Syntactical Truth Predicates for Second Order Arithmetic

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Abstract

We introduce a notion of *syntactical truth predicate* (s.t.p.) for the second order arithmetic $\mathbf{PA^2}$. An s.t.p. is a set T of *closed* formulas such that:

i) $T(t = u)$ iff the closed first order terms t and u are convertible, *i.e.* have the same value in the standard interpretation

ii) $T(A \to B)$ iff $(T(A) \Rightarrow T(B))$

iii) $T(\forall x A)$ iff $(T(A[x \leftarrow t])$ for any closed first order term $t)$

iv) $T(\forall X A)$ iff $(T(A[X \leftarrow \Delta])$ for any closed set definition $\Delta = \{x | D(x)\})$

S.t.p.'s can be seen as a counterpart to Tarski's notion of *(model-theoretical) validity* and have main model properties. In particular, their existence is equivalent to the existence of an ω-model of $\mathbf{PA^2}$, this fact being provable in $\mathbf{PA^2}$ with arithmetical comprehension only.

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Contents

1 Introduction

The language of second order Peano arithmetic $\mathbf{PA^2}$ (cf. Simpson [27], Takeuti [29] or Girard [12]) is that of a type theory with two types of objects: *natural numbers* and *sets* of natural numbers, respectively denoted by the letters $x, y, \ldots$ and $X, Y, \ldots$. Atomic formulas are *membership* statements $t \in X$ and *equality* statements $t = u$ where t, u are first order terms built from first order variables via the function symbols 0, S, $+$, $\times$ (of course, many variants are possible).

Notation 1.1 Let A, D be formulas. We denote $A[X \leftarrow \lambda x.D]$ the formula obtained by substituting $D[x \leftarrow t]$ for every atomic subformula $t \in X$ in A. (here we use Church's lambda notation for the propositional function $x \mapsto D(x)$).
We call $\lambda x.D$ a *set definition*. When x is the sole variable possibly free in D we say that $\lambda x.D$ is a *closed set definition*.

Clearly, $A[X \leftarrow \lambda x.D]$ corresponds to some normal form of an instantiation of the formula A on the set of numbers x such that $D(x)$ (the existence of which asks for some comprehension axiom).

Tarski's *semantical truth* predicate for the standard model $(\mathbf{N}, P(\mathbf{N}))$ of second order arithmetic is a third order object T_0 which is a set of pairs (F, ρ)

of formulas and environments, i.e. assignments of integers and sets of integers to first and second order variables. It can be defined by induction, the most interesting cases being :

- $T_0(t \in X, \rho) \Leftrightarrow n \in \rho(X)$ where n is the value of the term t under ρ.
- $T_0(A \to B, \rho) \Leftrightarrow (T_0(A, \rho) \Rightarrow T_0(B, \rho))$
- $T_0(\forall X A, \rho) \Leftrightarrow \forall S(T_0(A, (X, S) :: \rho))$ where $(X, S) :: \rho$ is an ML-like notation for the environment associating S to the variable X and $\rho(Y)$ to any other variable Y.

For *closed* formulas the environment is irrelevant so one can simply write $T_0(A)$. If A, B and $\forall XC$ are closed we have

- $T_0(A \to B) \Leftrightarrow (T_0(A) \Rightarrow T_0(B))$
- $T_0(\forall XC) \Rightarrow T_0(C[X \leftarrow \lambda x.D])$ for any closed set definition $\lambda x.D(x)$.

Guided by the above observation, we introduce the concept of *syntactical truth* predicate for second order arithmetic, which can be taken as a semantics for $\mathbf{PA^2}$ formalizable in the language of $\mathbf{PA^2}$. This notion is also connected with Dragalin's semi-formal system for the theory of definable sets of natural numbers (see [6, 7, 8]).
In the classical framework it is enough to consider universal first order and second order quantifiers and the entailment connective $\to$.

Definition 1.2 *A* syntactical truth *predicate (s.t.p.) is a set T of* **closed** *formulas of second order arithmetic such that*

i) $T(t = u)$ *iff the closed terms t and u are convertible,* i.e. *have the same value in the standard interpretation*

ii) $T(A \to B)$ *iff* $(T(A) \Rightarrow T(B))$

iii) $T(\forall x A)$ *iff* $(T(A[x \leftarrow t])$ *for any closed term t)*

iv) $T(\forall X A)$ *iff* $(T(A[X \leftarrow \lambda x.D(x)])$ *for any closed set definition* $\lambda x.D(x)$*)*

Note 1.3 1. Since s.t.p.'s deal with closed formulas, no membership case has to be considered in the previous definition.

2. In *iv)* $A[X \leftarrow \lambda x.D(x)]$ may be a bigger formula than $\forall X A$. Thus, s.t.p.'s are intrinsically circular objects which can not be handled with recursive constructions. Hence their existence is non-trivial.

3. The terminology *syntactical* truth predicates should be clear: here we interpret the universal quantification $\forall X$ by a quantification over *definitions* of sets, *i.e.* over a piece of syntax.

4. Whereas Tarski's semantical truth predicate is a third order object, syntactical truth predicates remain at the second order.

5. If the language is augmented with the connectives $\leftrightarrow$, $\vee$, $\wedge$, $\neg$ and the existential quantifier defined in the usual way (with $\perp$ being $0 = 1$ in the definition of $\neg$) then every s.t.p. T satisfies the following derived clauses:

ii bis) $T(A \wedge B)$ *iff* $(T(A)$ *and* $T(B))$
$T(A \vee B)$ *iff* $(T(A)$ *or* $T(B))$
$T(A \leftrightarrow B)$ *iff* $(T(A) \Leftrightarrow T(B))$
$T(\neg A)$ *iff (not* $T(A))$

iii bis) $T(\exists x A)$ *iff* $(T(A[x \leftarrow t])$ *for some closed term* $t)$

iv bis) $T(\exists X A)$ *iff* $(T(A[X \leftarrow \lambda x.D(x)])$ *for some closed set definition* $\lambda x.D(x))$

As for first order formulas a syntactical truth predicate is nothing but arithmetical truth:

Lemma 1.4 *Let* T *be a syntactical truth predicate. A closed first order formula is in* T *if and only if it is true in the standard interpretation with base* $\mathbf{N}$.

Proof We prove in fact that for every first order formula $F(x_1, \ldots, x_n)$ with free variables among $x_1, \ldots, x_n$ and for all closed terms $t_1, \ldots, t_n$ the formula $F(t_1, \ldots, t_n)$ is in T iff it is true in the standard interpretation with base $\mathbf{N}$. The proof is a straightforward induction on F. $\square$

The sequel of this paper is organized as follows.
In §2 we look at the status of Leibniz's equality axiom and the induction axiom, which happen to be equivalent with respect to s.t.p.'s. In §3, using Gödel's hierarchy of constructible sets, we prove in set theory the existence of an s.t.p. (which is in fact Leibniz, i.e. satisfies Leibniz's axiom). In §4 we recall the syntax of second order logic $\mathbf{L^2}$ and second order arithmetic $\mathbf{PA^2}$. We also introduce a second-order conservative extension $\mathbf{PA^1}$ of Peano arithmetic. In §5 we show that s.t.p.'s (resp. Leibniz s.t.p.'s) are an appropriate semantics of second order logic (resp. arithmetic), *i.e* that any closed provable formula is true under any such truth predicate. This is shown in $\mathbf{PA^1}$. In §6 we show that every Leibniz s.t.p. is the *semantical* truth predicate of some associated ω-model of $\mathbf{PA^2}$. From a (non Leibniz) s.t.p. we construct a semantical truth predicate of some ω-model of $\mathbf{PA^2}$. These results are also shown in $\mathbf{PA^1}$. Finally, in §7, refining the existence proof done in ZF, we construct an s.t.p. from the truth predicate of any ω-model of $\mathbf{PA^2}$. The construction is done in $\mathbf{PA^1}$. This proves in $\mathbf{PA^1}$ the equivalence between the existence of an s.t.p. and that of the semantical truth predicate of an ω-model of $\mathbf{PA^2}$. Also, a long Appendix details some developments of the ramified analytical hierarchy within the formal system $\mathbf{PA^2}$ necessary for the proof of the above last result.
All along the paper, provability in $\mathbf{PA^1}$ is refined to provability in $(I\Sigma^0_1, C\Delta^0_1)$-$\mathbf{PA^1}$, (i.e. Σ^0_1-induction plus Δ^0_1-comprehension).

2 Equality and syntactical truth predicates

2.1 Leibniz syntactical truth predicates

Recall that the second order characterization of equality

$$\forall x \forall y (x = y \leftrightarrow \forall X (x \in X \leftrightarrow y \in X)))$$

is a formula equivalent (modulo some weak comprehension axioms) to the conjunction of the axiom of equality $\forall x\ x = x$ and Leibniz's axiom

$(Leibniz)\quad \forall x \forall y (x = y \rightarrow \forall X (x \in X \rightarrow y \in X))$

(for the $\leftarrow$ direction, use the set definition $\lambda z.(x = z)$).

Proposition 2.1 *Let T be a syntactical truth predicate.*

1. *The following statements are equivalent*

   (a) $T(\forall x \forall y(x = y \to (A(x) \to A(y))))$ *for every second order formula* $A(x)$ *in which x is the sole free variable.*

   (b) $T(A(t)) \Rightarrow T(A(u))$ *for every second order formula A in which x is the sole free variable and every closed terms t, u with the same value.*

   (c) $T(\forall x A) \Leftrightarrow$ *(*$T(A[x \leftarrow S^n(0)])$ *for every* $n \in \mathbf{N}$*)* *for every second order formula A in which x is the sole free variable.*

   (d) $T(\forall x \forall y(x = y \to \forall X(x \in X \to y \in X)))$

2. *If A is a* first order *formula then statements (a),(b),(c) above are all true.*

Proof 1) (a) $\Leftrightarrow$ (b) and (a) $\Leftrightarrow$ (d) come from clauses *(iii)* and *(iv)* in the definition of s.t.p.'s.
(b) $\Rightarrow$ (c) Since every closed term is equal to some $S^n(0)$ we deduce (c) from clause *(iii)* and (b).
(c) $\Rightarrow$ (b) Clauses *(i),(ii)* yield $T(S^n(0) = S^m(0) \to (A(S^m(0) \leftrightarrow A(S^n(0))))$ for every $m, n \in \mathbf{N}$. Applying (c) with formula $(x = S^m(0) \to (A(S^m(0) \leftrightarrow A(x))$, we get $T(\forall x(x = S^m(0) \to (A(S^m(0) \leftrightarrow A(x)))$. Therefore, clause *(iii)* yields $T(A(S^m(0) \leftrightarrow A(t)))$, i.e. $T(A(S^m(0))) \Leftrightarrow T(A(t))$ for every term t with value m . Whence (b).
2) Apply Lemma 1.4 and use Point 1. □

This leads to the following Definition:

Definition 2.2 *A syntactical truth predicate is* Leibniz *if it contains Leibniz's axiom* $\forall x \forall y(x = y \to \forall X(x \in X \to y \in X))$.

Remark 2.3

1. Clearly, every s.t.p. satisfies the equality axiom $\forall x\ x = x$.
2. We shall see that Leibniz s.t.p.'s do exist. However, we do not know whether there exist non Leibniz s.t.p.'s.

3. In §7 we develop a complicate construction, formalizable in $\mathbf{PA^1}$, which gives a Leibniz s.t.p. from any s.t.p. We know no direct simple such construction: all simple attempts (using classical relativization techniques) fail due to the circular character of clause *(iv)* in the definition of s.t.p.'s which prevents any induction over formulas.

4. Clause *(iii)* for s.t.p.'s does depend on the language for first order terms. In particular, suppose we reduce that language to 0 and the sole successor function S. Second order quantifications over binary relations R allow to define $+$ and $\times$ from S as follows: $x + y = z$ iff

   $$\forall R\,(((0,x) \in R \wedge (\forall v \forall w((v,w) \in R \rightarrow (Sv,Sw) \in R))) \rightarrow (y,z) \in R)$$

   Similar for $\times$, using the above definition of $+$. If we extend in the obvious way clause *(iv)* for s.t.p.'s to quantifications over binary relations then every s.t.p. for that language is trivially Leibniz since clause *(iii)* then coincides with condition (c) of the above Proposition.

2.2 Inductive syntactical truth predicates

The following result relates Leibniz's axiom and the induction axiom

$(Ind)\quad \forall X(0 \in X \wedge \forall x(x \in X \rightarrow Sx \in X)) \rightarrow \forall x\; x \in X$

with respect to syntactical truth predicates.

Lemma 2.4 *An s.t.p. T is Leibniz if and only if it contains Ind.*

Proof $(\Rightarrow)$ Let A be a formula with at most one free variable x. We have to show that if $T(A[x \leftarrow 0])$ and $(T(A[x \leftarrow t]) \Rightarrow T(A[x \leftarrow S(t)])))$ for every closed term t then $T(A[x \leftarrow u])$ holds for any closed term u. But an easy induction on n shows $T(A[x \leftarrow S^n(0)])$ for all n. Let p be the value of u, from $T(A[x \leftarrow S^p(0)])$ we then get $T(A[x \leftarrow u])$ since T is Leibniz.
$(\Leftarrow)$ Suppose $T(Ind)$ and let $H(x,y)$ be $\forall X(x = y \rightarrow (x \in X \leftrightarrow y \in X))$. Given the fact that T is an s.t.p. we have to show that $T(\forall x \forall y H(x,y))$. Since T contains the induction axiom, we argue by induction on x and y.

- $T(\forall y H(0,y))$ is shown by induction on y : $T(H(0,0))$ trivially holds.

$T(H(0, S(u)))$ holds since $T(0 = S(u))$ means that 0 and $S(u)$ have the same value which is impossible.

- Assume that $T(\forall y H(t, y))$ holds. We show by induction on y that $T(\forall y H(S(t), y))$ holds:
  - $T(H(S(t), 0))$ holds since $T(S(t) = 0)$ is impossible
  - To prove $T(H(S(t), S(u)))$ let $D = \lambda x A$ be a closed set definition. Assume that $T(S(t) = S(u))$, we have to establish that $T(A[x \leftarrow S(t)] \Leftrightarrow T(A[x \leftarrow S(u)]$. But since T is an s.t.p. $S(t)$ and $S(u)$ must have the same value hence so do t and u. By induction hypothesis we have $T(H(t, u))$, thus letting $D' = \lambda x A[x \leftarrow S(x)]$ we can instantiate $H(t, u)$ on D' and we get $T(A[x \leftarrow S(t)] \Leftrightarrow T(A[x \leftarrow S(u)]$. $\Box$

3 Existence Theorem in ZF

We now prove an existence theorem as an application of Gödel's hierarchy of constructible sets ([13]).

