Intermediate submodels and generic extensions in set theory

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In this paper we show that various types of submodels of models of Zermelo-Fraenkel set theory (ZF) have the property that the passage from the submodel to the whole model is generic. These results lead to various applications such as Theorem A below on the iteration of the HOD operation. We also characterize the symmetric submodels of Cohen extensions of a model of ZF and show that these submodels have the above property.

Our main results are the following:

Let $L[x]$ denote the smallest transitive inner model of ZF containing all ordinals and $\{x\}$.

THEOREM A. *There exists a formula* $E(\alpha, v, w)$ *of the language of set theory* (α *is a variable ranging over the ordinals*) *such that the following are provable in the theory* ZF:

(i) $(\forall \alpha \, \exists y, E(\alpha, x, y)) \wedge (E(0, x, z) \Rightarrow L[x] = L[z])$;

(ii) $(E(\alpha, x, y) \wedge E(\alpha + 1, x, z)) \Rightarrow L[z] = (\mathrm{HOD})^{L(y)}$ (*where* $(\mathrm{HOD})^X$ *denotes the class constructed in* X *of sets hereditarily ordinal definable*);

(iii) $(0 \neq \lambda = \bigcup \lambda \wedge E(\lambda, x, y)) \Rightarrow L[y] = \bigcap \{L[z]\colon E(\alpha, x, z)$ *for some* $\alpha < \lambda\}$;

(iv) $(\alpha \neq \bigcup \alpha \vee (\alpha = 0 \wedge x \subset On)) \Rightarrow ((\beta \geqq \alpha \wedge E(\alpha, x, y) \wedge E(\beta, x, z)) \Rightarrow$ $L[y]$ *is a generic extension of* $L[z]$).

By $\mathfrak{N} = (N, \varepsilon)$ we denote a model of ZF and by M we denote an inner model of $\mathfrak{N}$, i.e., M is a transitive class in $\mathfrak{N}$, M contains the ordinals of $\mathfrak{N}$ and $(M, \varepsilon \restriction M^2)$ satisfies ZF. For $x \in N$ we let $M[x]$ denote the smallest inner model of $\mathfrak{N}$ which includes M and contains x as an element.

THEOREM B. *Suppose* $\mathfrak{N}$ *is a generic extension of* M (i.e., $N = M[G]$ *for some* M*-generic subset* G *of an ordered set in* M). *Let* $\mathfrak{U} = (U, \varepsilon \restriction U^2)$ *be a submodel of* $\mathfrak{N}$ *which satisfies* ZF *and includes* M. *Then* $\mathfrak{N}$ *is a generic extension of* $\mathfrak{U}$ *if and only if* $U = M[x]$ *for some* $x \in N$.

Let $\mathfrak{N}$ be a $\mathfrak{B}$-generic extension of M, where $\mathfrak{B}$ is a complete boolean algebra (c.b.a) in M. By the symmetric submodels of $\mathfrak{N}$ we mean those submodels associated (à la Fraenkel-Mostowski) to normal filters of subgroups of automorphisms of $\mathfrak{B}$.

By HOD X we denote the family of sets hereditarily ordinal definable from elements in X.

THEOREM C. *Let $\mathcal{B}$ be a* c.b.a *in M and suppose $\mathfrak{N}$ is a $\mathcal{B}$-generic extension of M.*

(i) *If $\mathfrak{X} = (X, \varepsilon \restriction X^2)$ is a submodel of $\mathfrak{N}$ which satisfies* ZF *and is such that $X = (\mathrm{HOD}\,(M \cup X))^{\mathfrak{N}}$, then $X = (\mathrm{HOD}\, M[x])^{\mathfrak{N}} = M[x]$ for some $x \in N$ and $\mathfrak{N}$ is a generic extension of $\mathfrak{X}$.*

(ii) *The symmetric submodels of $\mathfrak{N}$ are exactly the classes $(\mathrm{HOD}\, M[x])^{\mathfrak{N}}$, x varying over N.*

THEOREM D. *Suppose $N = M[a]$ for some $a \subset M$, $a \in N$. Let A be a subset of N such that $A \subset \{y \in N: y \varepsilon x\}$ for some $x \in N$ and such that $\mathfrak{N}$ is a generic extension of $M[z]$ for all $z \in A$. Set $U = \bigcap \{M[z]: z \in A\}$. Then $\mathfrak{U} = (U, \varepsilon \restriction U^2)$ satisfies* ZF *if and only if $\mathfrak{N}$ is a generic extension of $\mathfrak{U}$ and $U = M[t]$ for some $t \in U$.*

The basic tools that we employ are Cohen's forcing method and the machinery developed by R. Solovay in [13]. We also make essential use of two deep results of P. Vopěnka and P. Hájek [15] (which appear in our paper as 3.5, Theorem 1 and 9.1, Theorem 1). It can be noted that the technique of forcing is used not to produce models yielding relative consistency results but to construct auxiliary models which are tools to prove theorems in ZF or model-theoretic results about models of ZF.

Chapters 1 and 2 are devoted to a presentation of definitions and results in ordinal definability and forcing. The treatment of forcing follows Shoenfield (unramified forcing [11]). Proofs in these chapters are omitted (but for a few which have not appeared in the literature).

In Chapters 3, 4, 5 we prove further results on forcing which will be used in the sequel.

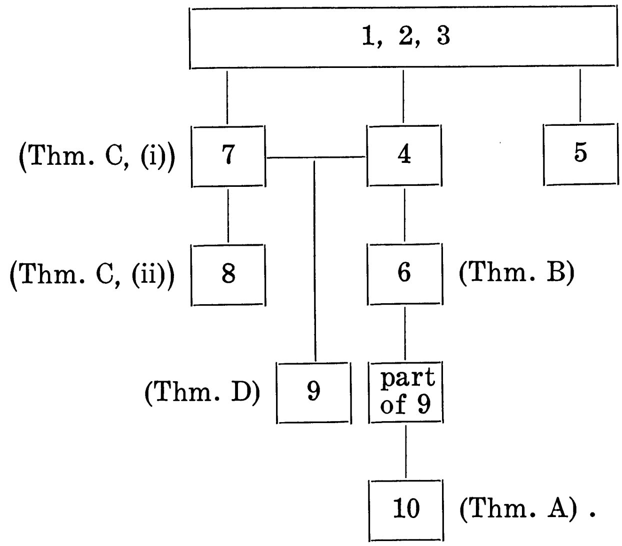
Chapter 6 gives the proof of Theorem B (cf. 6.1, Thm. 1).

In Chapter 7 we prove Theorem C, (i) (cf. 7.4, Thm. 3) which is basic for the proofs of Theorems A and D. Theorem C, (ii) is proved in Chapter 8.

Finally Theorems D and A are proved in Chapters 9 and 10 (cf. 9.4, Thm. 3 and 10.2, Thm. 1).

We would like to thank J-L. Krivine and K. McAloon with whom we had stimulating discussions on the subject of this paper and who brought to our attention many interesting questions. Also, we would like to thank the referee who suggested many interesting remarks.

Note: Some chapters can be read independently of others; the following diagram indicates the logical connections among the various chapters:



Contents

1. Families of hereditarily definable sets

1.1. We denote Zermelo-Fraenkel set theory by ZF, the axiom of choice by AC and ZF + AC by ZFC.

Let U, U' be unary predicate symbols; $(\mathrm{ZF})^U$ is the theory obtained by relativizing the axioms of ZF to U; $\mathrm{ZF}(U, U')$ is ZF plus all instances of the replacement scheme for formulas which involve the predicates U and/or U'.

1.2. Let $\mathfrak{N} = (N, \varepsilon)$ be a model of ZF. Given a subset X of N we let $(\mathfrak{N}, X)$ denote the expansion of $\mathfrak{N}$ in which X is the interpretation of the unary predicate symbol U. We say that X is a *class for* $\mathfrak{N}$ if $(\mathfrak{N}, X)$ is a model of $\mathrm{ZF}(U)$ and we then write $\mathfrak{N} \models \mathrm{ZF}(X)$.

1.3. A subset X of N is definable in $\mathfrak{N}$ (resp. $(\mathfrak{N}, Y)$) from $a_1, \cdots, a_n$ if for some formula of set theory F (resp. of the language $(=, \in, U)$) with free variables $v_0, v_1, \cdots, v_n$

$$X = \{a; \mathfrak{N} \models F(a, a_1, \cdots, a_n)\} \quad \left(\text{resp. } X = \{a; (\mathfrak{N}, Y) \models F(a, a_1, \cdots, a_n)\}\right) .$$

A subset X of N is definable in $\mathfrak{N}$ if it is definable from the empty sequence. Clearly if Y is a class for $\mathfrak{N}$ and X is definable in $(\mathfrak{N}, Y)$ from $a_1, \cdots, a_n$, then X is a class for $\mathfrak{N}$ (and even for $(\mathfrak{N}, Y)$, i.e., $(\mathfrak{N}, X, Y) \models \mathrm{ZF}(U, U')$).

1.4. The letters $\alpha, \beta, \gamma, \cdots$ vary over ordinals of $\mathfrak{N}$; V is the function on ordinals defined by $V_\alpha = \bigcup_{\beta<\alpha} P(V_\beta)$.

We say that M is an inner model of $\mathfrak{N}$ if M is a class for $\mathfrak{N}$ which is ε-transitive, contains the ordinals and such that $(M, \varepsilon \upharpoonright M^2)$ is a model of ZF.

LEMMA 1 (*Jech* [3]). *Let X be a class for $\mathfrak{N}$. If X is ε-transitive, closed under the eight Gödel operations and almost universal* (i.e., $\forall \alpha (\exists x \in X)(X \cap V_\alpha \subset x)$) *then X is an inner model of $\mathfrak{N}$.*

LEMMA 2. *Let M be an inner model of $\mathfrak{N}$ and let X be a class for $\mathfrak{N}$ such that $X \subset M$ and $\mathfrak{N} \models \mathrm{ZF}(M, X)$. If for all α, $(X \cap V_\alpha \in M)$ then $M \models \mathrm{ZF}(X)$, i.e., X is a class for M. In particular, if X is an inner model of $\mathfrak{N}$ then X is an inner model of M.*

Proof. Let $E(u, v)$ be a formula of the language $(\in, =, U)$ with parameters in M which defines a functional relation in (M, X) and let $a \in M$. The range of E on a is a set $b \in N$, $b \subset M$. Let β be larger than the ranks of a, b and the parameters in E. Using the reflection principle in $(\mathfrak{N}, M, X)$ we get an ordinal $\alpha > \beta$ such that $[E(x, y)]^{(M,X)} \Leftrightarrow [E(x, y)]^{(V_\alpha \cap M, V_\alpha \cap X)}$ for all $x, y \in (V_\alpha \cap M)$. Since $(V_\alpha \cap X) \in M$ we see that b is definable in M from elements of M, whence $b \in M$. Thus $M \models \mathrm{ZF}(X)$.

1.5. Let X be a class in $\mathfrak{N}$; we define in $(\mathfrak{N}, X)$ an inner model $L[X]$ of $\mathfrak{N}$ which is the smallest subset Y of N such that for all α $((X \cap V_\alpha) \in Y)$, Y is ε-transitive, $Y \supset On$ and $(Y, \varepsilon \upharpoonright Y^2)$ is a model of ZF.

The construction of $L[X]$ is as follows:

$$L[X] = \bigcup \{L_\alpha[X]; \alpha \in On\}$$

where the sequence $L_\alpha[X]$ is defined by induction in $(\mathfrak{N}, X)$:

$$L_\alpha[X] = \bigcup \{\{X \cap L_\beta[X]\} \cup \{x \cap L_\beta[X]; x \in TC(X)\} \cup \mathrm{Def}\,(L_\beta[X]); \beta < \alpha\}$$

(where $TC(X)$ is the class of sets in the transitive closure of some element in X and Def (a) is the set of subsets of a which are definable in a by a formula of the language $(\in, =)$ with parameters in a).

It is easy to see that $L[X]$ is definable in $(\mathfrak{N}, X)$ by a Σ_2 formula of the language $(\in, =, U)$. Also, there exists a surjective map from $On \times \mathrm{Seq}\,(TC(X))$. (Seq (A) is the class of finite sequences of elements in A) onto $L[X]$ which is definable in $(\mathfrak{N}, X)$ by a Σ_2 formula too.

Using 1.4, Lemma 2 we see that X is a class for $L[X]$.

If $X \in N$ then $X \in L[X]$ and $L[X]$ (as well as the surjective map from $On \times \mathrm{Seq}(TC(X))$ onto $L[X]$) is definable in $\mathfrak{N}$ by a Σ_1 formula of the language $(\in, =)$ with the parameter X.

Note that $L[X]$ satisfies AC when X is included in On.

1.6. Let M be an inner model of $\mathfrak{N}$, then $L[M] = M$. If $a \in N$ we write $M[a]$ in place of $L[M \cup \{a\}]$. Thus, $M[a]$ is the smallest subset Y of N which is ε-transitive, contains $M \cup \{a\}$ and such that $(Y, \varepsilon \restriction Y^2)$ is a model of ZF. The model $M[a]$ is definable in $(\mathfrak{N}, M)$ by a Σ_1 formula with the parameter a. Also, there exists a surjective map from $M \times \mathrm{Seq}(TC(a))$ onto $M[a]$ which is definable in $(\mathfrak{N}, M)$ by a Σ_1 formula with the parameter a.

We note that M is an inner model of $M[a]$ and that $M[a]$ satisfies AC when M does and a is a subset of M.

LEMMA 1. *Let M be an inner model of $\mathfrak{N}$. If $a, b \in N$ are such that $a \subset M$ and $b \subset M[a]$ then there exists $c \in N$ such that $c \subset M$ and $M[a][b] = M[c]$.*

1.7. We say that an element a in N is *definable* in $\mathfrak{N}$ *from* $a_1, \cdots, a_n$ if for some formula of set theory F with free variables $v_0, v_1, \cdots, v_n$, a is the unique set such that $\mathfrak{N} \models F(a, a_1, \cdots, a_n)$. We say that a is *definable* if it is definable from the empty sequence.

Note that a is definable in $\mathfrak{N}$ from $a_1, \cdots, a_n$ if and only if $\{b; b \varepsilon a\}$ is definable in $\mathfrak{N}$ as a subset of N.

We let $D^{\mathfrak{N}}$ be the subset of N consisting of elements which are definable in $\mathfrak{N}$, and we let $\mathrm{OD}^{\mathfrak{N}}$ be the subset of N consisting of elements which are ordinal definable in $\mathfrak{N}$. We shall omit the superscript $\mathfrak{N}$ when no confusion can arise.

It is shown in Myhill-Scott ([8]) that there exists a formula of set theory which defines OD in every ZF-model; there exists also a formula $E(v, w)$ such that in every ZF-model $\mathfrak{N}$, for all x in $\mathfrak{N}$, x is in $\mathrm{OD}^{\mathfrak{N}}$ if and only if there exists an ordinal α such that x is the only set satisfying in $\mathfrak{N}$ the formula $E(v, \alpha)$.

Hence D is definable in $\mathfrak{N}$ just in the case $D = \mathrm{OD}$ (if not, the first ordinal not in D would lead to a contradiction). Note that this case occurs in the minimal standard model (Cohen [1]). Also J. B. Paris has shown ([9]) that every theory containing ZF has a model in which $D = \mathrm{OD}$.

Furthermore $(D, \varepsilon \restriction D^2)$ is an elementary substructure of $(\mathrm{OD}, \varepsilon \restriction \mathrm{OD}^2)$.

1.8. If X is a subset of N we let $\mathrm{OD}\,X$ be the family of elements which are definable in $\mathfrak{N}$ from ordinals and elements of X. We let $\mathrm{HOD}\,X =$

$\{x \in N;\ TC(\{x\}) \subset \mathrm{OD}\,X\}$ where $TC(y)$ is the ε-transitive closure of y.

If X is a class for $\mathfrak{N}$ then $\mathrm{OD}\,X$ and $\mathrm{HOD}\,X$ are classes for $\mathfrak{N}$ since they are definable in $(\mathfrak{N}, X)$.

We note that there exists a formula $E(v, w, u)$ such that for all x in N, x is ordinal definable from y if and only if there exists an ordinal α such that x is the unique set satisfying in $\mathfrak{N}$ the formula $E(v, \alpha, y)$.

