent, commutative and associative and admits the empty set $\emptyset$ as an absorbing element.*
2. The operation $\uparrow$ has a neutral element if and only if $(L,\leq)$ has a minimum element 0. In this case, $\{0\}$ is the neutral element of $\uparrow$.
3. The operation $\uparrow$ distributes over the set union.

Proof. Straightforward.

Corollary 8. *1. The predicate $X = \emptyset$ is Π_1 expressible in $\langle \mathcal{P}(L);=,\uparrow\rangle$.*
2. If L has a minimum element 0 then the predicate $X = \{0\}$ is Π_1.

Proof. 1. Since $\emptyset$ is absorbing in $\langle \mathcal{P}(L);=,\uparrow\rangle$ and there is at most one absorbing element, $X = \emptyset$ holds if and only if $\forall Y\ X \uparrow Y = X$.
2. Similarly, $\{0\}$ is the unique neutral element in $\langle \mathcal{P}(L);=,\uparrow\rangle$, hence $X = \{0\}$ holds if and only if $\forall Y\ X \uparrow Y = Y$. $\square$

3.2 A characterization of the operation $\uparrow$

Because of Lemma 4, for two given final segments one is included into the other. Therefore the following result exhausts all possible cases. Its purpose is to work as much as possible with subsets rather than applying the original Definition 6 which mixes subsets and elements.

Lemma 9. *For all $X, Y \subseteq X$ we have*

$$X \uparrow Y = (X \cup Y) \cap X^{\geq} \cap Y^{\geq} = \begin{cases} Y \cup (X \cap Y^{>}) = Y \cup (X \cap Y^{\geq}) & \text{if } Y^{\geq} \subseteq X^{\geq} \\ X \cup (X^{>} \cap Y) = X \cup (X^{\geq} \cap Y) & \text{if } X^{\geq} \subseteq Y^{\geq} \end{cases}$$

Proof. If $z \in X \uparrow Y$ then $z = x \vee y$ for some $x \in X$ and $y \in Y$. If $x \leq y$ then $z = y \in X^{\geq} \cap Y$ and if $x \geq y$ then $z = x \in X \cap Y^{\geq}$. In both cases we have $z \in (X \cup Y) \cap X^{\geq} \cap Y^{\geq}$. Conversely, let $z \in (X \cup Y) \cap X^{\geq} \cap Y^{\geq}$. If $z \in X$ then $z \in X \cap Y^{\geq}$ hence $z \geq y$ for some $y \in Y$ and $z = z \vee y \in X \uparrow Y$. Similarly, if $z \in Y$ then z is also in $X \uparrow Y$. This proves equality $X \uparrow Y = (X \cup Y) \cap X^{\geq} \cap Y^{\geq}$. Since $X \subseteq X^{\geq}$ and $Y \subseteq Y^{\geq}$, the other stated equalities (under assumption $Y^{\geq} \subseteq X^{\geq}$ or $X^{\geq} \subseteq Y^{\geq}$) are derived by simple set computation. □

3.3 The partially ordered set $\langle \mathcal{P}(L), \preceq \rangle$

We consider the following binary relation on subsets which happens to be an ordering.

Definition 10. *For $X, Y \subseteq L$ we let $X \preceq Y \iff X \uparrow Y = Y$.*

E.g., if Y is the singleton $\{y\}$ then $X \preceq Y \iff \forall x \in X \; x \leq y$. In particular, if X, Y are the singletons $\{x\}, \{y\}$ then $\{x\} \preceq \{y\} \iff x \leq y$.

Proposition 11. *1. The relation $\preceq$ is a partial ordering on $\mathcal{P}(L)$ with $\emptyset$ as maximum element.*
2. The order $\preceq$ has a minimum element if and only if $(L, \leq)$ has a minimum element 0. In this case, $\{0\}$ is the minimum element of $\preceq$.
3. The order $\preceq$ restricted to $\mathcal{P}(L) \setminus \{\emptyset\}$ has a maximum element if and only if $(L, \leq)$ has a maximum element 1. In this case, $\{1\}$ is the maximum element of this restriction of $\preceq$.

Proof. 1. Reflexivity and antisymmetry are clear. We prove transitivity. Suppose $X \preceq Y \preceq Z$ then $X \uparrow Y = Y$ and $Y \uparrow Z = Z$ hence

$$X \uparrow Z = X \uparrow (Y \uparrow Z) = (X \uparrow Y) \uparrow Z = Y \uparrow Z = Z.$$

Since $\emptyset$ is absorbing for $\uparrow$ it is the maximum element of $(\mathcal{P}(L), \preceq)$.
Claims 2, 3 are straightforward. □

Lemma 12. *If $X \preceq Y$ then $Y \subseteq X^{\geq}$ hence $Y^{\geq} \subseteq X^{\geq}$.*

Proof. Apply equality $X \uparrow Y = Y$ and Lemma 9: $X \uparrow Y \subseteq X^{\geq}$. □

As usual, we denote by $X \prec Y$ the strict ordering defined by $X \preceq Y$ and $X \neq Y$.

Remark 13. It is a simple exercise to verify that $\langle \mathcal{P}(L), \preceq \rangle$ is a linear order if and only if L has at most two elements.

With a structure of linear ordering L is naturally associated a structure of lattice. We lifted the linear ordering to a partial ordering on the power set of L. This partial ordering is not associated with a structure of lattice, only with a structure of join semilattice.

Proposition 14. *$\langle \mathcal{P}(L); \preceq \rangle$ is a join semilattice: $X \uparrow Y$ is the join of X and Y.*

Proof. Since $(X \uparrow Y) \uparrow X = (X \uparrow Y) \uparrow Y = X \uparrow Y$ we have $X, Y \preceq X \uparrow Y$. Suppose $X, Y \preceq Z$. Then $(X \uparrow Y) \uparrow Z = (X \uparrow Z) \uparrow Y = Z \uparrow Y = Z$ hence $X \uparrow Y \preceq Z$. This proves that $X \uparrow Y$ is the join of X, Y. □

Remark 15. The $\preceq$ order may have no meet. For instance, consider the set $L = \omega^*$ of negative or null integers with the usual order and let $X = -2\mathbb{N}$ and $Y = -(2\mathbb{N}+1)$. Then $Z \preceq X$ if and only if Z is an infinite subset of X. Similarly with Y. Thus, X, Y have no common lower bound.

Considering the same sets as subsets of ω^* in the linear order $\omega + \omega^*$, $Z \preceq X$ if and only if $Z \subseteq \omega \cup X$ and $Z \cap \omega \neq \emptyset$ or $Z \cap \omega^*$ is infinite. Thus, X, Y have common lower bounds which are exactly the nonempty subsets of ω. However, any common lower bound Z is strictly upper bounded by another common lower bound T : if $z \in Z$ then let $T = \{t \in \omega \mid t > z\}$.

3.4 Final segments

It is clear that final subsets play a special rôle. Indeed, the partial order restricted to the final segments is linear and more importantly the binary operation $\uparrow$ and the partial order $\preceq$ between arbitrary subsets use final segments in their alternative definitions, such as Lemmas 9 and 18.