Theorem 3.1 *The existence of a Leibniz syntactical truth predicate is provable in Zermelo-Fraenkel set theory.*

Proof Let $\mathcal{P}(N)^L$ be the set of constructible sets of natural numbers in the sense of [13]. Let T be the restriction to closed formulas of Tarski's semantical truth predicate in the structure $(N, \mathcal{P}(N)^L)$. We claim that T is a syntactical truth predicate.

Clauses *(i)*, *(ii)* and *(iii)* for s.t.p.'s are trivially satisfied. We prove clause *(iv)* in its existential version (clearly equivalent in the presence of clause *(ii)*):

$$T(\exists X A) \text{ iff } (T(A[X \leftarrow \lambda x.D(x)]) \text{ for some closed set definition } D)$$

We first prove the $\Leftarrow$ implication. Since L is a model of the comprehension schema, there exists $U \in \mathcal{P}(N)^L$ such that $(N, \mathcal{P}(N)^L) \models \forall x(x \in U \leftrightarrow D(x))$. An easy induction over formulas F then shows that the semantical truth value in $(N, \mathcal{P}(N)^L)$ of $F[X \leftarrow \lambda x.D(x)]$ under environment ρ is that of F under environment $(X, U) :: \rho$. In particular, if (the closed formula) $A[X \leftarrow \lambda x.D(x)]$ is in T, *i.e.* is true, then A is true under environment (X, U), so that the closed formula $\exists X A$ is true, *i.e.* is in T.

We now prove the $\Rightarrow$ implication. It is established in [13], see also [1], that there exists a well-ordering on $\mathcal{P}(N)^L$ which is definable in $(N, \mathcal{P}(N)^L)$ by a second order formula. We will write $Less(X, Y)$ for such a formula. Let $A(X)$ be a formula with a second order variable X. If there exists in $\mathcal{P}(N)^L$ a set satisfying A then the least of such sets (in the sense of the well-order) is definable in $(N, \mathcal{P}(N)^L)$ by the following formula $D(x)$:

$$D(x) \Leftrightarrow \exists X(x \in X \wedge A(X) \wedge \forall Y(A(Y) \rightarrow \neg Less(Y, X)))$$

Thus, the formula $\exists X A(X) \rightarrow A[X \leftarrow \lambda x.D(x)]$ is in T. Applying clause *(ii)* we get $T(\exists X A) \Rightarrow T(A[X \leftarrow \lambda x.D(x)])$. □

Remark 3.2 Using generic models of set theory satisfying $V \neq L$ in which there are Δ^1_3 well-orderings of the continuum (see [16, 17, 18, 28]), the above construction leads to different syntactical truth predicates.

4 Second order logic and arithmetic

4.1 Syntax of second order logic in natural deduction

We follow Takeuti [29] (p.165-172, system **KC**) and Girard ([12], p.176-177), for the presentation of $\mathbf{L^2}$ and $\mathbf{PA^2}$. The second order language for these systems has already been introduced in §1. (*Note:* Set definitions introduced in Notation 1.1 are called *abstracts* in [29], p.168, and *abstraction terms* in [12], p.176).

Remark 4.1 Set definitions are not terms in the chosen syntax hence do not appear in formulas: there is no formula $t \in \lambda x.B$. In fact, substitution of set definition involves implicit β-reduction: we avoid any would-be formula $t \in \lambda x.B$ by immediately replacing it by its β-reduct $B[x \leftarrow t]$.

As usual, a *sequent* is a pair (Γ, A), also denoted $\Gamma \vdash A$, where Γ is a finite set of formulas and A is a formula. The intuitive meaning of $\Gamma \vdash A$ is that A holds under the set Γ of hypotheses. We write $\vdash A$ instead of $\emptyset \vdash A$.

Definition 4.2 The *axioms* and *inference rules* of $\mathbf{L^2}$ are:

$$\Gamma \cup \{A\} \vdash A \qquad (\text{A}ssumption)$$

$$\frac{\Gamma \cup \{A\} \vdash B}{\Gamma \vdash A \to B} \quad (\to i) \qquad\qquad (\to e) \quad \frac{\Gamma \vdash A \to B \qquad \Gamma \vdash A}{\Gamma \vdash B}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash \forall x A} \quad (\forall_1 i) \qquad\qquad (\forall_1 e) \quad \frac{\Gamma \vdash \forall x A}{\Gamma \vdash A[x \leftarrow t]}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash \forall X A} \quad (\forall_2 i) \quad (\forall_2 i) \qquad\qquad (\forall_2 e) \quad \frac{\Gamma \vdash \forall X A}{\Gamma \vdash A[X \leftarrow \lambda x.D]}$$

$$\Gamma \vdash (\neg\neg A) \to A \qquad (\text{C}lassic)$$

Here $\neg A$ stands for the intuitionistic negation $A \to (0 = 1)$. In these rules Γ is any finite set of formulas, A, B, D are formulas, x, X are any first, second order variables, t is a first order term. The left rules are *introduction rules* and the right ones *elimination rules*. The $\forall$ introduction rules are subject to the condition that the quantified variable is not free in a formula of Γ.

As usual, a *proof* is a finite sequence $J_1, \ldots, J_n$ of sequents such that for any $1 \leq i \leq n$ there exists $j_1, \ldots, j_k < i$ such that J_i can be obtained by one of the inference rules from $J_{j_1}, \ldots, J_{j_k}$. A *proof of* a sequent J is a proof ending by J. A sequent J is *provable* if there exists a proof of J. A formula A is *provable* if the sequent $\vdash A$ is provable.

Remark 4.3 Dual rules for $\exists_i$ are provable in $\mathbf{L^2}$. For instance, the $\forall_2$ elimination rule is equivalent to the $\exists_2$ introduction rule:

$$\frac{\Gamma \vdash A[X \leftarrow \lambda x.D]}{\Gamma \vdash \exists X A} \qquad (\exists_2 i)$$

This rule is also equivalent to the second order comprehension schema (cf.[12], p.177) $\forall \vec{y} \forall \vec{Y}\ \exists X\ \forall x (x \in X \leftrightarrow A(x, \vec{y}, \vec{Y}))$.

4.2 Second order arithmetic

Definition 4.4 1. Axioms and rules of $\mathbf{PA^2}$ are as follows:

(a) All axioms and rules of $\mathbf{L^2}$.

(b) Axioms for Robinson's first order elementary arithmetic (cf.[14] Def. 1.1 p.28 or [27], Def. I.2.4 i p.4).

(c) The axiom of equality $\forall x\ x = x$ and Leibniz's axiom

$(Leibniz) \quad \forall x \forall y (x = y \to \forall X (x \in X \to y \in X))$

(which imply $\forall x \forall y (x = y \leftrightarrow \forall X (x \in X \leftrightarrow y \in X))$, cf.§2.1).

(d) The axiom of induction, which is nothing but the definition of integers in second order logic,

$(Ind) \quad \forall X (0 \in X \wedge \forall x (x \in X \to Sx \in X)) \to \forall x\ x \in X)$

2. $C\Pi^1_1$-$\mathbf{PA^2}$ is obtained from $\mathbf{PA^2}$ by restricting the $\forall_2$ elimination rule to set definitions $\lambda x.A$ where A is a Π^1_1 formula. This corresponds exactly to restricting the comprehension schema to Π^1_1 formulas.

Remark 4.5 It is easy to see that $C\Pi^1_1$-$\mathbf{PA^2}$ yields the comprehension schema for boolean combinations of Π^1_1 formulas. In particular, $C\Pi^1_1$-$\mathbf{PA^2}$ does coincide with $C\Sigma^1_1$-$\mathbf{PA^2}$ (cf.[27], Def.I.5.2, p.16).

We shall also consider arithmetical theories with second order variables which are conservative extensions of usual first order arithmetical theories.

Definition 4.6 1. $\mathbf{PA^1}$ is obtained from $\mathbf{PA^2}$ by restricting the $\forall_2$ elimination rule to set definitions $\lambda x.A$ where A is a formula with no second order quantification (but which may contain second order free variables). This corresponds to restricting the comprehension schema to such formulas.

2. $(I\Sigma^0_1, C\Delta^0_1)$-$\mathbf{PA^1}$ is obtained from $\mathbf{PA^2}$ as follows:

(a) the comprehension schema is restricted to Δ^0_1-comprehension:

$$\forall x (A(x) \leftrightarrow \neg B(x)) \to \exists X\ \forall x (x \in X \leftrightarrow \forall x A(x)$$

where A, B run over Σ^0_1, i.e. formulas of the form $\exists \vec{z} C$ where C has no second order quantification and only bounded first order quantifications. This can also be expressed as some ad hoc $\forall_2$ elimination rule.

(b) the induction axiom is replaced by the Σ^0_1-induction schema

$$(A(0) \wedge \forall x(A(x) \rightarrow A(Sx))) \rightarrow \forall x A(x)$$

where A runs over Σ^0_1 formulas. (Observe that the induction axiom is a particular instance of this schema: take A to be $x \in X$).

Note 4.7 $\mathbf{PA^1}$ and $(I\Sigma^0_1, C\Delta^0_1)$-$\mathbf{PA^1}$ are respectively denoted ACA_0 and RCA_0 in papers about reverse mathematics, cf.[27].

Remark 4.8 1. $\mathbf{PA^1}$ (resp. $(I\Sigma^0_1, C\Delta^0_1)$-$\mathbf{PA^1}$) constitutes a conservative extension of usual first order Peano arithmetic $\mathbf{PA}$ (resp. $I\Sigma^0_1$-$\mathbf{PA}$, i.e. $\mathbf{PA}$ with the induction schema restricted to Σ^0_1 formulas). Cf.[27], Rk.I.3.3 p.7 and I.7.6 p.25.

2. The Σ^0_1-comprehension schema together with the induction axiom yield the Σ^0_1-induction schema. Thus,$(I\Sigma^0_1, C\Delta^0_1)$-$\mathbf{PA^1}$ is a restriction of $\mathbf{PA^1}$.

3. It is easy to deduce the comprehension schema for all formulas with no second order quantification from the comprehension schema for Σ^0_1 formulas (cf.[27], Rk.I.7.7 p.25). Whence the restriction to Δ^0_1-comprehension.

4. $(I\Sigma^0_1, C\Delta^0_1)$-$\mathbf{PA^1}$ coincides with $(I\Pi^0_1, C\Delta^0_1)$-$\mathbf{PA^1}$ (cf.[27], Def.I.7.2, p.24).

5. In the context of Δ^0_1 comprehension, the Σ^0_1-induction schema is equivalent to bounded Σ^0_1-comprehension, i.e. $\forall x\ \exists X\ \forall y(y \in X \leftrightarrow y < x \wedge A(y))$ where A is Σ^0_1 (cf.[27], Ex.II.3.13, p.72).

5 Soundness results for s.t.p.'s

Definition 5.1 *1. A* substitution *is a function σ associating to each first order variable x a term $\sigma(x)$ and to each second order variable X a set definition $\sigma(X)$.*

(a) We say σ is finite *if $\sigma(x) = x$ and $\sigma(X) = \lambda x.(x \in X)$ for all but finitely many variables x, X.*

(b) *We say σ is* closed *for a family of formulas $\mathcal{F}$ if all terms $\sigma(x)$ and all set definitions $\sigma(X)$ are closed for the variables x, X occuring free in some formula of $\mathcal{F}$.*

2. *If σ is a substitution and A is a formula with variables among $\vec{x} = x_1, \ldots, x_n$ and $\vec{X} = X_1, \ldots, X_p$, we let $\sigma(A)$ be $A[\vec{x} \leftarrow \sigma(\vec{x}), \vec{X} \leftarrow \sigma(\vec{X})]$ where $\sigma(\vec{x})$ stands for $\sigma(x_1), \ldots, \sigma(x_n)$ and similarly for $\sigma(\vec{X})$.*

Definition 5.2 *Let σ be a substitution and t be a first order term.*

- *We define the substitution $\sigma\{x \leftarrow t\}$ by the equations*
  $\sigma\{x \leftarrow t\}(x) = t \quad \sigma\{x \leftarrow t\}(y) = \sigma(y)$ *for* $y \neq x \quad \sigma\{x \leftarrow t\}(X) = \sigma(X)$
- *Similarly we define $\sigma\{X \leftarrow \lambda x.D\}$.*
- *If x is a first order variable we let $\sigma^x = \sigma\{x \leftarrow x\}$.*

Note 5.3 If σ is finite so are $\sigma\{x \leftarrow t\}$, $\sigma\{X \leftarrow \lambda x.D\}$, σ^x.
When σ is closed for $\mathcal{F}$, σ^x is no more closed in general.

We have the following elementary lemma concerning substitutions:

Lemma 5.4 *If σ is closed for A then $\sigma(\forall x A) = \forall x(\sigma^x(A))$ and $(\sigma^x(A))[x \leftarrow t] = \sigma\{x \leftarrow t\}(A)$ and $(\sigma^x(A))[x \leftarrow \sigma(t)] = \sigma(A[x \leftarrow t])$.
Similar results hold for second order substitutions.*

We can now state the main result of this section.

Theorem 5.5 (Soundness for $\mathbf{L^2}$) *Let T be a syntactical truth predicate. We let $T(\sigma(\Delta))$ mean that $T(\sigma(F))$ holds for any formula F in Δ.*

1. *If $\Gamma \vdash A$ is provable in $\mathbf{L^2}$ then for any finite substitution σ closed for $\Gamma \cup \{A\}$ we have $T(\sigma(\Gamma)) \Rightarrow T(\sigma(A))$.*
2. *If T is Leibniz then the same holds for sequents provable in $\mathbf{PA^2}$.*
3. *Points 1,2 are (formalizable and) provable in $(I\Sigma^0_1, C\Delta^0_1)$-$\mathbf{PA^1}$.*

Proof We argue by induction on the proof of $\Gamma \vdash A$:

- The case of the assumption axiom (Ass) is trivial.
- For the $(\rightarrow i)$ rule assume that $T(\sigma(\Gamma \cup \{A\})) \Rightarrow T(\sigma(B))$. Then $T(\sigma(\Gamma)) \Rightarrow (T(\sigma(A)) \Rightarrow T(\sigma(B)))$ *i.e.* (since T is a syntactical truth predicate) $T(\sigma(\Gamma)) \Rightarrow T(\sigma(A) \rightarrow \sigma(B))$ *i.e.* $T(\sigma(\Gamma)) \Rightarrow T(\sigma(A \rightarrow B))$.
- For the $(\rightarrow e)$ rule assume that $T(\sigma(\Gamma)) \Rightarrow T(\sigma(A \rightarrow B)))$ and $T(\sigma(\Gamma)) \Rightarrow T(\sigma(A))$. Then if $T(\sigma(\Gamma))$ holds we have $T(\sigma(A \rightarrow B)))$ *i.e.* $T(\sigma(A) \rightarrow \sigma(B))$ *i.e.* $T(\sigma(A)) \Rightarrow T(\sigma(B))$ and $T(\sigma(A))$ hence $T(\sigma(B))$.
- For the $(\forall_1 i)$ rule assume that $T(\sigma(\Gamma)) \Rightarrow T(\sigma(A))$ for any σ. Consider a term t and a particular substitution σ and define $\sigma' = \sigma\{x \leftarrow t\}$. We have $T(\sigma'(\Gamma)) \Rightarrow T(\sigma'(A))$ *i.e.* $T(\sigma(\Gamma)) \Rightarrow T(\sigma'(A))$ (since x does not appear in Γ) hence $T(\sigma(\Gamma)) \Rightarrow T((\sigma^x(A))[x \leftarrow t])$ by Lemma 5.4. Thus we get $T(\sigma(\Gamma)) \Rightarrow T(\forall x(\sigma^x(A)))$ *i.e.* $T(\sigma(\Gamma)) \Rightarrow T(\sigma(\forall x A))$ by Lemma 5.4.
- For the $(\forall_1 e)$ rule assume that $T(\sigma(\Gamma)) \Rightarrow T(\sigma(\forall x A))$. Then $T(\sigma(\Gamma)) \Rightarrow T(\forall x \sigma^x(A))$ hence for any term t $T(\sigma(\Gamma)) \Rightarrow T((\sigma^x(A))[x \leftarrow \sigma(t)])$ *i.e.* $T(\sigma(\Gamma)) \Rightarrow T(\sigma(A[x \leftarrow t]))$ by Lemma 5.4 since σ is closed.
- The $(\forall_2 i)$ and $(\forall_2 e)$ rules are treated in an essentially equivalent manner, replacing first order terms t by set definitions.
- Eventually note that $\neg\neg(A) \rightarrow A$ is true under T since $T(\neg A)$ is equivalent to $\neg T(A)$.

