Kolmogorov complexities K_{max}, K_{min} on computable partially ordered sets

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Contents

Abstract

We introduce a machine free mathematical framework to get a natural formalization of some general notions of infinite computation in the context of Kolmogorov complexity. Namely, the classes $Max_{PR}^{\mathbb{X}\to\mathcal{D}}$ and $Max_{Rec}^{\mathbb{X}\to\mathcal{D}}$ of functions $\mathbb{X} \to \mathcal{D}$ which are pointwise maximum of partial or total computable sequences of functions where $\mathcal{D} = (D, <)$ is some computable partially ordered set. The enumeration theorem and the invariance theorem always hold for $Max_{PR}^{\mathbb{X}\to\mathcal{D}}$, leading to a variant $K_{max}^{\mathcal{D}}$ of Kolmogorov complexity. We characterize the orders $\mathcal{D}$ such that the enumeration theorem (resp. the invariance theorem) also holds for $Max_{Rec}^{\mathbb{X}\to\mathcal{D}}$. It turns out that $Max_{Rec}^{\mathbb{X}\to\mathcal{D}}$ may satisfy the invariance theorem but not the enumeration theorem. Also, when $Max_{Rec}^{\mathbb{X}\to\mathcal{D}}$ satisfies the invariance theorem then the Kolmogorov complexities associated to $Max_{Rec}^{\mathbb{X}\to\mathcal{D}}$ and $Max_{PR}^{\mathbb{X}\to\mathcal{D}}$ are equal (up to a constant).
Letting $K_{min}^{\mathcal{D}} = K_{max}^{\mathcal{D}^{\mathrm{rev}}}$, where $\mathcal{D}^{\mathrm{rev}}$ is the reverse order, we prove that

either $K^{\mathcal{D}}_{min} =_{\rm ct} K^{\mathcal{D}}_{max} =_{\rm ct} K^D$ ($=_{\rm ct}$ is equality up to a constant) or $K^{\mathcal{D}}_{min}, K^{\mathcal{D}}_{max}$ are $\leq_{\rm ct}$ incomparable and $<_{\rm ct} K^D$ and $>_{\rm ct} K^{\emptyset',D}$. We characterize the orders leading to each case. We also show that $K^{\mathcal{D}}_{min}, K^{\mathcal{D}}_{max}$ cannot be both much smaller than K^D at any point.
These results are proved in a more general setting with two orders on D, one extending the other.

1 Introduction

1.1 Non halting programs for which the current output is eventually the wanted object, but one does not know when...

In this paper, we consider a particular kind of description methods in order to define variants of Kolmogorov complexity. Let's start with two paradigmatic examples. Given $n \in \mathbb{N}$ and $u \in \Sigma^*$ (where Σ be some finite alphabet), how do we get

- the value $BB(n)$ of the busy beaver function $BB : \mathbb{N} \to \mathbb{N}$,
- the value $K_{\Sigma^*}(u)$ of Kolmogorov complexity $K_{\Sigma^*} : \Sigma^* \to \mathbb{N}$?

The definitions of $BB(n)$ and $K_{\Sigma^*}(u)$ lead to the following mechanisms.

- run all Turing machines with $\leq n$ states and all programs with length $\leq n$,
- for each t, consider those machines and programs halting in $\leq t$ steps,
- look at the maximum number of cells visited by these machines,
- look at the minimum length of these programs.

In this way, one gets two computable functions $bb : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and $k : \Sigma^* \times \mathbb{N} \to \mathbb{N}$, with one more integer argument (for time steps) such that, for every fixed $n \in \mathbb{N}$ and $u \in \Sigma^*$, the maps $t \mapsto bb(n,t)$ and $t \mapsto k(u,t)$ are respectively monotone increasing and decreasing and are both eventually constant with respective values $BB(n)$ and $K_{\Sigma^*}(u)$. Since neither BB nor K_{Σ^*} is computable, there is no computable functions of n or u which bound the moment these maps become constant.

These examples lead us to introduce the following notion of description methods for objects of a partially ordered set $\mathcal{D}$ with a computable structure (cf. Definition 3, 4).
A computable approximation from below (resp. from above) of objects of $\mathcal{D}$ is a program for a computable function $f : \mathbb{X} \times \mathbb{N} \to \mathcal{D}$ (where $\mathbb{X}$ is some reasonable set such as $\mathbb{N}$ or $\mathbf{2}^*$, cf. §1.5 Notations) such that, for every fixed $x \in \mathbb{X}$, the map $t \mapsto f(x,t)$ is monotone increasing (resp. decreasing) and eventually constant. Nothing is assumed about the moment $t \mapsto f(x,t)$

becomes constant: there may be no computable function of x majorizing it. The associated decompressor — or description method — is the function $F : \mathbb{X} \to \mathcal{D}$ such that $F(x)$ is the limit value of $f(x,t)$ when $t \to +\infty$, i.e. the maximum (resp. minimum) value of the finite set $\{f(x,t) : t \in \mathbb{N}\}$. We shall call such functions F computably approximable from below (resp. from above).

Consider Σ^* with the prefix ordering. The context of non halting (hence infinite) computations, cf. Chaitin, 1975 [6], and Solovay, 1977 [21], leads to functions $F : \mathbf{2}^* \to \Sigma^*$ which are computably approximable from below. In fact, if the output alphabet is Σ, the current output $f(x,t)$ at time t is a function $f : \mathbf{2}^* \times \mathbb{N} \to \Sigma^*$ which is a computable approximation from below for words in Σ^* such that $F(x)$ is the max of the $f(x,t)$'s .

Observe that, in case the ordered set $\mathcal{D}$ is noetherian (resp. well-founded), the notion of approximation from below (resp. from above) of objects of $\mathcal{D}$ reduces to that of computable function $f : \mathbb{X} \times \mathbb{N} \to \mathcal{D}$ which is monotone increasing (resp. decreasing) with respect to its second argument. This is indeed the case with the approximation from above of the values of K_{Σ^*} since $(\mathbb{N}, <)$ is well-founded. Cf. also §2.4.6.

Other examples are developped in §2.4. In particular, there is one involving quotients of regular languages by a fixed computably enumerable language.

1.2 Functions approximable from below (resp. from above) as decompressors for variants of Kolmogorov complexity

The above mentioned context of non halting computations has recently led to interesting variants $K^{\infty}_{\Sigma^*} : \Sigma^* \to \mathbb{N}$, $K^{\infty}_{\mathbb{N}} : \mathbb{N} \to \mathbb{N}$ of Kolmogorov complexity introduced (in their prefix-complexity version H^{∞}) in Becher & Chaitin [2, 1], and developped in [3, 4].
This last Kolmogorov complexity $K^{\infty}_{\mathbb{N}}$ has also proved to be equal to the Kolmogorov complexity K_{card} introduced in Ferbus & Grigorieff, 2002 [11, 10] where we compare some natural set theoretical semantics of integers, namely Church iterators of functions, cardinals of computably enumerable sets, indexes of computably enumerable equivalence relations. Comparison of these semantics is done via associated Kolmogorov complexities which somehow constitute measures of their "abstraction degree" and are defined in terms of infinite or/and oracular computations.

The cornerstone of Kolmogorov complexity, namely the invariance theorem, really deals with partial computable functions, not Turing machines. In fact,

Turing machines do not constitute such an abstract structured mathematical framework as partial computable functions do. Going to this last framework opens new natural considerations which would not be simply viewed with Turing machines.
In this paper we abstract from non halting computations on Turing machines and develop a general machine-free mathematical framework using a partially ordered set $\mathcal{D}$. Namely, letting $\mathbb{X}$ be a basic space (cf. §1.5 Notations), we introduce the classes of functions $F : \mathbb{X} \to \mathcal{D}$

$$Max_{PR}^{\mathbb{X}\to\mathcal{D}} \quad , \quad Min_{PR}^{\mathbb{X}\to\mathcal{D}}$$

which are partial computably approximable from below (resp. from above). Which means that the $f : \mathbf{2}^* \times \mathbb{N} \to \mathcal{D}$ such that $F(x)$ is the max or min of the $f(x,t)$'s is partial computable rather than computable.
Of course, the *Min* classes are the *Max* classes associated to the reverse order.

We also introduce the subclasses of functions

$$Max_{Rec}^{\mathbb{X}\to\mathcal{D}} \quad , \quad Min_{Rec}^{\mathbb{X}\to\mathcal{D}}$$

which are computably approximable from below (resp. from above). It happens that the $Max_{Rec}^{\mathbb{X}\to\mathcal{D}}$ class is closely related with the class based on non halting Turing machines computations with outputs in $\mathcal{D}$ (modulo adequate coding of $\mathcal{D}$).

As for the above examples, the busy beaver function $BB : \mathbb{N} \to \mathbb{N}$ is in $Max_{Rec}^{\mathcal{D}}$and Kolmogorov complexity $K_{\Sigma^*} : \Sigma^* \to \mathbb{N}$ and its prefix-free variant $H_{\Sigma^*} : \Sigma^* \to \mathbb{N}$ are in $Min_{Rec}^{\mathcal{D}}$ with $\mathcal{D} = (\mathbb{N}, <)$.

These classes lead to new variants of Kolmogorov complexity which would just be ignored when considering Turing machines.

1.3 Main theorems

The development of Kolmogorov complexities $K_{max}^{\mathcal{D}}$, $K_{min}^{\mathcal{D}}$ associated to the classes $Max_{PR}^{\mathbf{2}^*\to\mathcal{D}}$ and $Min_{PR}^{\mathbf{2}^*\to\mathcal{D}}$ is straightforward (cf. §3). The main results of the paper deal with the comparison of $K_{max}^{\mathcal{D}}$, $K_{min}^{\mathcal{D}}$ with the classical Kolmogorov complexity K^D and its relativized version $K^{\emptyset',D}$ to oracle $\emptyset'$. In §4, we prove three theorems which give the main comparison relations (relative to the "up to a constant" order $\leq_{\mathrm{ct}}$, cf. §1.5 Notations) between these complexities.

The first theorem (Thm.24) is valid whatever be the partial order on D. It states that $K^{\emptyset',D} <_{\rm ct} \inf(K^{\mathcal{D}}_{max}, K^{\mathcal{D}}_{min})$ and that $K^{\mathcal{D}}_{max}$, $K^{\mathcal{D}}_{min}$, though obviously $\leq_{\rm ct} K^D$, cannot be simultaneously much smaller than K^D since

$$K^D \leq_{\rm ct} (K^{\mathcal{D}}_{max} + \log(K^{\mathcal{D}}_{max})) + (K^{\mathcal{D}}_{min} + \log(K^{\mathcal{D}}_{min}))$$

The second theorem (Thm.25) proves that either $K^{\mathcal{D}}_{max} =_{\rm ct} K^{\mathcal{D}}_{min} =_{\rm ct} K^D$ or $K^{\mathcal{D}}_{max}$,$K^{\mathcal{D}}_{min}$ are $\leq_{\rm ct}$ incomparable and both are $<_{\rm ct}$ to K^D. This dichotomy is also characterized by a simple property on the order.

The third theorem (Thm.26) considers two partial orders $<_{wk}$ and $<_{st}$ on D, the second extending the first. We give conditions $(*)$ and $(**)$ on the orders such that
- $(*)$ insures that $K^{\mathcal{D}_{st}}_{max} =_{\rm ct} K^{\mathcal{D}_{wk}}_{max}$ and $K^{\mathcal{D}_{st}}_{min} =_{\rm ct} K^{\mathcal{D}_{wk}}_{min}$,
- $(**)$ insures that $K^{\mathcal{D}_{st}}_{max} <_{\rm ct} K^{\mathcal{D}_{wk}}_{max}$ and $K^{\mathcal{D}_{st}}_{max} <_{\rm ct} K^{\mathcal{D}_{wk}}_{max}$ and neither $K^{\mathcal{D}_{st}}_{max}$ nor $K^{\mathcal{D}_{st}}_{min}$ is $\leq_{\rm ct} \min(K^{\mathcal{D}_{wk}}_{max}, K^{\mathcal{D}_{wk}}_{min})$.

These conditions are almost complementary: $(**)$ is an effective version of the negation of $(*)$.
An interesting case of this theorem is obtained when $\Sigma = \{1, ..., k\}$ with the obvious order and $<_{wk}$, $<_{st}$ are the prefix and the lexicographic orders on Σ^* (the last one being isomorphic to the order on k-adic rational reals in $[0, 1]$).

1.4 The $Max_{\boldsymbol{Rec}}$ and $Min_{\boldsymbol{Rec}}$ classes

In §5.2 and 5.3 we come back to the four classes $Max^{\mathbf{2}^*\to\mathcal{D}}_{PR}$, $Min^{\mathbf{2}^*\to\mathcal{D}}_{PR}$ and $Max^{\mathbf{2}^*\to\mathcal{D}}_{Rec}$, $Min^{\mathbf{2}^*\to\mathcal{D}}_{Rec}$. We compare them to that of partial computable functions $\mathbb{X} \to \mathcal{D}$ and look at the syntactical complexity of their domains and graphs.
In §5.4 we compute $Max^{\mathbb{X}\to\mathcal{D}}_{PR} \cap Min^{\mathbb{X}\to\mathcal{D}}_{PR}$ under simple conditions about the partial order on D.

In §6, we consider the possible development of Kolmogorov complexities based on the classes $Max^{\mathbf{2}^*\to\mathcal{D}}_{Rec}$ and $Min^{\mathbf{2}^*\to\mathcal{D}}_{Rec}$. This leads to look at the two following problems:
- the existence of an enumeration,
- the invariance theorem.

For each problem, we characterize the orders $\mathcal{D}$ for which there is a positive answer (cf. §6.1, 6.2).
It turns out (cf. Thm.43) that when the invariance theorem holds for $Max^{\mathbf{2}^*\to\mathcal{D}}_{Rec}$ then every function in $Max^{\mathbf{2}^*\to\mathcal{D}}_{PR}$ has an extension (not necessarily total) in $Max^{\mathbf{2}^*\to\mathcal{D}}_{Rec}$. This insures that the Kolmogorov complexities

associated to $Max^{\mathbf{2}^*\to\mathcal{D}}_{Rec}$ and $Max^{\mathbf{2}^*\to\mathcal{D}}_{PR}$ coincide. In particular, $K^{\infty}_{\Sigma^*}$ is the complexity associated to $Max^{\mathbf{2}^*\to\mathcal{D}}_{Rec}$ and $Max^{\mathbf{2}^*\to\mathcal{D}}_{PR}$ when $\mathcal{D}$ is Σ^* with the prefix order.
Surprisingly, there are orders such that the invariance theorem holds for $Max^{\mathbf{2}^*\to\mathcal{D}}_{Rec}$ whereas the enumeration theorem fails (compare Thm.42 and Thm.43).