Lemma 16. *If F is a final segment and $F \preceq Y$ then Y is a final segment.*

Proof. Suppose $y \in Y$ and $z \geq y$. Equality $F \uparrow Y = Y$ shows that $y \geq x$ for some $x \in F$. But then $z \geq x$ and since F is upwards closed we have $z \in F$ hence $z = \max\{z, y\} \in F \uparrow Y = Y$. Thus, Y is upwards closed. □

Lemma 17. *If F, G are final segments then $F \uparrow G = F \cap G$. In particular, $F \preceq G$ if and only if $F \supseteq G$.*

Proof. Since F and G are final segments we have $F = F^{\geq}$ and $G = G^{\geq}$. By Lemma 9 we obtain $F \uparrow G = (F \cup G) \cap F^{\geq} \cap G^{\geq} = (F \cup G) \cap F \cap G = F \cap G$. □

3.5 A characterization of the ordering $\preceq$

The following is an alternative definition of the relation $\preceq$ in set theoretical terms.

Lemma 18. *For all $X, Y \subseteq L$ we have $X \preceq Y$ if and only if*

$$X \cap Y^{\geq} \subseteq Y \subseteq X^{\geq}\,. \tag{3}$$

Note. Observe that the last occurrence of $X^{\geq}$ in the above expression cannot be replaced by $X^{>}$ (take $X = Y$ where X has a minimal element).

Proof. The statement follows from the next inclusions

$$\begin{cases} X \uparrow Y \subseteq Y \iff X \cap Y^{\geq} \subseteq Y \iff X \cap Y^{>} \subseteq Y \\ X \uparrow Y \supseteq Y \iff Y \subseteq X^{\geq} \end{cases}$$

Indeed, condition $X \uparrow Y \subseteq Y$ holds if and only if, for all $x \in X$ and $y \in Y$, $x > y \Rightarrow x \in Y$ (resp. $x \geq y \Rightarrow x \in Y$), which means $X \cap Y^{>} \subseteq Y$ (resp. $X \cap Y^{\geq} \subseteq Y$). Condition $X \uparrow Y \supseteq Y$ holds if and only if for all $y \in Y$ there exists $x \in X$ such that $x \leq y$, which means $Y \subseteq X^{\geq}$. ☐

The following "constructive" characterization of the relation $\prec$ will help when determining the immediate $\preceq$-predecessors of a subset (cf. §4.1).

Lemma 19. *The condition $X \prec Y$ holds if and only if one of the following two conditions is satisfied:*

$$X^{\geq} = Y^{\geq} \text{ and } X \subsetneq Y \tag{4}$$

$$X \setminus Y^{\geq} \neq \emptyset \text{ and } X \cap Y^{\geq} \subseteq Y \tag{5}$$

Proof. $\Rightarrow$. By Lemma 18 we know that $X \prec Y$ if and only if $X \neq Y$ and (3) above holds. The last inclusion $Y \subseteq X^{\geq}$ of (3) yields $Y^{\geq} \subseteq X^{\geq}$.
If $X^{\geq} = Y^{\geq}$ holds then $X = X \cap X^{\geq} = X \cap Y^{\geq} \subseteq Y$ by the first inclusion of (3) and thus $X \subsetneq Y$, showing that condition (4) is true. Otherwise $Y^{\geq} \subsetneq X^{\geq}$ hence $X \setminus Y^{\geq} \neq \emptyset$. Since we also have $X \cap Y^{\geq} \subseteq Y$ we see that condition (5) is true.
$\Leftarrow$. Conversely, suppose condition (4) is satisfied: $X^{\geq} = Y^{\geq}$ and $X \subsetneq Y$ holds. Then $X \cap Y^{\geq} \subseteq Y \cap Y^{\geq} = Y \subseteq Y^{\geq} = X^{\geq}$ and, by Lemma 18 $X \preceq Y$. Since $X \subsetneq Y$ we have $X \prec Y$.
Suppose condition (5) is satisfied. Then $X^{\geq} \setminus Y^{\geq} \neq \emptyset$ and Lemma 4 yields $Y^{\geq} \subseteq X^{\geq}$. Thus, using the assumption $X \cap Y^{\geq} \subseteq Y$, we get $X \cap Y^{\geq} \subseteq Y \subseteq Y^{\geq} \subseteq X^{\geq}$ hence $X \preceq Y$ (by Lemma 18). Now, $X \neq Y$ since $X \setminus Y^{\geq} \neq \emptyset$. Thus, $X \prec Y$. ☐

4 Defining single elements

As said in the introduction, the objective of this paper is to show that the two structures S and T (cf. (1) in §2.2) can be identified when properly encoded.

This requires in particular to prove that individual variables can be recovered in the structure T. This is achieved in Theorem 26.

We illustrate our approach by means of examples. With $L = \mathbb{N}$, one can convince oneself that every singleton $\{a\}$ can be defined by the number of subsets X such that $X \prec \{a\}$ (e.g., with $a = 0$ there is no strict predecessor, with $a = 1$ there are exactly 2 strict predecessors, namely $\{0\}$, $\{0, 1\}$, with $a = 2$ there are exactly 6 strict predecessors, namely $\{0\}$, $\{0, 1\}$, $\{0, 2\}$, $\{0, 1, 2\}$, $\{1\}$, $\{1, 2\}$). This however cannot be extended to linear orders such as $\mathbb{Z}$ and worse it suggests a new formula must be designed for each singleton. Luckily, whatever the linear order, the fact of being a singleton is defined by a unique formula asserting how many *immediate* predecessors it has. E.g., in $\mathbb{Z}$ it is the case for the three values of a above that there is exactly one immediate predecessor. The definability of singletons in established in Theorem 26. As we make no assumption on L in the investigation of the possible immediate predecessors we are led to consider different cases according to whether or not the given subset of L has a minimum, a greatest lower bound, a lower bound or no lower bound (as observed in Remark 5).

4.1 Immediate predecessors

The notion of immediate predecessors is as expected (cf. Definition 2).

Notation 20 *We denote by* $\mathtt{Suc}(X, Y)$ *the* Π_1*-predicate asserting that* Y *is an immediate successor of* X *(or* X *is an immediate predecessor of* Y*), i.e.*

$$X \prec Y \ \wedge\ \forall Z\ (X \preceq Z \preceq Y \iff (Z = X \ \vee\ Z = Y)) \tag{6}$$

We state the main result of this subsection.

Theorem 21. X *and* Y *are successive sets for* $\preceq$ *(i.e.* $\mathtt{Suc}(X, Y)$ *is true) if and only if one of the following conditions holds*

1. $Y = X \cup \{a\}$ *for some* $a \in X^{\geq} \setminus X$ *(in particular,* a *is not the minimum element of* Y*)*
2. $X = Y \cup \{b\}$ *where* b *is the maximum element of* $\{z \mid \forall y \in Y \ \ z < y\}$

We first inquire under which condition a subset X of Y is an immediate predecessor.

Lemma 22. *A subset* $X \subsetneq Y$ *is an immediate predecessor of* Y *if and only if* $X = Y \setminus \{a\}$ *where* $a \in Y$ *is not the minimum element in* Y*.*

Proof. $\Leftarrow$. Assume a is not minimum in Y. Then $Y^{\geq} = (Y \setminus \{a\})^{\geq}$. Because of $Y \setminus \{a\} \subsetneq Y$ Lemma 19 implies $Y \setminus \{a\} \prec Y$. Assume there exists Z such that $Y \setminus \{a\} \prec Z \prec Y$. Lemma 12 yields $Y^{\geq} \subseteq Z^{\geq} \subseteq (Y \setminus \{a\})^{\geq}$ which implies equalities $(Y \setminus \{a\})^{\geq} = Z^{\geq} = Y^{\geq}$ hence $Y \setminus \{a\} \subsetneq Z \subsetneq Y$ (by Lemma 19) which is impossible. This proves that $Y \setminus \{a\}$ is an immediate predecessor of Y.

