Cellular Automata on Infinigonal Grids of the Hyperbolic Plane

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Abstract

In this paper, we consider cellular automata on special grids of the hyperbolic plane: the grids constructed on infinigons, *i.e.* polygons with infinitely many sides. We show that the truth of arithmetical Σ_n formulas can be decided in finite time with infinite initial recursive configurations. Next, we define a new kind of cellular automata, endowed with data and more powerful operations that we call *register cellular automata*. This time, starting from finite configurations, it is possible to decide the truth of Σ_n formulas in linear time with respect to the size of the formula.

1 Introduction.

Much of the attraction of hyperbolic geometry comes from the strong esthetic impression given by tilings that can be obtained in the hyperbolic plane $I\!H^2$. That plane can in fact be tiled in infinitely many regular ways by tessellations starting from a convex regular polygon. This is well known from Poincaré's theorem. On that regard, hyperbolic geometry of the plane is much richer than its euclidean counterpart. However, that latter one has its revenge in high dimensions: there are always tilings of the euclidean space of dimension p for any $p \geq 2$ that are based on a regular polyhedron, while this is never true for the hyperbolic space of the same dimension p, already when $p \geq 5$.

The study of cellular automata on regular tessellations in the hyperbolic plane started with [3, 4]. Paper [1] gave a new impulse to the study by bringing new tools that solve the problem of locating cells in the rectangular regular grids of the hyperbolic plane. The simple tools of elementary arithmetics that are given in [1] strengthen the conviction that hyperbolic geometry should be more studied and that this could bring in a lot of new fascinating results.

The present paper considers cellular automata in a completely new settings of the hyperbolic plane: regular tessellations by special polygons that have infinitely many sides, which we call infinigons, see [2].

In such tessellations, cellular automata have infinitely many neighbours, so that some convention must be done on the exchange of information that should always be finite from the point of view of computer science.

In the first section, we review the main features of tessellations by infinigons.

In the second section, we extend the traditional definition of cellular automata to this new context. We show then that starting from infinite recursive configurations, it is possible to decide the truth of Σ_n formulas in time n. As a consequence, this shows that the n-th iterate of the transition function operating on configurations is Δ_{n+1} but not Σ_n.

In the third section, we propose a new model of CA$'s$, that will allow us to decide Σ_n formulas in linear time starting from finite configurations.

2 Infinigonal grids in $I\!H^2$.

2.1 Poincaré's model of $I\!H^2$.

For reasons that are connected with the homogeneity property of cellular automata, we shall consider Poincaré's unit disk as a model of $I\!H^2$. Recall that points of $I\!H^2$ are identified with points of the open unit disk, say U. Points of the unit circle ∂U do not belong to $I\!H^2$. However, for obvious reasons, they are called the points at infinity of $I\!H^2$. In U, a line is either the track in U of a diameter of ∂U or the track in U of a circle that is orthogonal to ∂U. Lines that have no common point, neither in U nor on ∂U are called *non-secant*.

Following [5] and [6], we shall argue in the south-western quarter of Poincaré unit disk. We refer to these papers for more details.

2.2 Infinigons.

It is well known that in the hyperbolic plane $I\!H^2$, there are *always* polygons with equal sides for which the vertex angle is a right angle at every vertex, provided that the number of sides is at least five.

As is indicated in [2], assume that we display these polygons for all possible number of sides s, with $s \geq 5$, in such a way that all these polygons have a common vertex O and their edges that meet in O are supported by the same orthogonal lines. As is suggested by Figure 1, below, we see that these polygons tend to a *limit* which has an infinite number of sides. It looks like a polygon, but it is not a finite figure: it has a point at infinity.

Following [2], we call such an object an *infinigon*. As is shown in [2], such an infinigon is circumscribed by a curve Γ which is a circle in the unit disk model of $I\!H^2$ but that is no more a circle in $I\!H^2$: indeed Γ is an euclidean circle that is tangent to the unit disk. It is called a *horocycle*. That horocycle is also the limit of the hyperbolic circles that circumscribe the polygons in the right part of Figure 2. Horocycles have also an important property. They are globally invariant trajectories of *ideal rotations*. This gives another way to define infinigons As regular polygons are attached to rotations of $\dfrac{2\pi}{k}$ for which they are invariant, infinigons are attached to *any* ideal rotation.