1.9. Let X be a subset of N. We say that an element $a \in N$ is definable in $(\mathfrak{N}, X)$ from $a_1, \cdots, a_n$ if for some formula of the language $(\in, =, U)$ with free variables $v_0, v_1, \cdots, v_n$, a is the unique set such that $(\mathfrak{N}, X) \models F(a, a_1, \cdots, a_n)$. We let $(\mathrm{OD}\,X)^*$ be the family of elements which are definable in $(\mathfrak{N}, X)$ from ordinals and elements of X. We let

$$(\mathrm{HOD}\,X)^* = \{x \in N;\ TC(\{x\}) \subset (\mathrm{OD}\,X)^*\}\ .$$

LEMMA 1. *If X is a class for $\mathfrak{N}$ then $(\mathrm{OD}\,X)^*$ and $(\mathrm{HOD}\,X)^*$ are classes for $\mathfrak{N}$. Also, $(\mathrm{HOD}\,X)^*$ satisfies* ZF, *hence it is an inner model of $\mathfrak{N}$.*

If X is non-empty then there exist surjective maps from $On \times \mathrm{Seq}\,(X)$ onto $(\mathrm{OD}\,X)^*$ and $(\mathrm{HOD}\,X)^*$ which are definable in $(\mathfrak{N}, X)$.

LEMMA 2. *If X is a class for $\mathfrak{N}$ such that $X \subset (\mathrm{HOD}\,X)^*$ (this occurs in particular if X is transitive) then*

$$(\mathrm{HOD}\,X)^* = \bigcup\,\{L[x];\ x \subset On \times \mathrm{Seq}\,(X)\ \textit{and}\ \ x \in (\mathrm{OD}\,X)^*\}\ .$$

1.10. Let X be a class for $\mathfrak{N}$. If $(X \cap V_\alpha) \in \mathrm{OD}\,X$ for all α then $\mathrm{OD}\,X = (\mathrm{OD}\,X)^*$ and therefore $\mathrm{HOD}\,X$ is an inner model of $\mathfrak{N}$. This is the case when X is definable in $\mathfrak{N}$ from parameters in $X \cup On$ or when X is a transitive almost universal class for $\mathfrak{N}$. Thus, if $X = M$ is an inner model of $\mathfrak{N}$ then we do not add new sets to $\mathrm{OD}\,M$ when we allow definitions involving a predicate for M and $\mathrm{HOD}\,M$ is an inner model of $\mathfrak{N}$. 1.4, Lemma 2 shows that M is a class for $\mathrm{HOD}\,M$ and 1.9, Lemma 2 shows that

$$\mathrm{HOD}\,M = \bigcup\,\{L[x];\ x \subset M\ \text{and}\ \ x \in \mathrm{HOD}\,M\}\ .$$

It is then clear that $\mathrm{HOD}\,M$ satisfies AC when M does.

2. Review of forcing

2.1. Let $\mathfrak{M} = (M, \varepsilon)$ be a model of ZF, $a \in M$ and $X \subset M$. We say that X is a subset of a if $X \subset \{b \in M;\ b\,\varepsilon\,a\}$. We say that X lies in $\mathfrak{M}$ if there exists an element x in $\mathfrak{M}$ such that $X = \{b \in M;\ b\varepsilon x\}$; in this case we need not distinguish between X and x.

Let $(C, \leqq)$ be an ordered set in $\mathfrak{M}$, let p, q, r vary over ε-elements of C. We say that p and q are *compatible* if $r \leqq p$ and $r \leqq q$ for some $r \in C$. A subset D of C is dense if for every $p \in C$ there exists $q \in D$ such that $q \leqq p$.

We say that a subset G of C is C-generic over $\mathfrak{M}$ if

(i) $p \in G,\ p \leqq q \Rightarrow q \in G$,

(ii) $p,\ q \in G \Rightarrow p,\ q$ compatible,

(iii) G meets all dense subsets of C which lie in $\mathfrak{M}$.

Remark 1. If $p \in C$ we let $C_p = \{q \in C;\ q \leqq p\}$. A subset D of C is said to be dense below p if $D \cap C_p$ is dense in C_p. Let G be C-generic over $\mathfrak{M}$; then G meets all subsets of C which lie in $\mathfrak{M}$ and are dense below some element of G.

Remark 2. We say that $X \subset C$ is open if $p \in X$ and $q \leqq p$ imply $q \in X$, for all $p,\ q \in C$. We can replace condition (iii) by (iii)$'$: G meets all dense open subsets of C which lie in $\mathfrak{M}$.

2.2. An ordered set C is *separative* if we have for all $p,\ q$ $\big(p \not\leqq q \Rightarrow \exists r \leqq p$ (r and q are incompatible)$\big)$.

If $\mathcal{B}$ is a boolean algebra we use $\mathbf{0},\ \mathbf{1},\ \wedge,\ \vee,\ -$ with their usual meanings. Recall that C is separative if and only if there exist in $\mathfrak{M}$ a complete boolean algebra $\mathcal{B}$ and an isomorphism of C into $\mathcal{B} - \{\mathbf{0}\}$ whose range is dense in $\mathcal{B} - \{\mathbf{0}\}$. Moreover such a boolean algebra is unique up to isomorphism and can be taken to be the family of regular open subsets of C, denoted $\mathcal{B}(C)$, C being given the order topology. Then the subsets of C which are C-generic over $\mathfrak{M}$ correspond to the $\mathfrak{M}$-complete ultrafilters of $\mathcal{B}(C)$.

Let $\mathcal{B}$ be a complete boolean algebra, we shall say that G is $\mathcal{B}$-generic over $\mathfrak{M}$ when G is $(\mathcal{B} - \{\mathbf{0}\})$-generic over $\mathfrak{M}$.

Remark 1. Let C be an ordered set; we define an equivalence relation $\sim_{\text{comp}}$ on C: $p \sim_{\text{comp}} q$ if and only if for all r (r is compatible with $p \Leftrightarrow r$ is compatible with q). We let C/comp be the quotient of C by $\sim_{\text{comp}}$. We put on C/comp the order induced by the following order on C:

$$p \leqq^* q \text{ if and only if } (\forall r \leqq p)\ (r \text{ is compatible with } q)\ .$$

Clearly, $p \sim_{\text{comp}} q$ if and only if $p \leqq^* q$ and $q \leqq^* p$. With the induced ordering C/comp is a separative ordered set and there is a natural correspondence between sets C-generic over $\mathfrak{M}$ and sets (C/comp)-generic over $\mathfrak{M}$.

2.3. We recall the definition of forcing: Forcing for atomic formulas, $p \Vdash x \in y$ and $p \Vdash x = y$ where p varies over C and $x,\ y$ vary over M, is defined by a joint induction on the ranks of x and y:

$$p \Vdash x \in y \text{ if and only if } (\forall q \leqq p)(\exists r \leqq q)\, \exists z (\exists s \geqq r)\big((z,\ s) \in y \text{ and } r \Vdash z = x\big)\ ,$$
$$p \Vdash x = y \text{ if and only if } (\forall q \leqq p)\, \forall z (\forall s \geqq q)\{\big((z,\ s) \in y \Longrightarrow q \Vdash z \in x\big) \text{ and } \big((z,\ s) \in x \Longrightarrow q \Vdash z \in y\big)\}\ .$$

For other formulas we follow the usual induction:

$$p \Vdash E \wedge E' \text{ if and only if } p \Vdash E \quad\text{and}\quad p \Vdash E' ,$$
$$p \Vdash \neg E \text{ if and only if } (\forall q \leqq p) \neg q \Vdash E ,$$
$$p \Vdash \forall v E(v) \text{ if and only if } (\forall x \in M) p \Vdash E(x) .$$

We define in $\mathfrak{M}$ a function $\hat{}: M \to M$ by induction on rank:

$$\hat{a} = \{(\hat{b}, p); p \in C, b \in a\} .$$

We can extend forcing to formulas of the language $(\in, =, U)$ putting $p \Vdash U(x)$ if and only if $(\forall q \leqq p)(\exists r \leqq q) \exists a r \Vdash x = \hat{a}$.

We let $\Gamma = \{(\hat{p}, p); p \in C\}$.

2.4. Main results about forcing: Let E be a formula of the language $(\in, =, U)$ with parameters in M. If $p \geqq q$ and $p \Vdash E$ then $q \Vdash E$. If E is an axiom of ZF(U) then $p \Vdash E$ for all $p \in C$. Also, $p \Vdash (E(\hat{a}_1, \cdots, \hat{a}_n))^U$ if and only if $\mathfrak{M} \models E(a_1, \cdots, a_n)$, and moreover $p \Vdash$ "U is transitive and contains the ordinals" for all $p \in C$.

Let C be separative, then $p \Vdash \hat{q} \in \Gamma$ if and only if $p \leqq q$ and $p \Vdash x \in \Gamma$ if and only if $(\forall q \leqq p)(\exists r \leqq q)(\exists s \geqq r) r \Vdash x = \hat{s}$.

The maximum principle: let M satisfy AC, if $p \Vdash \exists v E(v)$ then there exists a such that $p \Vdash E(a)$.

2.5. We define $[\![E]\!]$ as $\{p \in C; p \Vdash E\}$. This last set is a regular open subset of C; thus $[\![E]\!] \in \mathcal{B}(C)$.

Using the inductive definition of forcing we see that

$$[\![E \wedge E']\!] = [\![E]\!] \wedge [\![E']\!] , \quad [\![\neg E]\!] = -[\![E]\!],$$

and

$$[\![\forall v E(v)]\!] = \operatorname{Inf} \{[\![E(x)]\!]; x \in M\} .$$

Hence we can consider M as the field of the boolean valued model where $[\![\cdot]\!]$ gives the value of any formula. We write $M^{\mathcal{B}(C)}$ to denote this boolean model. Since $[\![E]\!] = \mathbf{1}$ when E is an axiom of ZF(U), $M^{\mathcal{B}(C)}$ satisfies ZF(U) as a boolean model. Moreover $[\![\Gamma$ is $\hat{C}$-generic over $U]\!] = \mathbf{1}$; if D is a dense subset of C then $p \Vdash p \in \Gamma \cap \hat{D}$ for all $p \in D$; hence $[\![\Gamma$ meets $\hat{D}]\!] = \mathbf{1}$. Also, $[\![U$ is transitive and contains the ordinals$]\!] = \mathbf{1}$. Finally, if $\mathfrak{M} \models E(a_1, \cdots, a_n)$ then $[\![(E(\hat{a}_1, \cdots, \hat{a}_n))^U]\!] = \mathbf{1}$.

Let T be a theory containing ZF (in the language $(\in, =, C)$). The above construction gives a finitistic proof of the following fact: a closed formula E is provable in T if and only if its relativization $(E)^U$ is provable in the theory ZF(U) plus "U is an inner model and $U(C)$, and there exists a set which is C-generic over U" plus $(T)^U$.

This will serve to convert proofs where we consider models and sets generic over these models into syntactic proofs (see 10.3).

2.6. Let G be C-generic over $\mathfrak{M}$. Following Easton [2], there exists a model $\mathfrak{M}_G = (M_G, \varepsilon)$ of ZF such that M is an inner model of $\mathfrak{M}_G$, G lies in $\mathfrak{M}_G$ and $M_G = M[G]$. Moreover, such a model is unique up to isomorphism and there is a surjective map Val_G from M onto $M[G]$ which has the following inductive definition inside $\mathfrak{M}_G$: $\mathrm{Val}_G(x) = \{\mathrm{Val}_G(y); \exists p \in G, (y, p) \in x\}$.

Let $E(v_1, \cdots, v_n)$ be a formula and $x_1, \cdots x_n$ be in M. The relation between forcing and truth is as follows:

$$\mathfrak{M}_G \models E(\mathrm{Val}_G(x_1), \cdots, \mathrm{Val}_G(x_n)) \text{ if and only if } \exists p \in G, p \Vdash E(x_1, \cdots, x_n) .$$

If $\mathfrak{M}$ is transitive then $\mathfrak{M}_G$ is well-founded and will be taken as the transitive model in the isomorphism class.

Remark 1. Let X be a class for $\mathfrak{M}$. We can extend forcing to formulas involving a predicate symbol U' to represent X, putting

$$p \Vdash U'(x) \text{ if and only if } (\forall q \leqq p)(\exists r \leqq q)(\exists a \in X) r \Vdash x = \hat{a} .$$

The equation between forcing and truth is still valid and we have $\mathfrak{M}_G \models$ ZF(M, X). We shall write $\hat{X}$ in place of U', where $\hat{X} = \{\hat{x}; x \in X\}$.

2.7. We define by induction a family V_α^C, $\alpha \in On$:

$$V_\alpha^C = \bigcup_{\beta<\alpha} P(V_\beta^C \times C) .$$

An easy induction shows that $\mathrm{Val}_G'' V_\alpha^C = (V_\alpha)^{M[G]}$.

LEMMA 1. *Let β be the rank of C, then $V_\alpha^C \subset V_{\beta+3\cdot\alpha}$ for all α.*

LEMMA 2. *Let β be the rank of C and suppose that α is a limit ordinal greater than $\beta\cdot\omega$; then* Val_G *maps $(V_\alpha)^M$ onto $(V_\alpha)^{M[G]}$.*

Proof. We note that the hypothesis on α implies $\beta + 3\cdot\alpha = \alpha$; thus $V_\alpha^C \subset (V_\alpha)^M$, whence $\mathrm{Val}_G'' (V_\alpha)^M = (V_\alpha)^{M[G]}$.

2.8. EXISTENCE LEMMA. *If $(P(C))^M$, the power set of C in M, is countable then there exist sets which are C-generic over $\mathfrak{M}$.*

Note that if M is countable so is $M[G]$.

2.9. Let $x \in M[G]$; we use $\bar{x}$ for an element of M such that $\mathrm{Val}_G(\bar{x}) = x$; $\bar{x}$ is called a term denoting x.

We shall need the following operation on terms. Let $p \in C$; we define $t \mid p$ by induction on the rank of the term t:

$$t \mid p = \{(t' \mid p, q) : q \leqq p \text{ and } \exists r \geqq q (t', r) \in t\} .$$

Clearly $p \Vdash t = t \mid p$; hence if $p \in G$ then $\mathrm{Val}_G(t) = \mathrm{Val}_G(t \mid p)$.

2.10. Let C' be a dense subset of C. If G' is C'-generic over $\mathfrak{M}$ then $\{p \in C: \exists p' \in G', p' \leqq p\}$ is C-generic over $\mathfrak{M}$; also, if G is C-generic over $\mathfrak{M}$ then $G \cap C'$ is C'-generic over $\mathfrak{M}$. This establishes a bijection between the set of C-generic filters over $\mathfrak{M}$, and the set of C'-generic filters over $\mathfrak{M}$. This allows us to replace the ordered set C by its complete boolean algebra $\mathcal{B}(C)$, more exactly by $\mathcal{B}(C) - \{\mathbf{0}\}$.

2.11. Let $(C_1, \leqq_1)$ and $(C_2, \leqq_2)$ be ordered sets of $\mathfrak{M}$. We define $C = (C_1 \times C_2, \leqq)$ as follows: $(p_1, p_2) \leqq (q_1, q_2)$ if and only if $p_1 \leqq_1 q_1$ and $p_2 \leqq_2 q_2$.

The following two facts are proved in Solovay [13]. A subset G of C is C-generic over M if and only if $G = G_1 \times G_2$ where G_1 is C_1-generic over M and G_2 is C_2-generic over $M[G_1]$. Moreover if $G = G_1 \times G_2$ is C-generic over M then $M[G_1] \cap M[G_2] = M$, $M[G_1]$, $M[G_2]$ being considered as subsets of $M[G]$.

The general iteration lemma is as follows:

LEMMA 1. *Let C be an ordered set in $\mathfrak{M}$, let G be C-generic over $\mathfrak{M}$, let D be an ordered set in $M[G]$, and let H be D-generic over $M[G]$. Then there exist an ordered set E in M and a subset K of E which is E-generic over $\mathfrak{M}$ and such that $M[G][H] = M[K]$.*

2.12. LEMMA 1. *Let $\mathfrak{N}$ be a model of* ZF, *let M, M' be inner models of $\mathfrak{N}$ such that $M \subset M'$, let C be an ordered set in M and let G be C-generic over M'. If $M[G] = M'[G]$ then $M = M'$.*

Proof. We first prove the lemma when M is a class for M'.