$\Rightarrow$. Conversely, assume X is an immediate predecessor of Y and $X \subsetneq Y$. This last inclusion implies $X \setminus Y^{\geq} = \emptyset$ hence the first case of Lemma 19 applies: $X^{\geq} = Y^{\geq}$. If $Y \setminus X$ contains two elements $b, c \neq a$ then $X \subsetneq (X \cup \{b\}) \subsetneq Y$ and $X^{\geq} \subseteq (X \cup \{b\})^{\geq} \subseteq Y^{\geq}$ hence $X^{\geq} = (X \cup \{b\})^{\geq} = Y^{\geq}$ which, again by Lemma 19, implies $X \prec (X \cup \{b\}) \prec Y$, contradicting the assumption that X is an immediate predecesor of Y. We conclude that $Y \setminus X$ has exactly one element, i.e. $Y = X \cup \{a\}$ for some $a \notin X$. Since $X^{\geq} = Y^{\geq}$. this element a cannot be the minimum element of Y. □

In the next lemma it is assumed that the set of strict lower bounds of Y has a maximum. This is for example the case if the linear order L is Noetherian (i.e. reverse of an ordinal) and the set Y is not coinitial in L.

Lemma 23. *Assume that the set $L \setminus Y^{\geq} = \{z \mid \forall y \in Y \; z < y\}$ has a maximum element b (i.e. either b is a predecessor of the minimum element of Y or Y has no minimum element but has a greatest lower bound which is b).*
Then $Y \cup \{b\}$ is an immediate predecessor of Y.

Proof. Since $b \in (Y \cup \{b\}) \setminus Y^{\geq}$ and $(Y \cup \{b\}) \cap Y^{\geq} = Y$ the second condition of Lemma 19 is satisfied hence $Y \cup \{b\} \prec Y$. Assume that Z satisfies

$$Y \cup \{b\} \prec Z \prec Y \tag{7}$$

By Lemma 12 we have

$$Y^{\geq} \subseteq Z^{\geq} \subseteq (Y \cup \{b\})^{\geq} = Y^{\geq} \cup \{b\}$$

hence $Z^{\geq} = (Y \cup \{b\})^{\geq}$ or $Z^{\geq} = Y^{\geq}$.

Assume first $Z^{\geq} = Y^{\geq}$. Applying Lemma 19 with inequality $Z \prec Y$, we get $Z \subsetneq Y$. Since condition $Z^{\geq} = Y^{\geq}$ implies $Z^{\geq} \subsetneq (Y \cup \{b\})^{\geq}$, applying Lemma 19 to inequality $Y \cup \{b\} \prec Z$ yields $(Y \cup \{b\}) \cap Z^{\geq} \subseteq Z$. Now, $(Y \cup \{b\}) \cap Z^{\geq} = (Y \cup \{b\}) \cap Y^{\geq} = Y$ hence $Y \subseteq Z$, contradicting the strict inclusion $Z \subsetneq Y$.

Assume now that $Z^{\geq} = (Y \cup \{b\})^{\geq}$. Then Lemma 19 applied to inequality $Y \cup \{b\} \prec Z$ yields $Y \cup \{b\} \subsetneq Z$. The same Lemma applied to inequality $Z \prec Y$ yields $Z \cap Y^{\geq} \subseteq Y$. Now, since $Z^{\geq} = (Y \cup \{b\})^{\geq} = Y^{\geq} \cup \{b\}$, we have $Y^{\geq} = Z^{\geq} \setminus \{b\}$ and inclusion $Z \cap Y^{\geq} \subseteq Y$ becomes $Z \setminus \{b\} \subseteq Y$ hence $Z \subseteq Y \cup \{b\}$ which contradicts the strict inclusion $Y \cup \{b\} \subsetneq Z$. □

Proof of Theorem 21 It suffices to prove that there exist no other predecessor than those defined in the previous two lemmas.

Let X be an immediate predecessor of Y. Lemma 19 insures that the two following cases are exhaustive.

Case $X \setminus Y^{\geq} = \emptyset$ and $X \subsetneq Y$. We conclude by Lemma 22 that X is as claimed in the first item of Theorem 21.

Case $X \setminus Y^{\geq} \neq \emptyset$ and $X \cap Y^{\geq} \subseteq Y$. We distinguish three subcases.

Subcase $X \cap Y^{\geq} \subsetneq Y$. We show that this subcase is impossible. Since X is the disjoint union of $X \setminus Y^{\geq}$ and $X \cap Y^{\geq}$, we have $X \subsetneq (X \setminus Y^{\geq}) \cup Y$. Also,

$$\begin{aligned}
X \uparrow ((X \setminus Y^{\geq}) \cup Y) &= \big(X \uparrow (X \setminus Y^{\geq})\big) \cup (X \uparrow Y) \\
&= \big(X \uparrow (X \setminus Y^{\geq})\big) \cup Y \\
&= \big((X \setminus Y^{\geq}) \uparrow (X \setminus Y^{\geq})\big) \cup \big((X \cap Y^{\geq}) \uparrow (X \setminus Y^{\geq})\big) \cup Y \\
&= (X \setminus Y^{\geq}) \cup (X \cap Y^{\geq}) \cup Y \\
&= (X \setminus Y^{\geq}) \cup Y \qquad \text{since } X \cap Y^{\geq} \subseteq X \uparrow Y = Y\,.
\end{aligned}$$

Thus, $X \prec (X \setminus Y^{\geq}) \cup Y$. We also have $(X \setminus Y^{\geq}) \cup Y \prec Y$ since $(X \setminus Y^{\geq}) \cup Y \neq Y$ and $((X \setminus Y^{\geq}) \cup Y) \uparrow Y = ((X \setminus Y^{\geq}) \uparrow Y) \cup (Y \uparrow Y) = Y$. This contradicts the fact that X is an immediate predecessor of Y.

Subcase $X \cap Y^{\geq} = Y$ *and* $L \setminus Y^{\geq}$ *has a maximum element* b. Then $X \setminus Y^{\geq} \subseteq \{z \mid z \leq b\}$. Observe that $X \preceq (\{b\} \cup Y) \prec Y$ since $(\{b\} \cup Y) \uparrow Y = Y$ and

$$X \uparrow (\{b\} \cup Y) = ((X \setminus Y^{\geq}) \cup Y) \uparrow (\{b\} \cup Y) = \{b\} \cup Y$$

Since X is an immediate predecessor of Y this implies $X = Y \cup \{b\}$. This case is covered by Lemma 23 and gives the second item of Theorem 21.