To prove Point 2, observe that first order axioms of $\mathbf{PA^2}$ are true in the standard model, hence are true in every s.t.p. (cf.Lemma1.4). Lastly, Leibniz's axiom and the induction axiom are true for every Leibniz s.t.p.(cf.2.4).
Using some reasonable primitive recursive coding of finite substitutions and proofs, we see that the above induction hypothesis is Π^0_1. From 4.8 point 4, we then get Point 3. $\Box$

The following corollary will be strengthened in §6.10.

Theorem 5.6 *Assume that there exists a syntactical truth predicate. Then second order arithmetic* $\mathbf{PA^2}$ *is consistent.*
Moreover, this is (formalizable and) provable in $(I\Sigma^0_1, C\Delta^0_1)$-$\mathbf{PA^1}$.

Proof First, we prove that $\mathbf{L^2}$ is consistent. Assume that there exists a proof of $\vdash 0 = 1$ then by Point 1 of the previous proposition one must have $T(0 = 1)$, a contradiction since 0 and 1 do not have the same value.
The same argument needs the existence of a Leibniz s.t.p. to get the consistency of $\mathbf{PA^2}$. However, using a relativization technique, one can interpret $\mathbf{PA^2}$ in $\mathbf{L^2}$ + Robinson's axioms (see [12] p.177-180 ii-iii), so that the consistency of $\mathbf{PA^2}$ reduces to that of $\mathbf{L^2}$ + Robinson. But Robinson's axioms are true in the standard model, hence are true in the s.t.p. T and we conclude as above. $\square$

6 Syntactical truth predicates and ω-models

6.1 Structures for the language of $\mathbf{PA^2}$ (within $\mathbf{PA^1}$)

A structure $\mathcal{M}$ for the language of second order arithmetic consists of a domain M for individuals and a domain $\mathcal{F}$ for sets which is (or can be identified with) a family of subsets of M. A priori, $\mathcal{F}$ is a third order object. Since we are working within $\mathbf{PA^2}$, we want $\mathcal{F}$ to be coded by a second order object, which leads to the following definitions.

Definition 6.1

1. Let M, U be sets. Using Cantor codes (cf.A.1), we let
$$U[n] = \{x \in M \mid < n, x > \in U\} \text{ for } n \in \mathbf{N}$$
$$Slices(U) = \{U[n] \mid n \in \mathbf{N}\} \subseteq P(M)$$

2. A structure $\mathcal{M}$ for the language of second order arithmetic consists of the following data: $\mathcal{M} = (M, Slices(U), 0_{\mathcal{M}}, S_{\mathcal{M}}, +_{\mathcal{M}}, \times_{\mathcal{M}}, =_{\mathcal{M}}, \in)$ where M, U are sets, $0_{\mathcal{M}}$ is an element of M, and $=_{\mathcal{M}}$ is a binary relation over M and $S_{\mathcal{M}}$ (resp. $+_{\mathcal{M}}, \times_{\mathcal{M}}$) is a unary (resp. binary) function over M.

3. We denote $Truth_{\mathcal{M}}(A, \rho)$ the semantical truth predicate for the structure $\mathcal{M}$, where A is a formula and ρ is an environment, i.e. an assignment of elements of the $\mathcal{M}$-domains to individuals and sets to first and second order variables occuring free in A.

4. $\mathcal{M}$ is recursive in a set X if M, U and all the operations and relations of $\mathcal{M}$ are recursive in X.

Using Gödel numbers of formulas, we can formalize in $\mathbf{PA^1}$ the notion of semantical truth predicate for a structure $\mathcal{M}$.

Definition 6.2 We denote $Semantic(T, \mathcal{M})$ the first order formula (with free second order variables) which expresses that T codes the semantical truth predicate of the structure $\mathcal{M}$, i.e. the set of Cantor codes (cf.A.1) $<f, x, u>$ such that f is the Gödel number of a second-order formula and x, u code assignments in the $\mathcal{M}$-domains for individuals and sets of the free first and second order variables of f for which f is true in the structure $\mathcal{M}$. (This is done via the usual inductive conditions defining semantical truth and also arithmetical translations of the adequate recursive operations on Gödel numbers).

The following Theorem is classical.

Theorem 6.3 *1. $(I\Sigma^0_1, C\Delta^0_1)$-$\mathbf{PA^1}$ proves that there is at most one set T such that $Semantic(T, \mathcal{M})$. We (abusively) keep the notation $Truth_{\mathcal{M}}$ from Def. 6.1 for this unique set T when it does exist.*

2. $C\Pi^1_1$-$\mathbf{PA^2}$ proves:

(a) $Truth_{\mathcal{M}}$ does exist for every $\mathcal{M}$.

(b) The relation $z \in Truth_{\mathcal{M}}$ is definable by Σ^1_1 and Π^1_1 formulas.

Proof 1) Without loss of generality, we can suppose that the natural ordering refines the subformula ordering on Gödel numbers. To prove Point 1 we show by induction on n that two solutions T_1, T_2 coincide up to n. Observe that the inductive conditions defining semantical truth are expressible as a Π^0_1 formula. Thus, this induction is formalizable in $(I\Sigma^0_1, C\Delta^0_1)$-$\mathbf{PA^1}$.
2) cf. Appendix, Application B.10.

Remark 6.4 1. For every n the restriction of $Truth_{\mathcal{M}}$ to formulas with at most n alternations of quantifiers (both first and second order ones) is $\Sigma^0_{Sup(1,n)}$ definable with parameter $\mathcal{M}$. Hence, its existence is provable in $\mathbf{PA^1}$ (cf. [14] p.78-81 or [19] p.119-127).

2. However, the existence of $Truth_{\mathcal{M}}$ can not (in general) be proved in $\mathbf{PA^1}$.

3. Since $Truth_{\mathcal{M}}$ consists of (Gödel numbers of) formulas and parameters in the structure, we see that $\mathcal{M}$ is always recursive in $Truth_{\mathcal{M}}$.

We mention an easy but useful Proposition.

Proposition 6.5 *1. $Truth_{\mathcal{M}}$ is closed by the $\forall_2$ elimination rule of $\mathbf{L^2}$ if and only if it closed by the comprehension schema*

$$\forall\vec{y}\forall\vec{Y}\ \exists X\ \forall x(x \in X \leftrightarrow \phi(x, \vec{y}, \vec{Y}))$$

where ϕ is any second order formula.

2. $Truth_{\mathcal{M}}$ satisfies all axioms and is closed by all other rules of $\mathbf{L^2}$.

3. Points 1,2 are formalizable and provable in $(I\Sigma^0_1, C\Delta^0_1)$-$\mathbf{PA^1}$ + $\exists T$ Semantic$(T, \mathcal{M})$

Proof Point 1 is the model theoretical version of the equivalence mentioned in Remark 4.3. Point 2 is easy.

6.2 The canonical model associated to an s.t.p.

To any s.t.p. T we shall associate a structure for which T is exactly the associated *semantical* truth predicate restricted to closed formulas.

Definition 6.6 Let T be a set of closed formulas of the language of second order arithmetic.

1. To each closed set definition $\lambda x.D$ we associate a set of closed terms $[\![D]\!]_T$ called the *semantics of $\lambda x.D$ relative to T*, as follows:

$$t \in [\![D]\!]_T \Leftrightarrow T(A[x \leftarrow t]) \text{ for any closed term } t.$$

2. We associate to T a structure of the language of second order arithmetic

$$\mathcal{M}_T = (\mathcal{CT}, \mathcal{D}ef(T), 0_T, S_T, +_T, \times_T, =_T, \in) \text{ where}$$

   (a) $\mathcal{CT}$ is the set of closed terms, 0_T is the term 0 and $S_T, +_T, \times_T$ are the obvious syntactical operations on $\mathcal{CT}$,

(b) $=_T$ is the relation such that $t =_T u$ iff $T(t = u)$,

(c) $\mathcal{D}ef(T) = Slices(\{(D,t) \mid t \in [\![D]\!]_T\})$, i.e. $\mathcal{D}ef(T)$ is the family of subsets of $\mathcal{CT}$ of the form $[\![D]\!]_T$ for closed set definitions $\lambda x.D$.

Remark 6.7 It is clear that $\{(D,t) \mid t \in [\![D]\!]_T\}$ is recursive in T. Thus, $\mathcal{M}_T$ is recursive in T and the whole construction of $\mathcal{M}_T$ is formalizable within $(I\Sigma_1^0, C\Delta_1^0)$-$\mathbf{PA^1}$.

Lemma 6.8 *Let T be a syntactical truth predicate.*

1. *Let A be a formula with free variables among $\vec{x} = x_1, \ldots, x_m$, $\vec{X} = X_1, \ldots, X_n$. Let $\vec{t} = t_1, \ldots, t_m$, $\lambda z.\vec{D} = \lambda z.D_1, \ldots, \lambda z.D_n$ be finite sequences of closed terms and closed set definitions. Then*
$$T(A[\vec{x} \leftarrow \vec{t}, \vec{X} \leftarrow \lambda z.\vec{D}]) \Leftrightarrow Truth_{\mathcal{M}_T}(A, (\vec{x}, \vec{t}) :: (\vec{X}, [\![\vec{D}]\!]_T))$$
In particular, T coincides with the restriction to closed formulas of $Truth_{\mathcal{M}_T}$.

2. *$\mathcal{M}_T$ and $Truth_{\mathcal{M}_T}$ are recursive in T.*

3. *$(I\Sigma_1^0, C\Delta_1^0)$-$\mathbf{PA^1}$ proves the existence of $Truth_{\mathcal{M}_T}$ and the formalizations of Points 1,2 (involving the Gödel number of A and codes for the assignments of variables).*

Proof We prove Point 1 by induction on A :

- The case where A is the atomic formula $u = v$ is trivial.

- If A is the atomic formula $u \in X_i$ then
$$\begin{aligned} Truth_{\mathcal{M}_T}(A, (\vec{x}, \vec{t}) :: (\vec{X}, [\![\vec{D}]\!]_T)) &\Leftrightarrow u[\vec{x} \leftarrow \vec{t}] \in [\![D_i]\!]_T \\ &\Leftrightarrow T(D_i[z \leftarrow u[\vec{x} \leftarrow \vec{t}]]) \\ &\Leftrightarrow T((u \in X_i)[\vec{x} \leftarrow \vec{t}, X_i \leftarrow \lambda z.D_i]) \\ &\Leftrightarrow T((u \in X_i)[\vec{x} \leftarrow \vec{t}, \vec{X} \leftarrow \lambda z.\vec{D}]) \end{aligned}$$

- If A is $B \rightarrow C$ then the equivalence is clear by clause *ii)* in the definition of syntactical truth predicates.

- If A is the formula $\forall y B$ then, applying the induction hypothesis and the ordinary substitution lemma for $Truth_{\mathcal{M}_T}$, we get

$$T(A[\vec{x} \leftarrow \vec{t}, \vec{X} \leftarrow \lambda z.\vec{D}]) \quad \Leftrightarrow \quad \forall u \in \mathcal{CT}\ T(B[\vec{x} \leftarrow \vec{t}, \vec{X} \leftarrow \lambda z.\vec{D}][y \leftarrow u])$$
$$\Leftrightarrow \quad \forall u \in \mathcal{CT}\ T(B[y \leftarrow u, \vec{x} \leftarrow \vec{t}, \vec{X} \leftarrow \lambda z.\vec{D}])$$
$$\Leftrightarrow \quad \forall u \in \mathcal{CT}\ Truth_{\mathcal{M}_T}(B, (y,u) :: (\vec{x},\vec{t}) :: (\vec{X}, [\![\vec{D}]\!]_T))$$
$$\Leftrightarrow \quad Truth_{\mathcal{M}_T}(\forall y B, (\vec{x},\vec{t}) :: (\vec{X}, [\![\vec{D}]\!]_T))$$

- If A is the formula $\forall Y B$ then applying the induction hypothesis (and denoting $\lambda z.D'$ a closed set definition) we get

$$T(A[\vec{x} \leftarrow \vec{t}, \vec{X} \leftarrow \lambda z.\vec{D}]) \quad \Leftrightarrow \quad \forall D'\ T(B[\vec{x} \leftarrow \vec{t}, \vec{X} \leftarrow \lambda z.\vec{D}][Y \leftarrow \lambda z.D']$$
$$\Leftrightarrow \quad \forall D'\ T(B[\vec{x} \leftarrow \vec{t}, \vec{X} \leftarrow \lambda z.\vec{D}, Y \leftarrow \lambda z.D']$$
$$\Leftrightarrow \quad \forall D'\ Truth_{\mathcal{M}_T}(B, (\vec{x},\vec{t}) :: (Y, [\![D']\!]_T) :: (\vec{X}, [\![\vec{D}]\!]_T))$$
$$\Leftrightarrow \quad \forall S \in \mathcal{D}ef(T)\ Truth_{\mathcal{M}_T}(B, (\vec{x},\vec{t}) :: (Y,S) :: (\vec{X}, [\![\vec{D}]\!]_T))$$
$$\Leftrightarrow \quad Truth_{\mathcal{M}_T}(\forall Y B, (\vec{x},\vec{t}) :: (\vec{X}, [\![\vec{D}]\!]_T))$$

Point 2 is trivial from Point 1. To get Point 3, use Remark 6.7 and observe that the above proof can be formalized as an induction over a Π^0_1 formula of the form $\forall\vec{t}\forall\vec{D}\ \phi(T,\vec{t},\vec{D},A)$. □

6.3 From an s.t.p. to an ω-model of $\mathbf{PA^2}$

Proposition 6.9 *Let T be a syntactical truth predicate. Then $\mathcal{M}_T$ is a model of $\mathbf{L^2}$ plus the axioms for Robinson's first order elementary arithmetic.*

Proof From Theorem 5.5 and Lemma 1.4 we know that T satisfies $\mathbf{L^2}$ and Robinson's axioms. We conclude using Lemma 6.8. □

We now prove the main Theorem of this section.