We reduce to the case M is countable. Using 2.6, Remark 1, we see that there exists $p \in G$ such that $p \Vdash \hat{M}[\Gamma] = \widehat{M'}[\Gamma]$ (C-forcing over M'). Let H be C-generic over $M'[G]$ such that $p \in H$, then $M[H] = M'[H]$ and $G \times H$ is $(C \times C)$-generic over M'. Applying 2.11, we get $M = M[G] \cap M[H]$ and $M' = M'[G] \cap M'[H]$, whence $M = M'$.

We now prove the lemma in the general case.

For all α we have $(V_\alpha \cap M') \in M[G]$ and M is a class for $M[V_\alpha \cap M']$ (see 1.6). Since $M[V_\alpha \cap M'] \subset M'$ we have $M[V_\alpha \cap M'][G] = M[G]$. Applying the previous case we get $M[V_\alpha \cap M'] = M$ for all α. Thus $M = \bigcup_\alpha M[V_\alpha \cap M'] = M'$, which proves the lemma.

2.13. Let $\mathcal{B}$ be a complete boolean algebra (in $\mathfrak{M}$) and let $\mathcal{A}$ be (in $\mathfrak{M}$) a complete subalgebra of $\mathcal{B}$. If $b, c \in \mathcal{B}$ we let $b \Delta c = (b - c) \vee (c - b)$ and $b^{\mathcal{A}} = \operatorname{Inf}\{x \in \mathcal{A}; x \geqq b\}$. We note that $(b \vee c)^{\mathcal{A}} = b^{\mathcal{A}} \vee c^{\mathcal{A}}$.

Let H be $\mathcal{A}$-generic over $\mathfrak{M}$. We define in $M[H]$ an equivalence relation $\sim_H$ on $\mathcal{B}$ as follows: $b \sim_H c$ if and only if $(b \Delta c)^{\mathcal{A}} \notin H$. We denote by $[b]_H$

the equivalence class of b and $\mathcal{B}/H$ the set of equivalence classes ($\mathcal{B}/H$ is a set in $M[H]$); $\mathcal{B}$ induces a structure of boolean algebra on $\mathcal{B}/H$.

The following theorem is implicit in Solovay-Tennenbaum [14, Sec. 5].

THEOREM 1. (i) *$\mathcal{B}/H$ is a complete boolean algebra in $M[H]$.*

(ii) *Set $S(H) = \{b \in \mathcal{B};\ b^{\mathcal{A}} \in H\}$; the separative quotient of $S(H)$ is canonically isomorphic to $(\mathcal{B}/H) - \{\mathbf{0}\}$ and this isomorphism is in $M[H]$.*

(iii) *Let K be $(\mathcal{B}/H)$-generic over $M[H]$ and let $G = \{b \in \mathcal{B};\ [b]_H \in K\}$. Then G is $\mathcal{B}$-generic over $\mathfrak{M}$ and $M[H][K] = M[G]$.*

(iv) *Let G be $\mathcal{B}$-generic over M. Then $G \cap \mathcal{A}$ is $\mathcal{A}$-generic over M and G is $S(G \cap \mathcal{A})$-generic over $M[G \cap \mathcal{A}]$.*

Proof of (i). Let $X \subset \mathcal{B}/H$, $X \in M[H]$. We show that X has a supremum in $\mathcal{B}/H$. Let $\bar{X}$ be a notation for X such that $\mathrm{Val}_H(\bar{X}) = X$. For $b \in \mathcal{B}$ we set $[\hat{b}] = \{(\hat{x}, -(b \Delta x)^{\mathcal{A}});\ x \in \mathcal{B}\}$. Clearly $\mathrm{Val}_H([\hat{b}]) = [b]_H$. We then define

$$u = \mathrm{Sup}\,\{b \wedge [\![[\hat{b}] \in \bar{X}]\!];\ b \in \mathcal{B}\}$$

(here $[\![\cdots]\!]$ is a boolean value in $\mathcal{A}$).

We show that $[u]_H$ is the supremum of X in $\mathcal{B}/H$. If $[b]_H \in X$ then $[\![[\hat{b}] \in \bar{X}]\!] \in H$ and $b \sim_H (b \wedge [\![[\hat{b}] \in \bar{X}]\!])$, whence $[u]_H \geqq [b]_H$. Thus $[u]_H$ is an upper bound of X. Now let $[v]_H$ be an upper bound of X: for every $b \in \mathcal{B}$, if $[b]_H \in X$ then $b - v \sim_H \mathbf{0}$.

Let $b \in \mathcal{B}$: (a) either $[b]_H \notin X$, therefore $(b \wedge [\![[\hat{b}] \in \bar{X}]\!]) \sim_H \mathbf{0}$ and $(b \wedge [\![[\hat{b}] \in \bar{X}]\!]) - v \sim_H \mathbf{0}$; or (b) $[b]_H \in X$, therefore $b \sim_H (b \wedge [\![[\hat{b}] \in \bar{X}]\!])$, and from $b - v \sim_H \mathbf{0}$ we get $(b \wedge [\![[\hat{b}] \in \bar{X}]\!]) - v \sim_H \mathbf{0}$. Thus $\mathrm{Sup}\,\{(b \wedge [\![[\hat{b}] \in \bar{X}]\!]) - v;\ b \in \mathcal{B}\} \sim_H \mathbf{0}$, i.e., $u - v \sim_H \mathbf{0}$ and $[u]_H \leqq [v]_H$. This shows that $[u]_H$ is the lowest upper bound of X in $\mathcal{B}/H$.

Proof of (ii). We first note that $[b]_H$ is not $\mathbf{0}$ if and only if $b^{\mathcal{A}} \in H$; i.e., if and only if $b \in S(H)$. Thus

$$\{[b]_H;\ b \in S(H)\} = (\mathcal{B}/H) - \{\mathbf{0}\}\ .$$

Now if p, q are in $S(H)$, keeping the notation of 2.2, Remark 1, we have

$$\begin{aligned} p \leqq^* q \text{ (in } S(H)) &\Longleftrightarrow (\forall p' \leqq p)(p' \in S(H) \\ &\qquad\qquad \Longrightarrow p' \text{ is compatible with } q \text{ in } S(H)) \\ &\Longleftrightarrow (\forall p' \leqq p)([p']_H \neq \mathbf{0} \Longrightarrow [p' \wedge q]_H \neq \mathbf{0}) \\ &\Longleftrightarrow [p - q]_H = \mathbf{0} \\ &\Longleftrightarrow [p]_H \leqq [q]_H\ . \end{aligned}$$

Therefore $p \sim_{\mathrm{comp}} q$ (in $S(H)$) if and only if $[p]_H = [q]_H$.

Thus the two equivalence relations $\sim_{\mathrm{comp}}$ and $\sim_H$ coincide on $S(H)$, whence the isomorphism between the separative quotient of $S(H)$ and $(\mathcal{B}/H) - \{\mathbf{0}\}$.

2.14. Let $\mathcal{B}$ be a complete boolean algebra and let G be $\mathcal{B}$-generic over M. Let $a \in M[G]$ be such that $a \subset M$, there exists a term $\bar{a}$ such that $\mathrm{Val}_G(\bar{a}) = a$. Let α be such that $a \subset (V_\alpha)^M$ and define $\mathcal{B}(\bar{a})$ as the smallest complete subalgebra of $\mathcal{B}$ which includes $\{[\![\hat{x} \in \bar{a}]\!]: x \in (V_\alpha)^M\}$. The following fact is proved in [14]:

PROPOSITION 1. $M[a] = M[G \cap \mathcal{B}(\bar{a})]$.

From the above proposition and 2.13, Theorem 1, part (iv) we get:

THEOREM 2 (*The Solovay basis result*). *Let C be an ordered set in a model $\mathfrak{M}$ of* ZF *and let G be C-generic over $\mathfrak{M}$. If $a \in M[G]$ is included in M, then $M[a]$ is a generic extension of M and $M[G]$ is a generic extension of $M[a]$.*

The following is a corollary of Proposition 1:

THEOREM 3. *Let $\mathcal{B}$ be a complete boolean algebra and let G be $\mathcal{B}$-generic over $\mathfrak{M}$. Let N be a model of* ZF *intermediate between M and $M[G]$*; i.e., *$M \subset N \subset M[G]$ and N is transitive in $M[G]$. If $N = \bigcup \{M[x]: x \subset M$ and $x \in N\}$ then there exists a complete subalgebra $\mathcal{A}$ of $\mathcal{B}$ such that $N = M[G \cap \mathcal{A}]$.*

Proof. Pick $x \subset M$ so that $(P(\mathcal{B}) \cap N) \in M[x] \subset N$. By Proposition 1, $N = M[x]$; whence, again by Proposition 1, the theorem.

COROLLARY 4. *Suppose that M satisfies the axiom of choice; let $\mathcal{B}$ and G be as in the previous theorem. If N is a model of* ZFC *intermediate between M and $M[G]$ then there exists a complete subalgebra $\mathcal{A}$ of $\mathcal{B}$ such that $N = M[G \cap \mathcal{A}]$.*

3. Automorphisms of complete boolean algebras

3.1. Let $\mathcal{B}$ be a complete boolean algebra in a model $\mathfrak{M}$ of ZF and let σ be an automorphism of $\mathcal{B}$, $\sigma \in \mathfrak{M}$.

We define $\tilde{\sigma}: M \to M$ by induction on rank:

$$\tilde{\sigma}(x) = (x - (V \times \mathcal{B})) \cup \{(\tilde{\sigma}(y), \sigma(b)); b \in \mathcal{B} \text{ and } (y, b) \in x\} .$$

LEMMA 1. *The map $\tilde{\sigma}$ is an automorphism of $M^{\mathcal{B}}$*; i.e., *$\tilde{\sigma}$ is a one-to-one map of M onto M and for every formula $E(v_1, \cdots, v_n)$ in $(\in, =, U)$*

$$[\![E(\tilde{\sigma}(x_1), \cdots, \tilde{\sigma}(x_n))]\!] = \sigma([\![E(x_1, \cdots, x_n)]\!]) .$$

We also note that $\tilde{\sigma}$ is the identity on the class $\hat{M}$ of $\hat{x}$, $x \in M$.

Clearly if σ, τ are automorphisms of $\mathcal{B}$ then $\widetilde{(\sigma \circ \tau)} = \tilde{\sigma} \circ \tilde{\tau}$. Hence $\widetilde{\sigma^{-1}} = \tilde{\sigma}^{-1}$.

Remark 2. Let $b \in \mathcal{B}$; an easy induction shows that $\tilde{\sigma}(x \mid b) = \tilde{\sigma}(x) \mid \sigma(b)$ (see 2.9).

3.2. Let G be $\mathcal{B}$-generic over $\mathfrak{M}$, then $\sigma''G$ (the image of G under σ) is $\mathcal{B}$-generic over $\mathfrak{M}$ too. Clearly $M[G] = M[\sigma''G]$. Moreover

$$\begin{aligned}\tilde{\sigma}\Gamma &= \{(\tilde{\sigma}\hat{b}, \sigma b); b \in \mathcal{B} - \{\mathbf{0}\}\} \\ &= \{(\hat{b}, \sigma b); b \in \mathcal{B} - \{\mathbf{0}\}\}\end{aligned}$$

and hence $\mathrm{Val}_G(\tilde{\sigma}\Gamma) = \overset{-1}{\sigma}{}''G$.

LEMMA 1. $\mathrm{Val}_{\sigma''G}(x) = \mathrm{Val}_G(\tilde{\overset{-1}{\sigma}}x)$ *for all* x *in* M.

Proof. By induction on rank,

$$\begin{aligned}\mathrm{Val}_{\sigma''G}(x) &= \{\mathrm{Val}_{\sigma''G}(y); \exists b \in \sigma''G, (y, b) \in x\} \\ &= \{\mathrm{Val}_G(\tilde{\overset{-1}{\sigma}}y); \exists b \in G, (y, \sigma b) \in x\} \\ &= \{\mathrm{Val}_G(\tilde{\overset{-1}{\sigma}}y); \exists b \in G, (\tilde{\overset{-1}{\sigma}}y, b) \in \tilde{\overset{-1}{\sigma}}x\} \\ &= \mathrm{Val}_G(\tilde{\overset{-1}{\sigma}}x)\ .\end{aligned}$$

3.3. Let $\mathcal{B} = \mathcal{B}(C)$, then every automorphism of the ordered set C extends canonically to an automorphism of $\mathcal{B}$.

3.4. DEFINITION 1. *We say that* σ *is involutive if* $\sigma = \overset{-1}{\sigma}$.

LEMMA 2. *Let* σ *be an automorphism of* $\mathcal{B}$ *and* $b \in \mathcal{B}$, $b \neq \mathbf{0}$. *There exists* $b' \leqq b$, $b' \neq \mathbf{0}$, *and an involutive automorphism* τ *of* $\mathcal{B}$ *such that* $\tau x = \sigma x$ *for all* $x \leqq b'$.

Proof. Either $\sigma x = x$ for all $x \leqq b$ and then we take $b' = b$, $\tau = \mathrm{Id}$, or there exists $b' \leqq b$ such that $\sigma b' \wedge b' = \mathbf{0}$ and then we define τ as follows:

$$\tau(x) = \sigma(x \wedge b') \vee \overset{-1}{\sigma}(x \wedge \sigma b') \vee \big(x - (b' \vee \sigma b')\big) \qquad \text{for all } x \in \mathcal{B}\ .$$

As a corollary we see that if G is $\mathcal{B}$-generic over $\mathfrak{M}$ then for every automorphism σ of $\mathcal{B}$ there exists an involutive automorphism τ of $\mathcal{B}$ such that $\sigma''G = \tau''G$.

3.5. The following theorem is due to P. Vopěnka and P. Hájek ([15]).

THEOREM 1. *Let* G *be* $\mathcal{B}$*-generic over* $\mathfrak{M}$ ($\mathcal{B}$ *being a complete boolean algebra*). *If a set* H *in* $M[G]$ *is* $\mathcal{B}$*-generic over* $\mathfrak{M}$ *and is such that* $M[H] = M[G]$ *then there exists an involutive automorphism* σ *of* $\mathcal{B}$ *such that* $H = \sigma''G$.

Before proving the theorem we state and prove a lemma:

LEMMA 2. *Let* $f\colon \mathcal{B} \to \mathcal{B}$, $f \in M$, *be such that* $f''G \subset G$. *Then there exists* $b \in G$ *such that* $f(c) \geqq c$ *for all* $c \leqq b$.

Proof of the lemma. Since $f''G \subset G$ we see that $x - f(x) \notin G$ for all $x \in \mathcal{B}$. Therefore $b_0 = \mathrm{Sup}\{x - f(x); x \in \mathcal{B}\} \notin G$. We then set $b = -b_0$.

Proof of the theorem. Let H be $\mathcal{B}$-generic over $\mathfrak{M}$ such that $M[H] = M[G]$, $H \neq G$. There are terms $\bar{G}$ and $\bar{H}$ in M such that $\mathrm{Val}_H(\bar{G}) = G$ and $\mathrm{Val}_G(\bar{H}) = H$. We define four functions k, l, m, n from $\mathcal{B}$ into $\mathcal{B}$: $k(x) = [\![\hat{x} \in \bar{G}]\!]$, $m(x) = [\![\hat{x} \in \bar{H}]\!]$, $l(x) = \mathrm{Inf}\,\{y \in \mathcal{B};\ x \Vdash \hat{y} \in \bar{H}\}$, $n(x) = \mathrm{Inf}\,\{y \in \mathcal{B};\ x \Vdash \hat{y} \in \bar{G}\}$. These functions are non-decreasing and clearly $(n \circ k)(x) \leqq x$ and $(l \circ m)(x) \leqq x$. Moreover $k''G \subset H$, $l''G \subset H$, $m''H \subset G$ and $n''H \subset G$. We then set $f(x) = k(x) \wedge l(x)$ and $g(x) = m(x) \wedge n(x)$. Clearly f and g are non-decreasing, $(g \circ f)''G \subset G$, $(f \circ g)''H \subset H$, and $(g \circ f)(x) \leqq x$ and $(f \circ g)(x) \leqq x$. Applying the lemma there are $b_0 \in G$ and $c_0 \in H$ such that $(g \circ f)(x) = x$ for all $x \leqq b_0$ and $(f \circ g)(x) = x$ for all $x \leqq c_0$.

We write $\mathcal{B}_b$ for $\{x \in \mathcal{B};\ x \leqq b\}$.