Subcase $X \cap Y^{\geq} = Y$ *and* $L \setminus Y^{\geq}$ *has no maximum element.* We show that this subcase is impossible. Recall an assumption of the case (of which this is a subcase): $X \setminus Y^{\geq} \neq \emptyset$. Let $d \in X \setminus Y^{\geq} \subseteq L \setminus Y^{\geq}$. Since $L \setminus Y^{\geq}$ has no maximum element, there exists some $c \notin Y^{\geq}$ such that $d < c$. Pose $X_0 = \{z \notin Y^{\geq} \mid z \geq c\}$ and observe that $Y \uparrow X_0 = Y$ (since X_0 is disjoint from $Y^{\geq}$) and also $(X \setminus Y^{\geq}) \uparrow X_0 = X_0$ (inclusion: use the fact that X_0 is a final segment, containment: inequality $d < c$ implies $\{d\} \uparrow X_0 = X_0$, conclude with the fact that $d \in X$). Using the assumption equality $X \cap Y^{\geq} = Y$, we obtain $X = (X \setminus Y^{\geq}) \cup Y$ hence

$$\begin{aligned}
X \uparrow (X_0 \cup Y) &= ((X \setminus Y^{\geq}) \cup Y) \uparrow X_0) \cup (X \uparrow Y) \\
&= \big((X \setminus Y^{\geq}) \uparrow X_0\big) \cup (Y \uparrow X_0) \cup (X \uparrow Y) = X_0 \cup Y
\end{aligned}$$

Since $X \neq X_0 \cup Y$ (witnessed by d) and $X_0 \cup Y \neq Y$ (witnessed by c) we get $X \prec X_0 \cup Y \prec Y$, which is a contradiction.

□

Corollary 24. *The set* $\{a\}$ *has an immediate predecessor in* $\langle \mathcal{P}(L), \preceq \rangle$ *if and only if* a *has a predecessor* c *(necessarily unique) in the linear order* $\langle L, \leq \rangle$*. In that case,* $\{c, a\}$ *is the unique immediate predecessor of* $\{a\}$*.*

4.2 Singleton sets

With the help of the previous inquiry on the immediate predecessors of a given subset, the characterization of the singletons is obtained by a simple bookkeeping on the number of their immediate predecessors. We start with listing all possible numbers of immediate predecessors of a given subset.

Proposition 25. *Let X be a nonempty subset with cardinality $|X|$.*
1. If X is infinite then it has infinitely many immediate predecessors.
2. If X is finite and nonempty then
- if $\min(X)$ is 0 or a limit, then X has $|X|-1$ immediate predecessors,
- otherwise (i.e., if $\min(X)$ is a successor) X has $|X|$ immediate predecessors.
The set of immediate predecessors is gathered in Table 1.

	$\min(X)$ is 0 or limit in L	$\min(X)$ is the successor of b in L
$X=\{x_1,\ldots,x_n\}$ with $x_1<\ldots<x_n$	$X\setminus\{x_i\},\ 2\le i\le n$	$X\setminus\{x_i\},\ 2\le i\le n$ $\{b\}\cup X$
$X=\{x\}$	no immediate predec.	$\{b,x\}$

Table 1. Immediate predecessors of a nonempty finite set X

Proof. This is a direct consequence of Theorem 21. □

Theorem 26. *The following families are definable with the stated complexity:*

$\texttt{HasPred0}(X)$	$\equiv$	*X has no predecessor*	Π_2
$\texttt{HasPred}n(X)$	$\equiv$	*X has exactly n predecessors*	$\Sigma_2\wedge\Pi_2$
$\texttt{SingLimit}(X)$	$\equiv$	*$X=\{x\}$ for some limit $x\in L$*	Π_2
$\texttt{SingSucc}(X)$	$\equiv$	*$X=\{x\}$ for some successor $x\in L$*	$\Sigma_2\wedge\Pi_2$
$\texttt{Single}(X)$	$\equiv$	*$X=\{x\}$ for some $x\in L$*	$\Sigma_2\wedge\Pi_2$

Proof. Recall that $X=\emptyset$, $X=\{0\}$ and $\texttt{Suc}(Z,X)$ are Π_1 (cf. Corollary 8 and Notation 20).
• For $\texttt{HasPred0}(X)$ consider the Π_2 formula $\forall Z\ \neg\texttt{Suc}(Z,X)$.
• When $n\ge 1$, for $\texttt{HasPred}n(X)$ consider the $\Sigma_2\wedge\Pi_2$ formula

$$\exists Z_1,\ldots,Z_n\ \Big(\,\big(\bigwedge_{1\le i\le n}\texttt{Suc}(Z_i,X)\big)\wedge\big(\bigwedge_{1\le i<j\le n}Z_i\neq Z_j\big)\,\Big)$$
$$\wedge\ \forall T_1,\ldots,T_{n+1}\ \Big(\,\big(\bigwedge_{1\le i\le n+1}\texttt{Suc}(T_i,X)\big)\Longrightarrow\bigvee_{1\le i<j\le n+1}T_i=T_j\,\Big)$$

Applying Proposition 25 and the above, we see that
• $\texttt{SingLimit}(X)$ can be taken to be the Π_2 conjunction of $\texttt{HasPred0}(X)$ with the formulas expressing that $X\neq\emptyset,\{0\}$.
• Observe that a set X has a unique predecessor in $\mathcal{P}(L)$ in only two cases:

(1) $X=\{u,v\}$ and $u<v$ and u is 0 or a limit element in L. Then in $\mathcal{P}(L)$ the unique predecessor of X is $\{u\}$ which itself has no predecessor in $\mathcal{P}(L)$.
(2) $X=\{x\}$ and x has a predecessor z in L. Then in $\mathcal{P}(L)$ the unique predecessor of X is $\{z,x\}$ which itself has a predecessor $\{z\}$ (it may also have another one, $\{v,z,x\}$ in case z has a predecessor v in L).

Thus, $\texttt{SingSucc}(X)$ can be taken to be the $\Sigma_2 \wedge \Pi_2$ formula

$$X \neq \emptyset, \{0\} \;\wedge\; \texttt{HasPred1}(X) \;\wedge\; \exists Z, T\ (\texttt{Suc}(T, Z) \wedge \texttt{Suc}(Z, X))$$

- $\texttt{Single}(X)$ is the formula $X = \{0\} \vee \texttt{SingLimit}(X) \vee \texttt{SingSucc}(X)$. □

4.3 Recovering the linear order

We already observed that the relation $\leq$ is expressible with the relation $\preceq$ on the singletons. For future use (in Proposition 38), we give an estimate of the complexity of the formula.

Lemma 27. *The following relations are* $\Sigma_2 \wedge \Pi_2$ *:*

$$\texttt{Leq} = \{(\{x\}, \{y\}) \mid x \leq y\} \qquad R = \{(\{x\}, \{y\}) \mid y \text{ is the successor of } x\}$$

Proof. Observe that $x \leq y$ if and only if $\{x\} \preceq \{y\}$. It suffices to define $\texttt{Leq}(X, Y)$ via the formula $\texttt{Single}(X) \wedge \texttt{Single}(Y) \wedge X \preceq Y$ and $R(X, Y)$ via the formula $\texttt{Single}(X) \wedge \texttt{Single}(Y) \wedge \texttt{Suc}(X, Y)$. □

4.4 Pairs

The operation $\uparrow$ is not appropriate to express that an element belongs to a subset. Indeed, $\{a\} \uparrow X = X$ holds if and only if a is a lower bound of X, i.e., if a is the minimum in which case it belongs to X or is a strict lower bound and then it does not belong to X. This ambiguity is lifted if instead of the singleton $\{a\}$ we use paris of the form $\{z, a\}$ as will be amply employed in section 5. The following result is the key to the proof that the membership predicate is definable with complexity Δ_4.