Theorem 6.10 *Let T be a syntactical truth predicate.*

1. *There exists an ω-model $\mathcal{M} = (\mathbf{N}, Slices(U), 0, S, +, \times, =, \in)$ of $\mathbf{PA^2}$ (where $S, +, \times$ are the usual operations on $\mathbf{N}$) such that U (hence $\mathcal{M}$) and $Truth_{\mathcal{M}}$ are recursive in T.*

2. *If T is Leibniz then $\mathcal{M}_T$ itself is (up to isomorphism) an ω-model of $\mathbf{PA^2}$.*

3. *Points 1,2 are formalizable and provable in* $(I\Sigma^0_1, C\Delta^0_1)$-$\mathbf{PA^1}$.

Proof We first prove Point 2. Theorem 5.5 insures that $\mathcal{M}_T$ is a model of $\mathbf{PA^2}$. Since T is Leibniz, the $\mathcal{M}_T$-equality relation on elements of $\mathcal{CT}$ is a congruence with respect to the $\in$-relation and the operations of $\mathcal{M}_T$ so that we can define a quotient model which satisfies exactly the same statements, hence is a model of $\mathbf{PA^2}$. Observe that each equivalence class contains a unique term of the form $S^n(0)$. Thus, the quotient is (up to isomorphism) an ω-model.
We now prove Point 1. Theorem 5.5 insures that $\mathcal{M}_T$ is a model of $\mathbf{L^2}$. We shall use a classical relativization technique (cf.[12], p.180 *(iii)*). Consider the following notions.

- $X \subseteq \mathcal{CT}$ is saturated if $((T(t = u) \wedge t \in X) \Rightarrow u \in X$ for all closed terms $t, u)$.
- Let $\widehat{\mathcal{D}ef(T)}$ be the family of saturated sets in $\mathcal{D}ef(T)$.

All these notions are clearly definable in $\mathcal{M}_T$. Also, $\widehat{\mathcal{M}_T} = (\mathcal{CT}, \widehat{\mathcal{D}ef(T)})$ constitutes a substructure of $\mathcal{M}_T$.

- By construction, all sets in $\widehat{\mathcal{D}ef(T)}$ are saturated. Thus, $\widehat{\mathcal{M}_T}$ satisfies Leibniz's axiom, whence also (via an easy induction on formulas) Leibniz's schema $\forall x \forall y(x = y \rightarrow (\phi(x) \rightarrow \phi(y)))$
- Now, we prove that $\widehat{\mathcal{M}_T}$ satisfies the comprehension schema. In fact, let Z be a subset of $\mathcal{CT}$ definable in $\widehat{\mathcal{M}_T}$ by a formula ϕ. Then Z is also definable in $\mathcal{M}_T$ by the formula obtained from ϕ by relativizing all second order quantifiers to $\widehat{\mathcal{D}ef(T)}$. Notice that we use here the fact that $\widehat{\mathcal{D}ef(T)}$ is second order definable in $\mathcal{M}_T$. Since $\mathcal{M}_T$ satisfies $\mathbf{L^2}$ hence comprehension, we have $Z \in \mathcal{D}ef(T)$. Now, since $\widehat{\mathcal{M}_T}$ satisfies Leibniz's schema and Z is definable over $\widehat{\mathcal{M}_T}$, Z is necessarily saturated. Using Proposition 6.5, we conclude that $\widehat{\mathcal{M}_T}$ satisfies $\mathbf{L^2}$.
- Let $X \in \widehat{\mathcal{D}ef(T)}$ be such that $(0 \in X) \wedge \forall x(x \in X \rightarrow S(x) \in X)$. Then every term $S^n(0)$ is in X. Now, any closed term t is convertible to some

$S^n(0)$ and $\widehat{\mathcal{M}_T}$ satisfies $t =_T S^n(0)$. Since X is saturated we see that t has to be in X. This proves that the induction axiom is true in $\widehat{\mathcal{M}_T}$.

- Finally, Robinson's axioms trivially hold.

This proves that $\widehat{\mathcal{M}_T}$ is a model of $\mathbf{PA^2}$. Of course, we don't have yet an ω-model. However, the argument about quotient model used in the above proof of Point 2 also works for $\widehat{\mathcal{M}_T}$. □

Remark 6.11 The above argument to prove Leibniz's axiom in $\widehat{\mathcal{M}_T}$ is a model-theoretic version of the proof that Leibniz s.t.p.'s are inductive (cf.Lemma 2.4).

7 The Equivalence Theorem

7.1 From an ω-model of $\mathbf{PA^2}$ to an s.t.p.

Theorem 7.1 (The Equivalence Theorem) $(I\Sigma^0_1, C\Delta^0_1)$-$\mathbf{PA^1}$ *proves the equivalence between the existence of a syntactical truth predicate and the existence of an ω-model of* $\mathbf{PA^2}$ *together with its semantical truth predicate.*

The $\Rightarrow$ direction of the equivalence comes from Theorem 6.10.
As for the other direction, which is the object of this section, we adapt the proof of Theorem 3.1 within a second order arithmetical framework. The natural idea is to replace Gödel's constructible hierarchy by its arithmetical version: the ramified analytical hierarchy (RAH).

The development of RAH within $\mathbf{PA^2}$ is rather long and technical and will be treated in the Appendix.

7.2 The Ramified Analytical Hierarchy

In set theory, the RAH is defined by induction on ordinals as follows:

- $RAH_0 = \emptyset$
- $RAH_{\alpha+1}$ is the family of subsets of $\mathbf{N}$ definable in $(\mathbf{N}, RAH_\alpha)$ by second order formulas with parameters in RAH_α

- RAH_λ is the union of the RAH_α 's, with $\alpha < \lambda$, whenever λ is a limit ordinal

- RAH is the union of all RAH_α

An obvious cardinality argument shows that there exists α such that $RAH_{\alpha+1} = RAH_\alpha$, hence $RAH = RAH_\alpha$. The smallest such α is denoted β_0 .

In relation with predicative analysis, this hierarchy was introduced by S.C. Kleene (1959, [20]) up to the ordinal ω_1^{CK} and studied by S. Feferman (1961,1964 [9, 11]), K. Schütte (1965 [26]) for the levels ω^ω and Γ_0 . The general construction was considered by P.J. Cohen (1963 [5]) who observed that the ordinal β_0 is countable.

Leeds & Putnam (1974 [23]) proved that (after step ω) the very same hierarchy is obtained if $RAH_{\alpha+1}$ is defined as the family of subsets of $\mathbf{N}$ definable in $(\mathbf{N}, RAH_\alpha)$ by second order formulas *without* parameters. Boolos & Putnam (1968 [3], p. 511, Thm 9) got the relation between RAH and the constructible hierarchy: for $\alpha < \beta_0$, $RAH_\alpha = L_{\omega+\alpha} \bigcap P(\mathbf{N})$.

RAH is much related to β-models of $\mathbf{PA^2}$ introduced by Mostowski (1959 [24], see also [25]): models $(M, \mathcal{F})$, with $\mathcal{F} \subseteq P(M)$, for which the notion of well-ordering is absolute: if $R \in \mathcal{F}$ codes a total ordering such that every non empty set $X \in \mathcal{F}$ has an R-smallest element then the same is true for X outside $\mathcal{F}$. Up to an isomorphism, the basis of such models is necessarily standard: $M = \mathbf{N}$ (just apply the hypothesis to the natural ordering on M). Gandy & Putnam (see Boyd & Hensel & Putnam, 1969 [4]) proved that

- $(\mathbf{N}, RAH)$ is the smallest β-model of $\mathbf{PA^2}$,

- β_0 is the supremum of the ordinal types of well-orders in RAH .

For a detailed review and some proofs, see Apt & Marek (1974 [2]), Kreisel (1968 [22], p.368).

7.3 RAH and $\mathbf{PA^2}$

As reviewed above, RAH has been extensively studied in set theory and in relation with β-models of $\mathbf{PA^2}$. As is the case with Gödel's constructible universe in set theory, the development of RAH within $\mathbf{PA^2}$, i.e. in a general model of $\mathbf{PA^2}$ (which is not a β-model nor even an ω-model), leads to an inner model of $\mathbf{PA^2}$ with a definable well-ordering on the family of sets, hence with a definable choice function.

Definition 7.2 *Let $\Theta(X,Y)$ be a second order formula. We denote WO_Θ the schema consisting of the obvious formula insuring that $\Theta(X,Y)$ defines a total ordering on sets together with the schema of axioms insuring that $\Theta(X,Y)$ is a well-ordering, namely*

$$\forall\vec{x}\,\forall\vec{U}(\exists X\,\Phi(X,\vec{x},\vec{U}) \rightarrow \exists X(\Phi(X,\vec{x},\vec{U}) \wedge \forall Y(\Phi(Y,\vec{x},\vec{U}) \rightarrow \Theta(X,Y))))$$

where Φ is any second order formula.

Proposition 7.3 *$\mathbf{PA^2}+WO_\Theta$ proves the following schema of axioms*

$$\forall\vec{x}\,\forall\vec{U}(\exists X(\Phi(\vec{x},\vec{U},X) \rightarrow \Phi(\vec{x},\vec{U},X)[X \leftarrow \lambda z.\rho(z,\vec{x},\vec{U})])$$

where Φ is any second order formula and $\rho(z,\vec{x},\vec{U})$ is the formula

$$\exists X(\Phi(X,\vec{x},\vec{U}) \wedge \forall Y(\Phi(Y,\vec{x},\vec{U}) \rightarrow \Theta(X,Y)) \wedge z \in X)$$

or the dual formula $\forall X(\ldots \rightarrow z \in X)$.

Definition 7.4 *As usual, the relativization of a formula F to a formula $\Phi(X)$ is the formula F^Φ obtained by replacing in F second order quantifications $\forall X \ldots$ and $\exists X \ldots$ by $\forall X(\Phi(X) \rightarrow \ldots)$ and $\exists X(\Phi(X) \wedge \ldots)$.*

Theorem 7.5

1. *There are Σ^1_2 formulas $RAH(X)$ and $\leq_{RAH}(X,Y)$ such that $\mathbf{PA^2}$ proves the relativization to $RAH(X)$ of every axiom of $\mathbf{PA^2}$ and of the schema of formulas $WO_{\leq_{RAH}}$.*
2. *Point 1 is formalizable and provable in $(I\Sigma^0_1, C\Delta^0_1)-\mathbf{PA^1}$.*

The above theorem is folklore, as are its applications to the conservativeness of (second order) choice axioms over $\mathbf{PA^1}$ and to the relative consistency of choice schema over $\mathbf{PA^2}$. This last point appears in Apt & Marek (1974, cf.Theorem

5.25) with no proof. Jakubovic (1981 [15]) avoids RAH and sketches in half a page a development of the constructible universe in an interpretation of a part of ZF set theory in $\mathbf{PA^2}$ (following Zbierski, 1971 [30]).
In the Appendix, we present to some extent the development of RAH in $\mathbf{PA^2}$:

- (Up to our knowledge) No reasonable direct development of RAH in $\mathbf{PA^2}$ has ever appeared in print. Only very recently, a detailed presentation of the set theoretic constructible hierarchy coded within $\mathbf{PA^2}$ became available in Simpson's book (1999 [27]).

- Though there is no surprise and all expected results just go through, such a development asks for some care (and more space than waving hands arguments). In particular, to prove in $\mathbf{PA^2}$ that RAH is a β-model (i.e. that well-orderings in the sense of RAH are true well-orderings, cf. Appendix, Proposition D.14) requires some work (whereas it is trivial in set theory). This last result is needed to get the key relativization properties of the definable well-ordering on RAH.

- Last but not least, in order to get Point 2 of Theorem 7.5 it is necessary to check the details of such a development as is done in D.3.

7.4 Proof of the Equivalence Theorem

Only the $\Leftarrow$ direction of the equivalence in Theorem 7.1 remains to be proved. Consider an ω-model $\mathcal{M}$ of $\mathbf{PA^2}$ and its semantical truth predicate $Truth_{\mathcal{M}}$. Apply Theorem 7.5 to get (recursively in $\mathcal{M}$ and $Truth_{\mathcal{M}}$) another ω-model $\mathcal{M}^{\mathcal{RAH}}$, together with its semantical truth predicate $Truth_{\mathcal{M}^{\mathcal{RAH}}}$, which satisfies $\mathbf{PA^2}$+ WO_{Θ}. Then Proposition 7.3 allows to use the very same argument developped in 3.1 to prove that $Truth_{\mathcal{M}^{\mathcal{RAH}}}$ is indeed a syntactical truth predicate. $\Box$

8 Conclusion

Among the questions raised by this paper let us mention the relation between Dragalin's semi-formal syntax [8] and the syntactical truth predicates proposed here as a possible semantics.

Let us remark that a syntactical semantics of third order arithmetic can be given in an analogous way, the treatment of third order quantification being similar to the second order case. Notice however that such syntactical truth predicates are still second order objects, whereas the semantical truth predicate is a fourth order object. This remark immediately generalizes to any order.

A Some naive set theory in $\mathbf{PA^1}$

This long Appendix is devoted to the proof of Theorem 7.5.

Conventions Implicit extensions by definitions are systematically used. Derivations in $\mathbf{PA^2}$ are presented in a more or less formal way, sometimes mixing vernacular descriptions and formal statements. Some arguments and assertions are also given model theoretic intuition.

A.1 Cantor pairing function and variations

We shall constantly code tuples, finite sequences and infinite eventually zero sequences of integers by integers. We cite below the useful properties of such codings. It is easy to explicit some recursive constructions and to define this coding by first order formulas (which can be taken Σ^0_1 or Π^0_1) such that all basic expected properties can be proved in **PA** (cf. [19] p105-108 or [27] p.65-69). Let $<x, y> = \frac{(x+y)(x+y+1)}{2} + x$ denote Cantor bijection from $\mathbf{N}^2$ onto $\mathbf{N}$.

1. *Finite sequences with fixed length* are coded via Cantor functions:

   $$<x> = x \quad , <x_0, \ldots, x_k> = <<x_0, \ldots, x_{k-1}>, x_k>$$

   For $k = 2$ we use the classical notations $x = <\pi_1(x), \pi_2(x)>$.

2. *Finite sequences with variable lengthes* are coded as follows: 0 codes the empty sequence, $1 + <k-1, <x_0, \ldots, x_{k-1}>>$ codes $(x_0, \ldots, x_{k-1})$.