Since $H \neq G$ we can suppose $b_0 \wedge c_0 = \mathbf{0}$. Set $b = b_0 \wedge g(c_0)$ and $c = f(b)$, then $c \leqq (f \circ g)(c_0) = c_0$ and $g(c) = b$. Moreover $b \in G$ and $c \in H$. Also $(g \restriction \mathcal{B}_c) \circ (f \restriction \mathcal{B}_b) = \mathrm{Id} \restriction \mathcal{B}_b$ and $(f \restriction \mathcal{B}_b) \circ (g \restriction \mathcal{B}_c) = \mathrm{Id} \restriction \mathcal{B}_c$. Hence $f \restriction \mathcal{B}_b$ is an isomorphism from $\mathcal{B}_b$ onto $\mathcal{B}_c$ and $g \restriction \mathcal{B}_c$ is the inverse of $f \restriction \mathcal{B}_b$. We then set $\sigma(x) = \big(f(x \wedge b)\big) \vee \big(g(x \wedge c)\big) \vee \big(x - (b \vee c)\big)$; σ is the desired automorphism of $\mathcal{B}$.

3.6. We shall need the following improvement of the previous theorem:

THEOREM 1. *Let G be $\mathcal{B}$-generic over $\mathfrak{M}$ and let H be a set in $M[G]$ which is $\mathcal{B}$-generic over $\mathfrak{M}$ and such that $M[H] = M[G]$. Let x be a term, b an element of $\mathcal{B}$ and $\mathcal{A}$ a complete subalgebra of $\mathcal{B}$, $\mathcal{A} \in M$. Suppose that* $\mathrm{Val}_G(x) = \mathrm{Val}_H(x)$, *$b \in G \cap H$ and $G \cap \mathcal{A} = H \cap \mathcal{A}$. Then there exists an involutive automorphism τ of $\mathcal{B}$ such that $\tau''G = H$ and*

(i) $[\![\tilde{\tau}x = x]\!] = \mathbf{1}$,

(ii) *$\tau c = c$ for all $c \leqq -b$,*

(iii) $\tau \restriction \mathcal{A} = \mathrm{Id} \restriction \mathcal{A}$.

Proof. We first show how to get (i).

Applying the previous theorem, let σ be such that $\sigma = \overset{-1}{\sigma}$ and $\sigma''G = H$. Using 3.2, Lemma 1 we have $\mathrm{Val}_H(x) = \mathrm{Val}_{\sigma''G}(x) = \mathrm{Val}_G(\tilde{\overset{-1}{\sigma}}x)$. Therefore $\mathrm{Val}_H(x) = \mathrm{Val}_G(x)$ yields $[\![x = \tilde{\overset{-1}{\sigma}}x]\!] \in G$. Let $b = [\![x = \tilde{\overset{-1}{\sigma}}x]\!]$; since σ is involutive $\sigma b = b$. Set $\tau(c) = \big(\sigma(x \wedge b)\big) \vee (x - b)$; τ is an involutive automorphism and $\tau''G = H$. We show that τ satisfies (i).

Using 3.1, Remark 2, we see that $\tilde{\tau}(x) \mid b = \tilde{\tau}(x) \mid \tau b = \tilde{\tau}(x \mid b)$, but an easy induction shows that $\tilde{\tau}(x \mid b) = \tilde{\sigma}(x \mid b)$ (σ and τ coinciding below b); also $\tilde{\sigma}(x) \mid b = \tilde{\sigma}(x \mid b)$, whence $\tilde{\tau}(x) \mid b = \tilde{\sigma}(x) \mid b$.

Using 2.9, we have $b \leqq [\![\tilde{\tau}(x) = \tilde{\tau}(x) \mid b]\!]$ and $b \leqq [\![\tilde{\sigma}(x) = \tilde{\sigma}(x) \mid b]\!]$; therefore $b \leqq [\![\tilde{\tau}(x) = \tilde{\sigma}(x)]\!]$. Since $b = [\![x = \tilde{\sigma}x]\!]$ we deduce $b \leqq [\![x = \tilde{\tau}(x)]\!]$.

Since τ is the identity on $\mathcal{B}_{-b}$ we have similarly

$$\tilde{\tau}(x) \mid -b = x \mid -b \quad \text{and} \quad -b \leqq [\![x = \tilde{\tau}(x)]\!] \,.$$

Thus we get $[\![x = \tilde{\tau}(x)]\!] = \mathbf{1}$, which is condition (i).

Let $\Gamma_{\mathcal{A}} = \{(\hat{a}, a); a \in \mathcal{A}\}$. The conditions $G \cap \mathcal{A} = H \cap \mathcal{A}$ and $b \in G \cap H$ amount to $\mathrm{Val}_G(\Gamma_{\mathcal{A}}) = \mathrm{Val}_H(\Gamma_{\mathcal{A}})$ and $\mathrm{Val}_G(\Gamma \mid -b) = \mathrm{Val}_H(\Gamma \mid -b)$. Noting that $[\![\tilde{\tau}(\Gamma_{\mathcal{A}}) = \Gamma_{\mathcal{A}}]\!] = \mathbf{1}$ if and only if τ is the identity on $\mathcal{A}$ and that $[\![\tilde{\tau}(\Gamma \mid -b) = (\Gamma \mid -b)]\!] = \mathbf{1}$ if and only if τ is the identity on $\mathcal{B}_{-b}$, we see that to satisfy (ii) and (iii) it suffices to satisfy (i) for x, $\Gamma \mid -b$ and $\Gamma_{\mathcal{A}}$.

3.7. We say that an ordered set C is homogeneous if for all p, q in C there exists an automorphism σ of C such that $\sigma(p)$ and q are compatible; we say that a complete boolean algebra $\mathcal{B}$ is homogeneous if $\mathcal{B} - \{\mathbf{0}\}$ is homogeneous. Clearly if C is homogeneous so is $\mathcal{B}(C)$.

The following lemma is well-known:

LEMMA 1. *Let C be a homogeneous ordered set and let G be C-generic over $\mathfrak{M}$; then* $(\mathrm{HOD}\, M)^{M[G]} = M$.

Note: The above definition of homogeneity for complete boolean algebras differs from the classical definition (see Sikorski [12]) which says that $\mathcal{B}$ is homogeneous if for all $a, b \in \mathcal{B} - \{\mathbf{0}, \mathbf{1}\}$ there exists an automorphism of $\mathcal{B}$ which maps a onto b. Let us say that $\mathcal{B}$ is strongly homogeneous if it has this last property. Obviously strong homogeneity implies homogeneity but the converse is false: consider the complete boolean algebra $P(X)$ where X is a set with more than two elements. However, R. Solovay has shown that if $\mathcal{B}$ is homogeneous there is a strongly homogeneous algebra $\mathcal{C}$ such that $\{x \in \mathcal{B}: \mathcal{B}_x \cong \mathcal{C}\}$ is dense in $\mathcal{B} - \{\mathbf{0}\}$ ($\mathcal{C}$ could be $\{\mathbf{0}, \mathbf{1}\}$); it follows that $\mathcal{B}$ is the direct sum of copies of $\mathcal{C}$.

3.8. If $\mathcal{B}$ is a complete boolean algebra we let $\mathcal{B}^* = \{b \in \mathcal{B}; b$ is fixed under all automorphisms of $\mathcal{B}\}$. For any complete subalgebra $\mathcal{A}$ of $\mathcal{B}$ we let $\mathcal{A}^+ = \{b \in \mathcal{B}; b$ is fixed under all automorphisms of $\mathcal{B}$ which are the identity on $\mathcal{A}\}$. Clearly, $\mathcal{B}^*$ and $\mathcal{A}^+$ are complete subalgebras of $\mathcal{B}$, $\mathcal{A}^{++} = \mathcal{A}^+$, $\mathcal{B}^* = (\{\mathbf{0}, \mathbf{1}\})^+$ and so $\mathcal{B}^{**} = \mathcal{B}^*$. Note that $\mathcal{A}^+$ depends on the algebra $\mathcal{B}$ of which $\mathcal{A}$ is a subalgebra.

Letting G be $\mathcal{B}$-generic over $\mathfrak{M}$, we set $\mathcal{G} = \{H \in M[G]; H$ is $\mathcal{B}$-generic over $\mathfrak{M}$ and $M[H] = M[G]\}$. By Vopěnka-Hájek's theorem (3.5, Thm. 1) $\mathcal{G} = \{\sigma''G; \sigma$ is an automorphism of $\mathcal{B}\}$. We set $S = \bigcup \mathcal{G}$ and, for any complete subalgebra $\mathcal{A}$ of $\mathcal{B}$, $S_{\mathcal{A}} = \bigcup \{H \in \mathcal{G}; H \cap \mathcal{A} = G \cap \mathcal{A}\}$. We put on S and $S_{\mathcal{A}}$ the orderings induced by that of $\mathcal{B}$.

The definitions of $\mathcal{B}^*$ and $\mathcal{A}^+$ are motivated by the following theorem whose part (i) is due to P. Vopěnka:

THEOREM 1. (i) $$(\mathrm{HOD}\,M)^{M[G]} = M[G \cap \mathcal{B}^*],$$
$$(\mathrm{HOD}\,M[G \cap \mathcal{A}])^{M[G]} = M[G \cap \mathcal{A}^+].$$

(ii) *$M[S] = M[G \cap \mathcal{B}^*]$ and G is S-generic over $M[S]$;*
$M[S_{\mathcal{A}}] = M[G \cap \mathcal{A}^+]$ and G is $S_{\mathcal{A}}$-generic over $M[S_{\mathcal{A}}]$.

Moreover S and $S_{\mathcal{A}}$ are homogeneous ordered sets.

Proof of (i). Since $G \cap \mathcal{A}^+ = \bigcap \{H \cap \mathcal{A}^+; H \cap \mathcal{A} = G \cap \mathcal{A} \wedge H \in \mathcal{G}\}$, $(G \cap \mathcal{A}^+) \in (\mathrm{HOD}\,M[G \cap \mathcal{A}])^{M[G]}$ and $M[G \cap \mathcal{A}^+] \subset (\mathrm{HOD}\,M[G \cap \mathcal{A}])^{M[G]}$. We now show the equality. To do this it suffices to show that if $a \in (\mathrm{HOD}\,M[G \cap \mathcal{A}])$ and $a \subset M[G \cap \mathcal{A}^+]$ then $a \in M[G \cap \mathcal{A}^+]$ (since if $(\mathrm{HOD}\,M[G \cap \mathcal{A}]) - M[G \cap \mathcal{A}^+] \neq \varnothing$ then an element in it of minimal rank would be a subset of $M[G \cap \mathcal{A}^+]$). As $\mathrm{Val}_{G \cap \mathcal{A}^+}$ maps M onto $M[G \cap \mathcal{A}^+]$ and is definable in $M[G]$ from M and $G \cap \mathcal{A}$ we can suppose that $a \subset M$. So, let α be such that $a \subset V_\alpha \cap M$, let $x_0 \in M$ and E be a formula such that, for all $x \in V_\alpha \cap M$, $x \in a$ if and only if $M[G] \models E(x, x_0, G \cap \mathcal{A})$. Since $[\![E(\hat{x}, \hat{x}_0, \Gamma_{\mathcal{A}})]\!] \in \mathcal{A}^+$ we see that $x \in a$ if and only if $[\![E(\hat{x}, \hat{x}_0, \Gamma_{\mathcal{A}})]\!] \in G \cap \mathcal{A}^+$. Therefore $a \in M[G \cap \mathcal{A}^+]$.

Proof of (ii). Since $S_{\mathcal{A}} = \bigcup \{H \in \mathcal{G}; H \cap \mathcal{A} = G \cap \mathcal{A}\}$, $S_{\mathcal{A}} \in (\mathrm{HOD}\,M[G \cap \mathcal{A}])$. Moreover, for all $H \in \mathcal{G}$, $H \cap \mathcal{A} = G \cap \mathcal{A}$ implies $H \cap \mathcal{A}^+ = G \cap \mathcal{A}^+$ (use 3.6, Thm. 1), and so $G \cap \mathcal{A}^+ = S_{\mathcal{A}} \cap \mathcal{A}^+$. Therefore $M[S_{\mathcal{A}}] = M[G \cap \mathcal{A}^+]$.

In fact $S_{\mathcal{A}}$ can be simply recovered from $G \cap \mathcal{A}^+$. Noting that $b^{\mathcal{A}^+} = \mathrm{Sup}\,\{\sigma b$: σ is an automorphism of $\mathcal{B}$ which is the identity on $\mathcal{A}\}$, we see that $b \in S_{\mathcal{A}}$ if and only if $b^{\mathcal{A}^+} \in G$. Thus, with the notation of 2.13, Theorem 1, (ii), $S_{\mathcal{A}} = S(G \cap \mathcal{A}^+)$. Therefore G is $S_{\mathcal{A}}$-generic over $M[S_{\mathcal{A}}]$.

3.9. Let $\mathcal{B}$ be a complete boolean algebra in $\mathfrak{M}$ and let G be $\mathcal{B}$-generic over $\mathfrak{M}$. Letting U be a unary predicate symbol, we consider $M[G]$ as a structure for the language $(\in, =, U)$ where M is the interpretation of U. Let $E(v, w)$ be a formula in $(\in, =, U)$, let $x \in M[G]$, and suppose that the class $X = \{y: M[G] \models E(x, y)\}$ is an inner model of $M[G]$ which includes M. We propose to evaluate $(\mathrm{HOD}\,M)^X$.

Let Form be the set of formulas of set theory with one free variable and let $\varphi_E(x, \alpha, F, u)$ be the formula which expresses "$F \in \mathrm{Form} \wedge (V_\alpha \cap \{y: E(x, y)\}) \models F(u)$".

DEFINITION 1. *Let $t \in M$; in $\mathfrak{M}$ we define $\mathcal{B}^{E,t}$ as the complete subalgebra of $\mathcal{B}$ generated by*

$$\{[\![\varphi_E(t, \hat{\alpha}, \hat{F}, \hat{u})]\!]: \alpha \in On, F \in \mathrm{Form}, u \in M\}\ .$$

PROPOSITION 2. *Suppose $X = \{y: M[G] \models E(x, y)\}$ is an inner model of $M[G]$ which includes M. Let $t \in M$ be such that* $\mathrm{Val}_G(t) = x$; *then* $(\mathrm{HOD}\,M)^X =$

$M[G \cap \mathcal{B}^{E,t}]$.

Proof. By the replacement scheme, there exists α_0 and $u_0 \in M$ such that in Definition 1 we can restrict α and u to lie in α_0 and u_0.

Let

$$A = \{(\alpha, F, u): \alpha < \alpha_0,\ u \in u_0,\ F \in \text{Form and } (V_\alpha)^X \models F(u)\} ;$$

since truth in $(V_{\alpha_0})^X$ is definable in X, we have $A \in (\text{HOD}\, M)^X$. Now, $[\![\varphi_E(t, \hat{\alpha}, \hat{F}, \hat{u})]\!] \in G$ if and only if $(\alpha, F, u) \in A$. Thus, $G \cap \mathcal{B}^{E,t} \in (\text{HOD}\, M)^X$ and hence $M[G \cap \mathcal{B}^{E,t}] \subset (\text{HOD}\, M)^X$.

To show that $(\text{HOD}\, M)^X \subset M[G \cap \mathcal{B}^{E,t}]$ it is sufficient to show that every subset of M which lies in $(\text{HOD}\, M)^X$ also lies in $M[G \cap \mathcal{B}^{E,t}]$. In fact, an element of minimal rank in $(\text{HOD}\, M)^X - M[G \cap \mathcal{B}^{E,t}]$ would be a subset of $M[G \cap \mathcal{B}^{E,t}]$; since there is a surjective map from M onto $M[G \cap \mathcal{B}^{E,t}]$ which is definable in X from $G \cap \mathcal{B}^{E,t}$, we would obtain a subset of M lying in $(\text{HOD}\, M)^X - M[G \cap \mathcal{B}^{E,t}]$.