Proposition 28. *The following predicates have the stated complexities :*

$R(Z, P)$	$\equiv Z = \{z\},\ P = \{z, a\}$ *for some* $z < a$	$\Sigma_2 \wedge \Pi_2$
$K(P)$	$\equiv P = \{0, a\}$ *for some* $a \in L$	Δ_2
$\texttt{Pair}_0(A, P)$	$\equiv A = \{a\},\ P = \{0, a\}$ *for some* $a \in L$	Π_3
$\texttt{Pair}(Z, A, P)$	$\equiv Z = \{z\},\ A = \{a\},\ P = \{z, a\}$ *for some* $z < a$	Π_3

Proof. Observe (Theorem 21 and Table 1) that $\{z\}$ is the immediate predecessor of a set P if and only if P is of the form $\{z, a\}$ for some $a > z$. This shows that the above predicate R is defined by the $\Sigma_2 \wedge \Pi_2$ formula $\texttt{Single}(Z) \wedge \texttt{Suc}(Z, P)$ whereas K is defined by the Σ_2 formula $\exists Z\ (Z = \{0\} \wedge \texttt{Suc}(Z, P))$ and the Π_2 formula $\forall Z\ (Z = \{0\} \Rightarrow \texttt{Suc}(Z, P))$.

Also, for all u we have $\{z, a\} \preceq \{u\}$ if and only if $a \leq u$. Thus, a triple (Z, A, P) is in $\texttt{Pair}$ if and only if $(Z, P) \in R$ and A is the smallest singleton set which dominates P. Considering the conjunction of the definition of $(Z, P) \in R$ with the formula $\forall U\ (\texttt{Single}(U) \Rightarrow (P \preceq U \Leftrightarrow A \preceq U))$ shows that $\texttt{Pair}$ is Π_3. Finally, $\texttt{Pair}_0$ can be Π_3 expressed as $\forall Z\ (Z = \{0\} \Rightarrow \texttt{Pair}(Z, A, P))$. □

5 Defining membership

In this section we solve the second ingredient of our proof, namely we show that the predicate $x \in X$ can be encoded in the structure T. More precisely we show that the membership predicate

$$\{(A, X) \mid A = \{a\} \text{ for some } a \in X\}$$

is Δ_4.

Before proving the general case (cf. Theorem 37) we consider the case where L has a minimum element 0 since we then get a simpler proof (cf. §5.3 and Theorem 35).

We give an intuition of the way we proceed in this simpler case. Let $a \in L$ and $X \subseteq L$. The condition $\{0, a\} \uparrow X = X$ is equivalent to $\{a\} \uparrow X \subseteq X$. This last condition is itself equivalent to the fact that a is a strict lower bound of X or that a belongs to X. In order to rule out the former condition, it suffices to say that $a^{\geq} \subseteq X^{\geq}$. This is the reason why the definability of the final segments and the upward closure of a subset take so much place in this section.

5.1 Defining final segments

Lemma 29. *Consider the Π_1 predicate $\Phi(X)$ which expresses that any two $\preceq$-upper bounds of X are $\preceq$ comparable.*

$$\Phi(X) \quad \equiv \quad \forall Y, Z \ \Big((X \preceq Y \ \wedge \ X \preceq Z) \Rightarrow (Y \preceq Z \vee Z \preceq Y) \Big)$$

Then $\Phi(X)$ holds if and only if X is a final segment or $X = \{a\}^{\geq} \setminus \{a^+\}$ where a^+ is the immediate successor of a in L (in case there is some).

Proof. $\Leftarrow$, 1st case. Assume X is a final segment. Conditions $X \preceq Y$ and $X \preceq Z$ imply that Y and Z are also final segments by Lemma 16 and these segments are $\preceq$-comparable by Lemma 4.
$\Leftarrow$, 2d case. Assume now that a has an immediate successor a^+ and $X = \{a\}^{\geq} \setminus \{a^+\} = \{a\} \cup \{a^+\}^{>}$. Consider some $X \prec U$. Since $U = (\{a\} \cup \{a^+\}^{>}) \uparrow U$ we have $\{a\} \uparrow U \subseteq U$ hence

$$U \subseteq \{a\}^{\geq} \qquad (*)$$

Subcase $a \in U$. Then $X = X \uparrow \{a\} \subseteq X \uparrow U = U$. Since $U \subseteq \{a\}^{\geq} = X \cup \{a^+\}$ and $U \neq X$ we see that $U = \{a\}^{\geq}$ is a final segment.
Subcase $a \notin U$ and $a^+ \in U$. Then $U = X \uparrow U \supseteq (\{a\} \cup \{a^+\}^{>}) \uparrow \{a^+\} = \{a^+\}^{\geq}$. Using (*) and the case assumption, we see that $U = \{a^+\}^{\geq}$ is a final segment.
Subcase $a \notin U$ and $a^+ \notin U$. Then (*) yields $U \subseteq \{a^+\}^{>}$ and $U = X \uparrow U = (\{a\} \cup \{a^+\}^{>}) \uparrow U = \{a^+\}^{>} \uparrow U$ hence $U = \{a^+\}^{>} \uparrow U$, i.e. $\{a^+\}^{>} \preceq U$. As an upper bound of the final segment $\{a^+\}^{>}$, the set U is also a final segment (cf. Lemma 16).
Thus, in all cases the set U is a final segment. Since all upper bounds U of X are final segments they are pairwise $\preceq$-comparable (by Lemma 4 and Proposition 11).

This proves that property $\Phi(X)$ is true.

$\Rightarrow$. We first show that condition $\Phi(X)$ implies that $X^{\geq} \setminus X$ has at most one element. By way of contradiction, assume there exist distinct $b, c \in X^{>} \setminus X$. Without loss of generality for some $a \in X$ we have $a < b < c$. Then $X \prec X \cup \{b\}$ and $X \prec X \cup \{c\}$ and b, c respectively witness that $(X \cup \{b\}) \uparrow (X \cup \{c\})$ is different from $X \cup \{b\}$ and $X \cup \{c\}$ which shows that $X \cup \{b\}$ and $X \cup \{c\}$ are incomparable, contradicting condition $\Phi(X)$.
At this point we know that if X is not a final segment but satisfies Φ then $X = X^{\geq} \setminus \{b\}$ where $b > a$ for some $a \in X$.
We claim that a is the minimum element of X. By way of contradiction, suppose $c \in X$ is such that $c < a$. Letting $U = X^{\geq} = X \cup \{b\}$ and $V = X \cap \{c\}^{\geq}$, we have $X \uparrow U = U$ and $X \uparrow V = V$ whereas $U \uparrow V = \{c\}^{\geq}$ is different from both U and V. Thus, U, V are incomparable upper bounds of X, contradicting $\Phi(X)$.
We now know that $X = \{a\}^{\geq} \setminus \{b\}$ where $a < b$. We claim that b is the successor in L of this minimum element a of X. By way of contradiction, suppose c is such that $a < c < b$. Letting $U = \{a\}^{\geq}$ and $V = X \setminus \{a\} = \{a\}^{>} \setminus \{b\}$, we again have $X \uparrow U = U$ and $X \uparrow V = V$ whereas $U \uparrow V = \{a\}^{\geq} \uparrow (\{a\}^{>} \setminus \{b\}) = \{a\}^{>}$ because $b = b \vee c \in \{a\}^{\geq} \uparrow (\{a\}^{>} \setminus \{b\})$. Thus, $U \uparrow V$ is different from both U and V hence U, V are incomparable upper bounds of X, contradicting $\Phi(X)$. □