1.5 Notations

1. Equality, inequality and strict inequality up to a constant between total functions $\mathbb{S} \to \mathbb{N}$ are denoted as follows:

$$\begin{array}{lcl}
f \ \leq_{\mathrm{ct}} \ g & \Leftrightarrow & \exists c\ \forall \mathtt{s}\ f(\mathtt{s}) \leq g(\mathtt{s}) + c \\
f \ =_{\mathrm{ct}} \ g & \Leftrightarrow & f \leq_{\mathrm{ct}} g \ \wedge \ g \leq_{\mathrm{ct}} f \quad \Leftrightarrow \quad \exists c\ \forall \mathtt{s}\ |f(\mathtt{s}) - g(\mathtt{s})| \leq c \\
f \ <_{\mathrm{ct}} \ g & \Leftrightarrow & f \leq_{\mathrm{ct}} g \ \wedge \ \neg(g \leq_{\mathrm{ct}} f) \quad \Leftrightarrow \quad f \leq_{\mathrm{ct}} g \ \wedge \ \forall c\ \exists \mathtt{s}\ g(\mathtt{s}) > f(\mathtt{s}) + c
\end{array}$$

2. *[Basic spaces]* $\mathbf{2}^*$ denotes the set of binary words. We call basic spaces the products of non empty finite families of spaces of the form $\mathbb{N}$ or $\mathbb{Z}$ or A^* where A is some finite alphabet. Basic spaces are denoted by $\mathbb{S}, \mathbb{X}, \mathbb{Y}, ...$

3. *[Partial recursive (or computable) functions]* $PR^{\mathbb{X}\to\mathbb{Y}}$ (resp. $Rec[\mathbb{X} \to \mathbb{Y}]$) denotes the family of partial (resp. total) computable functions from $\mathbb{X}$ to $\mathbb{Y}$.

2 The Max and Min classes of functions

2.1 Infinite computations and monotone machines

Recall that a Turing machine is monotone if its current output may only increase with respect to the prefix order on words: no overwriting is allowed. This is indeed Turing's original assumption [22], insuring that, in the limit of time, the output of a non halting computation always converges, either to a finite or to an infinite sequence. This concept was also considered by Levin [16] and Schnorr [18, 19], see [17] p.276. Such infinite computations with possibly infinite outputs can be used to obtain highly random reals, cf. Becher & Chaitin [2, 1] and Becher & Grigorieff [5].
In this paper, when considering infinite computations, we retain the sole limit outputs that are finite.

The following easy proposition links infinite computations, as considered for the definition of K^{∞} and its prefix version H^{∞} introduced in [1, 4], with the general approach which is the subject of this paper.

Proposition 1. *Let $F : \mathbf{2}^* \to \Sigma^*$ where Σ is some non empty finite alphabet. The following conditions are equivalent:*

i. F can be computed via possibly infinite computations on some monotone Turing machine with output alphabet Σ, according to the following convention: $F(\mathtt{s})$ is defined if and only if the output remains constant after some step.

ii. There exists a total computable function $f : \mathbf{2}^* \times \mathbb{N} \to \Sigma^*$ such that
- $f(\mathtt{s}, t)$ is monotone increasing in t with respect to the prefix order on Σ^*,
- $\mathtt{s} \in dom(F)$ if and only if $\{f(\mathtt{s}, t) : t \in \mathbb{N}\}$ is finite and non empty,
- $F(\mathtt{s})$ is the maximum value of $\{f(\mathtt{s}, t) : t \in \mathbb{N}\}$.

iii. Let λ denote the empty word. Idem as ii, with f such that

$$f(\mathtt{s}, 0) = \lambda \quad , \quad f(\mathtt{s}, t+1) \in \{f(\mathtt{s}, t)\} \cup \{f(\mathtt{s}, t)\sigma : \sigma \in \Sigma\}$$

Proof. $iii \Rightarrow ii$ is trivial; $i \Leftrightarrow iii$: let $f(\mathtt{s}, t)$ be the current output at time t when the input is $\mathtt{s}$. As for $ii \Rightarrow iii$, let $\widetilde{f}(\mathtt{s}, 0) = \lambda$ and $\widetilde{f}(\mathtt{s}, t+1)$ be the prefix of $f(\mathtt{s}, t+1)$ with length $\min(|\widetilde{f}(\mathtt{s}, t)| + 1, |f(\mathtt{s}, \mathtt{t+1})|)$. Then $\{\widetilde{f}(\mathtt{s}, t) : t \in \mathbb{N}\}$ and $\{f(\mathtt{s}, t) : t \in \mathbb{N}\}$ are simultaneously finite or infinite and, when finite, their maximum elements are equal. □

2.2 Mathematical modelization: the Max and Min classes

Proposition 1 and the argumentation in §1.1–1.2 invite to a mathematical, machine-free modelization of the notion of function defined by infinite computations. Namely that of function obtained as pointwise maximum of a computable sequence of total computable functions. *A construction which makes sense for maps from a basic set $\mathbb{X}$ into any computable partially ordered set $\mathcal{D} = (D, <)$*, and leads to the class $Max^{\mathbb{X}\to\mathcal{D}}_{Rec}$.
It is also quite natural – *in fact, it is even much more natural from a mathematical point of view* – to consider the version of the above modelization using *partial computable functions* instead of total computable ones. This leads to the class $Max^{\mathbb{X}\to\mathcal{D}}_{PR}$.
Natural and interesting important examples (cf. §2.4) are obtained when $\mathcal{D}$ is among the following (obviously computable) partially ordered sets:

$$(\mathbb{N}, <) \quad , \quad (\mathbb{Z}, <) \quad , \quad (\Sigma^*, <_{prefix}) \quad , \quad (\Sigma^*, <_{lexico})$$

and the reverse orders obtained by replacing $<$ by $>$, where $<_{lexico}$ on Σ^* depends on a total or partial order on the alphabet Σ.

Definition 2 (The $\max^{\mathcal{D}}$ and $\min^{\mathcal{D}}$ operators). *Let $\mathbb{X}$ be some basic set and $\mathcal{D} = (D, <)$ be some partially ordered set. Let $f : \mathbb{X} \times \mathbb{N} \to D$ be monotone increasing in its second argument on its domain. We define $\max^{\mathcal{D}} f : \mathbb{X} \to D$ (resp. $\min^{\mathcal{D}} f : \mathbb{X} \to D$) as the function*

i. defined on the $\mathtt{x}$'s in $\mathbb{X}$ for which the map $t \mapsto f(\mathtt{x}, t)$ has finite non empty range,

ii. and such that $(\max^{\mathcal{D}} f)(\mathtt{x})$ (resp. $(\min^{\mathcal{D}} f)(\mathtt{x})$) is the maximum (resp. minimum) element of $\{f(\mathtt{x}, t) : t \in \mathbb{N}\}$.

Definition 3.
1. A computable partially ordered set $\mathcal{D}$ is a triple $(D, <, \rho)$ such that $\rho : \mathbb{N} \to D$ is a bijective total map (in particular, D is infinite countable) and $<$ is a partial order on D such that $\{(m, n) : \rho(m) < \rho(n)\}$ is computable.
2. Let $\mathbb{X}$ be a basic space. A function $F : \mathbb{X} \to D$ is partial (resp. total) computable if so is $\rho^{-1} \circ F : \mathbb{X} \to \mathbb{N}$.
A set $Z \subseteq \mathbb{X} \times D^k$ is computable if so is $(Id_{\mathbb{X}}, \rho, ..., \rho)^{-1}(Z)$ as a subset of $\mathbb{X} \times \mathbb{N}^k$, where $Id_{\mathbb{X}}$ is the identity function on $\mathbb{X}$.

Of course, we shall omit any reference to ρ when $\mathcal{D}$ is $\mathbb{N}$ or $\mathbb{Z}$ with the natural order, or Σ^* with the prefix or the lexicographic order (with respect to some partial or total order of the elements of Σ).

Definition 4 (Max and Min classes). *Let $\mathbb{X}$ be a basic space and $\mathcal{D} = (D, <, \rho)$ be a computable partially ordered set. We let*

$$\begin{aligned} Max_{Rec}^{\mathbb{X}\to\mathcal{D}} &= \{\max^{\mathcal{D}} f : f : \mathbb{X} \times \mathbb{N} \to D \textit{ is total computable}\} \\ Max_{PR}^{\mathbb{X}\to\mathcal{D}} &= \{\max^{\mathcal{D}} f : f : \mathbb{X} \times \mathbb{N} \to D \textit{ is partial computable}\} \end{aligned}$$

We respectively denote by $Min_{PR}^{\mathbb{X}\to\mathcal{D}}$ and $Min_{Rec}^{\mathbb{X}\to\mathcal{D}}$ the analog classes defined with the $\min^{\mathcal{D}}$ operator, i.e. the classes $Max_{PR}^{\mathbb{X}\to\mathcal{D}^{rev}}$ and $Max_{PR}^{\mathbb{X}\to\mathcal{D}^{rev}}$ where $\mathcal{D}^{rev} = (D, >)$.

Proposition 1 can be rephrased in terms of the prefix ordering on Σ^*.

Proposition 5. *If Σ is a finite alphabet then $Max_{Rec}^{\mathbf{2}^*\to(\Sigma^*, <_{prefix})}$ is the class of functions computed via possibly infinite computations on monotone Turing machines (cf. Proposition 1 i) with Σ as output alphabet.*

2.3 Domains of functions in the Max/Min classes

We denote by $\Sigma^0_1 \wedge \Pi^0_1$ the family of conjunctions of Σ^0_1 and Π^0_1 formulas. Let $\mathbb{X}$ be a basic set and $\mathcal{D}$ be a computable ordered set. The arithmetical hierarchy on $\mathbb{N}$ induces a hierarchy on D and $\mathbb{X} \times D$: a relation $R \subseteq \mathbb{X} \times D$ is Σ^0_n or Π^0_n or $\Sigma^0_n \wedge \Pi^0_n$ if so is $(Id_{\mathbb{X}}, \rho)^{-1}(R) \subseteq \mathbb{X} \times \mathbb{N}$.

Proposition 6. *Let $\mathbb{X}$ be a basic set and $\mathcal{D}$ be a computable ordered set. Every partial function in $Max^{\mathbb{X}\to\mathcal{D}}_{PR}$ or in $Min^{\mathbb{X}\to\mathcal{D}}_{PR}$ has $\Sigma^0_1 \wedge \Pi^0_1$ graph and Σ^0_2 domain.*

Proof. Let $f : \mathbb{X} \times \mathbb{N} \to D$ be partial computable, monotone increasing in its second argument on its domain. Then

$$\begin{array}{rcl} (\max^{\mathcal{D}} f)(\mathtt{x}) = z & \Leftrightarrow & \exists t\ (f(\mathtt{x}, t) \text{ is defined } \wedge\ f(\mathtt{x}, t) = z) \\ & & \wedge\ \forall t\ (f(\mathtt{x}, t) \text{ is defined } \Rightarrow\ f(\mathtt{x}, t) \leq z) \\ \mathtt{x} \in dom(\max^{\mathcal{D}} f) & \Leftrightarrow & \exists z\ F(\mathtt{x}) = z \end{array}$$

Idem with $Min^{\mathbb{X}\to\mathcal{D}}_{PR}$. $\square$

2.4 Examples of functions in the Max and Min classes

The classes $Max^{\mathbb{X}\to\mathcal{D}}_{Rec}, Min^{\mathbb{X}\to\mathcal{D}}_{Rec}$ contain many fundamental non computable functions. To see that some functions are not in such classes, we shall use Theorem 40 below (the proof of which does not depend on any result of this §).

2.4.1 Kolmogorov and Chaitin-Levin program-size complexities

Proposition 7. *Let $\mathcal{D}$ be $(\mathbb{N}, <)$. Kolmogorov and Chaitin-Levin program-size complexities $K_{\mathbb{N}}, H_{\mathbb{N}} : \mathbb{N} \to \mathbb{N}$ (resp.$K_{\Sigma^*}, H_{\Sigma^*} : \Sigma^* \to \mathbb{N}$) are in $Min^{\mathbb{N}\to\mathcal{D}}_{Rec} \setminus Max^{\mathbb{N}\to\mathcal{D}}_{PR}$ (resp. in $Min^{\Sigma^*\to\mathcal{D}}_{Rec} \setminus Max^{\Sigma^*\to\mathcal{D}}_{PR}$).*

Proof. That K, H belong to $Min^{\mathbb{N}\to\mathcal{D}}_{Rec}$ is a mere reformulation of the well-known fact that they are computably approximable from above, i.e. they are limits of decreasing computable sequences of total computable functions. That these *total* functions are not in $Max^{\mathbb{N}\to\mathcal{D}}_{PR}$ is an obvious application of Theorem 40 below. $\square$

2.4.2 Busy beaver

Proposition 8. *Let $\mathcal{D}$ be $(\mathbb{N}, <)$. Let $BB : \mathbb{N} \to \mathbb{N}$ be the busy beaver function, i.e. $BB(n)$ is the maximum number of cells visited by the input*

head of a Turing machine with $n+1$ states which halts with no input.
Then $BB \in Max^{\mathbb{N}\to\mathcal{D}}_{Rec} \setminus Min^{\mathbb{N}\to\mathcal{D}}_{PR}$.

Proof. Observe that $BB = \max bb$ where bb is the total computable function such that $bb(n,t)$ is the maximum among 0 and the numbers of cells visited by Turing machines with $n+1$ states which halt in at most t steps.
An obvious application of Theorem 40 below shows that BB is not in $Min^{\mathbb{N}\to\mathcal{D}}_{PR}$. □

Remark 9. Variants of the busy beaver function can be very naturally defined with ranges over various types of data structures. For instance, finite graphs relative to the inclusion or embedding ordering.

2.4.3 Cardinality of finite computably enumerable sets

The following example is completely investigated in [11, 10].

Proposition 10. *Let $\mathcal{D}$ be $(\mathbb{N},<)$. Let $cardRE : \mathbb{N} \to \mathbb{N}$ be such that*

$$cardRE(n) \quad = \quad \begin{cases} card(W_n) & \text{if } W_n \text{ is finite} \\ \text{undefined} & \text{otherwise} \end{cases}$$

where $card(W_n)$ is the number of elements of the computably enumerable set W_n with code n.
Then $cardRE \in Max^{\mathbb{N}\to\mathcal{D}}_{Rec} \setminus Min^{\mathbb{N}\to\mathcal{D}}_{PR}$.