Let $z \subset (V_\alpha)^M$, $z \in (\text{HOD}\, M)^X$ and let F be a formula of set theory such that for some $s \in M$ we have $u \in z$ if and only if $X \models F(\langle u, s\rangle)$ for all $u \in (V_\alpha)^M$. Since X satisfies ZF, there exists an ordinal β such that $s \in (V_\beta)^X$ and $(V_\beta)^X$ reflects the formula F. Thus, for all $u \in (V_\alpha)^X$ we have $u \in z$ if and only if $(V_\beta)^X \models F(\langle u, s\rangle)$ if and only if $[\![\varphi_E(t, \hat{\beta}, \hat{F}, \widehat{\langle u, s\rangle})]\!] \in G$. This last equivalence shows that $z \in M[G \cap \mathcal{B}^{E,t}]$, which finishes the proof of the proposition.

Remark. Let E be the formula $w = w$; then $\mathcal{B}^{E,\varnothing} = \mathcal{B}^*$.

We give an application of Proposition 2.

Let λ be a limit ordinal, let $(\mathcal{B}_\alpha)_{\alpha \in \lambda}$ be in M a decreasing family of complete subalgebras of $\mathcal{B}$ and set $X = \bigcap \{M[G \cap \mathcal{B}_\alpha]: \alpha \in \lambda\}$. Using 1.4, Lemma 1 it is easy to see that X is an inner model of $M[G]$.

Let $E(v, w)$ be the following formula in the language $(\in, =, U)$: There exist v_1, v_2, v_3 $(v = (v_1, v_2, v_3)$ and v_3 is a function with domain v_2 and for all $\alpha \in v_2$, $w \in U[v_1 \cap v_3(\alpha)])$. For $\beta < \lambda$ we let $x_\beta = (G \cap \mathcal{B}_\beta, \lambda_\beta, (\mathcal{B}_{\beta+\alpha})_{\alpha \in \lambda_\beta})$ where λ_β is such that $\beta + \lambda_\beta = \lambda$.

It is clear that for all $\beta < \lambda$ we have $X = \{y: M[G \cap \mathcal{B}_\beta] \models E(x_\beta, y)\}$.

For $\beta < \lambda$ we let t_β be the canonical term such that

$$[\![t_\beta = (\Gamma_{\mathcal{B}_\beta}, \hat{\lambda}_\beta, \widehat{(\mathcal{B}_{\beta+\alpha})_{\alpha \in \lambda_\beta}})]\!]^{\mathcal{B}} = \mathbf{1} .$$

We note that t_β is also the canonical term such that

$$[\![t_\beta = (\Gamma_{\mathcal{B}_\beta}, \hat{\lambda}_\beta, \widehat{(\mathcal{B}_{\beta+\alpha})_{\alpha \in \lambda_\beta}})]\!]^{\mathcal{B}_\beta} = \mathbf{1}$$

(recall that $\Gamma_{\mathcal{A}} = \{(\hat{a}, a): a \in \mathcal{A}\}$).

We have

$$[\![\forall w(E(t_0, w)) \Longleftrightarrow E(t_\beta, w)]\!]^{\mathcal{B}} = \mathbf{1} ,$$

hence

$$[\![\varphi_E(t_0, \hat{\delta}, \hat{F}, \hat{u}) \Longleftrightarrow \varphi_E(t_\beta, \hat{\delta}, \hat{F}, \hat{u})]\!]^{\mathcal{B}} = \mathbf{1} \qquad \text{for all } \delta \in On,\ F \in \text{Form},\ u \in M .$$

Also,

$$[\![\varphi_E(t_\beta, \hat{\delta}, \hat{F}, \hat{u}]\!]^{\mathcal{B}} = [\![\varphi_E(t_\beta, \hat{\delta}, \hat{F}, \hat{u})]\!]^{\mathcal{B}_\beta}$$

whence

$$\mathcal{B}^{E,t_0} = \mathcal{B}^{E,t_\beta} = \mathcal{B}_\beta^{E,t_\beta} \qquad \text{for all } \beta \in \lambda .$$

DEFINITION 3. *We call the algebra $\mathcal{B}^{E,t_0}$ the derivative of the decreasing family $(\mathcal{B}_\alpha)_{\alpha\in\lambda}$ and we denote it by $((\mathcal{B}_\alpha)_{\alpha\in\lambda})^*$.*

From the above argument and Proposition 2 we deduce

PROPOSITION 4. (i) $(\mathrm{HOD}\,M)^X = M[G \cap ((\mathcal{B}_\alpha)_{\alpha\in\lambda})^*]$,

(ii) $((\mathcal{B}_\alpha)_{\alpha\in\lambda})^*$ *is included in* $\bigcap\{\mathcal{B}_\alpha\colon \alpha \in \lambda\}$ *and* $((\mathcal{B}_\alpha)_{\alpha\in\lambda})^* = ((\mathcal{B}_{\beta+\alpha})_{\alpha\in\lambda_\beta})^*$ *for all* $\beta \in \lambda$.

3.10. Let $\mathcal{B}$ be a complete boolean algebra in $\mathfrak{M}$. In $\mathfrak{M}$ we define by induction a decreasing family $\mathcal{B}^{(\alpha)}$, α varying over the non-limit ordinals, of complete subalgebras of $\mathcal{B}$: $\mathcal{B}^{(0)} = \mathcal{B}$, $\mathcal{B}^{(\alpha+2)} = (\mathcal{B}^{(\alpha+1)})^*$ for all α and $\mathcal{B}^{(\lambda+1)} = ((\mathcal{B}^{(\alpha+1)})_{\alpha\in\lambda})^*$ for all limit ordinals λ. We call the $\mathcal{B}^{(\alpha)}$'s the successive derivatives of $\mathcal{B}$.

The following theorem is a corollary of 3.8, Theorem 1 and 3.9, Proposition 4:

THEOREM 1. *Let G be $\mathcal{B}$-generic over $\mathfrak{M}$. Set $X_0 = M[G]$, $X_{\alpha+1} = M[G \cap \mathcal{B}^{(\alpha+1)}]$ for all α, and $X_\lambda = \bigcap\{M[G \cap \mathcal{B}^{(\alpha+1)}]\colon \alpha \in \lambda\}$ for all limit ordinals λ. The X_α's form a decreasing family of inner models of $M[G]$ such that $X_{\alpha+1} = (\mathrm{HOD}\,M)^{X_\alpha}$ for all α and $X_\lambda = \bigcap\{X_\alpha\colon \alpha \in \lambda\}$ for all limit ordinals λ.*

We shall write $(\mathrm{HOD}\,M)_\alpha^{M[G]}$ in place of X_α.

Remark 2. (i) In Chapter 9 we shall prove that for every limit ordinal λ there exists x such that $(\mathrm{HOD}\,M)_\lambda^{M[G]} = M[x]$ (9.5, Thm. 1).

(ii) The decreasing family $\mathcal{B}^{(\alpha)}$, $\alpha \in On$ is eventually constant; therefore the family $(\mathrm{HOD}\,M)_\alpha^{M[G]}$ is eventually constant.

(iii) The following result is easily deduced from K. McAloon's methods of [6]: for every $n \in \omega$ there exists a generic extension $L[x]$ of L such that

$$L[x] \supsetneqq (\mathrm{HOD})_1^{L[x]} \supsetneqq (\mathrm{HOD})_2^{L[x]} \supsetneqq \cdots \supsetneqq (\mathrm{HOD})_n^{L[x]} = L .$$

(iv) In [7] K. McAloon constructs two models $L[x_0]$, $L[x_1]$ which are generic extensions of L such that the sequence $(\mathrm{HOD})_n^{L[x_i]}$, $n \in \omega$, is strictly decreasing and $(\mathrm{HOD})_\omega^{L[x_i]} \neq L$; moreover $(\mathrm{HOD})_\omega^{L[x_0]}$ satisfies AC and $(\mathrm{HOD})_\omega^{L[x_1]}$

does not.

(v) In recent work ("Forcing with trees and ordinal definability") T. Jech has proved the following theorem: for every regular cardinal κ in L there exists a generic extension $L[G]$ of L such that the family $(\mathrm{HOD})_\alpha^{L[G]}$, $\alpha \in \kappa$, is strictly decreasing and $(\mathrm{HOD})_\kappa^{L[G]} = L$.

4. Collapsing algebras

4.1. Let C be an ordered set. An antichain of C is a subset of C whose elements are pairwise incompatible.

Let X, Y be two antichains of C; we say that $X \leqq Y$ if every element of X is less than an element of Y.

LEMMA 1. *Let C be well-orderable and let X, Y be two antichains of C. Then there exists a maximal antichain Z such that $Z \leqq X$ and $Z \leqq Y$.*

4.2. Let κ be a cardinal. We say that C satisfies the $<\kappa$-antichain condition if every antichain of C has cardinality strictly less than κ.

In ZFC one can prove that if C satisfies the $<\kappa$-antichain condition then $[\![\hat{\lambda}$ is a cardinal$]\!] = \mathbf{1}$ for all cardinals $\lambda \geqq \kappa$. Also the smallest κ such that C satisfies the $<\kappa$-antichain condition is finite or a regular cardinal.

4.3. Let C be a separative ordered set (see 2.2). An element p of C is an atom if no element of C is less than p. Clearly, if C has no atoms below p then there is an infinite antichain below p.

If C belongs to a model $\mathfrak{M}$ of ZF, a C-generic set G over $\mathfrak{M}$ lies in $\mathfrak{M}$ just in case there is an atom p in C and $G = \{q \in C;\, q \geqq p\}$.

4.4. Let A be a set with at least two elements. We denote by $C(A)$ the set of functions with domain an integer and range included in A. We order $C(A)$ by reverse inclusion.

If $s \in A$ we denote $|s|$ the domain of s. If $a \in A$ we let $s^\frown a$ be the function which contains s, has domain $|s| + 1$ and has value a at $|s|$. The ordered set $C(A)$ is separative, homogeneous and has no atoms.

We let $\mathcal{B}(A)$ be the complete boolean algebra associated to $C(A)$ (see 2.2).

The "effect" of $C(A)$ is to collapse A onto ω: there is a term f, namely $\{(\widehat{(n, x)}, [\![\exists\, p \in \Gamma\, \widehat{(n, x)} \in p]\!]);\, x \in A,\, n \in \omega\}$, such that $[\![f$ maps $\hat{\omega}$ onto $\hat{A}]\!] = \mathbf{1}$. R. Solovay [13] has shown that if A is transitive then the countable set $\{[\![f(\hat{m}) \in f(\hat{n})]\!];\, m, n \in \omega\}$ generates $\mathcal{B}(A)$.

4.5. Let κ be an infinite cardinal; $C(\kappa)$ has cardinality κ and collapses κ onto ω. The following theorem, due to K. McAloon, shows that this property somehow characterizes $C(\kappa)$. The theorem is proved in ZFC.

THEOREM 1. *Let C be a separative ordered set of cardinality κ. Suppose either $\kappa = \omega$ and C is atomless or κ is uncountable and C collapses κ onto ω* (i.e., $[\![\exists f(f \text{ maps } \hat{\omega} \text{ onto } \hat{\kappa})]\!] = \mathbf{1}$). *Then C contains a dense subset isomorphic to $C(\kappa)$.*

Proof. As C collapses κ, using 4.2, we see that if κ is uncountable then below any element of C there is an antichain of cardinality κ. If $\kappa = \omega$ the same conclusion comes from 4.3. and the hypothesis C is atomless.

By the maximum principle there is a term g such that $[\![g$ maps $\hat{\omega}$ onto $\Gamma]\!] = \mathbf{1}$. We say that p decides g at n if there is an $s \in C$ such that $p \Vdash g(\hat{n}) = \hat{s}$.

By induction we construct a family p_s, $s \in C(\kappa)$, of elements of C.

Let X be an antichain of C of cardinality κ and let Y be a maximal antichain of elements which decide g at 0. Use lemma 4.1. to get a maximal antichain $Z \leqq X, Y$; Z has cardinality κ. We let $\{p_s; s \in C(\kappa) \text{ and } |s| = \{0\}\}$ be an enumeration of Z.

Suppose p_s is defined. Let X be an antichain below p_s of cardinality κ and let Y be a maximal antichain below p_s of elements which decide g at $|s|$. Use Lemma 4.1. to get a maximal antichain Z below p_s, $Z \leqq X, Y$; Z has cardinality κ. We let $\{p_{s \frown \alpha}; \alpha \in \kappa\}$ be an enumeration of Z.

Clearly, for every $n \in \omega$, $\{p_s; |s| = n\}$ is a maximal antichain of C whose elements decide g at $0, \cdots, n-1$.

We set $D = \{p_s; s \in C(\kappa)\}$. The restriction to D of the ordering of C makes D an ordered set isomorphic to $C(\kappa)$. We now show that D is dense in C. Let p be an element of C. As $[\![g$ maps $\hat{\omega}$ onto $\Gamma]\!] = \mathbf{1}$ and $p \Vdash \hat{p} \in \Gamma$, we see that $p \Vdash \exists n \in \hat{\omega}, g(n) = \hat{p}$. Let $q \leqq p$ and $n \in \omega$ such that $q \Vdash g(\hat{n}) = \hat{p}$. There exists s such that $|s| = n + 1$ and p_s is compatible with q. Clearly $p_s \Vdash g(\hat{n}) = \hat{p}$. Hence $p_s \Vdash p \in \Gamma$ and, by 2.4, we have $p_s \leqq p$. Thus D is dense in C.

Remark 2. We note that it is provable in ZF alone that if C is a separative, countable and atomless ordered set then C contains a dense subset isomorphic to $C(\omega)$.

4.6. As a corollary of the preceding theorem we get the following result due to S. Kripke [4]:

THEOREM 1 (*in* ZFC). *Every complete boolean algebra can be embedded in a $\mathcal{B}(\kappa)$.*

Proof. Let κ be the cardinality of $\mathcal{B}$; $(\mathcal{B} - \{\mathbf{0}\}) \times C(\kappa)$ has cardinality κ and collapses κ onto ω. By 4.5, Theorem 1 we see that the complete boolean algebra associated to $(\mathcal{B} - \{0\}) \times C(\kappa)$ is $\mathcal{B}(\kappa)$. Hence $\mathcal{B}$ can be embedded in $\mathcal{B}(\kappa)$.

4.7. As another corollary of 4.5, Theorem 1 we get the following theorem due to J. L. Krivine [5]:

THEOREM 1. *Let κ be a cardinal in a model $\mathfrak{M}$ of* ZFC, *let G be $C(\kappa)$-generic over $\mathfrak{M}$, and let $X \in M[G]$, $X \subset M$. Then either $M[X] = M[G]$ or there exists in $M[G]$ a set H which is $C(\kappa)$-generic over $M[X]$ and such that $M[X][H] = M[G]$.*

Proof. Let $\mathcal{A}$ be a complete subalgebra of $\mathcal{B}(\kappa)$ such that $M[X] = M[G \cap \mathcal{A}]$ (see 2.14). Let $C = \{[p]_{G\cap\mathcal{A}};\, p \in C(\kappa)\}$ (hence we consider $C(\kappa)$ as a dense subset of $\mathcal{B}(\kappa)$) and $\tilde{G} = \{[p]_{G\cap\mathcal{A}};\, p \in G\}$. By 2.13, $\tilde{G}$ is C-generic over $M[X]$.

Let λ be, in $M[X]$, the cardinality of κ. Suppose $\lambda > \omega$. Then, in $M[X]$, C is a set of cardinality λ which collapses λ onto ω. Hence C contains a dense subset isomorphic to $C(\lambda)$ and so to $C(\kappa)$. As $C(\kappa)$ in $M[X]$ is the same as $C(\kappa)$ in M we get a set H in $M[G]$, $C(\kappa)$-generic over $M[X]$, such that $M[X][H] = M[G]$.

Suppose $\lambda = \omega$. Either $\tilde{G}$ contains an atom of C whence $\tilde{G} \in M[X]$ and $M[X] = M[G]$ or there is $p \in G$ such that C has no atoms below p, and as above we get a set H which is $C(\kappa)$-generic over $M[X]$ and such that $M[X][H] = M[G]$.

4.8. Since $C(\kappa)$ is a homogeneous ordered set, applying 3.7, Lemma 1 and 4.7, Theorem 1 we get:

THEOREM 1. *Let G be $C(\kappa)$-generic over $\mathfrak{M}$ and let $X \in M[G]$, $X \subset M$, then $(\mathrm{HOD}\, M[X])^{M[G]} = M[X]$.*

4.9. We now prove a result which does not require the axiom of choice in the ground model.

THEOREM 1. *Let C be an ordered set in a model $\mathfrak{M}$ of* ZF *and let G be C-generic over $\mathfrak{M}$. Let β be the rank of C, let α be a limit ordinal greater than $\beta \cdot \omega$ and let H be $C((V_\alpha)^{M[G]})$-generic over $M[G]$. Then there exists a set K which is $C((V_\alpha)^M)$-generic over M and such that $M[G][H] = M[K]$.*

To prove the theorem we need the following lemma.