Lemma 30. *The predicate X is a final segment is $\Sigma_2 \vee \Pi_2$. In case L has a minimum element it is Π_2.*

Proof. First, we consider the special case where L has a minimum element. The idea is to define the final segments X by saying: for all $\{0, a\}$, we have $\{0, a\} \uparrow X = X$, a property which is expressible by the Π_2 formula

$$\forall Y, Z\ ((Z = \{0\} \wedge \texttt{Suc}(Z, Y)) \Rightarrow Y \uparrow X = X)\,.$$

Assume X is a final segment, i.e. $X = X^{\geq}$. We have $\{0, a\} \uparrow X = \{0, a\} \uparrow X^{\geq} = X \cup (\{a\} \uparrow X^{\geq})$. Now, if $a \in X$ then $\{a\} \uparrow X = X \cap \{a\}^{\geq} \subseteq X$ and if $a \notin X = X^{\geq}$ then all elements of X dominate a hence $\{a\} \uparrow X = X$. In both cases, we see that $\{0, a\} \uparrow X = X$. Assume X is not a final segment. Then there exists $a < b$ with $a \in X$, $b \notin X$. Since $b = b \uparrow a \in \{0, b\} \uparrow X$ we see that $\{0, b\} \uparrow X \neq X$.

We now make no assumption on whether or not L has a minimum element. Consider the Π_1 predicate $\Phi(X)$ from Lemma 30. Rephrasing this last Lemma, there are three different possibilities for the set X to satisfy Φ :

1. $X = L$
2. X is a final segment different from L
3. $X = \{a\}^{\geq} \setminus \{a^{+}\}$, where a^{+} is the L-successor of a.

We discriminate case 3 from cases 1 and 2 as follows:
Case 1i: $X = L$ and there is no minimum element in L.
Then L has no immediate successor.

Case 1ii: $X = L$ and L has a minimum element 0 which admits a successor 0^+. Then $L \setminus \{0^+\} = \{0\}^{\geq} \setminus \{0^+\}$ is a strict predecessor of X which satisfies Φ.
Case 1iii: $X = L$ and L has a minimum element 0 which is right limit.
Then L has an immediate successor $L \setminus \{0\}$ which has no immediate successor.
Case 2. X is a final segment different from L
Then L is a strict predecessor of X which satisfies Φ.
Case 3. $X = \{a\}^{\geq} \setminus \{a^+\}$, where a^+ is the L-successor of a. Then X satisfies the following two properties:

(α) X has an immediate successor (namely, $\{a\}^{\geq}$) which itself has an immediate successor (namely $\{a\}^{>}$), unlike Cases 1i and 1ii,
(β) X has no strict predecessor which satisfies Φ, unlike Cases 1ii and 2.

Indeed, concerning (β), every (not necessarily immediate) predecessor Y of X is of one of the following two forms:

i. $Y = X \setminus Z$ with $\emptyset \neq Z \subseteq \{a^+\}^{>}$.

ii. $Y = Z \cup T$ where $\emptyset \neq Z$ and $Z \cap X^{\geq} = \emptyset$ and $T \subseteq X$.

Consequently, Y is not a final segment and $Y^{\geq} \setminus Y$ contains two elements except if (case (ii)) Y is of the form $Y^{\geq} \setminus \{a^+\}$. In this last case either Y has no minimum or it has a minimum and a^+ is not its immediate successor in L.

This proves that the $\Sigma_2 \vee \Pi_2$ formula

$$\Phi(X) \ \wedge\ \neg\Big((\exists U, V\ (\mathtt{Suc}(X,U) \wedge \mathtt{Suc}(U,V))\ \wedge\ \forall Y \prec X\ \neg\Phi(Y)\Big)$$

expresses that X is a final segment. □

Corollary 31. *The predicate $X = L$ is Π_3.*

Proof. Lemma 17 insures that L is the $\preceq$-minimal final segment:

$$X \text{ is final } \wedge\ \forall Y\ (Y \text{ is final} \Rightarrow X \preceq Y)$$

Since the predicate "is final" is $\Sigma_2 \vee \Pi_2$ this formula is Π_3. □

5.2 Upwards closure

Given $X \subseteq L$ we recall that $X^{\geq} = \{x \in L \mid \exists y \in X, y \leq x\}$.

Lemma 32. *The relation $\{(X,Y) \mid Y = X^{\geq}\}$ is Π_3.*

Proof. Observe that $X^{\geq}$ is the $\preceq$-minimum final set Z such that $X \preceq Z$. Thus, $Y = X^{\geq}$ is Π_3 expressible:

$$Y \text{ is final} \wedge\ X \preceq Y\ \wedge\ \forall Z\ ((Z \text{ is final} \wedge X \preceq Z) \Rightarrow Y \preceq Z)$$ □

5.3 Membership when L has a minimum element 0

Lemma 33. *For all $a \in L$ and $X \subseteq L$ it holds*

$$a \notin X^{>} \iff \{a\} \uparrow X = X.$$

Proof. By Lemma 18 the condition $\{a\} \uparrow X = X$ implies $X \subseteq \{a\}^{\geq}$, i.e., $a \notin X^{>}$. Conversely, $a \notin X^{>}$ implies $\emptyset = \{a\} \cap X^{>} \subseteq X \subseteq \{a\}^{\geq}$ and we conclude by the same lemma. □

Lemma 34. *For all $a \in L$ and $X \subseteq L$ we have*

$$a \in X \iff a \in X^{\geq} \ \wedge \ (\{0,a\} \uparrow X = X)$$

Proof. $\Rightarrow$. If $a \in X$ then $a \in X^{\geq}$ and $\{a\} \uparrow X \subseteq X$ hence $\{0,a\} \uparrow X = X \cup (\{a\} \uparrow X) = X$.
$\Leftarrow$. By contraposition it suffices to show that if $a \notin X$ and $a \in X^{\geq}$ then $\{0,a\} \uparrow X \neq X$. But this is clear since then $a \in \{a\} \uparrow X$ and a fortiori $a \in \{0,a\} \uparrow X$ whereas $a \notin X$. □

Theorem 35. *Assume L has a minimum element 0. Then the following membership predicate is Δ_4*

$$\texttt{IsIn}(A,X) \qquad \equiv \qquad A = \{a\} \text{ for some } a \in X$$

Proof. Let $\varphi(A,Z,U,X,Y)$ be the Π_3 conjunction of the formulas expressing that $A = \{a\}$ and $Z = \{0,a\}$ for some a (which is $\Sigma_2 \wedge \Pi_2$ by Proposition 28) and the formulas expressing that $U = A^{\geq}$ and $Y = X^{\geq}$ (which are Π_3 by Lemma 32). Observe that $a \in X^{\geq}$ if and only if $\{a\}^{\geq} \uparrow X^{\geq} = \{a\}^{\geq}$. Using Lemma 34, $\texttt{IsIn}(A,X)$ can be expressed by the following Σ_4 and Π_4 formulas:

$$\exists Z,U,Y \ (\varphi(A,Z,U,X,Y) \wedge \ U \uparrow Y = U \ \wedge \ Z \uparrow X = X)$$
$$\forall Z,U,Y \ (\varphi(A,Z,U,X,Y) \Rightarrow (U \uparrow Y = U \ \wedge \ Z \uparrow X = X))$$ □

5.4 Membership in the general case

The definition of membership we are looking for is based on the following characterization.