Proof. Observe that $cardRE = \max h$ where $h(n,t)$ is total computable and counts the number of elements of W_n obtained after t computation steps.
The domain of the partial function $cardRE$ is known to be Σ^0_2 complete, hence not $\Sigma^0_1 \wedge \Pi^0_1$. Applying Theorem 40 below, we see that $cardRE$ cannot be in $Min^{\mathbb{N}\to\mathcal{D}}_{PR}$. □

2.4.4 Interacting finite sets with a fixed computably enumerable set

If $X, Y \subseteq \mathbb{N}$, let's denote $X - Y$ and $X \setminus Y$ the sets

$$X - Y = \{x - y : x \in X \wedge y \in Y \wedge x \geq y\} \ , \ X \setminus Y = \{z : z \in X \wedge z \notin Y\}$$

Proposition 11. *Let $\mathcal{D}$ be be the family $P_{<\omega}(\mathbb{N})$ of finite subsets of $\mathbb{N}$, ordered by set inclusion. If $A \subseteq \mathbb{N}$ is a fixed computably enumerable set which is non computable then*

i. the maps $X \mapsto X \cap A$ and $X \mapsto X - A$ are in $Max^{\mathcal{D}\to\mathcal{D}}_{Rec} \setminus Min^{\mathcal{D}\to\mathcal{D}}_{PR}$.

ii. the map $X \mapsto X \setminus A$ is in $Min^{\mathcal{D}\to\mathcal{D}}_{Rec} \setminus Max^{\mathcal{D}\to\mathcal{D}}_{PR}$.

Proof. Let $A = \varphi(\mathbb{N})$ where $\varphi : \mathbb{N} \to \mathbb{N}$ is total computable. Define total computable maps $f, g, h : \mathcal{D} \times \mathbb{N} \to D$ such that

$$\begin{array}{rclcrcl} f(X,t) & = & X \cap \varphi(\{0,...,t\}) & \qquad & g(X,t) & = & X - \varphi(\{0,...,t\}) \\ h(X,t) & = & X \setminus \varphi(\{0,...,t\}) & & & & \end{array}$$

It is easy to see that $X \cap A = (\max^{\mathcal{D}} f)(X)$ and $X - A = (\max^{\mathcal{D}} g)(X)$ and $X \setminus A = (\min^{\mathcal{D}} h)(X)$. $\square$

2.4.5 Quotients of regular languages by a fixed computably enumerable language

We now come to a very different example.

The family *Reg* of regular languages over alphabet Σ can be defined by regular expressions which are words in the alphabet $\widetilde{\Sigma}$ obtained by enriching Σ with symbols $+, *, \cdot, (,)$.
Let $\zeta : \widetilde{\Sigma}^* \to Reg$ be the surjective map such that, if u is a regular expression then $\zeta(u)$ is the associated regular language, else $\zeta(u) = \emptyset$.
Since equality of regular languages is decidable, there exists a computable map $\eta : \mathbb{N} \to \widetilde{\Sigma}^*$ such that $\rho = \zeta \circ \eta : \mathbb{N} \to Reg$ is bijective.
Using decidability of inclusion of regular languages, we see that $(Reg, \subseteq, \rho)$ is a computable partially ordered set in the sense of Definition 3.

It is known that, if L is a regular language and $M \subseteq \Sigma^*$ is any language (even non computable) then

$$M^{-1}L = \{u \in \Sigma^* : \exists v \in M \ vu \in L\}$$

is always regular and $M^{-1}L = M'^{-1}L$ for some finite subset $M' \subseteq M$. Recall the core of the easy proof: if L is the set of words leading from state q_0 to a final state of automaton $\mathcal{A}$ and if the words in M lead from state q_0 to the states in X, then $M^{-1}L$ is the set of words leading from a state in X to a final state.

Proposition 12. *Let $M \subseteq \Sigma^*$ be a fixed computably enumerable language which is non computable. Let $F_M : Reg \to Reg$ be such that $F_M(L) = M^{-1}L$. Then F_M is in $Max^{Reg\to Reg}_{Rec} \setminus Min^{Reg\to Reg}_{PR}$.*

Proof. Let $M = \varphi(\mathbb{N})$ where $\varphi : \mathbb{N} \to \Sigma^*$ is a total computable function. Observe that $F_M = \max^{Reg} f_M$ where $f_M : Reg \times \mathbb{N} \to Reg$ is such that

$$f_M(L,t) = (\varphi(\{0,...,t\})^{-1}L$$

Observe that M is computable with oracle F since $u \in M$ if and only if $M^{-1}\{u\} = \{\lambda\}$. Since M is not computable, F cannot be computable. Using Theorem 40 point 1 (and the fact that F is total), we see that F is not in $Min_{PR}^{Reg \to Reg}$. □

Using the above surjection $\zeta : \widetilde{\Sigma}^* \to Reg$, one can reformulate the above result in terms of a partial computable preordering on words quite different of the usual ones. This necessitates a straightforward extension to preorderings of the material about the Max and Min classes.
Let $\mu : Reg \to \Delta^*$ be the map which associates to a regular language L the regular expression (obtained via some fixed algorithm) describing its minimal automaton. Observe that ζ is a retraction of the injective map μ, i.e. $\zeta \circ \mu$ is the identity map on Reg.

Proposition 13. *Let $\mathcal{D}$ be $\widetilde{\Sigma}^*$ with the following computable preordering:*

$$u \preceq v \;\Leftrightarrow\; \zeta(u) \subseteq \zeta(v)$$

Let $M \subseteq \widetilde{\Sigma}^*$ be a fixed computably enumerable language which is non re-cucomputablersive. Then the map $u \mapsto \mu(M^{-1}\zeta(u))$ (which maps a regular expression for L to one for $M^{-1}L$) is in $Max_{Rec}^{\mathcal{D} \to \mathcal{D}} \setminus Min_{PR}^{\mathcal{D} \to \mathcal{D}}$.

Proof. Let F, f be as in the proof of Proposition 12. Since $\zeta \circ \mu = Id_{Reg}$, we see that $\widetilde{F} = \mu \circ F \circ \zeta$ makes the following diagram commute:

$$\begin{tikzcd} Reg \arrow[r, "F"] & Reg \\ \widetilde{\Sigma}^* \arrow[u, "\zeta"] \arrow[r, "\widetilde{F}"] & \widetilde{\Sigma}^* \arrow[u, "\zeta"'] \end{tikzcd}$$

which allows to transfer the results of Proposition 12. □

2.4.6 Noetherian or well-founded orderings

Suppose $\mathcal{D}$ is Noetherian (resp. well-founded) and let $f : \mathbb{X} \times \mathbb{N} \to D$. If $t \mapsto f(x,t)$ is monotone increasing (resp. decreasing) then it is necessarily eventually constant. In that case, the considered notion of approximation from below (resp. from above) coincides with monotone approximation.
Fix $n \geq 1$. An important case is the noetherian set $(\mathcal{D}, \subseteq)$ of ideals in the ring of n-variables polynomials with real algebraic coefficients (this last hypothesis insures that $\mathcal{D}$ is countable with a computable ordering).

2.5 Normalized representations

It sometimes proves useful to normalize the f in $\max^{\mathcal{D}} f$.

Proposition 14. *Let $\mathbb{X}$ be a basic set and $\mathcal{D} = (D, <, \rho)$ be a computable ordered set.*
1. Every $F \in Max_{PR}^{\mathbb{X}\to\mathcal{D}}$ is of the form $F = \max^{\mathcal{D}} f$ for some partial computable $f : \mathbb{X} \times \mathbb{N} \to D$, monotone increasing in its second argument, such that $dom(f) = Z \times \mathbb{N}$ where Z is some Σ_1^0 subset of D.

2. If $F \in Max_{PR}^{\mathbb{X}\to\mathcal{D}}$ has Σ_1^0 domain then one can suppose $Z = dom(F)$.

Proof. 1. Let $g : \mathbb{X} \times \mathbb{N} \to D$ be partial computable, monotone increasing in its second argument, such that $F = \max^{\mathcal{D}} g$. Let $Z = \{\mathtt{x} : \exists t\ (\mathtt{x}, t) \in dom(g)\}$ be the first projection of $dom(g)$. Let $\theta : \mathbb{X} \to D$ be the partial computable function with domain Z such that $\theta(\mathtt{x})$ is the value first obtained in $\{g(\mathtt{s}, t) : t \in \mathbb{N}\}$ by dovetailing over computations of $g(\mathtt{s}, 0), g(\mathtt{s}, 1), \ldots$.
Let also

$$\Delta_{\mathtt{x},t} = \{g(\mathtt{x}, u) : u \leq t \wedge\ g(\mathtt{x}, u) \text{ halts in } \leq t \text{ steps}\}$$

and define f with domain $Z \times \mathbb{N}$ such that $f(\mathtt{x}, t)$ is the greatest element of $\{\theta(\mathtt{x})\} \cup \Delta_{\mathtt{x},t}$

2. Observe that Z necessarily contains $dom(F)$. If $dom(F)$ is Σ_1^0 then $\widehat{f} = f \upharpoonright (dom(F) \times \mathbb{N})$ is also partial computable and $\max^{\mathcal{D}} \widehat{f} = \max^{\mathcal{D}} f$. $\square$

3 Kolmogorov complexities $K^{\mathcal{D}}_{\boldsymbol{max}}$, $K^{\mathcal{D}}_{\boldsymbol{min}}$

Kolmogorov complexity theory goes through with the $Max_{PR}^{\mathbb{X}\to\mathcal{D}}$ and $Min_{PR}^{\mathbb{X}\to\mathcal{D}}$ classes with no difficulty.
First, we recall Kolmogorov complexity over elements of $\mathcal{D}$.

3.1 Kolmogorov complexity K^D

Classical Kolmogorov complexity for elements in D is defined as follows (cf. Kolmogorov, 1965 [13], or Li & Vitanyi [17], Downey & Hirschfeldt [8], Gàcs [12] or Shen [20]).

Definition 15. *Let $\varphi : \mathbf{2}^* \to D$. We denote $K_\varphi : D \to \mathbb{N}$ the partial function with domain $range(\varphi)$ such that*

$$K_\varphi(d) = \min\{|\mathtt{p}| : \varphi(\mathtt{p}) = d\}$$

I.e., considering words in $\mathbf{2}^*$ *as programs,* $K_\varphi(d)$ *is the shortest length of a program* $\mathtt{p}$ *mapped onto* d *by* φ.

Theorem 16 (Invariance theorem, Kolmogorov, 1965 [13]). *Let* $\mathbb{X}$ *be a basic space and* $\mathcal{D} = (D, <, \rho : \mathbb{N} \to D)$ *be a computable partially ordered set. When* φ *varies in the family* $PR^{\mathbf{2}^* \to D}$ *of partial computable functions* $\mathbf{2}^* \to D$*, there is a least* K_φ*, up to an additive constant:*

$$\exists \varphi \in PR^{\mathbf{2}^* \to D} \quad \forall \psi \in PR^{\mathbf{2}^* \to D} \quad K_\varphi \leq_{\mathrm{ct}} K_\psi$$

Such φ*'s are said to be optimal in* $PR^{\mathbf{2}^* \to D}$.

Definition 17. *Kolmogorov complexity* $K^D : D \to \mathbb{N}$ *is* K_φ *where* φ *is some fixed optimal function in* $PR^{\mathbf{2}^* \to D}$*. Thus,* K^D *is defined up to an additive constant.*

Of course, K^D and $K^{\mathbb{N}}$ are related.

Proposition 18. $K^D \circ \rho =_{\mathrm{ct}} K^{\mathbb{N}}$.

Proof. Since $PR^{\mathbf{2}^* \to D} = \{\rho \circ \psi : \psi \in PR^{\mathbf{2}^* \to \mathbb{N}}\}$ and $K_\psi(n) = K_{\rho \circ \psi}(\rho(n))$ for all $\psi \in PR^{\mathbf{2}^* \to \mathbb{N}}$, we see that if φ is optimal in $PR^{\mathbf{2}^* \to \mathbb{N}}$ then $\rho \circ \varphi$ is optimal in $PR^{\mathbf{2}^* \to D}$. ☐

We also observe the following simple fact:

Proposition 19. $\sup\{K^D(d) : d \in X\} = +\infty$ *for every infinite* $X \subseteq D$.

Proof. The result is well-known for $K^{\mathbb{N}}$ and it transfers to K^D using Proposition 18. ☐

3.2 Enumeration theorem for $Max^{\mathbb{X} \to \mathcal{D}}_{\boldsymbol{PR}}$

The classical enumeration theorem for partial computable functions goes through the max operator, leading to an enumeration of $Max^{\mathbb{X} \to \mathcal{D}}_{PR}$. First, we recall a folklore result on enumeration of *monotone* partial computable functions.

Proposition 20. *Let* $\mathbb{X}$ *be a basic set and* $\mathcal{D} = (D, <, \rho)$ *be a computable ordered set. Let* $PR^{\mathbb{X} \times \mathbb{N} \to D, \uparrow}$ *be the family of partial computable functions* $\mathbb{X} \times \mathbb{N} \to D$ *which are monotone increasing in their last argument. There exists a partial computable function* $\psi : \mathbb{N} \times \mathbb{X} \times \mathbb{N} \to D$ *such that*

$$\{\psi_n : n \in \mathbb{N}\} = PR^{\mathbb{X} \times \mathbb{N} \to \mathcal{D}, \uparrow}$$

where $\psi_n : \mathbb{X} \times \mathbb{N} \to D$ *denotes the function* $(\mathtt{x}, t) \mapsto \psi(n, \mathtt{x}, t)$.

Proof. Let $\phi : \mathbb{N} \times \mathbb{X} \times \mathbb{N} \to D$ be a partial computable function which enumerates the family $PR^{\mathbb{X}\times\mathbb{N}\to D}$ of partial computable functions $\mathbb{X} \times \mathbb{N} \to D$, i.e,

$$\{\phi_n : n \in \mathbb{N}\} = PR^{\mathbb{X}\times\mathbb{N}\to D}$$

We modify ϕ to ψ so as to get an enumeration of $PR^{\mathbb{X}\times\mathbb{N}\to D,\uparrow}$. Consider an injective computable enumeration $(n_i, \mathtt{x}_i, t_i, d_i)_{i\in\mathbb{N}}$ of the graph of ϕ. Let

$$Z = \{(n_i, \mathtt{x}_i, t_i, d_i) : \forall j < i \ (n_j = n_i \wedge x_j = x_i \wedge t_j < t_i \Rightarrow d_j \leq d_i)\}$$

Let $\psi : \mathbb{N} \times \mathbb{X} \times \mathbb{N} \to D$ be the partial computable function with graph Z. It is clear that ψ is monotone increasing in its last argument, so that so are all ψ_n's. Also, if ϕ_n is monotone increasing in its last argument then $\{n\} \times graph(\phi_n)$ is included in Z, so that $\psi_n = \phi_n$. Thus, the ψ_n's enumerate $PR^{\mathbb{X}\times\mathbb{N}\to D,\uparrow}$. □

Theorem 21 (Enumeration theorem for $Max_{PR}^{\mathbb{X}\to\mathcal{D}}$). *Let $\mathbb{X}$ be a basic set and $\mathcal{D} = (D, <, \rho)$ be a computable ordered set. There exists a function $E : \mathbb{N} \times \mathbb{X} \to D$ in $Max_{PR}^{\mathbb{N}\times\mathbb{X}\to D}$ such that*

$$\{E_n : n \in \mathbb{N}\} = Max_{PR}^{\mathbb{X}\to\mathcal{D}}$$

where $E_n : \mathbb{X} \to D$ denotes the function satisfying $E_n(\mathtt{x}) = E(n, \mathtt{x})$.