LEMMA 2. *In* ZF *it is provable that if α is a limit ordinal then there exists a bijection between V_α and $V_\alpha \times V_\alpha$.*

Proof. Since α is a limit, $V_\alpha \times V_\alpha$ is included in V_α. Now there is a natural injection of V_α into $V_\alpha \times V_\alpha$. Applying the Cantor-Bernstein theorem we see that V_α and $V_\alpha \times V_\alpha$ are equipotent.

Proof of Theorem 1. By Lemma 2, $(V_\alpha)^{M[G]}$ and $(V_\alpha)^{M[G]} \times (V_\alpha)^{M[G]}$ are equipotent in $M[G]$. Using the Cantor-Bernstein theorem we deduce that $(V_\alpha)^{M[G]}$ and $(V_\alpha)^M \times (V_\alpha)^{M[G]}$ are also equipotent in $M[G]$. Therefore $C((V_\alpha)^{M[G]})$ is isomorphic in $M[G]$ to a dense subset of $C((V_\alpha)^M) \times C((V_\alpha)^{M[G]})$, whence there exist H_0 and H_1 such that $H_0 \times H_1$ is $C((V_\alpha)^M) \times C((V_\alpha)^{M[G]})$-generic over $M[G]$ and $M[G][H] = M[G][H_0][H_1]$. In $M[G, H_0]$ the set $(V_\alpha)^M$ is countable.

The hypothesis on α makes it possible to apply 2.7, Lemma 2 and so $(V_\alpha)^{M[G]} = \mathrm{Val}_G''(V_\alpha)^M$. Therefore $(V_\alpha)^{M[G]}$ is also countable in $M[G, H_0]$. Thus $C((V_\alpha)^{M[G]})$ is isomorphic to $C(\omega)$ in $M[G, H_0]$, whence there exists H' which is $C(\omega)$-generic over $M[G, H_0]$ and such that $M[G, H_0][H'] = M[G, H_0][H_1]$.

Now $M[G, H_0, H'] = M[H_0, H'][G]$. Without loss of generality we can suppose that C is separative and atomless. Since $C \in (V_\alpha)^M$ and $(V_\alpha)^M$ is countable in $M[H_0, H']$, C is countable in $M[H_0, H']$. By 4.5, Remark 2 there exists G' which is $C(\omega)$-generic over $M[H_0, H']$ and such that $M[H_0, H'][G] = M[H_0, H'][G']$.

Finally we have $M[G][H] = M[H_0, H', G']$ and $H_0 \times H' \times G'$ is $C((V_\alpha)^M) \times C(\omega) \times C(\omega)$-generic over M. But in M the ordered set $C((V_\alpha)^M) \times C(\omega) \times C(\omega)$ contains a dense subset isomorphic to $C((V_\alpha)^M)$, whence there exists K which is $C((V_\alpha)^M)$-generic over M and such that $M[G, H] = M[K]$.

5. The way from $(\mathrm{HOD}\, M[x])^{M[G]}$ to $M[G]$ is generic

This section is devoted to the proof of Theorem 1 which generalizes 3.8, Theorem 1.

5.1. Let $\mathcal{B}$ be a complete boolean algebra in a model $\mathfrak{M}$ of ZF, let G be $\mathcal{B}$-generic over $\mathfrak{M}$, and set $\mathcal{G} = \{\sigma''G: \sigma \text{ is an automorphism of } \mathcal{B}\}$.

For $t \in M$ we define $\mathrm{rank}^*(t) = \inf\{\mathrm{rank}(\tilde{\sigma}(t)): \sigma \text{ is an automorphism of } \mathcal{B}\}$. Then if $(y, b) \in x$, $\mathrm{rank}^*(y) \leqq \mathrm{rank}^*(x)$, and $\mathrm{rank}^*(x) = \mathrm{rank}^*(\tilde{\sigma}(x))$. Also, $\{y: \mathrm{rank}^*(y) < \alpha\}$ is a set.

Let $x \in M[G]$ and $\bar{x} \in M$ be such that $\mathrm{Val}_G(\bar{x}) = x$. For $H \in \mathcal{G}$, set $T'(H) = \{(t, \mathrm{Val}_H(t)): t \in M, \mathrm{rank}^*(t) \leqq \mathrm{rank}^*(\bar{x}) \text{ and } [\![t \in TC(\{\bar{x}\})]\!] \in H\}$ and set

$$T(H) = \{(b, Z): b \in H,\ Z \text{ is a finite subset of } T'(H)\}\ .$$

Finally we set

$$T = \bigcup \{T(H): H \in \mathcal{G} \text{ and } \mathrm{Val}_H(\bar{x}) = x\}\ ,$$

and we put an ordering $\leqq$ on T: $(b, Z) \leqq (b', Z')$ if and only if $b \leqq b'$ in $\mathcal{B}$ and $Z \supset Z'$.

THEOREM 1. (i) $M[T] = (\mathrm{HOD}\, M[x])^{M[G]}$.

(ii) *In $M[T]$, $(T, \leqq)$ is a homogeneous ordered set. Moreover $T(G)$ is T-generic over $M[T]$ and $M[T][T(G)] = M[G]$.*

Note: The existence of a set T which has the above property is a particular case of a theorem of P. Vopěnka and P. Hájek (see 9.1). However the significance of the above T is interesting.

5.2. Let σ be an automorphism of $\mathcal{B}$ such that $[\![\tilde{\sigma}(\bar{x}) = \bar{x}]\!] = \mathbf{1}$; we define in $M[T]$ an automorphism f_σ of T: for $p \in T$, if $p = (b, \{(t, y), \cdots\})$ then $f_\sigma(p) = (\sigma(b), \{(\tilde{\sigma}(t), y), \cdots\})$.

Let $H, K \in \mathcal{G}$ be such that $\mathrm{Val}_H(\bar{x}) = \mathrm{Val}_K(\bar{x}) = x$ and let σ be an automorphism such that $K = \sigma''H$ and $[\![\tilde{\sigma}(\bar{x}) = \bar{x}]\!] = \mathbf{1}$; then f_σ maps $T(H)$ isomorphically onto $T(K)$.

We note that for all $H \in \mathcal{G}$ such that $\mathrm{Val}_H(\bar{x}) = x$ the elements of $T(H)$ are pairwise compatible (in fact $T(H)$ is a lattice).

Let p, q be elements of T; there exist $H, K \in \mathcal{G}$ and an automorphism σ of $\mathcal{B}$ such that $p \in T(H)$, $q \in T(K)$ and $\sigma''H = K$; $f_\sigma(p)$ and q both belong to $T(K)$ and therefore are compatible. Thus T is a homogeneous ordered set in $M[T]$.

5.3. Let $H \in \mathcal{G}$ be such that $\mathrm{Val}_H(\bar{x}) = x$; then $T'(H)$ and $T(H)$ are included in $M[x]$. Thus T is included in $M[x]$. But also $T \in (\mathrm{OD}\, M[x])^{M[G]}$; hence $T \in (\mathrm{HOD}\, M[x])^{M[G]}$ and $M[T] \subset (\mathrm{HOD}\, M[x])^{M[G]}$.

5.4. We now show that $T(G)$ is T-generic over $M[T]$.

Let Δ be a dense subset of T lying in $M[T]$; then there exists a formula E, $x_0 \in M$, $x_1, \cdots, x_n \in TC(\{x\})$ such that for all $p = (b, \{(t_1, y_1), \cdots, (t_m, y_m)\}) \in T$ we have $p \in \Delta$ if and only if $M[G] \models E(b, \{(t_1, y_1), \cdots, (t_m, y_m)\}, x_0, x_1, \cdots, x_n)$.

Let $\bar{x}_1, \cdots, \bar{x}_n \in M$ be such that $\mathrm{Val}_G(\bar{x}_1) = x_1, \cdots, \mathrm{Val}_G(\bar{x}_n) = x_n$ and set

$$D = \{a \in \mathcal{B}\colon a \Vdash E(\hat{b}, \{(\hat{t}_1, t_1), \cdots, (\hat{t}_m, t_m)\}^{\mathcal{B}}, \hat{x}_0, \bar{x}_1, \cdots, \bar{x}_n)$$

for some $b \geqq a$ and $t_1, \cdots, t_m \in M$ with rank* less than that of $\{\bar{x}\}\}$ (where $\{(\hat{t}_1, t_1), \cdots\}^{\mathcal{B}}$ is the canonical term s such that $[\![s = \{(\hat{t}_1, t_1), \cdots\}]\!] = \mathbf{1}$).

To show that Δ meets $T(G)$ it suffices to show that D meets G. Noting that D lies in M we can use the following lemma:

LEMMA 1. *Suppose that for all $a \in G$ there exists $b \in D$ such that $b \leqq a$; then D meets G.*

Proof. Let $a = \mathrm{Sup}(D)$, if $D \cap G = \varnothing$ then $(\mathbf{1} - a) \in G$ and there is no $b \in D$ such that $b \leqq (\mathbf{1} - a)$; contradiction!

Let $a \in G$; then $p = (a, \{(\bar{x}_1, x_1), \cdots, (\bar{x}_n, x_n)\}) \in T(G)$ and there exists $q \leqq p$, $q \in \Delta$. We let

$$q = \big(b, \{(\bar{x}_1, x_1), \cdots, (\bar{x}_n, x_n), (t_1, z_1), \cdots, (t_m, z_m)\}\big) ;$$

there exists $H \in \mathcal{G}$ such that $\mathrm{Val}_H(\bar{x}) = x$ and $q \in T(H)$, hence $b \leqq a$, $\mathrm{Val}_H(\bar{x}_1) = x_1, \cdots, \mathrm{Val}_H(\bar{x}_n) = x_n$, $b \in H$ and so $a \in G \cap H$. By 3.6, Theorem 1 there exists an automorphism σ of $\mathcal{B}$ such that

$$[\![\tilde{\sigma}(\bar{x}) = \bar{x}]\!] = \mathbf{1} , \quad [\![\tilde{\sigma}(\bar{x}_1) = \bar{x}_1]\!] = \mathbf{1}, \cdots, [\![\tilde{\sigma}(\bar{x}_n) = \bar{x}_n]\!] = \mathbf{1} ,$$

$H = \sigma''G$ and $\sigma(a) = a$.

Since $z_i = \mathrm{Val}_H(t_i) = \mathrm{Val}_G(\tilde{\overset{-1}{\sigma}}(t_i))$ for all $i \leqq n$, we have

$$q = \mathrm{Val}_G\big((\hat{b}, \{(\hat{\bar{x}}_1, \bar{x}_1), \cdots, (\hat{\bar{x}}_n, \bar{x}_n), (\hat{t}_1, \tilde{\overset{-1}{\sigma}}(t_1)), \cdots (\hat{t}_m, \tilde{\overset{-1}{\sigma}}(t_m))\}^{\mathcal{B}})^{\mathcal{B}}\big) .$$

Using the equation between forcing and truth we see that there exists $c \in G$ (we can suppose $c \leqq a$) such that $c \Vdash \hat{b} \in \tilde{\overset{-1}{\sigma}}(\Gamma)$ (recall that $H = \sigma''G = \mathrm{Val}_G(\tilde{\overset{-1}{\sigma}}(\Gamma))$) and

$$c \Vdash E\big(\hat{b}, \{(\hat{\bar{x}}_1, \bar{x}_1), \cdots, (\hat{\bar{x}}_n, \bar{x}_n), (\hat{t}_1, \tilde{\overset{-1}{\sigma}}(t_1)), \cdots, (\hat{t}_m, \tilde{\overset{-1}{\sigma}}(t_m))\}^{\mathcal{B}}, \hat{x}_0, \bar{x}_1, \cdots, \bar{x}_n\big) .$$

Applying σ and noting that $[\![\tilde{\sigma}(\bar{x}_i) = \bar{x}_i]\!] = \mathbf{1}$ for all $i \leqq n$, we get $\sigma(c) \Vdash \hat{b} \in \Gamma$ and

$$\sigma(c) \Vdash E\big(\hat{b}, \{(\hat{\bar{x}}_1, \bar{x}_1), \cdots, (\hat{\bar{x}}_n, \bar{x}_n), (\hat{t}_1, t_1), \cdots, (\hat{t}_m, t_m)\}^{\mathcal{B}}, \hat{x}_0, \bar{x}_1, \cdots, \bar{x}_n\big) .$$

But $\sigma(c) \Vdash \hat{b} \in \Gamma$ amounts to $b \geqq \sigma(c)$; therefore $\sigma(c) \in D$. Also, since $\sigma(a) = a$ and $c \leqq a$ we have $\sigma(c) \leqq a$.

Thus, using Lemma 1 we have shown that D meets G. Whence Δ meets $T(G)$ and so $T(G)$ is T-generic over $M[T]$.

Remark 2. In fact we can replace in this proof the hypothesis $\Delta \in M[T]$ by $\Delta \in (\mathrm{HOD}\, M[x])^{M[G]}$, thus showing that $T(G)$ is T-generic over $(\mathrm{HOD}\, M[x])^{M[G]}$.

5.5. Since T is homogeneous in $M[T]$, we have $(\mathrm{HOD}\, M[T])^{M[G]} = M[T]$ (3.7, Lemma 1). But $M[x] \subset M[T] \subset (\mathrm{HOD}\, M[x])^{M[G]}$ and so

$$(\mathrm{HOD}\, M[x])^{M[G]} \subset (\mathrm{HOD}\, M[T])^{M[G]} = M[T] ,$$

whence $M[T] = (\mathrm{HOD}\, M[x])^{M[G]}$.

Remark. This last equality can also be obtained using the facts that $M[T] \subset (\mathrm{HOD}\, M[x])^{M[G]}$, $T(G)$ is T-generic over $(\mathrm{HOD}\, M[x])^{M[G]}$, and $M[T][T(G)] = (\mathrm{HOD}\, M[x])[T(G)]$ (2.12, Lemma 1).

6. The models $M[x]$

6.1. THEOREM 1. *Let C be an ordered set in a model $\mathfrak{M}$ of* ZF *and let G be C-generic over $\mathfrak{M}$. Let X be a model of* ZF *intermediate between M and $M[G]$*; i.e., *$M \subset X \subset M[G]$ and X is transitive in $M[G]$. Then $M[G]$ is a generic extension of X if and only if there exists $x \in X$ such that $X = M[x]$.*

Proof. Suppose that $M[G]$ is a generic extension of X; let D, K be such that $D \in X$ and K is D-generic over X and $X[K] = M[G]$. By the reflection principle applied in $X[K]$ there exists α greater than the rank of D such that $G \in (X \cap V_\alpha)[K]$. Then we have $M[X \cap V_\alpha][K] = M[G]$. Since $M[X \cap V_\alpha] \subset X$ we can apply 2.12, Lemma 1 and we get $X = M[X \cap V_\alpha]$.

Now let $a \in M[G]$; we show that $M[G]$ is a generic extension of $M[a]$. We can suppose that M is countable. Let α be greater than the rank of a and let H be $C(V_\alpha \cap M[a])$-generic over $M[G]$ (recalling that $C(A)$ is the set of functions with domain an integer and range included in A, we order $C(A)$ by reverse inclusion). Then it is easy to see that $M[a][H] = M[r]$ with $r \subset \omega$ and $M[G][H]$ is a generic extension of M. Applying the Solovay basis result (2.14, Thm. 2) we see that $M[G][H]$ is a generic extension of $M[r]$. Since $M[r]$ is a generic extension of $M[a]$ we obtain that $M[G][H]$ is a generic extension of $M[a]$ (2.11, Lemma 1). Now, $G \subset M[a]$ and $M[a] \subset M[a][G] = M[G] \subset M[G][H]$; applying the Solovay basis result we see that $M[G]$ is a generic extension of $M[a]$.

Remark 2. Using the fact that M is an inner model of $M[x]$ (a corollary of 1.4, Lemma 4) we get the following consequence of Theorem 1: let M, X, N be models of ZF such that $M \subset X \subset N$; if N is a generic extension of both M and X then M is a class in X.

Question: Can there exist a model N of ZF intermediate between M and $M[G]$ such that $N \neq M[x]$ for all $x \in N$?