Lemma 36. *Let $a \in L$ and $X \subseteq L$. The following three conditions are equivalent:*

1. $a \in X$
2. *either* $\big(\{a\}^{\geq} \subsetneq X^{\geq}$ *and* $\forall z \in (X^{\geq} \setminus \{a\}^{\geq}) \ \ X \uparrow \{z\} = X \uparrow \{z,a\}\big)$
   or $\{a\}^{\geq} = X^{\geq}$
3. *either* $\big(\{a\}^{\geq} \subsetneq X^{\geq}$ *and* $\exists z \in (X^{\geq} \setminus \{a\}^{\geq}) \ \ X \uparrow \{z\} = X \uparrow \{z,a\}\big)$
   or $\{a\}^{\geq} = X^{\geq}$

Proof. (1) ⇒ (2). Assume $a \in X$. Then $\{a\} \subseteq X$ hence $\{a\}^{\geq} \subseteq X^{\geq}$. If $\{a\}^{\geq} = X^{\geq}$ then we are done so we assume $\{a\}^{\geq} \subsetneq X^{\geq}$. Let $z \notin \{a\}^{\geq}$, i.e. $z < a$. Since $a \in X$ we have equality $X \uparrow \{a\} = X \cap \{a\}^{\geq}$ and since $z < a$ we have $X \cap \{a\}^{\geq} \subseteq X \cap \{z\}^{\geq} \subseteq X \uparrow \{z\}$. Thus, $X \uparrow \{a\} \subseteq X \uparrow \{z\}$ and

$$X \uparrow \{z\} \ \subseteq \ X \uparrow \{z,a\} \ = \ (X \uparrow \{z\}) \cup (X \uparrow \{a\}) \ = \ X \uparrow \{z\}$$

which implies $X \uparrow \{z\} = X \uparrow \{z,a\}$. This proves the first disjunct in the expression of point 2 (even a little more since we do not need the constraint $z \in X^{\geq}$).
(2) ⇒ (3). Trivial.
¬(1) ⇒ ¬(3). Assume $a \notin X$. Since $\{a\}^{\geq}$ and $X^{\geq}$ are final segments, Lemma 4 insures that $\{a\}^{\geq}$ and $X^{\geq}$ are comparable for inclusion.
Case $\{a\}^{\geq} = X^{\geq}$. Then a is the minimum element of X hence $a \in X$, contradiction.
Case $X^{\geq} \subsetneq \{a\}^{\geq}$. Then (3) trivially fails (as wanted).
Case $\{a\}^{\geq} \subsetneq X^{\geq}$. Let $z \in X^{\geq} \setminus \{a\}^{\geq}$. Then there exists $b \in X$ such that $b \leq z < a$. We have $a = \max\{b,a\} \in X \uparrow \{z,a\}$ whereas $a \notin X \uparrow \{z\}$ (since $a \notin X$ and $z < a$). Thus, $X \uparrow \{z,a\} \neq X \uparrow \{z\}$ and (3) fails. □

Theorem 37. *The following membership predicate is* Δ_4

$$\texttt{IsIn}(A,X) \qquad \equiv \qquad A = \{a\} \textit{ for some } a \in X$$

Proof. Let $\alpha(T,U)$ be a Π_3 formula expressing that $U = T^{\geq}$ (cf. Lemma 32). Recall that, for final segments F, G we have $F \subseteq G$ if and only if $G \preceq F$ (cf. Lemma 17). Also, $z \in F$ if and only if $\{z\}^{\geq} \subseteq F$ if and only if $F \preceq \{z\}^{\geq}$. Let $\theta^{\exists}(X,Y,A,U)$ and $\theta^{\forall}(X,Y,A,U)$ be the following Σ_4 and Π_4 formulas

$$\begin{aligned} &\exists Z,V,P\ \big(\texttt{Single}(Z) \ \wedge\ \alpha(Z,V) \ \wedge\ Y \preceq Z \ \wedge\ U \not\preceq Z \ \wedge \texttt{Pair}(Z,A,P) \\ &\hspace{18em} \wedge X \uparrow Z = X \uparrow P\big) \\ &\forall Z,V,P\ \big(\texttt{Single}(Z) \ \wedge\ \alpha(Z,V) \ \wedge\ Y \preceq Z \ \wedge\ U \not\preceq Z \ \wedge \texttt{Pair}(Z,A,P) \\ &\hspace{18em} \Longrightarrow X \uparrow Z = X \uparrow P\big) \end{aligned}$$

Recall that
- if F is a final segment then $z \in F \iff \{z\}^{\geq} \subseteq F$,
- $\texttt{Pair}(Z,A,P)$ means that $Z = \{z\}$, $A = \{a\}$ and $P = \{z,a\}$ for some $z < a$.
Let $\alpha(Z,V)$ be a Π_3 formula expressing that $V = Z^{\geq}$ (cf. Lemma 32) and let $\theta^{\exists}(X,Y,A,U)$ and $\theta^{\forall}(X,Y,A,U)$ be the following Σ_4 and Π_4 formulas

$$\begin{aligned} &\exists Z,V,P\ (\texttt{Pair}(Z,A,P) \wedge \alpha(Z,V) \wedge\ Y \preceq Z \ \wedge\ U \not\preceq Z \ \wedge\ X \uparrow Z = X \uparrow P) \\ &\forall Z,V,P\ (\texttt{Pair}(Z,A,P) \wedge \alpha(Z,V) \wedge\ Y \preceq Z \ \wedge\ U \not\preceq Z) \Longrightarrow X \uparrow Z = X \uparrow P) \end{aligned}$$

which, applied to $Y = X^{\geq}$, $A = \{a\}$ and $U = \{a\}^{\geq}$ express respectively

$$\begin{aligned} &\exists z \in (X^{\geq} \setminus \{a\}^{\geq})\ \ X \uparrow \{z\} = X \uparrow \{z,a\} \\ &\forall z \in (X^{\geq} \setminus \{a\}^{\geq})\ \ X \uparrow \{z\} = X \uparrow \{z,a\} \end{aligned}$$

Let $\Phi(X, Y, A, U)$ be the Π_3 conjunction of $\alpha(X, Y)$ and $\alpha(A, U)$. Using the $\Sigma_2 \wedge \Pi_2$ predicate `Single` from Theorem 26, consider the Σ_4 and Π_4 formulas

$$\texttt{Single}(A) \wedge\ \exists Y, U\ \big(\Phi(X, Y, A, U) \wedge ((Y \prec U\ \ \wedge \theta^{\exists}(X, Y, A, U))\,)\ \vee\ U = Y\big)$$
$$\texttt{Single}(A) \wedge\ \forall Y, U\ \big(\Phi(X, Y, A, U) \Rightarrow ((Y \prec U\ \ \wedge \theta^{\forall}(X, Y, A, U))\,)\ \vee\ U = Y\big)$$

Conditions (2) and (3) of Lemma 36 show that these formulas define the predicate `IsIn`. □