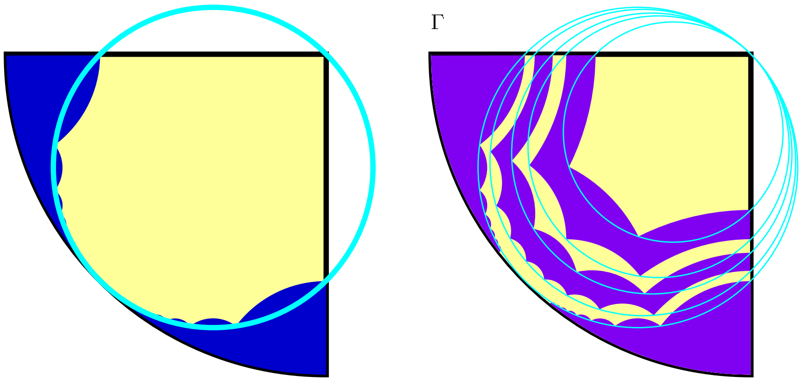

Figure 1 The infinigon with angle $\frac{\pi}{2}$ and some of its finite sided approximations.

As indicated in [2], there are always infinigons with angle α, whatever be α.

2.3 Infinigrids and their Canonical Enumerations.

Another remarkable property of the infinigons is that when the interior angle between consecutive sides is $\frac{2\pi}{k}$ for $k \geq 3$, it is possible to tile the plane with copies of the initial infinigon, say $\mathcal{I}$ by reflecting $\mathcal{I}$ in its sides and, recursively, the new infinigons in their sides. This result is sometimes considered as folklore or being implicit in classical works on hyperbolic geometry, possibly in Poincaré's himself. As far as we know, the only explicit and detailed construction is [2].

We shall investigate cellular automata in such grids. In order to simplify the study, we shall only consider *rectangular* grids, i.e. grids with rectangular infinigons. But [2] gives tools to deal with any infinigonal grid.

As is clear from Figure 1, the family of (closest) neighbours of any infinigon of the grid is naturally indexed by $\mathbb{Z}$ (as is its family of sides).

Also, in the case of rectangular grids, two consecutive neighbours of any infinigon I have a common neighbour different from I (the picture is similar to that of the hexagonal grid of the euclidean plane).

Consider the family $Seq(\mathbb{Z})$ of all finite sequences of integers in $\mathbb{Z}$. Let's denote $s \frown j$ the extension of s by j. Let $\mathcal{R}$ be the symmetric binary relation over $Seq(\mathbb{Z})$ which contains the pairs

$$(s \frown i \frown -1, s \frown i-1)\ ,\ (s \frown i \frown 0, s)$$
$$(s \frown i \frown 1, s \frown i+1)\ ,\ (s \frown i \frown 2, s \frown i+1 \frown -2)$$

where $i \neq -2$ or s is empty, and all pairs

$$(s\frown j \frown -2 \frown -1, s \frown j-1 \frown 3)\ ,\ (s\frown j \frown -2 \frown 0, s \frown j-1)$$
$$(s\frown j \frown -2 \frown 1, s \frown j)\ ,\ (s\frown j \frown -2 \frown 2, s \frown j \frown 3)$$

Identifying sequences $s, t \in Seq(\mathbb{Z})$ if $(s,t) \in \mathcal{R}$, we get the set σ.

Let $\mathcal{N}$ be the symmetric binary relation over σ which contains the pairs $(s, s\frown j)$ and $(s \frown j, s\frown j+1)$ modulo $\mathcal{R}$ identifications.