3. For *infinite eventually zero sequences* (useful for assignments of variables) we use $proj_\infty(k,x) = \pi_2(\pi_1^{(k)}(x))$. In particular, the everywhere zero sequence is coded by 0.

A.2 Relations in $\mathbf{PA^1}$

The following notions and notations take place in $\mathbf{PA^1}$, i.e. correspond to first order formulas (with second order variables) such that $\mathbf{PA^1}$ proves all basic expected properties.

1. *Sets coding binary relations* To a set α we associate the relation
$$rel(\alpha) = \{(i,j) \mid <i,j> \in \alpha\}$$
with $Domain(\alpha) = \{x \mid \exists y(<x,y> \in \alpha \vee <y,x> \in \alpha)\}$

2. *Total orderings* Suppose $rel(\alpha)$ is a reflexive total ordering relation on $Domain(\alpha)$. We shall write $i \leq_\alpha j$ (resp. $i <_\alpha j$) in place of $<i,j> \in \alpha$ (resp. $<i,j> \in \alpha \wedge i \neq j$). Proper initial segments are denoted
$$\alpha_{\leq p} = \{i \mid <i,p> \in \alpha\} \ , \ \alpha_{<p} = \{i \mid <i,p> \in \alpha \wedge i \neq p\}$$
Successor and limit orderings are defined in the usual way.

3. *Lexicographic product and ω power of total orderings.*
$\alpha \otimes \beta = \{<<i,m>,<j,n>> \mid i <_\alpha j \vee (i = j \wedge m <_\beta n\}$
$\alpha^\omega = \{<x,y> \mid \forall i(proj_\infty(i,x), proj_\infty(i,y) \in Domain(\alpha))$
$\wedge \exists i((proj_\infty(i,x) <_\alpha proj_\infty(i,y)) \wedge \forall j{<}i(proj_\infty(i,x) = proj_\infty(i,y)))\}$

A.3 Sets coding families of sets

With sets it is possible to code countable families of sets as slices of the binary relations associated to sets, cf. Def.6.1. Such countable families will be sufficient to develop RAH in $\mathbf{PA^2}$. In the sequel equality for sets $X = Y$ is to be interpreted as $\forall x(x \in X \leftrightarrow x \in Y)$.

Definition A.1 (Total orderings on families of sets in $\mathbf{PA^1}$) 1. Let $rel(\xi)$ be a total ordering relation and X be a set. We associate to ξ

a relation

$$REL(\xi, X) = \{<u,v> \mid \exists u' \exists v' \ (u' \leq_\xi v' \wedge X[u'] = X[u] \wedge X[v'] = X[v])$$
$$\wedge \forall w((X[w] = X[u] \rightarrow u' \leq_\xi w) \wedge (X[w] = X[v] \rightarrow v' \leq_\xi w))\}$$

which is to be interpreted as follows:

- $REL(\xi, X)$ is a total ordering relation with domain the family of sets Y such that the set $\{u \in Domain(\xi) \mid X[u] = Y\}$ is non empty and has a ξ-smallest element,
- sets are compared via their ξ-smallest representatives.

Thus, $X[u] \leq^{sets}_{\xi,X} X[v]$ is just $<u,v> \in REL(\xi, X)$, whereas $Y \leq^{sets}_{\xi,X} Z$ is expressed by the formula

$$\exists u \exists v (Y = X[u] \wedge Z = X[v] \wedge <u,v> \in REL(\xi, X))$$

This notion is particularly suited for the case ξ is a well-ordering.

2. We write $\leq^{sets}_{\xi,X} = \leq^{sets}_{\eta,Y}$ for the first order formulas expressing that these orderings have the very same domain of sets and coïncide on this domain.

3. $INITSEGM(\leq^{sets}_{\xi,X}, \leq^{sets}_{\eta,Y})$ is the first order formula expressing

   - the inclusion of $Domain(\leq^{sets}_{\xi,X})$ as an initial segment of $Domain(\leq^{sets}_{\eta,Y})$,
   - and that $\leq^{sets}_{\xi,X}$ is the restriction of $\leq^{sets}_{\eta,Y}$ to $Domain(\leq^{sets}_{\xi,X})$.

Proposition A.2 $\mathbf{PA^1}$ *proves all basic expected properties about* $\leq^{sets}_{\xi,X}$.

Note A.3 Variables x, y,… will be used as 1st order variables representing elements considered as themselves whereas variables u, v,… will be used as 1st order variables for elements encoding subsets. Variables α, β,… will be used as 2d order variables representing relations whereas variables ξ, η,… will be used as 2d order variables for sets encoding total orderings on subsets.

B Well-orderings and induction in $\mathbf{PA^2}$

B.1 Well-orderings in $\mathbf{PA^2}$

Well-orderings are defined in the obvious way in second order arithmetic by a Π^1_1 formula. Observe that if $\mathcal{M} = (M, Slices(U), S_{\mathcal{M}}, +_{\mathcal{M}}, \times_{\mathcal{M}}, =_{\mathcal{M}}, \in)$ is a model of $\mathbf{PA^2}$ and $rel(\alpha) \in Slices(U)$ is a well-ordering (briefly written wo) in the sense of $\mathcal{M}$ then $rel(\alpha)$ is a reflexive total ordering on $Domain(\alpha)$ such that every non empty subset of $Domain(\alpha)$ lying in $Slices(U)$ has an α-smallest element. Of course, there can be subsets of $Domain(\alpha)$ outside $Slices(U)$ with no α-smallest element.

Usual results about well-orderings are provable in $\mathbf{PA^2}$ and even in $C\Pi^1_1$-$\mathbf{PA^2}$, but not in $\mathbf{PA^1}$, proofs being trivial adaptations of the usual set theoretical ones.

Proposition B.1 *1.* (Lexicographic product and ω power of orderings) $\mathbf{PA^1}$ *proves that if α and β are well-orderings so are $\alpha \otimes \beta$ and α^ω.*

2. (Rigidity of well-orderings) $\mathbf{PA^1}$ *proves that if ϕ, ψ code isomorphisms from α onto β then $\phi = \psi$.*

3. (Comparing well-orderings) $C\Pi^1_1$-$\mathbf{PA2}$ *proves that for all well-orderings α, β there exists a unique ϕ which is either an isomorphism from α to an initial segment of β or from β to an initial segment of α.*

4. (Longest well-ordered initial segments in total orderings) $C\Sigma^1_2$-$\mathbf{PA2}$ *proves that every total ordering δ contains a greatest initial segment of δ which is well-ordered.*

Remark B.2 Whereas there are canonical representatives for isomorphism types of well-orderings in set theory, there are none in second order arithmetic.

B.2 Well-orderings on families of sets

Notation B.3 1. We write $\leq^{sets}_{\xi,X}$ *is a well-ordering* (cf. Def.A.1) for the Π^1_1 formula which expresses that this total ordering has domain $Slices(X)$

and that there exists a smallest element in every non empty subset of $Slices(X)$ of the form $Slices(Y)$.

2. We denote $Equiv_X$ the set $\{< u, v > \mid \ X[u] = X[v]\}$.

Proposition B.4 $\mathbf{PA^1}$ *proves that if* $rel(\xi)$ *is a well-ordering and the domain of* $rel(\xi)$ *meets every equivalence class of* $Equiv_X$ *then* $\leq^{sets}_{\xi,X}$ *is a well-ordering on* $Slices(X)$.

Remark B.5 Let $Least(\xi, X)$ be the set of elements which are $rel(\xi)$-minimal in their equivalence classes for $Equiv_X$ and let $\phi_{\xi,X} : Least(\xi, X) \mapsto Slices(X)$ be defined by $\phi_{\xi,X}(u) = X[u]$. Then $\mathbf{PA^1}$ proves that $\phi_{\xi,X}$ is an isomorphism between $rel(\xi \bigcap \{<u, v> \mid u, v \in Least(\xi, X)\})$ and $\leq^{sets}_{\xi,X}$.

B.3 Definition by induction on well-orderings in $\mathbf{PA^2}$

Definition by induction in second order arithmetic is essentially the same as in set theory. The main difference is due to the fact that *there are no canonical representatives for ordinals in second order arithmetic*. Since we have not included any form of the axiom of choice in $\mathbf{PA^2}$, some care is necessary to get prenex formulas with second order quantifications ahead. A drastic way to overcome this problem is to get as many first order formulas (with second order parameters) as possible. This is the reason why, in the next Theorem, we introduce simultaneously a code $R(\alpha)$ for the family of sets obtained at step α of the induction and a code $Z(\alpha)$ for the sequence $(R(\alpha_{<i}))_{i \in Domain(\alpha)}$ of families of sets obtained at previous steps.
Along with the above notations $T[i]$ and $\alpha_{\leq p}$, we shall also use

Notation B.6 We denote $T|W = \{<u, v> \in T \mid j \in W\}$, so that

$$(T|Domain(\alpha_{<i}))[j] = \textit{if } j <_\alpha i \textit{ then } T[j] \textit{ else } \emptyset.$$

Theorem B.7 (Definition by induction)
To every formula $A(\alpha, Z, R)$ *(possibly with some other first order and second order parameters) we associate the formula* $IND_A(\alpha, Z, R)$

α *is a total ordering of* $Domain(\alpha) \wedge A(\alpha, Z, R)$

$$\wedge \forall i \in Domain(\alpha)\ A(\alpha_{<i}, Z|Domain(\alpha_{<i}), Z[i])$$
$$\wedge \forall i \notin Domain(\alpha)\ Z[i] = \emptyset$$

1. *Let $Func_A$ be the formula $\forall\alpha\forall Z\exists! R\ A(\alpha, Z, R)$. The following assertions are provable in $\mathbf{PA^2}$+($rel(\alpha)$ is a wo) $+Func_A$*

   - $\exists! Z\ \exists! R\ IND_A(\alpha, Z, R)$

     We denote $Z(\alpha), R(\alpha)$ the unique such Z, R associated to α.

   - $\forall i{\in}Domain(\alpha)(Z(\alpha_{<i}) = Z(\alpha)|Domain((\alpha_{<i}) \wedge R((\alpha_{<i}) = Z(\alpha)[i])$

2. *If Φ is a formula we let $Inductive_{A,\Phi}$ be*

   $$\forall\alpha\forall Z\forall R((A(\alpha, Z, R) \wedge \forall m \in Domain(\alpha)\Phi(Z[m])) \Rightarrow \Phi(R))$$

   Then $\mathbf{PA^2}$+($rel(\alpha)\, is\, a\, wo$) $+ Func_A + Inductive_{A,\Phi}$ proves $\Phi(R(\alpha))$.

3. *Let $Iso(I, \alpha, \beta)$ be a first order formula which expresses that $rel(\alpha)$, $rel(\beta)$ are total orderings and I is an isomorphism between $rel(\alpha)$ and $rel(\beta)$. If Ψ is a formula we let $IsoInd_{A,\Psi}$ be the formula*

   $$\forall\alpha\forall Z\forall R\ \forall\alpha'\forall Z'\forall R'\ \forall I((A(\alpha, Z, R) \wedge A(\alpha', Z', R)$$
   $$\wedge Iso(I, \alpha, \alpha') \wedge \forall m \in Domain(\alpha)\Psi(Z[m], Z'[I(m)])) \Rightarrow \Psi(R, R'))$$

   Then $\mathbf{PA^2}$+($rel(\alpha)$ and $rel(\beta)$ are isomorphic wo) $+Func_A + IsoInd_{A,\Psi}$ proves $\Psi(R(\alpha), R(\beta))$.

4. *If A is a first order formula (with free second order variables) then*

   (a) *In all previous items, one can replace $\mathbf{PA^2}$ by $C\Pi^1_1$-$\mathbf{PA^2}$.*

   (b) *There are Σ^1_1 (resp. Π^1_1) formulas*

   $$\Phi(\alpha, X),\ \varphi(\alpha, x)\ \Psi(\alpha, X),\ \psi(\alpha, x)$$

   such that $C\Pi^1_1$-$\mathbf{PA^2}$+($rel(\alpha)$ is a wo) proves

   $$\begin{cases} \forall X(\Phi(\alpha, X) \leftrightarrow X \in Slices(Z(\alpha))) \\ \forall z(\varphi(\alpha, z) \leftrightarrow z \in Z(\alpha)) \end{cases} \tag{1}$$

   $$\begin{cases} \forall X(\Psi(\alpha, X) \leftrightarrow X \in Slices(R(\alpha))) \\ \forall z(\psi(\alpha, z) \leftrightarrow z \in R(\alpha)) \end{cases} \tag{2}$$

(c) *Let* $\Theta(X, \vec{p}, \vec{P})$ *be a formula, where* $\vec{p}, \vec{P}$ *are first order and second order parameters. Let* $Good_\Theta(\beta)$ *be the formula*

$$\Theta(\beta, \vec{p}, \vec{P}) \wedge \Theta(R(\beta), \vec{p}, \vec{P}) \wedge \Theta(Z(\beta), \vec{p}, \vec{P})$$

If second order quantifications in Φ, Φ', Ψ, Ψ' *are relativized to* $\Theta(X, \vec{p}, \vec{P})$ *then*

i. *equivalences* (1),(2) *with relativized formulas are provable in* $C\Pi^1_1$-$\mathbf{PA^2}$+$rel(\alpha)$ *is a wo* +$Good_\Theta(\alpha)$
$$+\forall i \in Domain(\alpha) Good_\Theta(\alpha_{<i})$$

ii. *equivalences* (1) *with relativized formulas are provable in* $C\Pi^1_1$-$\mathbf{PA^2}$+$rel(\alpha)$ *is a limit wo* +$\forall i \in Domain(\alpha) Good_\Theta(\alpha_{<i})$

Proof 1) Let $rel(\alpha)$ be some fixed well-ordering and suppose $IND_A(\alpha, Z, R)$ and $IND_A(\alpha, Z', R')$. Using hypothesis $Func_A$ and the very definition of IND_A, an easy induction over $rel(\alpha)$ shows $\forall i \in Domain(\alpha)\, Z[i] = Z'[i]$ and therefore $R = R'$. This gives the unicity statement in the first item.
As for the existence, apply the comprehension schema to get

$$E = \{i \in Domain(\alpha) \mid \exists Z_i\, \exists R_i\, IND_A(\alpha_{<i}, Z_i, R_i)\}$$

We show that $E = Domain(\alpha)$.

- From the definition of IND_A it is clear that if $i \in E \wedge IND_A(\alpha_{<i}, Z_i, R_i)$ then $IND_A(\alpha_{<j}, Z_i|Domain(\alpha_{<j}), Z_i[j])$ for every $j <_\alpha i$. Thus, E is an α-initial segment.