Proof. Let $\psi : \mathbb{N} \times \mathbb{X} \times \mathbb{N} \to D$ be a partial computable function which enumerates $PR^{\mathbb{X}\times\mathbb{N}\to D,\uparrow}$. Let $E : \mathbb{N}\times\mathbb{X} \to D$ be such that $E_n = \max^{\mathcal{D}} \psi_n$ for all n. For any $F : \mathbb{X} \to D$ in $Max_{PR}^{\mathbb{X}\to\mathcal{D}}$ there exists n such that $F = \max^{\mathcal{D}} \psi_n$. We then have

$$\begin{array}{rcl}
\mathtt{x} \in dom(F) & \Leftrightarrow & \{\psi_n(\mathtt{x}, t) : t \text{ s.t. } \psi_n(\mathtt{x}, t) \text{ is defined}\} \text{ is finite non empty} \\
& \Leftrightarrow & \{\psi(n, \mathtt{x}, t) : t \text{ s.t. } \psi(n, \mathtt{x}, t) \text{ is defined}\} \text{ is finite non empty} \\
& \Leftrightarrow & (n, \mathtt{x}) \in dom(E) \\
& \Leftrightarrow & \mathtt{x} \in dom(E_n) \\
F(\mathtt{x}) & = & \text{greatest element of } \{\psi_n(\mathtt{x}, t) : t \text{ s.t. } \psi_n(\mathtt{x}, t) \text{ is defined}\} \\
& = & \text{greatest element of } \{\psi(n, \mathtt{x}, t) : t \text{ s.t. } \psi(n, \mathtt{x}, t) \text{ is defined}\} \\
& = & E(n, \mathtt{x}) \\
& = & E_n(\mathtt{x})
\end{array}$$

Which proves that E enumerates $Max_{PR}^{\mathbb{X}\to\mathcal{D}}$. □

3.3 Kolmogorov complexity $K^{\mathcal{D}}_{max}$ and $K^{\mathcal{D}}_{min}$

The invariance theorem extends easily to $Max_{PR}^{\mathbf{2}^*\to\mathcal{D}}$, leading to Kolmogorov complexity $K^{\mathcal{D}}_{max} : D \to \mathbb{N}$.

Theorem 22 (Invariance theorem for $Max_{PR}^{\mathbf{2}^*\to\mathcal{D}}$). *Let $\mathbb{X}$ be a basic space and $\mathcal{D} = (D, <, \rho : \mathbb{N} \to D)$ be a computable partially ordered set. When F varies in the family $Max_{PR}^{\mathbf{2}^*\to\mathcal{D}}$ there is a least K_F, up to an additive constant:*

$$\exists U \in Max_{PR}^{\mathbf{2}^*\to\mathcal{D}} \quad \forall F \in Max_{PR}^{\mathbf{2}^*\to\mathcal{D}} \quad K_U \leq_{\mathrm{ct}} K_F$$

Such U's are said to be optimal in $Max_{PR}^{\mathbf{2}^*\to\mathcal{D}}$.

Proof. The usual proof works. Let $E : \mathbb{N} \times \mathbf{2}^* \to D$ in $Max_{PR}^{\mathbb{N}\times\mathbf{2}^*\to\mathcal{D}}$ be an enumeration of $Max_{PR}^{\mathbf{2}^*\to\mathcal{D}}$. Define $U : \mathbf{2}^* \to D$ such that $U(0^n 1p) = E(n, p)$ and $U(q)$ is undefined if q is not of the form $0^n 1p$ for some $n \in \mathbb{N}$ and $p \in \mathbf{2}^*$. If $F \in Max_{PR}^{\mathbf{2}^*\to\mathcal{D}}$ and $F = E_n$ then

$$\begin{array}{rcl}
K_F(d) & = & \min\{|\mathtt{p}| : F(\mathtt{p}) = d\} \\
& = & \min\{|\mathtt{p}| : E(n, \mathtt{p}) = d\} \\
& = & \min\{|\mathtt{p}| : U(0^n 1p) = d\} \\
& = & \min\{|0^n 1p| : U(0^n 1p) = d\} - n - 1 \\
& \geq & \min\{|\mathtt{q}| : U(\mathtt{q}) = d\} - n - 1 \\
& = & K_U(d) - n - 1
\end{array}$$

□

Definition 23. *Kolmogorov complexity $K^{\mathcal{D}}_{max}$ is defined up to an additive constant as any K_U where U is optimal in $Max_{PR}^{\mathbf{2}^*\to\mathcal{D}}$.*
Kolmogorov complexity $K^{\mathcal{D}}_{min}$ is $K^{\mathcal{D}'}_{max}$ where $\mathcal{D}'$ is the reverse order of $\mathcal{D}$.

4 Main theorems: comparing K, $K^{\mathcal{D}}_{max}$, $K^{\mathcal{D}}_{min}$, $K^{\emptyset'}$

4.1 The $<_{\mathrm{ct}}$ hierarchy theorem

The main motivation of this section is to compare the Kolmogorov complexities

$$K^D,\ K^{\mathcal{D}}_{max},\ K^{\mathcal{D}}_{min},\ K^{\emptyset'} : D \to \mathbb{N}$$

Comparisons of $K^{\mathcal{D}}_{max}, K^{\mathcal{D}}_{min}$ and K^D turn out to be a particular application of more general results dealing with both $K^{\mathcal{D}}_{max}$ and $K^{\mathcal{D}}_{min}$ complexities relative to two computable orders $\mathcal{D}_{st} = (D, <_{st}, \rho)$ and $\mathcal{D}_{wk} = (D, <_{wk}, \rho)$ on the same set D, the strong one $<_{st}$ being an extension of the weak one $<_{wk}$. A question with naturally arises when considering for instance the prefix and lexicographic orders on Σ^*.

In the case of $\mathbb{N}$ with the natural order or of Σ^* with the prefix order, the inequalities $K^{\emptyset',D} <_{\mathrm{ct}} K^{\mathcal{D}}_{max} <_{\mathrm{ct}} K^D$ were obtained (modulo Proposition 5) for the prefix version H^∞ in Becher & Chaitin, 2002 [1], see also Becher & Figueira & Nies & Picci, 2005 [4].
We state our results as three theorems, the proofs of which are given in §4.5 to 4.10.

Theorem 24 (1st hierarchy theorem). *Let $\mathcal{D} = (D, <, \rho)$ be a computable ordered set.*
1. $K^{\emptyset',D} <_{\mathrm{ct}} \inf(K^{\mathcal{D}}_{max}, K^{\mathcal{D}}_{min})$
2. $K^{\mathcal{D}}_{max}, K^{\mathcal{D}}_{min}$ *are $\leq_{\mathrm{ct}}$ smaller than K^D but not simultaneously much smaller:*

$$K^D \leq_{\mathrm{ct}} (K^{\mathcal{D}}_{max} + \log(K^{\mathcal{D}}_{max})) + (K^{\mathcal{D}}_{min} + \log(K^{\mathcal{D}}_{min}))$$

Theorem 25 (2d hierarchy theorem).
1. If $(D, <)$ contains arbitrarily large finite chains then $K^{\mathcal{D}}_{max}$ and $K^{\mathcal{D}}_{min}$ are $\leq_{\mathrm{ct}}$ incomparable and both are $<_{\mathrm{ct}}$ smaller than K^D.
In fact, a much stronger property holds:

- *i.* K^D *is not majorized by a computable function of* $K^{\mathcal{D}}_{min}$,
- *ii.* K^D *is not majorized by a computable function of* $K^{\mathcal{D}}_{max}$,
- *iii.* $K^{\mathcal{D}}_{max}$ *is not majorized by a computable function of* $K^{\mathcal{D}}_{min}$,
- *iv.* $K^{\mathcal{D}}_{min}$ *is not majorized by a computable function of* $K^{\mathcal{D}}_{max}$,

I.e., for any total computable function $\alpha : \mathbb{N} \to \mathbb{N}$, the following sets are infinite

$$\begin{array}{ccc} \{d \in D : K^{\mathcal{D}}_{max}(d) \geq \alpha(K^{\mathcal{D}}_{min}(d))\} & , & \{d \in D : K^D(d) \geq \alpha(K^{\mathcal{D}}_{min}(d))\} \\ \{d \in D : K^{\mathcal{D}}_{min}(d) \geq \alpha(K^{\mathcal{D}}_{max}(d))\} & , & \{d \in D : K^D(d) \geq \alpha(K^{\mathcal{D}}_{max}(d))\} \end{array}$$

2. If $(D, <)$ does not contain arbitrarily large finite chains then

$$K^{\mathcal{D}}_{min} =_{\mathrm{ct}} K^{\mathcal{D}}_{max} =_{\mathrm{ct}} K^D$$

Theorem 26 (3d hierarchy theorem). *Let $\mathcal{D}_{st} = (D, <_{st}, \rho)$ and $\mathcal{D}_{wk} = (D, <_{wk}, \rho)$ be two computable orders on the same set D ("wk" and "st" stand for "weak" and "strong") such that $<_{st}$ is an extension of $<_{wk}$.*
1. Let $(*)$ be the following condition

(∗) *For all k there exists a strong chain with k elements which is a weak antichain.*

If (∗) holds then $K^{\mathcal{D}_{st}}_{max} <_{\rm ct} K^{\mathcal{D}_{wk}}_{max}$ and $K^{\mathcal{D}_{st}}_{min} <_{\rm ct} K^{\mathcal{D}_{wk}}_{min}$.
In fact, a much stronger property holds: $\inf(K^{\mathcal{D}_{wk}}_{min}, K^{\mathcal{D}_{wk}}_{max})$ is not majorized by a computable function of $K^{\mathcal{D}_{st}}_{max}$ or $K^{\mathcal{D}_{st}}_{min}$. I.e., for any total computable function $\alpha : \mathbb{N} \to \mathbb{N}$, the following sets are infinite

$$\{d \in D : \inf(K^{\mathcal{D}_{wk}}_{min}(d), K^{\mathcal{D}_{wk}}_{max}(d)) \geq \alpha(K^{\mathcal{D}_{st}}_{max}(d))\}$$

$$\{d \in D : \inf(K^{\mathcal{D}_{wk}}_{min}(d), K^{\mathcal{D}_{wk}}_{max}(d)) \geq \alpha(K^{\mathcal{D}_{st}}_{min}(d))\}$$

2. Let (∗∗) be the following condition (which is an effective version, tailored for infinite computations, of the negation of (∗), cf. §4.2).

(∗∗) *There exists k such that for every partial computable $f : \mathbf{2}^* \times \mathbb{N} \to D$ which is monotone increasing in its second argument relative to the strong order $<_{st}$ there exist partial computable functions $f_1, ..., f_k : \mathbf{2}^* \times \mathbb{N} \to D$ which are monotone increasing in their second argument relative to the weak order $<_{wk}$ such that*

$$\{f(p,t) : t \in \mathbb{N}\} = \bigcup_{i=1,\ldots,k} \{f_i(p,t) : t \in \mathbb{N}\}$$

If (∗∗) holds then $K^{\mathcal{D}_{st}}_{min} =_{\rm ct} K^{\mathcal{D}_{wk}}_{min}$ and $K^{\mathcal{D}_{st}}_{max} =_{\rm ct} K^{\mathcal{D}_{wk}}_{max}$.

Corollary 27. *Let Σ be a finite or infinite countable alphabet, let $<_1, <$ be computable orders on Σ such that $<_1$ is partial but non trivial and $<$ is a total extension of $<_1$. Consider on Σ^* the following orders: the prefix order $<_{prefix}$, the lexicographic orders $<_{lexico_1}$ and $<_{lexico}$ associated to $<_1$ and $<$. Then*

$$K^{\mathcal{D}_{prefix}}_{max} <_{\rm ct} K^{\mathcal{D}_{<_{lexico_1}}}_{max} <_{\rm ct} K^{\mathcal{D}_{<_{lexico}}}_{max} \quad , \quad K^{\mathcal{D}_{prefix}}_{min} <_{\rm ct} K^{\mathcal{D}_{<_{lexico_1}}}_{min} <_{\rm ct} K^{\mathcal{D}_{<_{lexico}}}_{min}$$

Proof. Let $a, b, c, d \in \Sigma$ be such that $a <_1 b$ and $c < d$ but $c \not<_1 d$. Since $<$ extends $<_1$, c and d are $<_1$ incomparable. Observe that $\{a^n b : n \in \mathbb{N}\}$ is an infinite increasing chain for $<_{lexico_1}$ and an antichain for the prefix order. Also, $\{c^n d : n \in \mathbb{N}\}$ is an infinite increasing chain for $<_{lexico}$ and an antichain for the $<_{lexico_1}$ order. This gives condition (∗) relative to the pairs $(<_{prefix}, <_{lexico_1})$ and $(<_{lexico_1}, <_{lexico})$ of orders on Σ^*. □

4.2 $(*)$ is an effective version of the negation of $(**)$

Recall Dilworth's theorem.

Theorem 28 (Dilworth, 1950 [7]). *Let $\mathcal{D} = (D, <)$ be an ordered set and $k \in \mathbb{N}$. If every antichain in D has at most k elements then D is the union of k chains.*

Dilworth's theorem leads to an equivalent form $(\dagger)$ of $(*)$ and condition $(**)$ appears as an effective version of $\neg(\dagger)$, tailored for infinite computations.