Fix a particular infinigon I_ϵ of a rectangular grid and two consecutive neighbours I_0, I_1 of I_ϵ. To this triple (I_ϵ, I_0, I_1) we can associate a canonical enumeration $s \mapsto I_s$ of the infinigrid by σ such that the neighbourhood graph of the infinigrid coincides with $\mathcal{N}$. The intuitive idea is as follows:

i) A priori the neighbours of I_s are the $I_{s\frown i}$'s.
ii) The neighbourhood relation being symmetric, one of the $I_{s\frown i\frown j}$'s has to be identified with I_s. We choose to identify I_s and $I_{s\frown i\frown 0}$.
iii) Considering the successive neighbours $I_{s\frown i-1}$, $I_{s\frown i}$ and $I_{s\frown i+1}$ of I_s, we see that $I_{s\frown i-1}$ and $I_{s\frown i+1}$ are neighbours of $I_{s\frown i}$. Thus, some of the $I_{s\frown i\frown j}$'s have to be identified with $I_{s\frown i-1}$ and $I_{s\frown i+1}$. We choose to identify $I_{s\frown i-1}$ with $I_{s\frown i\frown -1}$ and to identify $I_{s\frown i+1}$ with $I_{s\frown i\frown 1}$.
iv) As already observed, consecutive neighbours $I_{s\frown i}$, $I_{s\frown i+1}$ of I_s have a common neighbour different from I_s. Thus, some of the $I_{s\frown i\frown j}$'s has to be identified with some of the $I_{s\frown i+1\frown j}$'s. We choose to identify $I_{s\frown i\frown 2}$ with $I_{s\frown i+1\frown -2}$.
v) Points *i)* to *iv)* give the four first families of pairs in $\mathcal{R}$. The other four families are obtained by considering the neighbours of $I_{s\frown i\frown -2}$ and fixing the needed identifications.

See Figure 2 (cf. [2] for details).

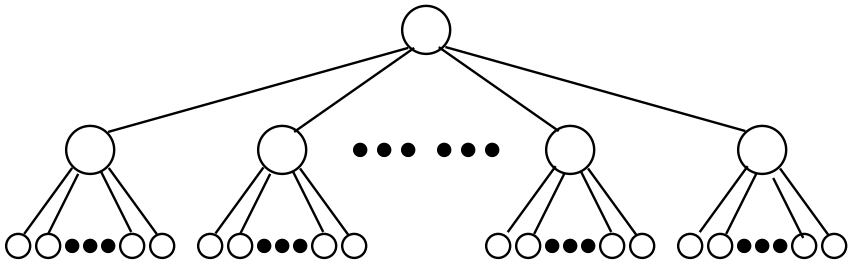

Figure 2 The infinite tree associated to the infinigrid. Two levels are indicated on the representation. It is not difficult to see that $\mathbb{Z}$ provides natural *addresses* to the sons of a node.

3 Cellular Automata on an infinigonal grid.

Now, consider a rectangular infinigrid, which we identify to the set σ introduced above. As usual in the theory of cellular automata, we associate a finite automaton to each cell of the infinigrid, the same automaton for each cell.

But there is a problem: how to define the exchange of information between the cells? As indicated in the introduction, computer science asks that whatever be the objects that we consider, they must exchange only finite amounts of information. And when we mean *finite*, we really mean *bounded by a fixed constant.*

Before defining the local table of transition, we have to define the neighbourhood of a cell. As a cell has an infinite number of neighbours, it is difficult to select a finite set of them that could be natural and uniform for all cells. On another side, we have to comply to the limitation set upon the amount of information exchange between cells. And so, we suggest the following notion.

Definition 1 – A cellular automata on the infinigrid is *k-isotropic* if each cell s can only know its own state and, for each state $q \in Q$, whether there are $0, 1, \ldots, k$ or $> k$ of its neighbours which are in state q.

Let Q be the set of states, let $X(s,i,t)$ (resp. $X(s,i^+,t)$) denote the set of states $q \in Q$ such that at time t there are exactly i (resp. at least $i+1$) neighbours of s in state q, and let $< s,t >$ denote the state of cell s at time t. The above definition expresses that the transition function δ maps $Q \times (2^Q)^{k+2}$ into Q, so that

$$< s,t+1 >= \delta(< s,t >, X(s,0,t), \ldots, X(s,k,t), X(s,k^+,t))$$

Notation 1 – We denote Δ the global transition function which operates on the space of configurations of the infinigrid, i.e $\Delta : Q^\sigma \to Q^\sigma$.