- If E were not the whole of $Domain(\alpha)$ there would exist an α-smallest element m in $Domain(\alpha) \setminus E$. We define Z_m, R_m as follows:

  Case 1 *m is the first element of α* In this case $\alpha_{<m} = \emptyset$. We set $Z_m = \emptyset$ and let R_m be the unique set such that $A(\emptyset, \emptyset, R_m)$.

  Case 2 *m is α-limit* We define Z_m as follows.

  Let $j <_\alpha m$. The very definition of E insures that all $Z_i[j]$, for i such that $j <_\alpha i <_\alpha m$, have the same value, we let Z_m be this common value.

  If $j \geq_\alpha m$ or $j \notin Domain(\alpha)$ we let $Z_m = \emptyset$.

We also let R_m be such that $A(\alpha_{<m}, Z_m, R_m)$.

Case 3 *m is α-successor of p.* We let Z_m, R_m be such that

$$\forall i <_\alpha p \; Z_m[i] = Z_p[i] \;,\; Z_m[p] = R_p \text{ and } A(\alpha_{<m}, Z_m, R_m).$$

Using hypothesis $Func_A$, it is clear that $IND_A(\alpha_{<m}, Z_m, R_m)$ holds in all three cases, whence $m \in E$, contradiction. This proves that $E = Domain(\alpha)$.

Now, let Z be such that

- if $j \in Domain(\alpha)$ is not α-greatest then $Z[j]$ is the common value of $Z_i[j]$ for i such that $j <_\alpha i$
- if $j \in Domain(\alpha)$ is α-greatest then $Z[j] = R_j$
- if $j \notin Domain(\alpha)$ then $Z[j] = \emptyset$.

Let R be such that $A(\alpha, Z, R)$. It is clear that $IND_A(\alpha, Z, R)$, which proves the existence assertion in item 1.
2)-3), 4) (c) are easy.
4) (a) Observe that E is defined by a Σ^1_1 formula.
4) (b) Observe that $x \in R(\alpha) \Leftrightarrow \exists Z \exists R(IND_A(\alpha, Z, R) \wedge x \in R)$

$$\Leftrightarrow \forall Z \forall R(IND_A(\alpha, Z, R) \rightarrow x \in R) \; \square$$

Theorem B.8 (Definition by simultaneous induction)
If $A(\alpha, Z_1, R_1, Z_2, R_2)$ is a formula (possibly with some other first order and second order parameters) we let $IND_A(\alpha, Z_1, R_1, Z_2, R_2)$ be the formula

$$rel(\alpha) \text{ is a total ordering of } Domain(\alpha) \wedge A(\alpha, Z_1, R_1, Z_2, R_2)$$
$$\wedge \forall i \in Domain(\alpha) A(\alpha_{<i}, Z_1|Domain(\alpha_{<i}), Z_1[i], Z_2|Domain(\alpha_{<i}), Z_2[i])$$
$$\wedge \forall i \notin Domain(\alpha) Z_1[i] = Z_2[i] = \emptyset$$

Obvious adaptations of all assertions in Theorem B.7 are valid.

As an easy application of Theorem B.7, we prove in $C\Pi^1_1$-$\mathbf{PA^2}$ the statement in Point 2 of Theorem 6.3. First, we need some conventions about Gödel numbers.

Convention B.9 1. Without loss of generality, we can suppose that the Gödel numbering of formulas is such that

- 0 is the Gödel number of the formula $x_0 \in X_0$,
- every element is a Gödel number (for some second order formula),
- the natural ordering refines the subformula ordering. We denote ω_{Godel} the natural ordering when considered for Gödel numbers.

2. We denote $GN(F)$ the Gödel number of the formula F and $GN_{\rightarrow}(f, g)$, $GN_{\forall_1}(i, f)$ and $GN_{\forall_2}(i, f)$ the arithmetical operations on Gödel numbers associated to the logical operations on formulas corresponding to implication and quantifications $\forall x_i, \forall X_i$.

Application B.10 (Proof of Theorem 6.3) We apply Theorem B.7 to the well-ordering ω_{Godel} and the obvious adequate first order formula $A(\alpha, Z, R)$, so as to get $Z_{\mathcal{M}}(\omega_{Godel})$ such that

- $Z_{\mathcal{M}}(\omega_{Godel})[GN(t_1 = t_2)] =$
  $\{<x, u> \mid \exists z(Val(\mathcal{M}, t_1, x, z) \wedge Val(\mathcal{M}, t_2, x, z))\}$
  where $Val(\mathcal{M}, t, x, z)$ is a first order formula insuring that z is the value of the term t in $\mathcal{M}$ for the assignment of first order variables coded by x (cf. [19] p.119-127 or [14] p.78; the expected properties of this formula being provable in **PA**).
- $Z_{\mathcal{M}}(\omega_{Godel})[GN(t \in X_i)] =$
  $$\{<u, x> \mid \exists z(Val(\mathcal{M}, t, x, z) \wedge (z \in U[proj_\infty(i, u)])\}$$
- $Z_{\mathcal{M}}(\omega_{Godel})[GN_{\rightarrow}(f, g)]$,
  $Z_{\mathcal{M}}(\omega_{Godel})[GN_{\forall_1}(i, f)]$,
  $Z_{\mathcal{M}}(\omega_{Godel})[GN_{\forall_2}(i, f)]$ are respectively
  $$\{z \mid z \in Z_{\mathcal{M}}(\omega_{Godel})[f] \rightarrow z \in Z_{\mathcal{M}}(\omega_{Godel})[f])\}$$
  $$\{<x, u> \mid \forall z <Subst_\infty(z, i, x), u> \in Z_{\mathcal{M}}(\omega_{Godel})[f]\}$$
  $$\{<x, u> \mid \forall z <x, Subst_\infty(z, i, u)> \in Z_{\mathcal{M}}(\omega_{Godel})[f]\}$$
  where $Subst_\infty(z, i, x)$ denotes the unique y such that
  $$proj_\infty(i, y) = z \wedge \forall j(j \neq i \Rightarrow proj_\infty(j, y) = proj_\infty(j, x))$$

We get the desired truth predicate by rearrangement in Cantor coding:

$$Truth_{\mathcal{M}} = \{<f, x, u> \mid <f, <x, u>> \in Z_{\mathcal{M}}(\omega_{Godel})\} \ \Box$$

C Extensions of ω-models

C.1 Extension by definitions of an ω-model

We shall now focus on the sole ω-structures.

Definition C.1 1. To every set U we associate an ω-structure

$$\Omega(U) = (\mathbf{N}, Slices(U), 0, S, +, \times, =, \in)$$

and families of (codes of) definable relations over $\Omega(U)$:

$$Def^{\Sigma^0_\infty}(U) = \{<f, m, u, x> \mid \phi_0(f, m, x) \wedge \exists y(Restr(x, y, f, m) \wedge <f, y, u> \in Truth_{\Omega(U))}\}$$

$$Def^{\Sigma^1_\infty}(U) = \{<f, m, u, x> \mid \phi_1(f, m, x) \wedge \exists y(Restr(x, y, f, m) \wedge <f, y, u> \in Truth_{\Omega(U))}\}$$

where $Truth_{\Omega(U))}$ is as in Def.6.2 and

- $\phi_1(f, m, x)$ (resp. ϕ_0) is a first order formula expressing that f is the Gödel number of a formula (resp. with no second order quantification) with free first order variables among $x_0, \ldots, x_m$ and that x codes an m-sequence.
- $Restr(x, y, f, m)$ is a first order formula expressing that y codes the assignment obtained by restriction of the m-sequence x to the free first order variables occuring in the formula with Gödel number f.

Intuition: The families of slices of these sets are the families of relations (coded via Cantor functions for tuples) which are respectively second order and first order definable in the model $(\mathbf{N}, Slices(U))$, with second order parameters in $Slices(U)$.

2. Let τ be 1 or 2. We let $True^{\mathbf{PA}^\tau}(U)$ be the Π^0_1 formula expressing via Gödel numbers that every axiom of $\mathbf{PA}^\tau$ is true in $\Omega(U)$. Otherwise said, this formula expresses the validity of formulas obtained from axioms of $\mathbf{PA^2}$ by restricting second order quantifications to $Slices(U)$.

Proposition C.2 *Let m be 0 or 1. The following are provable in $C\Pi^1_1$-$\mathbf{PA^2}$.*

1. $Slices(U) \subseteq Slices(Def^{\Sigma^0_\infty}(U))$

2. $Slices(U) = Slices(V) \rightarrow (Slices(Def^{\Sigma^m_\infty}(U)) = Slices(Def^{\Sigma^m_\infty}(V)))$

3. *The relation* $z \in Def^{\Sigma^m_\infty}(U)$ *is definable by* Σ^1_1 *and* Π^1_1 *formulas.*

4. $True^{\mathbf{PA^1}}(U) \Leftrightarrow (Slices(U) = Slices(Def^{\Sigma^0_\infty}(U)))$
   $True^{\mathbf{PA^2}}(U) \Leftrightarrow (Slices(U) = Slices(Def^{\Sigma^1_\infty}(U)))$

5. $\forall \vec{z}\ \forall u_1 \ldots \forall u_k\ \exists p\ \forall x_1 \ldots \forall x_l\ (<x_0, \ldots, x_l>) \in Def^{\Sigma^m_\infty}(U)[p] \leftrightarrow$
   $$(\Phi^{Slices(U)}(x_0, \ldots, x_l, \vec{z}, U[u_1], \ldots, U[u_k])))$$
   where Φ *is any* Σ^m_∞ *formula and* $\Phi^{Slices(U)}$ *is its relativization.*

6. $True^{\mathbf{PA^1}}(Def^{\Sigma^0_\infty}(U))$ *and* $True^{\mathbf{PA^1}}(Def^{\Sigma^1_\infty}(U))$.

Proof 1) Trivial. 2) By induction on f we show that

$$\forall m \forall u \exists v \exists w\ ((Def^{\Sigma^m_\infty}(U))[<f, m, u>] = Def^{\Sigma^m_\infty}(V))[<f, m, v>])$$
$$\wedge (Def^{\Sigma^m_\infty}(V))[<f, m, u>] = Def^{\Sigma^m_\infty}(U))[<f, m, w>]))$$

We consider the sole case $f = GN(x_i \in X_j)$ and only treat the first equality. If u assigns value p to variable X_j then

$$Def^{\Sigma^m_\infty}(U))[<f, m, u>] = \{(x_1, \ldots, x_m)\ \mid\ x_i \in U[p]\}$$

From $Slices(U) = Slices(V)$ we get q such that $U[p] = V[q]$. If v assigns value q to X_j then $Def^{\Sigma^m_\infty}(U)[<f, m, u>] = Def^{\Sigma^m_\infty}(V)[<f, m, v>]$.

3)-4) are easy. 5) Just set $p = <GN(\Phi(x_0, \ldots, x_l), u>$ where u assigns values $u_1, \ldots, u_k$ to the free second order variables of Φ.

6) If a set $W \subseteq \mathbf{N}$ is definable in $\Omega(Def^{\Sigma^m_\infty}(U))$ by a *first order* formula F with parameters in $Slices(Def^{\Sigma^m_\infty}(U))$ then it is also definable in $\Omega(U)$ by a formula with parameters in $Slices(U)$ which is first order if $m = 0$ and second order if $m = 1$: just substitute in F each second order parameter by its definition in $\Omega(U)$. Formalizing this argument with Gödel numbers we get

$$Slices(Def^{\Sigma^0_\infty}(Def^{\Sigma^m_\infty}(U))) = Slices((Def^{\Sigma^m_\infty}(U))$$

whence Point 6 as an application of Point 4. $\Box$

Proposition C.3 (Crucial relation with Def and Slices) *The following are provable in* $C\Pi^1_1$-$\mathbf{PA^2}$ $+ True^{\mathbf{PA^1}}(U)$.

1. $\forall p\ Slices(Def^{\Sigma^1_\infty}(U[p])) \subseteq Slices(U)$

2. (Commutation of $Def^{\Sigma^1_\infty}(\ldots)$ and $Slice(\ldots)$)
$$\forall p \exists q\ Def^{\Sigma^1_\infty}(U[p]) = (Def^{\Sigma^1_\infty}(U))[q]$$

Proof 1) Observe that $True^{\mathbf{PA^1}}(U)$ insures closure properties of $Slices(U)$ under all arithmetical functions. Therefore, it suffices to prove that the sets $Z_{Model\Omega(U[p])}(\omega_{Godel})[f]$ constructed in the proof given in Application B.10 are all in $Slices(U)$. This is done by induction over Gödel numbers:

- If $f = GN(x_i \in X_j)$ then
$$\begin{aligned} Z_{\Omega(U[p])}(\omega_{Godel})[f] &= \{<u,x> \mid proj_\infty(i,x) \in (U[p])[proj_\infty(j,u)]\} \\ &= \{<u,x> \mid \exists v \exists y(<v,y> \in U[p] \\ &\qquad \wedge proj_\infty(i,x) = y \wedge proj_\infty(j,u) = v)\} \end{aligned}$$
which gives a simple definition of $(Def^{\Sigma^1_\infty}(U[p])[<f,m,u>]$ from $U[p]$, f, m, u. We conclude using closure properties of the family $Slices(U)$.

- Other atomic cases and the induction steps are trivial applications of closure properties of $Slices(U)$.

2) First, recall that $Z_{\Omega(U[p])}((\omega_{Godel})_{<f})$ is the finite sequence $(Z_{\Omega(U[p])}(\omega_{Godel})[g])_{g<f}$. ¿From item 1) an easy induction proves that, for every Gödel number f, there exists q such that $Z_{\Omega(U[p])}((\omega_{Godel})_{<f}) = U[q]$.
Applying item 4 (b) in Theorem B.7 (and expliciting all parameters which were omitted in the statement of this theorem), we get a second order formula $\psi(\omega_{Godel}, z, U[p])$ which defines the relation $z \in Z_{\Omega(U[p])}(\omega_{Godel})$. Now, in ψ we can eliminate the parameter ω_{Godel} by replacing it by its first order definition and some extra existential (or universal) second order quantification, and get an equivalent formula $\widehat{\psi}(z, U[p])$ with the sole parameter $U[p]$.
Since ω_{Godel} is limit, we can apply item 4 (c) ii in Theorem B.7 which proves that the definition of the relation $z \in Z_{\Omega(U[p])}(\omega_{Godel})$ relativizes to $Slices(U)$. Thus, letting u be such that $proj_\infty(0,u) = p$ and $r = <GN(\widehat{\psi}(x_0, X_0)), 1, u>$, we get $Z_{\Omega(U[p])}(\omega_{Godel}) = (Def^{\Sigma^1_\infty}(U))[r]$. Now, $Truth^{\Sigma^1_\infty}(\Omega(U[p])$

is obtained by rearrangement of $Z_{\Omega(U[p])}(\omega_{Godel})$ in Cantor coding (cf. end of Application B.10), and we can use closure properties of $Slices(Def^{\Sigma^1_\infty}(U))$ (given by item 5 in Proposition C.2) to get s such that $Truth^{\Sigma^1_\infty}(\Omega(U[p])) = (Def^{\Sigma^1_\infty}(U))[s]$. Lastly, $Def^{\Sigma^1_\infty}(U[p])$ is simply obtained from $Truth^{\Sigma^1_\infty}(\Omega(U[p]))$, so that we get q such that $Def^{\Sigma^1_\infty}(U[p]) = (Def^{\Sigma^1_\infty}(U))[q]$. □

C.2 Well-ordered models

We now consider well-orderings on ω-models as introduced in Definition A.1.