Proposition 29. *Let $\mathcal{D}_{st} = (D, <_{st})$ and $\mathcal{D}_{wk} = (D, <_{wk})$ be two orders on the same set D such that $<_{st}$ is an extension of $<_{wk}$.*
Then $(*)$ is equivalent to the following condition $(\dagger)$:

- $(\dagger)$ *For all k there exists a finite strong chain X which is not the union of k weak chains*

Proof. $(*) \Rightarrow (\dagger)$. Apply $(*)$ with $k+1$ and observe that a weak antichain with $k+1$ elements cannot be the union of k weak chains.
$\neg(*) \Rightarrow \neg(\dagger)$. Let k be an integer which contradicts $(*)$. Then, in any strong chain, any weak antichain has $< k$ elements. Apply Dilworth's theorem to get $\neg(\dagger)$. □

Remark 30. 1. Clearly $(**) \Rightarrow \neg(\dagger)$. We do not know whether the converse implication holds or not. The problem is that the proof of Dilworth's theorem is not incremental as we now detail. Let $X \cup \{d\}$ be a strong chain with $d >_{st} x$ for all $x \in X$ and such that every weak antichain included in X has at most k elements. If X is covered by k weak chains $C_1, ..., C_k$ then d may be incomparable to the top elements of all these k chains. Thus, though $X \cup \{d\}$ is also the union of k weak chains, such chains may be quite different from the C_i's.
Condition $(**)$ (as contrasted to $\neg(\dagger)$), does insure such an incremental character.
2. In case $<_{wk}$ has a smallest element d, condition $(**)$ is equivalent to the analog condition in which functions $f, f_1, ..., f_k$ are replaced by total computable $g, g_1, ..., g_k$. This can be seen by defining $g, g_1, ..., g_k$ from $f, f_1, ..., f_k$ as follows

$$g(p, 0) = d \quad , \quad g(p, t+1) = \begin{cases} f(p,t) & \text{if } f(p,t) \text{ converges in} \leq t \text{ steps} \\ g(p,t) & \text{otherwise} \end{cases}$$

and the same with $g_1, ..., g_k$ from $f_1, ..., f_k$.

4.3 $K^{\mathcal{D}}_{max}, K^{\mathcal{D}}_{min}$ are not simultaneously much smaller than K^D

Lemma 31. *Let $\mathcal{D} = (D, <, \rho)$ be a computable ordered set. Let $c : \mathbf{2}^* \times \mathbf{2}^* \to \mathbf{2}^*$ be a total computable injective map and let $J : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and $M \in \mathbb{N}$ be such that $|c(p, q)| \leq J(|p|, |q|) + M$ for all $p, q \in \mathbf{2}^*$. Then*

$$K^D \leq_{\rm ct} J(K^{\mathcal{D}}_{min}, K^{\mathcal{D}}_{max})$$

In particular (with the special convention $\log(0) = 0$*),*

$$K^D \leq_{\rm ct} (K^{\mathcal{D}}_{max} + \log(K^{\mathcal{D}}_{max})) + (K^{\mathcal{D}}_{min} + \log(K^{\mathcal{D}}_{min}))$$

Proof. Let $U, V : \mathbf{2}^* \to D$ be optimal in $Max^{\mathbf{2}^* \to \mathcal{D}}_{PR}$ and $Min^{\mathbf{2}^* \to \mathcal{D}}_{PR}$, i.e. $K^{\mathcal{D}}_{max} = K_U$ and $K^{\mathcal{D}}_{min} = K_V$. Let $f, g : \mathbf{2}^* \times \mathbb{N} \to D$ be partial computable, respectively monotone increasing and decreasing with respect to their 2d argument such that $U = \max^{\mathcal{D}} f$ and $V = \min^{\mathcal{D}} g$.

Define a partial computable function $\varphi : \mathbf{2}^* \to D$ as follows:

- If r is not in $range(c)$ then $\varphi(r)$ is undefined. Else, from input r, get p and q such that $c(p, q) = r$.
- Dovetail computations of the $f(p, t)$'s and $g(q, t)$'s for $t = 0, 1, 2, ...$
- If and when there are t', t'' such that $f(p, t')$ and $g(q, t'')$ are both defined and have the same value then output their common value and halt.

By the invariance theorem, there is a constant N such that $K^D \leq K_\varphi + N$. Let $d \in D$ and let p, q be shortest programs such that $U(p) = V(q) = d$, i.e. $K^{\mathcal{D}}_{max}(d) = |p|$ and $K^{\mathcal{D}}_{min}(d) = |q|$.
Observe that, whenever $f(p, t')$ and $g(q, t'')$ are both defined, we have $f(p, t') \leq d \leq g(p, t'')$. Also, since $U(p) = \max^{\mathcal{D}}\{f(p, t) : t \in \mathbb{N}\}$ and $V(q) = \min^{\mathcal{D}}\{g(q, t) : t \in \mathbb{N}\}$, there are t', t'' such that $f(p, t') = d = g(q, t'')$. Therefore, $\varphi(c(p, q))$ halts and outputs d. Therefore

$$K^D(d) \leq K_\varphi(d) + N \leq |c(p, q)| + N \leq J(K^{\mathcal{D}}_{min}(d), K^{\mathcal{D}}_{max}(d)) + N$$

The last assertion of the Lemma is obtained with the injective map

$$c(p, q) = \begin{cases} 0^{|Bin(|p|)|}1Bin(|p|)pq & \text{if } |p| \leq |q| \\ 1^{|Bin(|q|)|}0Bin(|q|)pq & \text{if } |p| > |q| \end{cases}$$

(where $Bin(x)$ denotes the binary representation of x) since

$$|c(p, q)| = |p| + |q| + 2\lfloor \log(\min(|p|, |q|)) \rfloor + 3 \leq (|p| + \log(|p|)) + (|q| + \log(|q|)) + 3$$

$\square$

4.4 $K^{\mathcal{D}}_{max}, K^{\mathcal{D}}_{min}$ and the jump

Proposition 32. *1. Let $\mathbb{X}$ be a basic space. All functions in $Max^{\mathbb{X}\to\mathcal{D}}_{PR}$ and $Min^{\mathbb{X}\to\mathcal{D}}_{PR}$ are partial computable in $\emptyset'$. In particular, K^D is recurcomputablesive in $\emptyset'$.*

2. $K^{\mathcal{D}}_{min}$ and $K^{\mathcal{D}}_{max}$ are computable in $\emptyset'$.

Proof. 1. Proposition 6 insures that any $F : \mathbb{X} \to \mathcal{D}$ in $Max^{\mathbb{X}\to\mathcal{D}}_{PR}$ or $Min^{\mathbb{X}\to\mathcal{D}}_{PR}$ has $\Sigma^0_1 \wedge \Pi^0_1$ graph. Therefore two calls to oracle $\emptyset'$ suffice to decide $F(\mathtt{x}) = d$.

2. Let $p_0, p_1, \ldots$ be a *length increasing* enumeration of $\mathbf{2}^*$ and let $U : \mathbf{2}^* \to \mathcal{D}$ be optimal in $Max^{\mathbf{2}^*\to\mathcal{D}}_{PR}$, i.e. $K_U = K^{\mathcal{D}}_{max}$. One can compute $K^{\mathcal{D}}_{max}(d)$ with oracle $\emptyset'$ as follows:

- i. Using oracle $\emptyset'$, test successive equalities $U(p) = d$ (cf. Point 1) for programs $p = p_0, p_1, \ldots$.
- ii. When such an equality holds (which necessarily does happen) then output $|p|$ and halt.

Idem with $K^{\mathcal{D}}_{min}$. □

4.5 Proof of Theorem 24 (1st hierarchy theorem)

1. *Large inequality* $K^{\emptyset',D} \leq_{\mathrm{ct}} \inf(K^{\mathcal{D}}_{min}, K^{\mathcal{D}}_{max})$. Point 1 of Proposition 32 insures that $Max^{\mathcal{D}}_{PR}$ and $Min^{\mathcal{D}}_{PR}$ are included in $PR^{\emptyset'}$. Therefore $K^{\emptyset',D} \leq_{\mathrm{ct}} K^{\mathcal{D}}_{min}$ and $K^{\emptyset',D} \leq_{\mathrm{ct}} K^{\mathcal{D}}_{max}$, i.e. $K^{\emptyset',D} \leq_{\mathrm{ct}} \inf(K^{\mathcal{D}}_{min}, K^{\mathcal{D}}_{max})$.
Strict inequality $K^{\emptyset',D} <_{\mathrm{ct}} \inf(K^{\mathcal{D}}_{min}, K^{\mathcal{D}}_{max})$. Point 2 of Proposition 32 insures that $\inf(K^{\mathcal{D}}_{min}, K^{\mathcal{D}}_{max})$ is computable in $\emptyset'$. Now, the well-known fact that if $\psi =_{\mathrm{ct}} K^D$ then ψ is not computable relativizes: if $\psi =_{\mathrm{ct}} K^{\emptyset',D}$ then ψ is not computable in $\emptyset'$. In particular, $\inf(K^{\mathcal{D}}_{min}, K^{\mathcal{D}}_{max}) \neq_{\mathrm{ct}} K^{\emptyset',D}$.
2. This is the contents of Lemma 31. □

4.6 Inequalities $K^{\mathcal{D}_{st}}_{max} \leq_{\mathrm{ct}} K^{\mathcal{D}_{wk}}_{max}$ and $K^{\mathcal{D}_{st}}_{min} \leq_{\mathrm{ct}} K^{\mathcal{D}_{wk}}_{min}$

The following result is straightforward.

Proposition 33. *With the notations of Theorem 26,*

$$K^{\mathcal{D}_{st}}_{min} \leq_{\mathrm{ct}} K^{\mathcal{D}_{wk}}_{min} \quad , \quad K^{\mathcal{D}_{st}}_{max} \leq_{\mathrm{ct}} K^{\mathcal{D}_{wk}}_{max}$$

Proof. Since $<_{st}$ extends $<_{wk}$, every partial computable function $\mathbf{2}^* \to D$ which is monotone increasing in its second argument relative to $<_{wk}$ is also monotone increasing relative to $<_{st}$. So that $Max^{\mathcal{D}_{wk}}_{PR} \subseteq Max^{\mathcal{D}_{st}}_{PR}$. Which yields $K^{\mathcal{D}_{st}}_{max} \leq_{\mathrm{ct}} K^{\mathcal{D}_{wk}}_{max}$. □

4.7 If $(*)$ holds: proof of Point 1 of Theorem 26 (3rd hierarchy theorem)

We use the notations of Theorem 26.

Lemma 34. *Let $\alpha : \mathbb{N} \to \mathbb{N}$ be a total computable function. If condition $(*)$ holds then there exists total functions $F, G : \mathbb{N} \to D$ respectively in $Max_{Rec}^{\mathbb{N}\to\mathcal{D}_{st}}$ and $Min_{Rec}^{\mathbb{N}\to\mathcal{D}_{st}}$ and a constant c such that, for all $i \in \mathbb{N}$,*

$$\begin{array}{ccccc} K_{max}^{\mathcal{D}_{wk}}(F(i)) \geq \alpha(i) & , & K_{min}^{\mathcal{D}_{wk}}(F(i)) \geq \alpha(i) & , & K_{max}^{\mathcal{D}_{st}}(F(i)) \leq \log(i) + c \\ K_{max}^{\mathcal{D}_{wk}}(G(i)) \geq \alpha(i) & , & K_{min}^{\mathcal{D}_{wk}}(G(i)) \geq \alpha(i) & , & K_{min}^{\mathcal{D}_{st}}(G(i)) \leq \log(i) + c \end{array}$$

Proof. 1. Since $(*)$ holds, for all $i \in \mathbb{N}$, there exists a finite strong chain with $2^{\alpha(i)+1}$ elements which is a weak antichain. Dovetailing over subsets of D with $2^{\alpha(i)+1}$ elements, one can effectively find such a strong chain Z_i. Thus, there exists a total computable function $\sigma : \mathbb{N} \times \mathbb{N} \to D$ such that, for all $i \in \mathbb{N}$,

- $\sigma(i, 0) <_{st} \sigma(i, 1) <_{st} \dots <_{st} \sigma(i, 2^{\alpha(i)+1} - 1)$
- $Z_i = \{\sigma(i, j) : j = 0, \dots, 2^{\alpha(i)+1} - 1\}$ is a weak antichain.

2. Let f and g are partial computable functions $\mathbf{2}^* \times \mathbb{N} \to D$ such that $U = \max^{\mathcal{D}_{wk}} f$ and $V = \min^{\mathcal{D}_{wk}} g$ are optimal in $Max_{PR}^{\mathbf{2}^*\to\mathcal{D}_{wk}}$ and $Min_{PR}^{\mathbf{2}^*\to\mathcal{D}_{wk}}$, i.e. $K_U = K_{max}^{\mathcal{D}_{wk}}$ and $K_V = K_{min}^{\mathcal{D}_{wk}}$.
We observe that inequalities $K_{max}^{\mathcal{D}_{wk}}(F(i)) \geq \alpha(i)$ and $K_{min}^{\mathcal{D}_{wk}}(F(i)) \geq \alpha(i)$ are equivalent to disequalities $U(p) \neq F(i)$ and $V(p) \neq F(i)$ for every p such that $|p| < \alpha(i)$.
We define $F, G : \mathbb{N} \to D$ as $F = \max^{\mathcal{D}_{st}} \ell$ and $F = \min^{\mathcal{D}_{st}} \ell$ for some total computable $\ell : \mathbb{N} \times \mathbb{N} \to D$. Let

$$\begin{array}{rcl} X_p & = & \{f(p,t) : t \text{ s.t. } f(p,t) \text{ converges}\} \\ Y_p & = & \{g(p,t) : t \text{ s.t. } g(p,t) \text{ converges}\} \\ X_p^t & = & \{f(p,t') \in Z_i : t' \leq t \text{ and } f(p,t') \text{ converges in} \leq t \text{ steps}\} \\ Y_p^t & = & \{g(p,t') \in Z_i : t' \leq t \text{ and } g(p,t') \text{ converges in} \leq t \text{ steps}\} \end{array}$$

Since Z_i is a weak antichain and X_p, Y_p are weak chains, each one of the sets $Z_i \cap X_p$ and $Z_i \cap Y_p$ has at most one element. Thus, $\bigcup_{|p|<\alpha(i)} (X_p \cup Y_p)$ has at most $2(2^{\alpha(i)} - 1) = 2^{\alpha(i)+1} - 2$ elements in Z_i. Since Z_i has $2^{\alpha(i)+1}$ elements and the $\sigma(i,j)$'s are in Z_i, the following definition makes sense:

$$\ell(i,t) = \sigma(i,j) \text{ where } j \text{ is least such that } \sigma(i,j) \notin \bigcup_{|p|<\alpha(i)} (X_p^t \cup Y_p^t)$$

Now, $F(i) = (\max^{\mathcal{D}_{st}} \ell)(i)$ and $G(i) = (\min^{\mathcal{D}_{st}} \ell)(i)$ are of the form $\ell(i, t'_i)$ and $\ell(i, t''_i)$ for some t'_i, t''_i, hence they are not in $\bigcup_{|p|<\alpha(i)}(X_p \cup Y_p)$. In particular, since $U(p) = \max^{\mathcal{D}_{wk}} X_p$ is in X_p and $V(p) = \min^{\mathcal{D}_{wk}} Y_p$ is in Y_p, we see that $F(i)$ and $G(i)$ are not in $\{U(p), V(p)\}$ for any $|p| < \alpha(i)$. Which proves that $K^{\mathcal{D}_{wk}}_{max}(F(i))$, $K^{\mathcal{D}_{wk}}_{min}(F(i))$, $K^{\mathcal{D}_{wk}}_{max}(G(i))$ and $K^{\mathcal{D}_{wk}}_{min}(G(i))$ are all $\geq \alpha(i)$.