Theorem 1 – *1) There is a boolean combination $\Phi(x,X)$ of Σ^0_n formulas which defines the n-th iterate $\Delta^{(n)}(C)$ of the global transition function applied to an initial configuration $C : Q^\sigma \to Q$, i.e. the relation*

$$x \in \Delta^{(n)}(C)$$

2) There is an 0-isotropic cellular automata such that any Σ^0_n or Π^0_n arithmetical formula $F(x_1,\ldots,x_k)$ with k free variables is recursively encoded in $\Delta^{(n+\min(k,1))}(C_F)$ where $C_F : \sigma \to Q$ is a (recursive) configuration of σ and $F \mapsto C_F$ is recursive. In particular, the above formula $\Phi(x,X)$ can not be Σ^0_n.

Proof. *1)* The set $X(s,i^+,t)$ is definable as follows:

$$\bigwedge_{q\in Q}(q \in X(s,i,t) \Leftrightarrow \exists s_0 \ldots \exists s_i \text{ (the } s_j\text{'s are distinct neighbours of } s))$$

Also, $X(s,i,t) = X(s,(i-1)^+,t) \setminus X(s,i^+,t)$. This shows that the global transition function Δ is a boolean combination of Σ^0_1 relations. By composition, we get a Δ^0_{n+1} definition of the n-th iterate $\Delta^{(n)}$.

2) We only consider closed formulas; the general case being simple adaptation. Fix n and consider a closed Σ_n formula

$$F = \exists\, x_1\, \forall\, x_2\ \ldots\ \xi\, x_n\ G(x_1,\ldots,x_n)$$

where $G(x_1,y_1,\ldots,x_n,y_n)$ is a primitive recursive term with value 0 for false and 1 for true and ξ is the quantifier $\forall$ if n is even (resp. $\exists$) if n is odd).

Initialize the infinigrid as follows: in cell with address $a_1 \ldots a_n$, we put the value of $G(a_1,\ldots,a_n)$ as a state (in $\{0,1\}$). We also put (as a second component of the state) the parity of the depth of the cell so as to indicate the quantifier that corresponds to the depth of the cell. The root is in a starting state D. Beyond the n^{th} level of the tree, all cells are in a quiescent state $\#$.

We can also view the display as indicated in Figure 3, where each cell appears as a *supervisor* of a line constituted of infinitely many cells. By definition of 0-isotropic cellular automata, the supervisor s can see whether or not (at time t) there is some cell in state 1 (it checks whether $X(s,0^+,t)$ is non empty or empty).

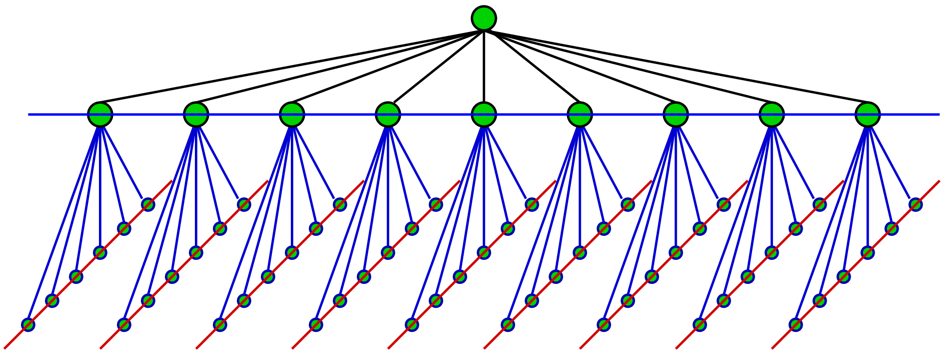

Figure 3 A subgraph of the infinigrid neighbourhood

It is now easy to see that the cells that supervise a line of #'s enter a *flashing* state 0 or 1, depending on their initial value. The supervisor knows whether it is an existential or a universal quantifier and it enters state 0 or 1 according to the nature of this quantifier. This parallel bottom-up process goes on until the root is reached. According to the global state of the line, the root displays a final state 0 or 1, depending on whether there is an 0 on its supervised line or there are only 1's. ■