6 Final proofs

6.1 Defining the downarrow operation with uparrow

Proposition 38. *The predicate $X \downarrow Y = Z$ is Π_5.*

Proof. Recall the $\Sigma_2 \wedge \Pi_2$ predicate $\texttt{Leq} = \{(\{a\}, \{b\}) \mid a \leq b\}$ (cf. Lemma 27). Let $\theta(A, B, C)$ be the $\Sigma_2 \vee \Pi_2$ formula expressing that $A = \{a\}$, $B = \{b\}$ and $C = \{\min(a, b)\}$, for some $a, b \in L$:

$$(\texttt{Leq}(A, B) \Rightarrow C = A) \wedge (\texttt{Leq}(B, A) \Rightarrow C = B)$$

Using a Σ_4 definition of `IsIn` (cf. Theorem 37), the following formula is Π_5

$$\forall C\ (\texttt{IsIn}(C, Z) \Rightarrow \exists A, B\ (\texttt{IsIn}(A, X) \wedge \texttt{IsIn}(B, Y) \wedge \theta(A, B, C)))$$
$$\wedge\ \forall A, B, C\ (\texttt{IsIn}(A, X) \wedge \texttt{IsIn}(B, Y) \wedge \theta(A, B, C) \Rightarrow \texttt{IsIn}(C, Z))$$

and defines the predicate $X \downarrow Y = Z$.

6.2 Defining the uparrow operation with the order

Proposition 39. *The $\uparrow$ operation is Π_1 definable in $\langle \mathcal{P}(L); \preceq \rangle$.*

Proof. Proposition 14 insures that $\uparrow$ is the join operation in $\langle \mathcal{P}(L); \preceq \rangle$. Thus, the Π_1 formula

$$\forall U\ ((X \preceq U\ \ \wedge\ \ Y \preceq U) \iff Z \preceq U)$$

is a definition of the predicate $X \uparrow Y = Z$ in $\langle \mathcal{P}(L); \preceq \rangle$. □

6.3 Equivalent first-order structures

With the notion of equivalence of structure defined in paragraph 2.2, we may state the main result. We also give, at no cost, an easy extension by considering not only the $(x, y) \mapsto \max\{x, y\}$ function lifted to sets but also the $(x, y) \mapsto \min\{x, y\}$ function lifted to sets.

Theorem 40. *For a given linear ordering L the three structures*

$$\mathcal{S}_1 = \langle \mathcal{P}(L); =, \uparrow \rangle \qquad \mathcal{S}_2 = \langle \mathcal{P}(L); =, \downarrow \rangle \qquad \mathcal{S}_3 = \langle \mathcal{P}(L); =, \preceq \rangle$$

are first-order interpretable one from each other and, in each of them, one can define a structure isomorphic to

$$\mathcal{S}_4 = \langle L, \mathcal{P}(L); =_L, <, \in \rangle$$

for the isomorphism mapping a subset of L to itself and an element $a \in L$ to the singleton set $\{a\}$.

Proof. Theorem 37 and Lemma 27 show that the map $x \mapsto \{x\}$ and $X \mapsto X$ defines an isomorphism between the multisorted structure $\mathcal{S}_4$ and a multisorted structure $\mathcal{S}'_1$ expressible in $\mathcal{S}_1$:

$$\mathcal{S}'_1 = \langle U, \mathcal{P}(L); \mathrm{Eq}, \mathtt{Leq}, \mathtt{IsIn} \rangle$$

where $\begin{cases} U = \{X \mid X \subseteq L, \mathtt{Single}(X)\} \\ \mathrm{Eq} = \{(X,Y) \mid \mathtt{Single}(X) \wedge \mathtt{Single}(Y) \wedge X = Y\} \\ \mathtt{Leq} = \{(X,Y) \mid \mathtt{Single}(X) \wedge \mathtt{Single}(Y) \wedge X \prec Y\} \\ \mathtt{IsIn} = \{(X,Y) \mid \mathtt{Single}(X) \wedge \mathtt{IsIn}(X,Y)\} \end{cases}$.

Proposition 14 (and the fact that $\preceq$ is defined with $\uparrow$) shows that the operation of $\mathcal{S}_1$ is interpretable in $\mathcal{S}_3$ and vice-versa. Thus, $\mathcal{S}_1$ and $\mathcal{S}_3$ are equivalent.

Proposition 38 shows that the operation of $\mathcal{S}_2$ is interpretable in $\mathcal{S}_1$. Observing that $\uparrow$ and $\downarrow$ considered in the reverse linear order $(L, \geq)$ are respectively $\downarrow$ and $\uparrow$ in $(L, \leq)$, we see that the operation of $\mathcal{S}_1$ is interpretable in $\mathcal{S}_2$. Thus, $\mathcal{S}_1$ and $\mathcal{S}_2$ are equivalent.

References

1. Julius Richard Büchi. *On a decision problem in restricted second-order arithmetic.* Proc. 1960 International Congress for Logic, Methodology and Philosophy, p. 1-11. Stanford Univ. Press, 1962.
2. Julius Richard Büchi. The monadic second-order theory of all countable ordinals. *Lecture Notes in Math.*, 328:1-127. Springer, 1973.
3. Julius Richard Büchi and Charles Zaiontz. Deterministic automata and the monadic theory of ordinals $< \omega_2$. *Zeitschrift für math. Logik und Grundlagen der Mat..*, 29:313-336, 1983.
4. Christian Choffrut and Serge Grigorieff. Logical theory of the additive monoid of subsets of natural integers. In *Automata, Universality, Computation. A tribute to Maurice Margenstern*, p. 39-74. Springer Verlag, 2015.
5. Christian Choffrut and Serge Grigorieff. Logical theory of the monoid of languages over a nontally alphabet. *Fundamenta Informaticae*, 138(1-2):159–177, 2015.
6. Yuri Gurevich. Elementary properties of ordered abelian groups. *Algebra and Logic*, 3(1):5-39, 1964. English version in *AMS Translations*, 46:165-192, 1965.
7. Yuri Gurevich. Monadic theory of order and topology. *Israel Journal of Mathematics*, 34:45-71, 1979.

8. Yuri Gurevich, Menachem Magidor and Saharon Shelah. The monadic theory of ω_2. *The Journal of Symbolic Logic*, 48(2):387-398, 1983.
9. Yuri Gurevich and Saharon Shelah. Modest theory of short chains II. *The Journal of Symbolic Logic*, 44:491-502, 1979.
10. Yuri Gurevich and Saharon Shelah. Monadic theory of order and topology in ZFC. *Annals of Mathematical Logic*, 23:179-198, 1982.
11. Yuri Gurevich and Saharon Shelah. Interpreting second-order logic in the monadic theory of order. *The Journal of Symbolic Logic*, 48:816-828, 1983.
12. Yuri Gurevich and Saharon Shelah. The monadic theory and the 'next world'. *Israel Journal of Mathematics*, 49:55-68, 1984.
13. Shmuel Lifsches and Saharon Shelah. The monadic theory of ω_2 may be complicated. *Archive for Mathematical Logic*, 31:207-213, 1992.
14. Joseph G. Rosenstein. *Linear orderings.* Academic Press, 1982.
15. Saharon Shelah. The monadic theory of order. *Annals of Mathematics*, 102:379–419, 1975.