3. Since $F \in Max^{\mathbb{N}\to\mathcal{D}_{st}}_{Rec}$, the invariance theorem insures that $K^{\mathcal{D}_{st}}_{max} \leq_{\rm ct} K_F$. Now, $K_F(F(i)) \leq_{\rm ct} \log(i)$, hence the inequality $K^{\mathcal{D}_{st}}_{max}(F(i)) \leq \log(i) + c$ for some constant c. Idem with $K^{\mathcal{D}_{st}}_{min}(G(i))$. □

Proof of Point 1 of Theorem 26. Apply Lemma 34 with α' such that α' is monotone increasing and $\alpha'(i) \geq \max(\alpha(i), i)$ for all i. Since $\alpha'(i)$ tends to $+\infty$ with i, so does $F(i)$. Let i_0 be such that $\log(i) + c \leq i$ for all $i \geq i_0$. Since α' is increasing and $\alpha' \geq \alpha$, for all $i \geq i_0$ we have

$$K^{\mathcal{D}_{wk}}_{max}(F(i)) \geq \alpha'(i) \geq \alpha'(\lfloor \log(i) + c \rfloor) \geq \alpha'(K^{\mathcal{D}_{st}}_{max}(F(i))) \geq \alpha(K^{\mathcal{D}_{st}}_{max}(F(i)))$$

Similarly, we have $K^{\mathcal{D}_{wk}}_{min}(F(i)) \geq \alpha(K^{\mathcal{D}_{st}}_{max}(F(i)))$ and $K^{\mathcal{D}_{wk}}_{max}(G(i)) \geq \alpha(K^{\mathcal{D}_{st}}_{min}(G(i)))$ and $K^{\mathcal{D}_{wk}}_{min}(G(i)) \geq \alpha(K^{\mathcal{D}_{st}}_{min}(G(i)))$.
Finally, observe that $\{F(i) : i \geq i_0\}$ and $\{G(i) : i \geq i_0\}$ are infinite. Which concludes the proof of Point 1 of Theorem 26. □

4.8 Proof of Point 1 of Theorem 25 (2d hierarchy theorem)

Comparing K^D to $K^{\mathcal{D}}_{max}$ and $K^{\mathcal{D}}_{min}$.
Let $<_{st}$ be $<$ and $<_{wk}$ be the empty order. Then

$$K^{\mathcal{D}_{st}}_{max} = K^{\mathcal{D}}_{max} \quad , \quad K^{\mathcal{D}_{st}}_{min} = K^{\mathcal{D}}_{min} \quad , \quad K^{\mathcal{D}_{wk}}_{max} = K^{\mathcal{D}_{wk}}_{min} = K^D$$

The condition (in Point 1 of Theorem 25) that $\mathcal{D}$ contains arbitrarily large chains insures condition $(*)$ about $<_{st}$ and $<_{wk}$. Thus, we can apply (the just proved) Point 1 of Theorem 26. This gives properties i and ii of Point 1 of Theorem 25.

Comparing $K^{\mathcal{D}}_{max}$ and $K^{\mathcal{D}}_{min}$.
We shall prove properties iii and iv of Point 1 of Theorem 25 using properties i and ii and also Lemma 31.
Applying Lemma 31, let c be such that,

$$(\dagger) \qquad K^D \leq 2\,(K^{\mathcal{D}}_{max} + K^{\mathcal{D}}_{min}) + c$$

Property iii applied to $\alpha'(i) = 2\,(\alpha(i) + i) + c$ insures that the set

$$X = \{d : K^D(d) \geq 2\,(\alpha(K^{\mathcal{D}}_{min}(d)) + K^{\mathcal{D}}_{min}(d)) + c\}$$

is infinite. Now, using (†), we see that, for $d \in X$,

$$2\,(\alpha(K^{\mathcal{D}}_{min}(d)) + K^{\mathcal{D}}_{min}(d)) + c \leq K^D(d) \leq 2\,(K^{\mathcal{D}}_{max}(d) + K^{\mathcal{D}}_{min}(d)) + c$$

hence $K^{\mathcal{D}}_{max}(d) \geq \alpha(K^{\mathcal{D}}_{min}(d))$. Which proves iii. The proof of iv is similar.
□

4.9 If $(**)$ holds: proof of Point 2 of Theorem 26 (3d hierarchy theorem)

Lemma 35. *With the notations of Theorem 26, if condition* $(**)$ *holds then*

$$K^{\mathcal{D}_{st}}_{min} \geq_{\rm ct} K^{\mathcal{D}_{wk}}_{min} \quad , \quad K^{\mathcal{D}_{st}}_{max} \geq_{\rm ct} K^{\mathcal{D}_{wk}}_{max}$$

Proof. 1. Let k be as in $(**)$. Let U_{st} be optimal in $Max^{\mathcal{D}_{st}}_{PR}$ and $f : \mathbf{2}^* \times \mathbb{N} \to D$ be partial computable such that $\max^{\mathcal{D}_{st}} f = U_{st}$.
Due to Proposition 14, we can suppose that f has domain of the form $Z \times \mathbb{N}$ and is monotone increasing in its second argument, with respect to the strong order.
Applying $(**)$ to f, we get k partial computable functions $f_1, ..., f_k$, monotone increasing in their second argument, with respect to the weak order, such that

$$(\sharp) \qquad \{f(p,t) : t \in \mathbb{N}\} = \bigcup_{i=1,\ldots,k} \{f_i(p,t) : t \in \mathbb{N}\}$$

Define $g : \mathbf{2}^* \times \mathbb{N} \to D$ such that

$$g(q,t) = \begin{cases} f_i(p,t) & \text{if } q = 0^i 1^{k-i} p \text{ for some } p \text{ and } 1 \leq i \leq k \\ \text{undefined} & \text{otherwise} \end{cases}$$

Clearly, g is partial computable and monotone increasing in its second argument relative to the weak order $<_{wk}$.
If $p \in dom(U_{st})$, then $\{f(p,t) : t \in \mathbb{N}\}$ is finite and non empty. Let $f(p,t_p)$ be its $<_{st}$ greatest element. Condition $(\sharp)$ insures that there exists i such that $\{g(0^i 1^{k-i} p, t) : t \in \mathbb{N}\}$ is finite and contains $f(p,t_p)$. Since g is $<_{wk}$ increasing in t, the set $\{g(0^i 1^{k-i} p, t) : t \in \mathbb{N}\}$ is a weak chain. Since $<_{st}$ extends $<_{wk}$, $f(p,t_p)$ is necessarily its $<_{wk}$ greatest element. Thus,

$$U_{st}(p) = f(p,t_p) = (\max{}^{wk} g)(0^i 1^{k-i} p)$$

This proves that, for all $d \in D$,

$$\begin{array}{rcl} K^{D_{st}}_{max}(d) & = & \text{least } |p| \text{ such that } U_{st}(p) = d \\ & = & \text{least } |p| \text{ such that } (\max^{wk} g)(0^i 1^{k-i} p) = d \text{ for some } i \\ & \geq & \text{least } |q| - k \text{ such that } (\max^{wk} g)(q) = d \\ & = & K_{\max^{\mathcal{D}_{wk}} g}(d) - k \end{array}$$

Since, by the invariance theorem, $K_{\max^{\mathcal{D}_{wk}} g} \geq_{\rm ct} K^{D_{wk}}_{max}$, we get the desired inequality $K^{D_{st}}_{max} \geq_{\rm ct} K^{D_{wk}}_{max}$.
2. Considering the reverse orders, we get the inequality $K^{\mathcal{D}_{st}}_{min} \geq_{\rm ct} K^{\mathcal{D}_{wk}}_{min}$. □

Proof of Point 2 of Theorem 26. Straightforward from the above Lemma 35 and Proposition 33. □

4.10 Proof of Point 2 of Theorem 25 (2d hierarchy theorem)

As in §4.8, let $<_{st}$ be $<$ and $<_{wk}$ be $\emptyset$, so that

$$K^{\mathcal{D}_{st}}_{max} = K^{\mathcal{D}}_{max} \quad , \quad K^{\mathcal{D}_{st}}_{min} = K^{\mathcal{D}}_{min} \quad , \quad K^{\mathcal{D}_{wk}}_{max} = K^{\mathcal{D}_{wk}}_{min} = K^{D}$$

Suppose all chains in $(D, <)$ have length $\leq k$. We shall prove condition $(**)$ for the above orders $<_{st}$ and $<_{wk}$.
Let $f : \mathbf{2}^* \times \mathbb{N} \to D$ be partial computable, monotone increasing in its 2d argument for the strong order, i.e. for the $<$ order. Compute $f(p,t)$ for $t = 0, 1, ...$ to get the $\leq k$ distinct elements of the chain $\{f(p,t) : t \in \mathbb{N}\}$ (not necessarily in increasing order) and let $f_i(p)$ be the i-th element so obtained (if there is some). Then $f_0, ..., f_k : \mathbf{2}^* \to D$ are partial computable and

$$\{f(p,t) : t \in \mathbb{N}\} = \{f_i(p) : \ i \text{ s.t. } f_i(p) \text{ is defined}\}$$

which insures condition $(**)$.
Applying Point 2 of Theorem 26 (proved above), we get $=_{\rm ct}$ equalities which are exactly those of Point 2 of Theorem 25. □

5 Complementary results about the Max and Min classes

In this section we further investigate the different Max and Min classes. The results do not involve as many technicalities as those of §4.

5.1 Total functions in $Max_{\boldsymbol{Rec}}^{\mathbb{X}\to\mathcal{D}}$ and $Max_{\boldsymbol{PR}}^{\mathbb{X}\to\mathcal{D}}$

As a straightforward corollary of Point 2 of Proposition 14, we get the following result.

Theorem 36. *The classes $Max_{Rec}^{\mathbb{X}\to\mathcal{D}}$ and $Max_{PR}^{\mathbb{X}\to\mathcal{D}}$ contain the same total functions:*

$$Max_{PR}^{\mathbb{X}\to\mathcal{D}} \cap D^{\mathbb{X}} = Max_{Rec}^{\mathbb{X}\to\mathcal{D}} \cap D^{\mathbb{X}}$$

5.2 Comparing $Max_{\boldsymbol{PR}}^{\mathbb{X}\to\mathcal{D}}, Max_{\boldsymbol{Rec}}^{\mathbb{X}\to\mathcal{D}}$ and $PR^{\mathbb{X}\to\mathcal{D}}, Rec^{\mathbb{X}\to\mathcal{D}}$

Proposition 37. *Let $\mathbb{X}$ be a basic set and $\mathcal{D} = (D, <, \rho)$ be a computable ordered set.*
1. If $<$ is empty then $PR^{\mathbb{X}\to\mathcal{D}} = Max_{PR}^{\mathbb{X}\to\mathcal{D}}$ and $Rec^{\mathbb{X}\to\mathcal{D}} = Max_{Rec}^{\mathbb{X}\to\mathcal{D}}$.

2. If $<$ is not empty then $Max_{Rec}^{\mathbb{X}\to\mathcal{D}}$ contains non computable total functions. In particular, $PR^{\mathbb{X}\to\mathcal{D}} \subset Max_{PR}^{\mathbb{X}\to\mathcal{D}}$ and $Rec^{\mathbb{X}\to\mathcal{D}} \subset Max_{Rec}^{\mathbb{X}\to\mathcal{D}}$ (where $\subset$ denotes strict inclusion).

3. Whatever be $<$, $PR^{\mathbb{X}\to\mathcal{D}}$ is not included in $Min_{Rec}^{\mathbb{X}\to\mathcal{D}} \cup Max_{Rec}^{\mathbb{X}\to\mathcal{D}}$.

Proof. 1. Straightforward.
2. Inclusions $PR^{\mathbb{X}\to\mathcal{D}} \subseteq Max_{PR}^{\mathbb{X}\to\mathcal{D}}$ and $Rec^{\mathbb{X}\to\mathcal{D}} \subseteq Max_{Rec}^{\mathbb{X}\to\mathcal{D}}$ are obvious.
Suppose there exists comparable distinct elements $a < b$ in D. Let Z be some computably enumerable non computable subset of $\mathbb{X}$ and let $\theta : \mathbb{N} \to \mathbb{X}$ be a total computable map with range Z. Define $f : \mathbb{X} \times \mathbb{N} \to D$ total computable, monotone increasing in t, such that

$$f(\mathtt{x}, t) = \begin{cases} a & \text{if } \mathtt{x} \notin \{\theta(n) : n \leq t\} \\ b & \text{otherwise} \end{cases}$$

Then $\max f$ is total and $(\max f)^{-1}(b) = Z$ and $(\max f)^{-1}(a) = \mathbb{X} \setminus Z$. Since Z is not computable, $\max f$ is not computable. Which proves $Rec^{\mathbb{X}\to\mathcal{D}} \subset Max_{Rec}^{\mathbb{X}\to\mathcal{D}}$
3. First, we consider the case where $(D, <)$ has a minimal element d. Let $\pi_d^Z : \mathbb{X} \to D$ be the partial computable function with domain Z (as in Point 2 of this proof) which is constant on Z with value d. We show that π_d^Z is not in $Max_{Rec}^{\mathcal{D}}$. Suppose $f : \mathbb{X} \times \mathbb{N} \to D$ is total computable, monotone in its second argument, such that $\max^{\mathcal{D}} f = \pi_d^Z$. Since d is minimal in D, $(\max^{\mathcal{D}} f)(\mathtt{x}) = d$ if and only if $\forall t\ f(\mathtt{x}, t) = d$. Thus, the computably enumerable set Z would be Π_1^0, hence computable, contradiction.
We now consider the case where $(D, <)$ has no minimal element. Let $\gamma : D \to D$ be the total computable function which associates to each $d \in D$

the element $\rho(k_d)$ where n_d is the least k such that $\rho(k) < d$. Let $(\phi)_{\mathtt{e}\in\mathbb{X}}$ be an enumeration of $PR^{\mathbb{X}\times\mathbb{N}\to D}$ which is partial computable as a function $\Phi : \mathbb{X}\times\mathbb{X}\times\mathbb{N} \to D$. We consider an enumeration $(\mathtt{e}_n, \mathtt{x}_n, t_n, d_n)_{n\in\mathbb{N}}$ of the graph of Φ and define a partial computable function $\varphi : \mathbb{X} \to \mathcal{D}$ as follows:

$$\varphi(\mathtt{x}) = \begin{cases} \gamma(d_n) & \text{if } n \text{ is least such that } \mathtt{e}_n = \mathtt{x}_n = \mathtt{x} \\ \text{undefined} & \text{if there is no such } n \end{cases}$$