Proposition C.4 *1. The following are provable in* $C\Pi^1_1$-$\mathbf{PA^2}$*:*

(a) $rel(\xi)$ *is a wo* $\rightarrow INITSEGM(\leq^{sets}_{\xi,U}, \leq^{sets}_{\omega_{Godel}\otimes\omega\otimes\xi^\omega, Def^{\Sigma^1_\infty}(U)})$

(b) $rel(\xi)$ *is a wo* $\wedge Slices(U) = Slices(V) \wedge \leq^{sets}_{\xi,U} = \leq^{sets}_{\eta,V}$

$$\rightarrow \leq^{sets}_{\omega_{Godel}\otimes\omega\otimes\xi^\omega, Def^{\Sigma^1_\infty}(U)} = \leq^{sets}_{\omega_{Godel}\otimes\omega\otimes\eta^\omega, Def^{\Sigma^1_\infty}(V)}$$

2. The following is provable in $C\Pi^1_1$-$\mathbf{PA^2}$ $+ True^{\mathbf{PA^1}}(U)$ *:*

$\forall p\ \exists r\ (rel(\xi)$ *is a wo* $\wedge \xi \in Slices(U)$

$$\rightarrow REL(\omega_{Godel} \otimes \omega \otimes \xi^\omega, Def^{\Sigma^1_\infty}(U[p]) = (Def^{\Sigma^1_\infty}(U))[r])$$

Proof 1) The fact that $Slices(U)$ is an initial segment is due to the definition of the lexicographic product and to the equality:

$$\begin{aligned} U[u] &= Def^{\Sigma^1_\infty}(U))[<\ulcorner x_0 \in X_0 \urcorner, 0, u>] \\ &= Def^{\Sigma^1_\infty}(U))[<0, 0, u>] \end{aligned}$$

(recall our choice $GN(x_0 \in X_0) = 0$, cf. Convention B.9). 2) Easy from the closure properties insured by $True^{\mathbf{PA^1}}(U)$ and Proposition C.3. □

D Ramified Analytical Hierarchy in $\mathbf{PA^2}$

D.1 Defining RAH in $\mathbf{PA^2}$

As a direct application of arithmetization of syntax and definition by induction in $\mathbf{PA^2}$, we get the desired construction of the ramified analytical hierarchy.

To handle limit cases, we first explicit a Definition and state an easy Proposition.

Definition D.1 *Let* $rel(\alpha)$ *be a total ordering. We say that* (Z,ζ) *is an* α*-chain of total orderings if*

- $\forall i \in Domain(\alpha)(\leq^{sets}_{\zeta[i],Z[i]}$ *is a total ordering with domain* $Slices(Z[i]))$
- $\forall i \forall j (i \leq_\alpha j \rightarrow INITSEGM(\leq^{sets}_{\zeta[i],Z[i]}, \leq^{sets}_{\zeta[j],Z[j]}))$

Proposition D.2 $\mathbf{PA^1}$ *proves that for every limit total ordering* $rel(\alpha)$ *and every* α*-chain* (Z,ζ)*, there exists a unique pair* (R,η) *such that*

- $Domain(\leq^{sets}_{\eta,R}) = Slices(R) = \bigcup_{i \in Domain(\alpha)} Slices(Z[i])$
- $\leq^{sets}_{\eta,R} = \bigcup_{i \in Domain(\alpha)} \leq^{sets}_{\zeta[i],Z[i]}$
- $\forall i \in Domain(\alpha) INITSEGM(\leq^{sets}_{\zeta[i],Z[i]}, \leq^{sets}_{\eta,R})$

We call (R,η) *the* chain-union *of the* α*-chain* (Z,ζ).

Theorem D.3 (Ramified Analytical Hierarchy in $\mathbf{PA^2}$**)** *1. There exists a first order formula (with second order free variables)*

$$Hierarchy(\alpha, R, \eta, Z, \zeta)$$

such that the following are provable in $C\Pi^1_1$-$\mathbf{PA^2}$ $+(rel(\alpha)$ *is a wo).*

(a) $\exists! R\, \exists! \eta\, \exists! Z\, \exists! \zeta\, Hierarchy(\alpha, R, \eta, Z, \zeta)$

We denote RAH_α , λ_α, Seq-RAH_α, Seq-λ_α the unique such R, η, Z, ζ associated to α.

(b) $\leq^{sets}_{\lambda_\alpha, RAH_\alpha}$ *is a wo with domain* $Slices(RAH_\alpha)$

$$\wedge(Seq\text{-}RAH_\alpha, Seq\text{-}\lambda_\alpha) \text{ is an } \alpha\text{-chain of well-orderings}$$

$$\wedge \forall i \in Domain(\alpha)\ (Seq\text{-}RAH_\alpha[i] = RAH_{\alpha_{<i}} \wedge Seq\text{-}\lambda_\alpha[i] = \lambda_{\alpha_{<i}})$$

$$\wedge \forall i \notin Domain(\alpha)\ (Seq\text{-}RAH_\alpha[i] = Seq\text{-}\lambda_\alpha[i] = \emptyset)$$

(c) i. $rel(\alpha) = \emptyset \rightarrow RAH_\alpha = \lambda_\alpha = \emptyset$

ii. $rel(\alpha)$ *is a successor of the wo* $rel(\beta) \rightarrow$

$$(RAH_\alpha = Def^{\Sigma^1_\infty}(RAH_\beta) \wedge \lambda_\alpha = \omega_{Godel} \otimes \omega \otimes {\lambda_\beta}^{\omega})$$

iii. $rel(\alpha)$ *is a limit wo* $\rightarrow$

$$(RAH_\alpha, \lambda_\alpha) \text{ is the chain-union of } (Seq\text{-}RAH_\alpha, Seq\text{-}\lambda_\alpha)$$

(d) $rel(\beta)$ *is a wo isomorphic to* $rel(\alpha) \rightarrow$

$$Slices(RAH_\beta) = Slices(RAH_\alpha) \wedge \leq^{sets}_{\lambda_\beta, RAH_\beta} = \leq^{sets}_{\lambda_\alpha, RAH_\alpha}$$

(This is the key property to overcome the lack of canonical representatives for well-orderings.)

(e) $rel(\beta)$ *is a wo with smaller order type than* $rel(\alpha) \rightarrow$

$$Slices(RAH_\beta) \subseteq Slices(RAH_\alpha)$$
$$\wedge INITSEGM(\leq^{sets}_{\lambda_\beta, RAH_\beta}, \leq^{sets}_{\lambda_\alpha, RAH_\alpha})$$

2. *There are* Σ^1_1 *(resp.* Π^1_1*) formulas* $\Psi(\alpha, X)$, $\psi(\alpha, X)$ *such that*

(a) $C\Pi^1_1$-$\mathbf{PA^2}$ $+(rel(\alpha)$ *is a wo) proves*

$\forall X(\Psi(\alpha, X) \leftrightarrow X \in Slices(RAH_\alpha))$, $\forall z(\psi(\alpha, z) \leftrightarrow z \in RAH_\alpha)$

(b) *Let* $\Theta(X, \vec{p}, \vec{P})$ *be a formula, where* $\vec{p}, \vec{P}$ *are first order and second order parameters.*

$C\Pi^1_1$-$\mathbf{PA^2}$ $+(rel(\alpha)$ *is a limit wo)* $+\forall i \in Domain(\alpha)$

$$(\Theta(RAH_{\alpha_{<i}}, \vec{p}, \vec{P}) \wedge \Theta(\lambda_{\alpha_{<i}}, \vec{p}, \vec{P})$$
$$\wedge \Theta(Seq\text{-}RAH_{\alpha_{<i}}, \vec{p}, \vec{P}) \wedge \Theta(Seq\text{-}\lambda_{\alpha_{<i}}, \vec{p}, \vec{P}))$$

proves

$$\forall X(\Psi^\Theta(\alpha, X) \leftrightarrow X \in Slices(RAH_\alpha))$$
$$\forall z(\psi^\Theta(\alpha, z) \leftrightarrow z \in RAH_\alpha)$$

where Ψ^Θ *is* Ψ *with second order quantifications relativized to* $\Theta(X, \vec{p}, \vec{P})$.

Idem with λ_α.

3. *If* $rel(\alpha) \neq \emptyset$ *then* $True^{\mathbf{PA^1}}(RAH_\alpha)$. *In particular,* $Slices(RAH_\alpha)$ *benefits strong closure properties.*

Proof 1) Just apply Theorem B.8 to the formula $A(\alpha, Z, R, \sigma, \rho)$ which expresses that

- $rel(\alpha)$ is a total ordering
- if $rel(\alpha)$ has a greatest element i and $REL(\sigma[i], Z[i])$ is a total ordering on $Slices(Z[i])$ then $R = Def^{\Sigma^1_\infty}(Z[i])$ and $\eta = \omega_{Godel} \otimes \omega \otimes \sigma[i]^\omega$
- if $rel(\alpha)$ is limit and (Z, ζ) is an α-chain then (R, η) is its chain-union else $R = \emptyset$.

2) Item (b) uses the fact that when α is limit we consider the chain-union, so that a definition for Seq-RAH_α induces one for RAH_α.
3) is a direct application of item 4 of Proposition C.2 in case $rel(\alpha)$ is successor. The limit case is easy. □

D.2 Some properties of the RAH

Remark D.4 The definition of RAH given below may seem somewhat cumbersome since it does not use extensions by definitions valid in $\mathbf{PA^2}$ (but not in $\mathbf{PA^1}$) such as the function $\alpha \mapsto RAH_\alpha$. The reason for such a choice is that it allows to prove relativizations to domains (like RAH itself) for which we do not yet know that they satisfy the axioms of $\mathbf{PA^2}$.

Definition D.5 *We consider the following Σ^1_2 formulas*

- $RAH(X)$ *is* $\exists\alpha\exists R\exists\eta\exists Z\exists\zeta((rel(\alpha)$ *is a wo*
$$\wedge Hierarchy(\alpha, R, \eta, Z, \zeta) \wedge X \in Slices(R))$$
- $X \leq_{RAH} Y$ *is* $\exists\alpha\exists R\exists\eta\exists Z\exists\zeta\exists v((rel(\alpha)$ *is a wo*
$$\wedge Hierarchy(\alpha, R, \eta, Z, \zeta) \wedge X \leq^{sets}_{\eta,R} Y)$$

Intuitive interpretation of these formulas in a model $(M, \mathcal{F})$ of $\mathbf{PA^2}$:

- $RAH \subseteq \mathcal{F}$ is the union of all families $Slices(RAH_\alpha)$ where $\alpha \in \mathcal{F}$ is such that $rel(\alpha)$ is a well-ordering,
- $\leq_{RAH} \subseteq RAH \times RAH$ is the union of all orderings $\leq^{sets}_{\lambda_\alpha, RAH_\alpha}$.

Proposition D.6 *The following are provable in $C\Pi^1_1$-$\mathbf{PA^2}$:*

1. $\leq_{RAH}$ *is a total reflexive ordering relation on RAH*

2. *all formulas of the schema* $WO_{\leq_{RAH}}$ *(i.e.* $\leq_{RAH}$ *is a well-ordering in the sense of Definition7.2)*

3. *If* $rel(\alpha)$ *is a wo then* $Slices(RAH_\alpha)$ *is an initial segment of RAH and* $\leq^{sets}_{\lambda_\alpha,RAH_\alpha}$ *is the restriction of* $\leq_{RAH}$ *to* $Slices(RAH_\alpha)$

Proof 1) and 3) are easy from Theorem D.3. 2) is a direct consequence of 1). □

Proposition D.7 $C\Pi^1_1$-$\mathbf{PA^2}$ *proves* $RAH_\alpha \notin Slices(RAH_\alpha)$.

Proof This is a version of Russell's paradox. Consider the set $X = \{u \mid <u,u> \notin RAH_\alpha\}$. By closure properties of $Slices(RAH_\alpha)$, cf.item 3 in Theorem D.3) we have $RAH_\alpha \in Slices(RAH_\alpha) \Rightarrow X \in Slices(RAH_\alpha)$. We get the usual contradiction when considering the truth value of $u \in RAH_\alpha[u]$. □

Definition D.8 *If* $rel(\alpha)$ *is a wo we let* $\alpha+1$ *denote any set such that* $rel(\alpha+1)$ *is (up to isomorphism) a wo successor of* $rel(\alpha)$.
We say that α *is RAH-contributive if*

$$Slices(RAH_{\alpha+1}) \setminus Slices(RAH_\alpha) \neq \emptyset$$

Proposition D.9 *The following are provable in*

$$C\Pi^1_1\text{-}\mathbf{PA^2} + (\alpha \textit{ is an RAH-contributive wo})$$

1. *If* $rel(\beta)$ *is a well-ordering with smaller order type than* $rel(\alpha)$ *then* β *is also RAH-contributive.*

2. *The order type of* $rel(\alpha)$ *is at most that of* $\leq^{sets}_{\lambda_\alpha,RAH_\alpha}$ *i.e. there exists an isomorphism from* $(Domain(\alpha), rel(\alpha))$ *onto an initial segment of* $(Slices(RAH_\alpha), \leq^{sets}_{\lambda_\alpha,RAH_\alpha})$.

3. RAH_α, λ_α, *Seq-*RAH_α, *Seq-*λ_α *all belong to* $Slices(RAH_{\alpha+1})$.

4. *There exists* $\alpha' \in Slices(RAH_{\alpha+1})$ *such that* $rel(\alpha)$ *and* $rel(\alpha')$ *are isomorphic.*

Proof 1) Let γ be a wo such that β, $\beta+1$, α, $\alpha+1$ are isomorphic to initial segments $\gamma_{<m}$, $\gamma_{<n}$, $\gamma_{<p}$, $\gamma_{<q}$. Suppose β were not RAH-contributive, so that $Slices(RAH_{\gamma_{<m}}) = Slices(RAH_{\gamma_{<n}})$. An easy induction shows that

$$\forall i \geq_\gamma m \; Slices(RAH_{\gamma_{<i}}) = Slices(RAH_{\gamma_{<m}})$$

whence $\gamma_{<p}$ is not RAH-contributive, i.e. α is not RAH-contributive. Contradiction.