To stress the role of the hyperbolic plane in the previous result, let's consider diverse notions of cellular automata in the euclidean plane which also use sets of states of infinite subfamilies of cells. Fix a finite neighbourhood $\mathcal{V} = \{\vec{v_1}, \ldots, \vec{v_k}\}$ of the euclidean plane. Let's denote $< (x,y) + \mathcal{V}, t >$ the sequence $(< (x,y) + \vec{v_1}), t >, \ldots, < (x,y) + \vec{v_k}), t >)$
i) Let $\delta_1 : Q^k \times 2^Q \rightarrow Q$ be a transition function such that
$< (x,y), t+1 >= \delta_1(< (x,y) + \mathcal{V}, t >, \{< (z, y+1), t >: \ z \in \mathbb{Z}\}$ *ii)*
Let $\delta_2 : Q \times 2^Q \rightarrow Q$ be the transition function such that
$< (x,y), t+1 >= \delta_2(< (x,y) + \mathcal{V}, t >, \{< (x', y'), t >: \ (x', y') \neq (x,y)\}$
One can prove the n-th iterations $\Delta_1^{(n)}$ and $\Delta_2^{(n)}$ of the global transition functions of such cellular automata in the euclidean plane are always Δ_2^0. Which contrasts with the result of the above theorem relative to the hyperbolic plane.

4 Register Cellular Automata on an infinigonal grid.

The previous construction to decide Σ_n^0 arithmetical truth has the inconvenient to use infinite initial configurations. It should be nice to initialize an infinite configuration in finite time, which seems to be possible with infinigons. Indeed, a cell may start infinitely many computations at the same time by sending an appropriate signal to its neighbours which shall be seen by all of them at the same top of the clock.

However, sending a signal is not enough. The (neighbour) cells must be able to perform distinct computations. To that purpose, it is reasonable that they know their address and have the possibility to use it for computation. As the address may be encoded in arbitrary large natural numbers, we give the cell decoding functions that are considered as working in one step of computation. For the same reason, as the cell has at its disposal finitely many states only, we give it the possibility to translate the result of its computation into an appropriate state in one step of computation.

Accordingly, we introduce an extension of the notion of cellular automata that is adapted to infinigrids:

Definition 2 – Register cellular automata *on the rectangular infinigrid are variants of isotropic cellular automata such that*

- each cell is fitted with a fixed finite automaton A; one of the states of the cell is called **quiescent***; two states are called* **final***, one for* **acceptation***, the other one for* **rejection***.*
- each cell is fitted with two **registers***,* **a** *and* **x***;* **a** *is read-only and contains an integer which encodes the address of the cell;* **x** *is a read-write register that contains an integer and that the cell uses for its computations;*
- each cell is also endowed with the following facilities:
  - *it may freely copy the contents of* **a** *into* **x***;*
  - *it computes the following functions in one step:*
    *+, -, /, *,* mod*, sg, $\overline{sg}$ and $(n)_i$ for all i with $1 \leq i \leq |n|$, where $|n|$ is the number of terms encoded in n, the address of the cell or the content of* **x***.*
- data are given to the root in unary and at the initial time, all cells but the root are in the quiescent state;
- the computation ends when the root enters a final state;

Notice that (apart from the distribution of adresses) register cellular automata have finite initial configurations: the sole root may be in non quiescent state.

Taking as guidelines the proof of Theorem 1, it is possible to prove the following result:

Theorem 2 – *Register cellular automata on the rectangular infinigrid are able to decide the truth of any Σ_n formula. Moreover, they can perform the needed computation in time linear in the length of the formula.*

Proof. To the proof of Theorem 1, we add the following ingredient, taken from [5]:

There is a polynomial P such that for all e, n and y, $\varphi_e(n) \simeq y$ if and only if there are integers $x_1, \ldots, x_9$ such that $P(e, n, y, x_1, \ldots, x_9) = 0$, where φ_e is the partial recursive function with number e.

From the definition, after a time which is linear in $e+n+y$, all cells with address $<e, n, y, x_1, \ldots, x_9>$ compute the value of $P(e, n, y, x_1, \ldots, x_9)$. This needs the same time for all of them. And so, it is possible to perform the computation of $P(e, n, y, x_1, \ldots, x_9)$ on the infinigrid in constant time. From the argument of Theorem 1, the backward signal reaches the root in 12 steps. And so, our claim is proved for Σ_1 formulas. Now, we repeat the argument of Theorem 1 and, by induction on n, we get that the register cellular automaton is able to decide the truth of any formula of Σ_n. ■

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