It is clear that, for every $\mathtt{e}$, if $\phi_{\mathtt{e}}(\mathtt{e}, t)$ is defined for some t then $\varphi(\mathtt{e})$ is defined and $\varphi(\mathtt{e}) < \phi_{\mathtt{e}}(\mathtt{e}, t)$. In particular, if $\phi_{\mathtt{e}}$ is total then $\varphi(\mathtt{e}) < (\max^{\mathcal{D}} \phi_{\mathtt{e}})(\mathtt{e})$, hence $\varphi \neq \max^{\mathcal{D}} \phi_{\mathtt{e}}$. Which proves that φ is not in $Max^{\mathbb{X}\to\mathcal{D}}_{Rec}$.
Arguing with $\mathcal{D}^{\text{rev}}$ we get some function in $PR^{\mathbb{X}\to\mathcal{D}}$ which is not in $Min^{\mathbb{X}\to\mathcal{D}}_{Rec}$. Considering $\varphi_0, \varphi_1 \in PR^{\mathbb{X}\to\mathcal{D}}$ such that $\varphi_0 \notin Max^{\mathbb{X}\to\mathcal{D}}_{Rec}$ and $\varphi_1 \notin Min^{\mathbb{X}\to\mathcal{D}}_{Rec}$ and a computable bijection $\sigma : \mathbb{X}\times\{0,1\} \to \mathbb{X}$ we get a partial computable function $\varphi : \mathbb{X} \to \mathcal{D}$ which is not in $Max^{\mathbb{X}\to\mathcal{D}}_{Rec} \cup Min^{\mathbb{X}\to\mathcal{D}}_{Rec}$ by setting $\varphi(\sigma(\mathtt{x}, 0)) = \varphi_0(\mathtt{x})$ and $\varphi(\sigma(\mathtt{x}, 1)) = \varphi_1(\mathtt{x})$. □

5.3 Post hierarchy and the Max/Min classes

We keep notations of §2.3.

Theorem 38. *Let $\mathbb{X}$ be a basic set and $\mathcal{D}$ be a computable ordered set.*
1. Let D' be an initial segment of D (i.e. $d' \in D' \wedge e < d' \Rightarrow e \in D'$). Suppose D' is Π^0_1 and does not contain any strictly increasing infinite sequence $d'_0 < d'_1 < \dots$. Then
i. Every D'-valued function in $Max^{\mathbb{X}\to\mathcal{D}}_{PR}$ has $\Sigma^0_1 \wedge \Pi^0_1$ domain.
ii. Every D'-valued function in $Max^{\mathbb{X}\to\mathcal{D}}_{Rec}$ has Π^0_1 domain.

2. Let D' be a final segment of D (i.e. $d' \in D' \wedge e > d' \Rightarrow e \in D'$). Suppose D' is Σ^0_1 and does not contain any strictly increasing infinite sequence. Then
i. Every D'-valued function in $Max^{\mathbb{X}\to\mathcal{D}}_{PR}$ has Σ^0_1 domain.
ii. Every D'-valued function in $Max^{\mathbb{X}\to\mathcal{D}}_{Rec}$ is total.

Proof. 1. Suppose that $\max^{\mathcal{D}} f$ is D'-valued. Since D' is an initial segment and f can be supposed monotone increasing in its second argument, if $(\max^{\mathcal{D}} f)(\mathtt{x})$ is defined then, for all t, $f(\mathtt{x}, t)$ is either undefined or in D'. Now, since D' has no infinite increasing sequence, the set $\{f(\mathtt{x}, t) : t \in \mathbb{N} \text{ s.t. } f(\mathtt{x}, t) \in D'\}$ cannot be infinite. Thus, $\mathtt{x} \in dom(\max^{\mathcal{D}} f)$ if and only if

$$\exists t\ f(\mathtt{x}, t) \text{ is defined } \wedge\ \forall t\ (f(\mathtt{x}, t) \text{ is defined } \Rightarrow\ f(\mathtt{x}, t) \in D')$$

In case f is total computable, then the above equivalence is simply

$$\mathtt{x} \in dom(\max^{\mathcal{D}} f)\ \Leftrightarrow\ \forall t\ f(\mathtt{x}, t) \in D'$$

2. Since D' is a final segment and f can be supposed monotone increasing in its second argument, if $(\max^{\mathcal{D}} f)(\mathtt{x})$ is defined then, for all t large enough, $f(\mathtt{x},t)$ is either undefined or in D'. Now, since D' has no infinite increasing sequence, the set $\{f(\mathtt{x},t) : t \in \mathbb{N} \text{ s.t. } f(\mathtt{x},t) \in D'\}$ cannot be infinite. Thus,

$$\mathtt{x} \in dom(\max^{\mathcal{D}} f) \quad \Leftrightarrow \quad \exists t\ (f(\mathtt{x},t) \text{ is defined } \wedge f(\mathtt{x},t) \in D')$$

$\square$

The next corollary is an application of the above theorem with the reverse of the following $\mathcal{D}$'s:

- $\mathcal{D}$ is the natural order on $\mathbb{Z}$ and $D' = \mathbb{N}$,
- $\mathcal{D}$ is the natural order on $\mathbb{N}$ or of the prefix order on Σ^* and $D' = D$,

Corollary 39. *1. Every $\mathbb{N}$-valued function in $Min_{PR}^{\mathbb{X}\to\mathbb{Z}}$ (resp. $Min_{Rec}^{\mathbb{X}\to\mathbb{Z}}$) has $\Sigma^0_1 \wedge \Pi^0_1$ (resp. Π^0_1) domain.*
2. Let $\mathcal{D}$ be $\mathbb{N}$ with the natural order or Σ^* with the prefix partial order. Then every function in $Min_{PR}^{\mathbb{X}\to\mathcal{D}}$ (resp. $Min_{Rec}^{\mathbb{X}\to\mathcal{D}}$) has Σ^0_1 domain (resp. is total).

5.4 $Max \cap Min$ classes

Theorem 40. *Let $\mathbb{X}$ be a basic set and $\mathcal{D} = (D, <, \rho)$ be a computable ordered set.*
1. Every function $F : \mathbb{X} \to D$ in $Max_{PR}^{\mathbb{X}\to\mathcal{D}} \cap Min_{PR}^{\mathbb{X}\to\mathcal{D}}$ is the restriction of a partial computable function $\mathbb{X} \to \mathcal{D}$ to some $\Sigma^0_1 \wedge \Pi^0_1$ subset of $\mathbb{X}$.
In particular, every total function in $Max_{PR}^{\mathbb{X}\to\mathcal{D}} \cap Min_{PR}^{\mathbb{X}\to\mathcal{D}}$ is computable.

2. Suppose $\mathcal{D}$ has no maximal (resp. minimal) element. Then the restriction of any partial computable function $\mathbb{X} \to \mathcal{D}$ to any $\Sigma^0_1 \wedge \Pi^0_1$ subset of $\mathbb{X}$ is in $Max_{PR}^{\mathbb{X}\to\mathcal{D}}$ (resp. $Min_{PR}^{\mathbb{X}\to\mathcal{D}}$).

3. Suppose $\mathcal{D}$ has no maximal or minimal element. Then $Max_{PR}^{\mathbb{X}\to\mathcal{D}} \cap Min_{PR}^{\mathbb{X}\to\mathcal{D}}$ coincides with the family of restrictions of partial computable functions $\mathbb{X} \to \mathcal{D}$ to $\Sigma^0_1 \wedge \Pi^0_1$ subsets of $\mathbb{X}$.

Proof. 1. Let $F = \max^{\mathcal{D}} f = \min^{\mathcal{D}} g$ where $f, g : \mathbb{X} \times \mathbb{N} \to D$ are partial computable and f (resp. g) is monotone increasing (resp. decreasing) in its second argument. Let's check that $F(\mathtt{x})$ is defined if and only if

$$(*) \qquad (\exists t', t''\ f(\mathtt{x},t') = g(\mathtt{x},t'')) \wedge (\forall u, v\ f(\mathtt{x},u) \leq g(\mathtt{x},v))$$

In fact, if $F(\mathtt{x})$ is defined then

$F(\mathtt{x}) = f(\mathtt{x}, t') = g(\mathtt{x}, t'')$ for some t', t'',

$g(\mathtt{x}, u) \leq F(\mathtt{x}) \leq f(\mathtt{x}, v)$ for all u, v such that $g(\mathtt{x}, u), f(\mathtt{x}, v)$ are defined.

Conversely, from $(*)$ we see that, for $u \geq t'$ and $v \geq t''$, $f(\mathtt{x}, u) = f(\mathtt{x}, t') = g(\mathtt{x}, t'') = g(\mathtt{x}, v)$. Hence the finiteness of $\{f(\mathtt{x}, u) : u\}$ and $\{g(\mathtt{x}, v) : v\}$.
This proves that the domain of F is $\Sigma^0_1 \wedge \Pi^0_1$.
Let $G : \mathbb{X} \to \mathcal{D}$ be the partial computable function defined as follows:

> Dovetail computations of $f(\mathtt{x}, 0), f(\mathtt{x}, 1), \ldots, g(\mathtt{x}, 0), g(\mathtt{x}, 1), \ldots$ until we get t', t'' such that $f(\mathtt{x}, t'), g(\mathtt{x}, t'')$ are both defined and equal. Output this common value.

Applying $(*)$, if $F(\mathtt{x})$ is defined, then so is $G(\mathtt{x})$ and $F(\mathtt{x}) = G(\mathtt{x})$. Thus, F is the restriction of a partial computable function to some $\Sigma^0_1 \wedge \Pi^0_1$ set.

2. Suppose there is no maximal element. Since the order $<$ is computable, by dovetailing, one can define a total computable function $\gamma : D \to D$ such that $\gamma(d) > d$ for all $d \in D$. Let $F : \mathbb{X} \to \mathcal{D}$ be partial computable and let $Z \subseteq \mathbb{X}$ be $\Sigma^0_1 \wedge \Pi^0_1$ definable:

$$\mathtt{x} \in Z \Leftrightarrow (\exists t\ R(\mathtt{x}, t)) \wedge (\forall t\ S(\mathtt{x}, t))$$

where $R, S \subseteq \mathbb{X} \times \mathbb{N}$ are computable. Letting $\gamma^{(t)}$ denote the t-th iterate of γ, we define $f : \mathbb{X} \times \mathbb{N} \to D$ as follows:

$$f(\mathtt{x}, t) \quad = \quad \begin{cases} F(\mathtt{x}) & \text{if } F(\mathtt{x}) \text{ converges in } \leq t \text{ steps} \\ & \text{and } (\exists t' \leq t\ R(\mathtt{x}, t')) \wedge (\forall t' \leq t\ S(\mathtt{x}, t')) \\ \gamma^{(t)}(F(\mathtt{x})) & \text{if } F(\mathtt{x}) \text{ converges in } \leq t \text{ steps} \\ & \text{and } \exists t' \leq t\ \neg S(\mathtt{x}, t') \\ \text{undefined} & \text{otherwise} \end{cases}$$

It is easy to check that $\max^{\mathcal{D}} f$ is the restriction of F to Z.
The assertion with $Min^{\mathbb{X} \to \mathcal{D}}_{PR}$ is obtained with the order reverse to $\mathcal{D}$.
3. Straightforward from Points 1 and 2. □

Remark 41. Theorem 38 shows that Points 2, 3 of the above theorem do not hold for general ordered sets $\mathcal{D}$.

6 $Max^{\mathbf{2}^*\to\mathcal{D}}_{\boldsymbol{Rec}}$ and $Min^{\mathbf{2}^*\to\mathcal{D}}_{\boldsymbol{Rec}}$ and Kolmogorov complexity

Since there is no computable enumeration of total computable functions, it seems a priori desperate to get an invariance theorem for the class $Max^{\mathbb{X}\to\mathcal{D}}_{Rec}$. Nevertheless, there are important cases where such a result does hold. For instance, when $\mathcal{D}$ is $\mathbb{N}$ with its usual ordering.
The purpose of this section is to characterize the orders $\mathcal{D}$ such that an invariance theorem holds for the class $Max^{\mathbf{2}^*\to\mathcal{D}}_{Rec}$ (resp. $Min^{\mathbf{2}^*\to\mathcal{D}}_{Rec}$).
First, we deal with the enumeration theorem.

6.1 $Max^{\mathbb{X}\to\mathcal{D}}_{\boldsymbol{Rec}}$ and the enumeration theorem

Theorem 42 (Enumeration theorem for $Max^{\mathbb{X}\to\mathcal{D}}_{\boldsymbol{Rec}}$). *Let $\mathbb{X}$ be a basic set and $\mathcal{D} = (D, <, \rho)$ be a computable ordered set. The following conditions are equivalent:*
i. There exists a smallest element in $\mathcal{D}$.
ii. There exists a function $\widetilde{E} : \mathbb{N} \times \mathbb{X} \to D$ in $Max^{\mathbb{N}\times\mathbb{X}\to D}_{Rec}$ such that

$$\{\widetilde{E}_n : n \in \mathbb{N}\} = Max^{\mathbb{X}\to\mathcal{D}}_{Rec}$$

where $\widetilde{E}_n : \mathbb{X} \to D$ denotes the function $\mathtt{x} \mapsto \widetilde{E}(n, \mathtt{x})$.