Note: The above question can be generalized as follows: let $\mathfrak{U} = (U, \varepsilon)$ be a model of ZF and let M be an inner model of $\mathfrak{U}$ such that $U = M[a]$ for some $a \subset M$; can there exist a model N of ZF intermediate between M and U such that $N \neq M[x]$ for all $x \in N$? This question can be answered positively. In fact, Solovay has shown that if $U = L[0^\#]$ then there exists a model N of ZFC with $L \subset N \subset L[0^\#]$, and in N, $\{\alpha: 2^{\aleph_\alpha} = \aleph_{\alpha+2}\}$ is cofinal in On (N is an Easton type Cohen extension of L); it follows that $N \neq L[x]$ for any $x \in N$.

6.2. DEFINITION 1. *Let M, M' be models of* ZF, *M' being an inner model of M. We say that M is a quasi-generic extension of M' if for some complete boolean algebra $\mathfrak{B}$ in M we have $[\![V$ is a generic extension of $\hat{M}']\!]^{\mathfrak{B}} =$* 1 *(see notation in* 2.12*), the boolean value referring to the $\mathfrak{B}$-forcing over M.*

LEMMA 2. *Let M, M' be countable models of* ZF. *Then M is a quasi-generic extension of M' if and only if $M' \subset M$ (we do not have to assume that M' is a class in M) and there exists a model N of* ZF *which is a generic extension of both M and M'* (i.e., *there exist $C \in M$ and $C' \in M'$ such that N*

is a C-generic extension of M, and N is a C′-generic extension of M′).

Proof. The assertion within parentheses comes from 6.1, Remark 2. Other points are easy.

PROPOSITION 3. *If M is a quasi-generic extension of M' then $M = M'[x]$ for some $x \in M$.*

Proof. It suffices to prove the proposition for countable M. In that case, using Lemma 2, it is a corollary of 6.1, Theorem 1.

PROPOSITION 4. *Let M, M' be models of* ZF. *Then M is a generic extension of M' if and only if M is a quasi-generic extension of M' and $M = M'[a]$ for some $a \subset M'$.*

Proof. Again it suffices to prove the proposition for countable M. One implication is trivial; using Lemma 2 the other one follows from the Solovay basis result (2.14, Thm. 2).

THEOREM 5. *Let M, M', M'' be models of* ZF. *If M is a quasi-generic extension of M' and M' is a quasi-generic extension of M'' then M is a quasi-generic extension of M''.*

Proof. It suffices to prove the theorem for countable M. Let $\mathcal{B}$ be a complete boolean algebra in M' such that

$$[\![V \text{ is a generic extension of } \hat{M}'']\!]^{\mathcal{B}} = \mathbf{1}$$

($\mathcal{B}$-forcing over M'). Using Lemma 2 let N be a generic extension of both M' and M. Considering $\mathcal{B} - \{\mathbf{0}\}$ as an ordered set in N let G be $(\mathcal{B} - \{\mathbf{0}\})$-generic over N. We have $M'[G] \models V$ is a generic extension of M'', i.e., $M'[G]$ is a generic extension of M''. Also, $N = M'[H]$ where H is D-generic over M' for some ordered set $D \in M'$. Since G is $(\mathcal{B} - \{\mathbf{0}\})$-generic over N, we have (by 2.11) $N[G] = M'[H][G] = M'[G][H]$. Thus $N[G]$ is a generic extension of $M'[G]$. But $M'[G]$ is a generic extension of M''; therefore $N[G]$ is a generic extension of M''. Since N is a generic extension of M, so is $N[G]$. Finally we see that $N[G]$ is a generic extension of both M and M''. This proves that M is a quasi-generic extension of M''.

COROLLARY 6. *Let M be a quasi-generic extension of M'. If N is a generic extension of M such that $N = M'[a]$ for some $a \subset M'$ then N is a generic extension of M'.*

Proof. By Theorem 5, N is a quasi-generic extension of M'; now Proposition 4 yields the conclusion.

Finally we note the following fact:

PROPOSITION 7. *Let M, M' be inner models of $\mathfrak{N}$ such that $M' \subset M$. If $\mathfrak{N}$ is a quasi-generic extension of both M and M' then M is a quasi-generic extension of M'.*

Proof. We reduce to the countable case. Let $\mathcal{B}$, $\mathcal{B}'$ be complete boolean algebras in $\mathfrak{N}$ such that

$[\![V$ is a generic extension of $M]\!]^{\mathcal{B}} = \mathbf{1}$ and

$[\![V$ is a generic extension of $M']\!]^{\mathcal{B}'} = \mathbf{1}$.

Let $G \times H$ be $(\mathcal{B} - \{\mathbf{0}\}) \times (\mathcal{B}' - \{\mathbf{0}\})$-generic over $\mathfrak{N}$. Applying 2.11, we see that $N[G][H]$ is a generic extension of both M' and M, whence the desired conclusion.

Remark 8. The notion of quasi-generic extension is strictly larger than that of generic extension. In fact, Cohen's original models for the negation of AC ([1]) are quasi-generic (but not generic) extensions of their constructible parts.

7. Groups of automorphisms

7.1. Let $\mathfrak{M}$ be a model of ZF. Let $\mathcal{B}$ be a complete boolean algebra in $\mathfrak{M}$; we denote by $\mathrm{Aut}(\mathcal{B})$ the group of automorphisms of $\mathcal{B}$ and by Id the identity automorphism.

Let $\{b_i: i \in I\}$ be a subset of $\mathcal{B}$ whose supremum is $\mathbf{1}$ and let $\{\sigma_i: i \in I\}$ be a subset of $\mathrm{Aut}(\mathcal{B})$; we say that the σ_i's are summable on the b_i's if there exists an automorphism σ such that $\sigma \restriction \mathcal{B}_{b_i} = \sigma_i \restriction \mathcal{B}_{b_i}$ for all $i \in I$ (recall that $\mathcal{B}_b = \{x \in \mathcal{B}: x \leqq b\}$). Such an automorphism σ is unique and we call it the sum of the σ_i's on the b_i's and we write $\sigma = \sum \{\sigma_i \restriction b_i: i \in I\}$.

PROPOSITION 1. *Let $\sigma = \sum \{\sigma_i \restriction b_i: i \in I\}$, F be a formula and $x, y \in M$; and suppose that for all $i \in I$ we have $[\![F(x, \tilde{\sigma}_i(y))]\!] = \mathbf{1}$; then $[\![F(x, \tilde{\sigma}(y))]\!] = \mathbf{1}$.*

Proof. Since $[\![\tilde{\sigma}_i(y) = \tilde{\sigma}_i(y \mid b_i)]\!] \geqq \sigma_i(b_i)$, $[\![\tilde{\sigma}(y) = \tilde{\sigma}(y \mid b_i)]\!] \geqq \sigma(b_i)$ and $\tilde{\sigma}_i(y \mid b_i) = \tilde{\sigma}(y \mid b_i)$, we have $[\![\tilde{\sigma}_i(y) = \tilde{\sigma}(y)]\!] \geqq \sigma(b_i)$. Therefore $[\![F(x, \tilde{\sigma}(y))]\!] \geqq \sigma(b_i)$ for all $i \in I$. Thus $[\![F(x, \tilde{\sigma}(y))]\!] = \mathbf{1}$.

LEMMA 2. *Every automorphism σ of $\mathcal{B}$ is a sum of involutive automorphisms which are themselves sums of $\{\sigma, \overset{-1}{\sigma}, \mathrm{Id}\}$.*

Proof. Let $\{b_i: i \in I\}$ be the family of $b \in \mathcal{B}$ such that either $\sigma(x) = x$ for all $x \leqq b$, or $\sigma(b) \wedge b = \mathbf{0}$; the supremum of $\{b_i: i \in I\}$ is $\mathbf{1}$. Set

$$\sigma_i(x) = \sigma(x \wedge b_i) \vee \overset{-1}{\sigma}(x \wedge \sigma(b_i)) \vee (x - (b_i \vee \sigma(b_i)))$$

for $x \in \mathcal{B}$. Clearly the σ_i's are involutive automorphisms which are sums of $\{\sigma, \overset{-1}{\sigma}, \mathrm{Id}\}$ and $\sigma = \sum \{\sigma_i \restriction b_i: i \in I\}$.

Let X be a subset of $\mathrm{Aut}(\mathcal{B})$; we note $\bar{X}$ the family of sums of elements of X. Clearly $X \subset \bar{X}$ and $\bar{\bar{X}} = \bar{X}$. We say that X is closed if $X = \bar{X}$.

The following lemma is a corollary of Lemma 2:

LEMMA 3. *Let* Ω_0 *be a closed subgroup of* $\mathrm{Aut}(\mathcal{B})$ *and let* $\mathrm{Invol}(\Omega_0)$ *be the set of involutive automorphisms which are in* Ω_0*; then* $\Omega_0 = \overline{\mathrm{Invol}(\Omega_0)}$.

7.2. For $x \in M$ we set $\Omega(x) = \{\sigma \in \mathrm{Aut}(\mathcal{B}) \colon [\![\tilde{\sigma}(x) = x]\!] = \mathbf{1}\}$. For $x, y \in M$ we denote by $(x, y)^{\mathcal{B}}$ the canonical term z such that $[\![z = (x, y)]\!] = \mathbf{1}$.

PROPOSITION 1. (i) $\Omega(x)$ *is a closed subgroup of* $\mathrm{Aut}(\mathcal{B})$.

(ii) $\Omega(\tilde{\sigma}(x)) = \sigma \cdot \Omega(x) \cdot \overset{-1}{\sigma}$ *for all* $\sigma \in \mathrm{Aut}(\mathcal{B})$.

(iii) $\Omega((x, y)^{\mathcal{B}}) = \Omega(x) \cap \Omega(y)$.

Proof. Part (i) is a corollary of 7.1, Proposition 1; (iii) is obvious. Now let $\tau \in \mathrm{Aut}(\mathcal{B})$; we have

$$\begin{aligned}
\tau \in \Omega(\tilde{\sigma}(x)) &\text{ if and only if } [\![(\tilde{\tau}\tilde{\sigma})(x) = \tilde{\sigma}(x)]\!] = \mathbf{1} \\
&\text{ if and only if } [\![(\widetilde{\overset{-1}{\sigma}}\tilde{\tau}\tilde{\sigma})(x) = x]\!] = \mathbf{1} \\
&\text{ if and only if } (\overset{-1}{\sigma}\tau\sigma) \in \Omega(x) \\
&\text{ if and only if } \tau \in (\sigma \cdot \Omega(x) \cdot \overset{-1}{\sigma}) \,.
\end{aligned}$$

Thus (ii) is proved.

Letting Ω_0 be a subgroup of $\mathrm{Aut}(\mathcal{B})$, we define $\Gamma(\Omega_0) = \{(\tilde{\sigma}(\Gamma), \mathbf{1}) \colon \sigma \in \Omega_0\}$; clearly if G is $\mathcal{B}$-generic over $\mathfrak{M}$ then $\mathrm{Val}_G(\Gamma(\Omega_0)) = \{\sigma''G \colon \sigma \in \Omega_0\}$.

LEMMA 2. $\Omega(\Gamma(\Omega_0)) = \bar{\Omega}_0$.

Proof. First we note the following fact: if $[\![\tilde{\sigma}(\Gamma) = \Gamma]\!] = a$ then $\sigma(x) = x$ for all $x \leqq a$ (if not, supposing $a \neq \mathbf{0}$, there would exist $b \leqq a$, $b \neq \mathbf{0}$, such that $\sigma(b) \wedge b = \mathbf{0}$; since $[\![\hat{b} \in \Gamma]\!] = b$ and $[\![\hat{b} \in \tilde{\sigma}(\Gamma)]\!] = \sigma(b)$, we would have $[\![\hat{b} \in \Gamma \Leftrightarrow \hat{b} \in \tilde{\sigma}(\Gamma)]\!] = \mathbf{1} - (b \vee \sigma(b))$, contradicting $b = b \wedge a \leqq [\![\hat{b} \in \Gamma$ and $\Gamma = \tilde{\sigma}\Gamma]\!]$).

Now let $\sigma \in \Omega(\Gamma(\Omega_0))$; then $[\![\tilde{\sigma}(\Gamma(\Omega_0)) = \Gamma(\Omega_0)]\!] = \mathbf{1}$ and $[\![\tilde{\sigma}\Gamma \in \Gamma(\Omega_0)]\!] = \mathbf{1}$. Let $\{b_i \colon i \in I\}$ be a subset of $\mathcal{B}$ whose supremum is $\mathbf{1}$ and $\{\tau_i \colon i \in I\}$ a family of automorphisms belonging to Ω_0 such that $[\![\tilde{\sigma}\Gamma = \tau_i\Gamma]\!] = b_i$ for all $i \in I$. By the above fact we have $\sigma(x) = \tau_i(x)$ for all $x \leqq \overset{-1}{\sigma}(b_i)$; therefore $\sigma = \sum \{\tau_i \restriction \overset{-1}{\sigma}(b_i) \colon i \in I\}$ and $\sigma \in \bar{\Omega}_0$.

Since $\Omega(\Gamma(\Omega_0))$ is closed and contains Ω_0 we conclude that $\bar{\Omega}_0 = \Omega(\Gamma(\Omega_0))$.

COROLLARY 3. *A subgroup* Ω_0 *of* $\mathrm{Aut}(\mathcal{B})$ *is closed if and only if there exists an* x *such that* $\Omega_0 = \Omega(x)$.

7.3. We let $T(b) = \{\sigma \in \mathrm{Aut}(\mathcal{B}) \colon \sigma(x) = x \text{ for all } x \leqq (\mathbf{1} - b)\}$.

We write $x \in \mathrm{OD}\,My$ in place of the formula "x is ordinal definable from elements in $M \cup y$" and we write $x \in \mathrm{OD}\,M\{y\}$ in place of the formula "x is ordinal definable from y and elements in M".

THEOREM 1. *For all x, $y \in M$ we have*

$$[\![x \in \mathrm{OD}\, M\{y\}]\!] = \mathrm{Sup}\,\{b \colon \Omega(x) \supset \Omega(y) \cap T(b)\} .$$

Proof. Let $b \in \mathcal{B}$ be such that $\Omega(x) \supset \Omega(y) \cap T(b)$. Let G be $\mathcal{B}$-generic over $\mathfrak{M}$, $b \in G$, and set $\mathcal{G} = \{\sigma'' G \colon \sigma \in \mathrm{Aut}\,(\mathcal{B})\}$. By 3.5, Theorem 1, $\mathcal{G}$ is M-definable in $M[G]$. By 3.6, Theorem 1 the following is true in $M[G]$:

$$\forall v(v = \mathrm{Val}_G(x) \Longleftrightarrow \exists H \in \mathcal{G}(b \in H \text{ and } \mathrm{Val}_H(y) = \mathrm{Val}_G(y) \text{ and } v = \mathrm{Val}_H(x))) .$$

In fact, if $H \in \mathcal{G}$ is such that $b \in H$ and $\mathrm{Val}_H(y) = \mathrm{Val}_G(y)$ then there exists $\sigma \in \Omega(y) \cap T(b)$ such that $H = \sigma'' G$; since $\Omega(x) \supset \Omega(y) \cap T(b)$ we have $[\![\tilde{\sigma}(x) = x]\!] = \mathbf{1}$, whence $\mathrm{Val}_H(x) = \mathrm{Val}_G(x)$.

Thus we have shown that

$$\mathrm{Sup}\,\{b \colon \Omega(x) \supset \Omega(y) \cap T(b)\} \leqq [\![x \in \mathrm{OD}\, M\{y\}]\!] .$$

Now let $X = \{b \in \mathcal{B}$: for some α and $t \in M$ we have $b \models \forall v(v = x \Leftrightarrow E(v, y, \hat{t}, \hat{\alpha}))\}$ where E is the formula given in 1.7. Clearly we have $\mathrm{Sup}\, X = [\![x \in \mathrm{OD}\, M\{y\}]\!]$.