2) As a particular successor for α we let $Suc(\alpha_{<r})$ be either $\alpha_{<s}$ if s is the α-successor of r or α if r is the α-greatest element. Consider the function f defined for $r \in Domain(\alpha)$ as follows:

$f(r) =$ the smallest integer i such that

$$RAH_\alpha[i] \in Slices(RAH_{Suc(\alpha_{<r})}) \setminus Slices(RAH_{\alpha_{<r}})$$

Since $Slices(RAH_{\alpha_{<r}})$ is an initial segment of $Slices(RAH_{Suc(\alpha_{<r})})$ for $\leq_{RAH}$, we see that f is a strictly increasing embedding from $rel(\alpha)$ into $\leq^{sets}_{\lambda_\alpha, RAH_\alpha}$. We conclude using Proposition B.1 item 3, which gives an embedding of the range of f onto an initial segment.

3) It suffices to prove these properties for initial segments $\alpha = \delta_{<n}$, $n \in Domain(\delta)$, of any wo $rel(\delta)$. Suppose that the desired properties are true for all $\delta_{<p}$ such that $p <_\delta n$, we prove that they are valid for $\delta_{<n}$.

We can suppose $\delta_{<n}$ to be RAH-contributive, else there would be nothing to prove. Applying item 1), all $\delta_{<p}$ such that $p <_\delta n$ are also RAH-contributive, hence satisfy properties 3-4 according to the induction hypothesis.

Case n is the δ-smallest element, i.e. $\delta_{<n} = \emptyset$

Trivial, since $RAH_\emptyset = \lambda_\emptyset = \text{Seq-}RAH_\emptyset = \text{Seq-}\lambda_\emptyset = \emptyset$.

Case n is the δ-successor of m, hence $RAH_{\delta_{<n}} = Def^{\Sigma^1_\infty}(RAH_{\delta_{<m}})$

By induction hypothesis there are p, q such that

$$RAH_{\delta_{<m}} = RAH_{\delta_{<n}}[p] \text{ , Seq-}RAH_{\delta_{<m}} = RAH_{\delta_{<n}}[r]$$

Using commutation of $Def^{\Sigma^1_\infty}$ and $Slice$ (cf.Proposition C.3), we get

$$\begin{array}{rcl} RAH_{\delta_{<n}} & = & Def^{\Sigma^1_\infty}(RAH_{\delta_{<m}}) \quad = \quad Def^{\Sigma^1_\infty}(RAH_{\delta_{<n}}[p]) \\ & = & (Def^{\Sigma^1_\infty}(RAH_{\delta_{<n}})[p'] \quad = \quad RAH_{Suc(\delta_{<n})}[p'] \\ & \in & Slices(RAH_{Suc(\delta_{<n})}) \end{array}$$

If $i \neq m$ then $\text{Seq-}RAH_{\delta_{<n}}[i] = \text{Seq-}RAH_{\delta_{<m}}[i] = (RAH_{\delta_{<n}}[r])[i]$.

Also, Seq-$RAH_{\delta_{<n}}[m] = RAH_{\delta_{<m}} = RAH_{\delta_{<n}}[p]$

Using closure properties of $Slices(RAH_{\delta_{<n}})$ we can mix these two representations and get r' such that Seq-$RAH_{\delta_{<n}} = RAH_{\delta_{<n}}[r']$. A fortiori, we get r'' such that Seq-$RAH_{\delta_{<n}} = RAH_{Suc(\delta_{<n})}[r'']$.

Idem with $\lambda_{\delta_{<n}}$ and Seq-$\lambda_{\delta_{<n}}$.

Case n is limit

Using the fact that the RAH_α's are increasing, the induction hypothesis shows that $RAH_{\delta_{<i}} \in Slices(RAH_{Suc(\delta_{<n})})$ for $i <_\alpha n$. According to item 2 (b) in Theorem D.3, the Σ^1_2 definitions of $RAH_{\delta_{<n}}$, Seq-$RAH_{\delta_{<n}}$, $\lambda_{\delta_{<n}}$ and Seq-$\lambda_{\delta_{<n}}$ relativize to $RAH_{\delta_{<n}}$. Consequently, these sets are all slices of $Def^{\Sigma^1_\infty}(RAH_{\delta_{<n}}) = RAH_{Suc(\delta_{<n})}$.

4) Item 2) shows that $rel(\alpha)$ can be imbedded onto an initial segment of $\leq^{sets}_{\lambda_\alpha, RAH_\alpha}$. We conclude using item 3) and first order closure properties of $RAH_{\alpha+1}$. □

Proposition D.10 (Reflexion Property) *The following are provable in* $C\Pi^1_1$-$\mathbf{PA^2}$.

1. $\forall X(RAH(X) \Rightarrow \exists \alpha(rel(\alpha)$ *is a wo* $\wedge RAH(\alpha) \wedge X \in Slices(RAH_\alpha))$

2. $\forall P_1 \ldots \forall P_l\ \exists \alpha\ \forall x_1 \ldots \forall x_k (\exists X(RAH(X) \wedge \Phi(X, x_1, \ldots, x_k, P_1, \ldots, P_l)$
$$\leftrightarrow (\exists X(X \in Slices(RAH_\alpha) \wedge \Phi(X, x_1, \ldots, x_k, P_1, \ldots, P_l)))$$

Proof 1) Mere reformulation of the last item of the previous Proposition D.9.
2) If X is in RAH we consider γ such that

$X \in Slices(RAH_{\gamma+1})$ and $Slices(RAH_\gamma)$, $\gamma + 1$ are in RAH .

We let $Rank(X)$ be the $<_{RAH}$ smallest set β in RAH such that $rel(\delta)$ is isomorphic to $rel(\gamma + 1)$. Let $f(x_1, \ldots, x_k, P_1, \ldots, P_l)$ be the $<_{RAH}$ smallest set X in RAH such that $\Phi(X, x_1, \ldots, x_k, P_1, \ldots, P_l)$ if there exists such a set and $\emptyset$ if there is none. Let $\xi(x_1, \ldots, x_k, P_1, \ldots, P_l) = Rank(f(x_1, \ldots, x_k, P_1, \ldots, P_l))$. A convenient α is the ordinal sum of the $\xi(x_1, \ldots, x_k, P_1, \ldots, P_l)$'s , indexed by the $(x_1, \ldots, x_k)$'s, ordered lexicographically. □

Proposition D.11 *1.* $C\Pi^1_1$-$\mathbf{PA^2}$ +(*α is a non RAH-contributive wo) proves*

$$\forall X(RAH(X) \leftrightarrow X \in Slices(RAH_\alpha))$$

2. $C\Pi^1_1$-$\mathbf{PA^2}$ + *(there exists some non RAH-contributive wo) proves the relativization to RAH of all axioms of* $\mathbf{PA^2}$*.*

3. If $k \geq 1$ *and* $k \in \mathbf{N}$ *then* $C\Pi^1_k$-$\mathbf{PA^2}$ + *(every wo is RAH-contributive) proves the relativization to RAH of all axioms of* $C\Pi^1_k$-$\mathbf{PA^2}$*.*

Proof 1) Easy from (the contraposition of) item 1 in Proposition D.9.
2) If $Slices(RAH_\alpha) = Slices(RAH_{\alpha+1})$ then

$$Slices(RAH_\alpha) = Slices(Def^{\Sigma^1_\infty}(RAH_\alpha))$$

and item 3 of Proposition C.2 implies $True^{\mathbf{PA^2}}(RAH_\alpha)$. Whence all axioms of $\mathbf{PA^2}$ are valid in RAH_α; we conclude using item 1.
3) Using the *Rank* function introduced in the proof of item 2 of the previous Proposition, an easy induction up to k shows that $C\Pi^1_k$-$\mathbf{PA^2}$ proves a reflection property for Σ^1_k formulas: second order quantifiers can be bounded in the relativization to RAH of Σ^1_k formulas, namely

$$\Phi^{RAH}(x_1, \ldots, x_k) \leftrightarrow \exists\delta(rel(\delta) \text{ is a wo } \wedge \Phi^{RAH_\delta}(x_1, \ldots, x_k))$$

for every Σ^1_k formula Φ. Thus,

$$\{<x_1, \ldots, x_k> \mid \Phi^{RAH}(x_1, \ldots, x_k)\} = \{<x_1, \ldots, x_k> \mid \Phi^{RAH_\delta}(x_1, \ldots, x_k)\} \in Slices(RAH_{\delta+1})$$

This yields the relativization to RAH of every instance of the second order comprehension schema. □

As a corollary of the above Proposition, we obtain

Theorem D.12 $\mathbf{PA^2}$ *(resp.* $C\Pi^1_k$-$\mathbf{PA^2}$*) proves the relativization to RAH of all axioms of* $\mathbf{PA^2}$ *(resp.* $C\Pi^1_k$-$\mathbf{PA^2}$*).*

Proposition D.13 $C\Pi^1_1$-$\mathbf{PA^2}$ *proves that* RAH, RAH_α *are absolute for* RAH*:*

1. $\forall X(RAH(X) \to RAH^{RAH}(X))$

2. $\forall\alpha((RAH(\alpha) \wedge rel(\alpha) \text{ is a wo}) \to \forall x(x \in RAH_\alpha \Leftrightarrow (x \in (RAH_\alpha)^{RAH}))$

Proof 1) Using item 3 of Proposition D.9 and , we see that the following are provable in $C\Pi^1_1$-$\mathbf{PA^2}$ $+RAH(X)$:

- $\exists\alpha(rel(\alpha)$ is a wo $\wedge X \in Slices(RAH_\alpha))$
- $\exists\alpha\exists R\exists\eta\exists Z\exists\rho(RAH(\alpha) \wedge RAH(R) \wedge RAH(\eta) \wedge RAH(Z) \wedge RAH(\rho)$

  $\wedge rel(\alpha)$ is a wo $\wedge X \in Slices(R) \wedge Hierarchy(\alpha, R, \eta, Z, \rho))$

which is exactly $RAH^{RAH}(X)$.
2) is similar. Observe that we can consider $(RAH_\alpha)^{RAH}$ since we now know that RAH satisfies $\mathbf{PA^2}$. □

Proposition D.14 $C\Pi^1_1$-$\mathbf{PA^2}$ *proves that the notion of well-ordering is absolute for RAH, i.e.* $RAH(\alpha) \rightarrow ((rel(\alpha)$ *is a wo*$) \leftrightarrow (rel(\alpha)$ *is a wo*$)^{RAH})$

Proof Obviously, if $rel(\alpha)$ is a wo (a real one) lying in RAH then it is a wo in the sense of RAH.
Conversely, suppose α in RAH is such that $rel(\alpha)$ is a wo in the sense of RAH. Since we know that $C\Pi^1_1$-$\mathbf{PA^2}$ relativizes to RAH, we can consider the set $(RAH_\alpha)^{RAH}$ which is in RAH.
Let $rel(\eta)$ be a real wo in RAH such that α, $(RAH_\alpha)^{RAH}$ are all in $Slices(RAH_\eta)$. Due to Proposition D.13, $(RAH_\eta)^{RAH} = RAH_\eta$. Thus, $(RAH_\alpha)^{RAH} \in Slices((RAH_\eta)^{RAH})$, i.e. $(RAH_\alpha \in Slices(RAH_\eta))^{RAH}$. But RAH satisfies $\mathbf{PA^2}$, hence Proposition D.7. Thus, $rel(\alpha)$ has smaller order type than – i.e. is embeddable in – $rel(\eta)$ in the sense of RAH. Such an embedding is a real one. Since $rel(\eta)$ is a real wo, so is $rel(\alpha)$. □

Proposition D.15 $C\Pi^1_1$-$\mathbf{PA^2}$ *proves that* $\leq_{RAH}$ *and* λ_α *are absolute for RAH:*

1. $(RAH(X) \wedge RAH(Y)) \rightarrow (X \leq_{RAH} Y \leftrightarrow (X \leq_{RAH} Y)^{RAH})$
2. $rel(\alpha)$ *is a wo* $\rightarrow \forall z(z \in \lambda_\alpha \leftrightarrow (z \in \lambda_\alpha)^{RAH})$

Proof 1) The following are provable in $C\Pi^1_1$-$\mathbf{PA^2}$ $+RAH(X) + RAH(Y)$.

- $X \leq_{RAH} Y \leftrightarrow \exists\alpha(rel(\alpha)$ is a wo $\wedge X \leq^{sets}_{\lambda_\alpha, RAH_\alpha} Y)$

- $X \leq_{RAH} Y \leftrightarrow \exists\alpha\exists R\exists\eta\exists Z\exists\rho(rel(\alpha)$ is a wo $\wedge X \leq^{sets}_{\eta,R} Y$

$$\wedge RAH(\alpha) \wedge RAH(R) \wedge RAH(\eta) \wedge RAH(Z) \wedge RAH(\rho))$$

which is exactly $X \leq_{RAH} Y \leftrightarrow (X \leq_{RAH} Y)^{RAH}$ (due to the absoluteness of well-ordering).

2) As in Proposition D.13. □

As a corollary of the above Proposition, we obtain

Theorem D.16 *$C\Pi^1_1$-$\mathbf{PA^2}$ proves the relativizations to RAH of the formulas of the schema $WO_{(\leq_{RAH})^{RAH}}$ (see Definition7.2).*

D.3 RAH in $\mathbf{PA^2}$ formalized in $\mathbf{PA^1}$

The next Theorem internalizes previous results in $\mathbf{PA^1}$.

Theorem D.17 *All Theorems and Propositions in subsections B.3 to D.2 are formalizable and provable in $(I\Sigma^0_1, C\Delta^0_1)$−$\mathbf{PA^1}$.*

Proof Clearly, if $\mathbf{PA^2}$ proves F then primitive recursive arithmetic, a fortiori $(I\Sigma^0_1, C\Delta^0_1)$−$\mathbf{PA^1}$, proves the usual arithmetical formula (involving the Gödel number of F) which expresses that $\mathbf{PA^2}$ proves F. What is to be checked is that the same holds for schemas of formulas. There are 3 cases to consider.

Case Point 2 of Proposition D.6

Easy, since the schema $WO_{\leq_{RAH}}$ is proved directly: if there exists X in RAH such that $\Phi(X, \vec{U})$ then there exists such X in some RAH_α. The $\leq_{RAH}$-smallest such X is then the $\leq^{sets}_{\lambda_\alpha, RAH_\alpha}$-smallest such X in $Slices(RAH_\alpha)$. So, we only apply some comprehension axiom to get a set, the $\leq^{sets}_{\lambda_\alpha, RAH_\alpha}$-smallest element of which is the desired one.

Case Proposition D.10, Point 2. Idem as above.

Case Proposition D.11, Points 2,3

Point 2 is trivial since we did get $True^{\mathbf{PA^2}}(RAH_\alpha)$, hence an arithmetized result. Point 3 is proved by induction. The induction formula $F(k)$ is Π^0_1 ($\forall\phi(\phi$ is an axiom of $C\Pi^1_k$-$\mathbf{PA^2} \rightarrow (\phi^{RAH}$ is provable in $\ldots)$ hence the induction goes through arithmetization in $(I\Sigma^0_1, C\Delta^0_1)$−$\mathbf{PA^1}$. □

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