Proof. $i \Rightarrow ii$. Let $\alpha \in D$ be the smallest element of D. As in §3.2, let $\psi : \mathbb{N} \times \mathbb{X} \times \mathbb{N} \to D$ be partial computable monotone increasing in its last argument such that $E = \max^{\mathcal{D}} \psi$ is an enumeration of $Max^{\mathbb{X}\to\mathcal{D}}_{PR}$. Consider an injective computable enumeration $(n_i, \mathtt{x}_i, t_i, d_i)_{i\in\mathbb{N}}$ of the graph of ψ. Since α is the smallest element, we can define a *total* computable function $\widetilde{\psi} : \mathbb{N} \times \mathbb{X} \times \mathbb{N} \to D$ as follows:

$$\begin{aligned} X(n, \mathtt{x}, t) &= \{d_i : i \leq t \wedge n_i = n \wedge \mathtt{x}_i = \mathtt{x} \wedge t_i \leq t\} \\ \widetilde{\psi}(n, \mathtt{x}, t) &= \text{greatest element of } \{\alpha\} \cup X(n, \mathtt{x}, t) \end{aligned}$$

Suppose ψ_n is total, we show that $\max^{\mathcal{D}} \widetilde{\psi}_n = \max^{\mathcal{D}} \psi_n$. Fix some $\mathtt{x}$. Observe that $\{\widetilde{\psi}_n(\mathtt{x}, t) : t \in \mathbb{N}\}$ is $\{\psi_n(\mathtt{x}, t) : t \in \mathbb{N}\}$ or $\{\alpha\} \cup \{\psi_n(\mathtt{x}, t) : t \in \mathbb{N}\}$. Thus, $\{\widetilde{\psi}_n(\mathtt{x}, t) : t \in \mathbb{N}\}$ and $\{\psi_n(\mathtt{x}, t) : t \in \mathbb{N}\}$ are simultaneously finite or infinite, and when finite they have the same greatest element. Since ψ_n is total, this proves that $(\max^{\mathcal{D}} \widetilde{\psi}_n)(\mathtt{x}) = (\max^{\mathcal{D}} \psi_n)(\mathtt{x})$. Thus, every function in $Max^{\mathbb{X}\to D}_{Rec}$ is of the form $\max^{\mathcal{D}} \widetilde{\psi}_n$ for some n.
Set $\widetilde{E} = \max^{\mathcal{D}} \widetilde{\psi}$. Then $\widetilde{E}$ is in $Max^{\mathbb{N}\times\mathbb{X}\to D}_{Rec}$ and the $\widetilde{E}_n$'s enumerate $Max^{\mathbb{X}\to D}_{Rec}$.

$ii \Rightarrow i$. We prove $\neg i \Rightarrow \neg ii$. Suppose $\mathcal{D}$ has no minimum element. By dovetailing one can define a total computable map $\gamma : D \to D$ such that $d \not\leq \gamma(d)$ for all d.
Let $E = \max^{\mathcal{D}} g : \mathbb{N} \times \mathbb{X} \to D$ where $g : \mathbb{N} \times \mathbb{X} \times \mathbb{N} \to D$ is total computable monotone increasing in its last argument. We define a total computable map $f : \mathbb{X} \to D$ such that $f \neq E_n$ for all n. Let $\theta : \mathbb{N} \to \mathbb{X}$ be some computable bijection. Set $f(\theta(n)) = \gamma(g(n, \theta(n), 0))$. Then

$$g(n, \theta(n), 0) \not\leq f(\theta(n)) \text{ and } g(n, \theta(n), 0) \leq (\max{}^{\mathcal{D}} g)(\theta(n)) = E_n(\theta(n))$$

Thus, $f(\theta(n)) \neq E_n(\theta(n))$. Hence $f \neq E_n$ for all n. □

6.2 $Max^{\mathbf{2}^* \to \mathcal{D}}_{\boldsymbol{Rec}}$ and the invariance theorem

If $\mathcal{D}$ contains a smallest element then the enumeration theorem of §6.1 allows to get an invariance result for the class $Max^{\mathbf{2}^* \to \mathcal{D}}_{Rec}$.
Surprisingly, it turns out that an invariance result can be proved for partially ordered sets with no smallest element, hence which fail the enumeration theorem.
Also, in case the class $Max^{\mathbf{2}^* \to \mathcal{D}}_{Rec}$ has optimal functions then they prove to be also optimal for the bigger class $Max^{\mathbf{2}^* \to \mathcal{D}}_{PR}$.

Theorem 43. *Let $\mathbb{X}$ be a basic space and $\mathcal{D} = (D, <, \rho : \mathbb{N} \to D)$ be a computable partially ordered set. Let $(*)$ be the following condition on $\mathcal{D}$:*

- $(*)$ *The set of minimal elements of D is finite and every element of D dominates a minimal element*

1. If $\mathcal{D}$ satisfies $(*)$ then

- *i. Every function in $Max^{\mathbf{2}^* \to \mathcal{D}}_{PR}$ has an extension (not necessarily total) in $Max^{\mathbf{2}^* \to \mathcal{D}}_{Rec}$.*
- *ii. The invariance theorem holds for $Max^{\mathbf{2}^* \to \mathcal{D}}_{Rec}$.*
- *iii. Every U in $Max^{\mathbf{2}^* \to \mathcal{D}}_{Rec}$ which is optimal for $Max^{\mathbf{2}^* \to \mathcal{D}}_{Rec}$ is also optimal for the class $Max^{\mathbf{2}^* \to \mathcal{D}}_{PR}$.*
  In particular, the Kolmogorov complexity associated to $Max^{\mathbf{2}^* \to \mathcal{D}}_{Rec}$ coincides (up to a constant) with that associated to $Max^{\mathbf{2}^* \to \mathcal{D}}_{PR}$.

2. If $\mathcal{D}$ does not satisfy $(*)$ then the invariance theorem fails for $Max^{\mathbf{2}^* \to \mathcal{D}}_{Rec}$. Moreover, counterexamples can be taken in the class $Rec^{\mathbf{2}^* \to \mathcal{D}}$ of total computable functions $\mathbf{2}^* \to \mathcal{D}$:

$$\forall G \in Max^{\mathbf{2}^* \to \mathcal{D}}_{Rec} \;\; \exists F \in Rec^{\mathbf{2}^* \to \mathcal{D}} \quad K_G \not\leq_{\mathrm{ct}} K_F$$

Proof. 1. Suppose $(*)$ holds and let $M = \{m_0, ..., m_k\}$ be the set of minimal elements. For $i \le k$, let $D_i = \{d \in D : d \ge m_i\}$. Some of the D_i's may be finite, though not all of them (else D would be finite). Let $\ell \le k$ be such that D_i is infinite for $i \le \ell$ and finite for $\ell < i \le k$. Since the D_i are computable, for $i \le \ell$, there exists a computable map $\rho_i : \mathbb{N} \to D_i$ such that $\mathcal{D}_i = (D_i, < \cap (D_i \times D_i), \rho_i)$ is a computable partially ordered set.
A. Since D_i has a smallest element, namely m_i, $Max^{\mathcal{D}_i}_{Rec}$ satisfies the enumeration theorem (cf. Theorem 42). The proof of Theorem 22 applies, insuring that $Max^{\mathcal{D}_i}_{Rec}$ satisfies the invariance theorem.
Let $g_i : \mathbf{2}^* \times \mathbb{N} \to D_i$ be total computable such that $\max^{\mathcal{D}_i} g_i = U_i : \mathbf{2}^* \to D_i$ is optimal in $Max^{\mathcal{D}_i}_{Rec}$.
Let's check that U_i is also optimal in $Max^{\mathcal{D}_i}_{PR}$. Let $F_i \in Max^{\mathcal{D}_i}_{PR}$ and $F_i = \max^{\mathcal{D}_i} f_i$ where $f_i : \mathbf{2}^* \times \mathbb{N} \to D_i$ is partial computable monotone increasing in its second argument and has domain $Z_i \times \mathbb{N}$ where $Z_i \subseteq \mathbf{2}^*$ is computably enumerable (cf. Proposition 14). Define a total computable map $\widetilde{f_i} : \mathbf{2}^* \times \mathbb{N} \to D_i$ such that

$$\widetilde{f_i}(p,t) = \begin{cases} f_i(p,t) & \text{if } p \text{ is seen to be in } Z_i \text{ in} \le t \text{ steps} \\ m_i & \text{otherwise} \end{cases}$$

Set $\widetilde{F_i} = \max^{\mathcal{D}_i} \widetilde{f_i}$. If $p \in Z$ then $\widetilde{f_i}(p,t) = f_i(p,t)$ for t large enough, so that $F_i(p) = (\max^{\mathcal{D}_i} f_i)(p) = (\max^{\mathcal{D}_i} \widetilde{f_i})(p) = \widetilde{F_i}(p)$. Thus, $\widetilde{F_i}$ extends F_i. Which trivially yields $K_{\widetilde{F_i}} \le K_{F_i}$. Since $\widetilde{F_i} \in Max^{\mathcal{D}_i}_{Rec}$, we have $K_{U_i} \le_{\mathrm{ct}} K_{\widetilde{F_i}}$. Hence $K_{U_i} \le_{\mathrm{ct}} K_{F_i}$.
B. We group the functions g_i and U_i of Point A to get a total computable $g : \mathbf{2}^* \times \mathbb{N} \to D$ and the associated $U = \max^{\mathcal{D}} g$ in $Max^{\mathcal{D}}_{Rec}$. Define g as follows:

$$g(q,t) = \begin{cases} g_i(p,t) & \text{if } q \text{ is of the form } 0^i 1p \text{ with } i \le \ell,\ p \in \mathbf{2}^* \\ m_0 & \text{otherwise} \end{cases}$$

For $i \le \ell$ and $d \in D_i$, we have

$$K_U(d) \le K_{U_i}(d) + i + 1 \text{ for all } i \le \ell \text{ and } d \in D_i$$

Suppose F is in $Max^{\mathcal{D}}_{PR}$ is of the form $F = \max^{\mathcal{D}} f$ where $f : \mathbf{2}^* \times \mathbb{N} \to D$ is partial computable. For $i \le \ell$, let $F_i = \max^{\mathcal{D}_i} f_i$ where $f_i : \mathbf{2}^* \to D_i$ is such that

$$f_i(p,t) = \begin{cases} f(p,t) & \text{if } f(p,t) \text{ is defined and is in } D_i \\ \text{undefined} & \text{otherwise} \end{cases}$$

Clearly, F_i is the restriction of F to $F^{-1}(D_i)$. Thus, $K_F(d) = K_{F_i}(d)$ for all $d \in D_i$.
Since $F_i \in Max_{PR}^{\mathcal{D}_i}$ and U_i is optimal in $Max_{PR}^{\mathcal{D}_i}$, there exists c_i such that $K_{U_i} \leq K_{F_i} + c_i$. Thus, for $d \in D_i$, we have

$$K_U(d) \leq K_{U_i}(d) + i + 1 \leq K_{F_i}(d) + c_i + i + 1 \leq K_F(d) + c_i + i + 1$$

Let a be the maximum value of K_F on the finite set $\bigcup_{\ell < j \leq k} D_j$. Set $c = \sup(\{c_i + i + 1 : i \leq \ell\} \cup \{a\})$. Then $K_U(d) \leq K_F(d) + c$ for all $d \in D$. Which proves that U, which is in $Max_{Rec}^{\mathcal{D}}$, is optimal in $Max_{PR}^{\mathcal{D}}$.
C. If V in $Max_{Rec}^{\mathbf{2}^* \to \mathcal{D}}$ is optimal for $Max_{Rec}^{\mathbf{2}^* \to \mathcal{D}}$ then $K_V \leq_{\mathrm{ct}} K_U$ (where U is as in B). Since U is is optimal in $Max_{PR}^{\mathcal{D}}$, so is V.

2. Suppose $(*)$ fails. Observe that, for every finite subset Z of D, there exists d such that $z \not\leq d$ for all $z \in Z$. Else, the set of minimal elements of Z would satisfy $(*)$.
Let $D^{<\omega}$ be the set of finite sequences of elements of D. By dovetailing we can define a total computable function $\gamma : D^{<\omega} \to D$ such that, for all $(d_0, ..., d_k) \in D^{<\omega}$,

$$d_i \not\leq \gamma(d_0, ..., d_k) \text{ for all } i = 0, ..., k$$

Let $b : \mathbb{N} \to \mathbf{2}^*$ be such that $b(0)$ is the empty word and $b(2n+1) = b(n)0$ and $b(2n+2) = b(n)1$. As is well known (cf. Li & Vitanyi [17], p.12), b is a total computable bijection which is length increasing: $i < j \Rightarrow |b(i)| \leq |b(j)|$, so that

$$\{b_i : i \leq 2^k - 2\} = \{q \in \mathbf{2}^* : |q| < k\}$$

Let $G = \max^{\mathcal{D}} g$ where $g : \mathbf{2}^* \times \mathbb{N} \to D$ is total computable. Define a total computable $F : \mathbf{2}^* \to \mathcal{D}$ as follows:

$$F(p) = \gamma(g(b_0, 0), ..., g(b_{2^{2|p|}-2}, 0))$$

By definition of F, we see that $g(q, 0) \not\leq F(p)$ for all q such that $|q| < 2|p|$. In particular, if $|q| < 2|p|$ and $G(q)$ is defined, since $g(q, 0) \leq G(q)$ we have $F(p) \neq G(q)$. This insures that $K_G(F(p)) \geq 2|p|$. Since, obviously, $K_F(F(p)) \leq |p|$, we get $K_G(F(p)) \geq K_F(F(p)) + |p|$. Which proves that $K_G - K_F$ takes arbitrarily large values, hence G cannot be optimal in $Max_{Rec}^{\mathbf{2}^* \to \mathcal{D}}$. Since F is total computable, this also proves the last assertion of Point 2. $\square$

Applying Theorem 43 to $\mathbb{N}$ and $\mathbb{Z}$ with the natural orderings, we get the following result. It is interesting to compare Point 1 with Proposition 5.

Corollary 44. *1. The invariance theorem holds for the class* $Max_{Rec}^{\mathbf{2}^*\to\mathbb{N}}$*. Moreover, optimal functions in* $Max_{Rec}^{\mathbf{2}^*\to\mathbb{N}}$ *are optimal for the class* $Max_{PR}^{\mathbf{2}^*\to\mathbb{N}}$*. In particular, the Kolmogorov complexity associated to* $Max_{Rec}^{\mathbf{2}^*\to\mathbb{N}}$ *coincides (up to a constant) with that associated to* $Max_{PR}^{\mathbf{2}^*\to\mathbb{N}}$*.*

2. The invariance theorem fails for the classes $Min_{Rec}^{\mathbf{2}^*\to\mathbb{N}}$*,* $Max_{Rec}^{\mathbf{2}^*\to\mathbb{Z}}$ *and* $Min_{Rec}^{\mathbf{2}^*\to\mathbb{Z}}$*.*

Since *Reg* with the inclusion ordering (cf. §2.4.5) has a minimum and a maximum element (namely $\emptyset$ and $\widetilde{\Sigma}$), we get:

Corollary 45. *The invariance theorem holds for the classes* $Max_{Rec}^{\mathbf{2}^*\to Reg}$ *and* $Min_{Rec}^{\mathbf{2}^*\to Reg}$*. In particular, the associated Kolmogorov complexities coincide (up to a constant) with those associated to* $Max_{PR}^{\mathbf{2}^*\to Reg}$ *and* $Min_{PR}^{\mathbf{2}^*\to Reg}$*.*

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