Let $b \in X$ and let $\sigma \in \Omega(y) \cap T(b)$; then $\sigma(b) = b$, $\tilde{\sigma}(\hat{\alpha}) = \hat{\alpha}$, $\tilde{\sigma}(\hat{t}) = \hat{t}$ and $[\![\tilde{\sigma}(y) = y]\!] = \mathbf{1}$. Since $b \Vdash \forall v(v = x \Leftrightarrow E(v, y, \hat{t}, \hat{\alpha}))$, applying σ we obtain $b \Vdash \forall v(v = \tilde{\sigma}(x) \Leftrightarrow E(v, y, \hat{t}, \hat{\alpha}))$, whence $b \Vdash x = \tilde{\sigma}(x)$. Also, $\sigma \in T(b)$ and therefore $(\mathbf{1} - b) \Vdash \tilde{\sigma}(x) = x$. Thus $[\![x = \tilde{\sigma}(x)]\!] = \mathbf{1}$ and $\sigma \in \Omega(x)$; whence $\Omega(y) \cap T(b) \subset \Omega(x)$ and so

$$\mathrm{Sup}\, X = [\![x \in \mathrm{OD}\, M\{y\}]\!] \leqq \mathrm{Sup}\,\{b \colon \Omega(x) \supset \Omega(y) \cap T(b)\} .$$

This establishes the theorem.

Let Ω be a subgroup of $\mathrm{Aut}\,(\mathcal{B})$ and $y \in M$; we set

$$[\![\Omega, y]\!] = \mathrm{Sup}\,\{[\![z_1 \in y]\!] \wedge \cdots \wedge [\![z_n \in y]\!] \colon z_1, \cdots, z_n \in TC(y) \text{ and } \Omega((z_1, \cdots, z_n)^{\mathcal{B}}) \subset \Omega\} .$$

THEOREM 2. *For all x, $y \in M$ we have*

$$[\![x \in \mathrm{OD}\, My]\!] = \mathrm{Sup}\,\{b \wedge [\![\Omega, y]\!] \colon \Omega \textit{ is a closed subgroup of } \mathrm{Aut}\,(\mathcal{B}),\ b \in \mathcal{B} \textit{ and } \Omega(x) \supset \Omega \cap T(b)\} .$$

Proof. We denote by $\vec{z}$ the boolean finite sequence $(z_1, \cdots, z_n)^{\mathcal{B}}$ and by $[\![\vec{z} \in y]\!]$ the expression $[\![z_1 \in y]\!] \wedge \cdots \wedge [\![z_n \in y]\!]$.

Using the preceding theorem and the fact that

$$[\![x \in \mathrm{OD}\, My \Longleftrightarrow (\exists z_1, \cdots, z_n \in y)(x \in \mathrm{OD}\, M\{(z_1, \cdots, z_n)^{\mathcal{B}}\})]\!] = \mathbf{1} ,$$

we have

$$\begin{aligned}
[\![x \in \mathrm{OD}\, My]\!] &= \operatorname{Sup}\{[\![x \in \mathrm{OD}\, M\{(z_1, \cdots, z_n)^{\mathcal{B}}\}]\!] \wedge [\![\vec{z} \in y]\!] : z_1, \cdots, z_n \in TC(y)\} \\
&= \operatorname{Sup}\{\operatorname{Sup}\{b \wedge [\![\vec{z} \in y]\!] : b \in \mathcal{B} \\
&\qquad \text{such that } \Omega(x) \supset \Omega(\vec{z}) \cap T(b)\} : \vec{z} \in TC(y)\} \\
&= \operatorname{Sup}\{b \wedge \operatorname{Sup}\{[\![\vec{z} \in y]\!] : \vec{z} \in TC(y) \\
&\qquad \text{such that } \Omega(x) \supset \Omega(\vec{z}) \cap T(b)\} : b \in \mathcal{B}\} \\
&= \operatorname{Sup}\{b \wedge \operatorname{Sup}\{\operatorname{Sup}\{[\![\vec{z} \in y]\!] : \vec{z} \in TC(y) \\
&\qquad \text{such that } \Omega = \Omega(\vec{z})\} : \Omega \text{ such that } \Omega(x) \supset \Omega \cap T(b)\} : b \in \mathcal{B}\} \\
&= \operatorname{Sup}\{b \wedge \operatorname{Sup}\{\operatorname{Sup}\{[\![\vec{z} \in y]\!] : \vec{z} \in TC(y) \text{ such that } \Omega \supset \Omega(\vec{z})\} : \Omega \\
&\qquad \text{is closed and such that } \Omega(x) \supset \Omega \cap T(b)\} : b \in \mathcal{B}\} \\
&= \operatorname{Sup}\{b \wedge [\![\Omega, y]\!] : \Omega \text{ is a closed subgroup of } \operatorname{Aut}(\mathcal{B}),\ b \in \mathcal{B} \\
&\qquad \text{such that } \Omega(x) \supset \Omega \cap T(b)\}\ .
\end{aligned}$$

7.4. Let G be $\mathcal{B}$-generic over $\mathfrak{M}$. If y, z are terms in M such that $[\![\Omega, y]\!] = [\![\Omega, z]\!]$ for all closed subgroups Ω of $\operatorname{Aut}(\mathcal{B})$, then by 7.3, Theorem 2 we have $(\mathrm{OD}(M \cup \mathrm{Val}_G(y)))^{M[G]} = (\mathrm{OD}(M \cup \mathrm{Val}_G(z)))^{M[G]}$.

Therefore by replacement we see that there exists an ordinal η such that for all $t \in M[G]$ there exists $u \in (V_\eta)^{M[G]}$ such that $(\mathrm{OD}(M \cup t))^{M[G]} = (\mathrm{OD}(M \cup u))^{M[G]}$ and so

$$(\mathrm{HOD}(M \cup t))^{M[G]} = (\mathrm{HOD}(M \cup u))^{M[G]}\ .$$

We shall write $\mathrm{HOD}(M \cup X)$ in place of $(\mathrm{HOD}(M \cup X))^{M[G]}$.

Now let $u \in M[G]$; then

$$\mathrm{HOD}(M \cup u) = \bigcup \{\mathrm{HOD}(M \cup t) : t \in \mathrm{HOD}(M \cup u)\}\ .$$

By replacement there exists an ordinal $\alpha(u)$ such that for all $t \in \mathrm{HOD}(M \cup u)$ there exists $s \in ((\mathrm{HOD}(M \cup u)) \cap (V_{\alpha(u)})^{M[G]})$ such that $\mathrm{OD}(M \cup t) = \mathrm{OD}(M \cup s)$ and $\mathrm{HOD}(M \cup t) = \mathrm{HOD}(M \cup s)$. Thus,

$$\begin{aligned}
\mathrm{HOD}(M \cup u) &= \bigcup \{\mathrm{HOD}(M \cup s) : s \in ((\mathrm{HOD}(M \cup u)) \cap (V_{\alpha(u)})^{M[G]})\} \\
&= \mathrm{HOD}(M \cup ((\mathrm{HOD}(M \cup u)) \cap (V_{\alpha(u)})^{M[G]}))\ .
\end{aligned}$$

Set $\xi = \bigcup \{\alpha(u) : u \in (V_\eta)^{M[G]}\}$; then for every set $t \in M[G]$ we have

$$\mathrm{HOD}(M \cup t) = \mathrm{HOD}(M \cup ((\mathrm{HOD}(M \cup t)) \cap (V_\xi)^{M[G]}))\ .$$

We note that if X is a class for $M[G]$ then by replacement we have $\mathrm{HOD}(M \cup X) = \bigcup \{\mathrm{HOD}(M \cup (X \cap V_\alpha)) : \alpha \in On\}$; thus

$$\mathrm{HOD}(M \cup X) = \bigcup \{\mathrm{HOD}(M \cup ((\mathrm{HOD}(M \cup (X \cap V_\alpha))) \cap V_\xi)) : \alpha \in On\}$$

and therefore $\mathrm{HOD}(M \cup X) = \mathrm{HOD}(M \cup (V_\xi \cap (\mathrm{HOD}(M \cup X))))$.

LEMMA 1. *There exists an ordinal ξ such that for all $X \subset M[G]$, if X is a class for $M[G]$ (in particular if $X \in M[G]$), then* $\mathrm{HOD}(M \cup X) = \mathrm{HOD}(M \cup (V_\xi \cap (\mathrm{HOD}(M \cup X))))$.

We now replace t by $TC(\{t\})$, the transitive closure of $\{t\}$; $\mathrm{HOD}(M \cup TC(\{t\}))$ which is equal to $\mathrm{HOD}\,M[t]$ is an inner model of $M[G]$ (see 1.9), hence $(V_\xi \cap \mathrm{HOD}\,M[t]) \in \mathrm{HOD}\,M[t]$ and $\mathrm{HOD}\,M[t] = \mathrm{HOD}\,M[(V_\xi \cap \mathrm{HOD}\,M[t])]$. Thus we get the following theorem:

THEOREM 2. *Let G be $\mathcal{B}$-generic over M. There exists an ordinal ξ such that for every $t \in M[G]$ there exists $u \in (V_{\xi+1})^{M[G]}$ such that* $\mathrm{HOD}\,M[t] = \mathrm{HOD}\,M[u]$.

As a corollary of Theorem 2 we get the following basic result:

THEOREM 3. *Let G be $\mathcal{B}$-generic over $\mathfrak{M}$. Let X be a subset of $M[G]$ such that $(X, \varepsilon \restriction X^2)$ is a model of* ZF *and $X = (\mathrm{HOD}\,(M \cup X))^{M[G]}$; then X is a class for $M[G]$ and there exists a set $x \in X$ such that $X = M[x]$. Also, the rank of x can be bounded independently of X.*

Proof. We emphasize that we do not make the hypothesis that X is a class for $M[G]$; thus in the proof we do not use the replacement scheme associated with the structure $(M[G], X)$.

We first note that X is transitive and

$$(X \cap (V_\alpha)^{M[G]}) \in X \text{ for all ordinals } \alpha \text{ in } M\,.$$

Now $X = \bigcup \{X \cap V_\alpha : \alpha\}$; so

$$X \subset \bigcup \{\mathrm{HOD}\,M[X \cap V_\alpha] : \alpha\} \subset \mathrm{HOD}\,(M \cup X) = X\,.$$

Let ξ be as in Theorem 2. Then for all α

$$\mathrm{HOD}\,M[X \cap V_\alpha] = \mathrm{HOD}\,M[(V_{\xi+1} \cap \mathrm{HOD}\,M[X \cap V_\alpha])]$$

and

$$(V_{\xi+1} \cap \mathrm{HOD}\,M[X \cap V_\alpha]) \in \mathrm{HOD}\,M[X \cap V_\alpha] \subset X\,,$$

therefore

$$(V_{\xi+1} \cap \mathrm{HOD}\,M[X \cap V_\alpha]) \in (X \cap V_{\xi+2})\,;$$

hence

$$\bigcup \{\mathrm{HOD}\,M[X \cap V_\alpha] : \alpha\} \subset \mathrm{HOD}\,M[X \cap V_{\xi+2}] \subset \mathrm{HOD}\,M \cup X = X\,.$$

Thus $X = \mathrm{HOD}\,M[X \cap V_{\xi+2}]$.

By 5.1, Theorem 1 there exists $x \in X$ such that $X = M[x]$.

COROLLARY 4. *Let G be $\mathcal{B}$-generic over $\mathfrak{M}$; then, for every $x \in M[G]$, $(\mathrm{HOD}\,(M \cup x))^{M[G]}$ is a model of* ZF *if and only if there exists $y \in M[G]$ such that $(\mathrm{HOD}\,(M \cup x))^{M[G]} = (\mathrm{HOD}\,M[y])^{M[G]}$.*

Remark 5. The following is an example of an x such that $\mathrm{HOD}\,(M \cup x)$ is not a model of ZF. Let $G = \langle a_i, i \in \omega\rangle$ be a Cohen generic sequence of generic subsets of ω. Let $x = \{a_i : i \text{ even}\}$. Then $P(\omega) \cap (\mathrm{HOD}(M \cup x))^{M[G]} = P(\omega) \cap (\bigcup_{n \in \omega} M[a_0, a_2, \cdots, a_{2n}])$. But if $Y \subset P(\omega)$ is ordinal definable from elements of x, and $x \subset Y$, then for all odd n, $a_n \in Y$. This shows that $P(\omega) \cap (\mathrm{HOD}(M \cup x))^{M[G]}$ does not belong to $(\mathrm{HOD}(M \cup x))^{M[G]}$ so that this class is not a model of ZF.

8. Symmetric submodels

8.1. Let $\mathcal{B}$ be a complete boolean algebra in a model $\mathfrak{M}$ of ZF.

Let $\mathcal{F}$ be a filter on the set of subgroups of Aut ($\mathcal{B}$).

We set

$$N(\mathcal{F}) = \{\sigma \in \mathrm{Aut}(\mathcal{B}) : \sigma\Omega\overset{-1}{\sigma} \in \mathcal{F} \text{ and } \overset{-1}{\sigma}\Omega\sigma \in \mathcal{F} \text{ for all } \Omega \in \mathcal{F}\} ;$$

$N(\mathcal{F})$ is a subgroup of Aut ($\mathcal{B}$). We say that $\mathcal{F}$ is normal if $N(\mathcal{F}) \in \mathcal{F}$.

Set $S(\mathcal{F}) = \{x \in \mathfrak{M} : \Omega(x) \in \mathcal{F}\}$; we define in $\mathfrak{M}$ the class $HS(\mathcal{F})$ by recursion on rank:

$$x \in HS(\mathcal{F}) \text{ if and only if } x \in S(\mathcal{F}) \text{ and}$$
$$\forall y, b(b \in \mathcal{B} \text{ and } (y, b) \in x \Longrightarrow y \in HS(\mathcal{F})) .$$

DEFINITION 1. *Let G be $\mathcal{B}$-generic over $\mathfrak{M}$. The subclasses of $M[G]$ which have the form $\mathrm{Val}''_G\, HS(\mathcal{F})$ for some normal filter $\mathcal{F}$ are called the symmetric submodels of $M[G]$.*

The following result is well-known (see Scott-Solovay [10] or T. Jech [3]):

PROPOSITION 2. *All symmetric submodels of $M[G]$ are models of* ZF *(with the $\in$-relation induced by that of $M[G]$).*

The remainder of this subsection is devoted to the proof of the following theorem:

THEOREM 3. *The symmetric submodels of $M[G]$ are exactly the classes $(\mathrm{HOD}\, M[x])^{M[G]}$, x varying over $M[G]$.*

DEFINITION 4. *Let $\mathcal{F}$ be a filter on the set of subgroups of* Aut ($\mathcal{B}$); *we define in $\mathfrak{M}$ the boolean values of the atomic formulas $x \in S(\mathcal{F})$ and $x \in HS(\mathcal{F})$:*

$$[\![x \in S(\mathcal{F})]\!] = \mathrm{Sup}\,\{b ; [\![x = x']\!] \geqq b \textit{ for some } x' \in S(\mathcal{F})\}$$
$$[\![x \in HS(\mathcal{F})]\!] = \mathrm{Sup}\,\{b ; [\![x = x']\!] \geqq b \textit{ for some } x' \in HS(\mathcal{F})\} .$$

It is easy to see that the axioms of ZF still hold (i.e., have boolean value 1) for the formulas involving these new atomic formulas.

We note the following lemma which is a corollary of 7.2, Proposition 1, (ii):

LEMMA 5. *Let $x \in S(\mathcal{F})$, $y \in HS(\mathcal{F})$ and let $\sigma \in N(\mathcal{F})$; then $\tilde{\sigma}(x) \in S(\mathcal{F})$ and $\tilde{\sigma}(y) \in HS(\mathcal{F})$.*

PROPOSITION 6. *If $\mathcal{F}$ is a normal filter then*

$$[\![x \in HS(\mathcal{F})]\!] = [\![(\forall y \in TC(\{x\}))(y \in S(\mathcal{F}))]\!] \qquad \textit{for all } x \in M .$$

Proof. The proof is by induction on the rank of x. Let $(y, b) \in x$, $b \in \mathcal{B}$, then by the induction hypothesis we have $[\![(\forall z \in TC(\{y\}))(z \in S(\mathcal{F}))]\!] = \operatorname{Sup}\{c$: $[\![y = y']\!] \geqq c$ for some $y' \in HS(\mathcal{F})\}$.

By replacement we can restrict y' to lie in some V_α for all y.

We set $x_0 = \{(y', b \wedge c)$: $y' \in HS(\mathcal{F}) \cap V_\alpha$ and $[\![y = y']\!] \geqq c$ for some y such that $(y, b) \in x\}$. It is not difficult to see that

$$[\![x = x_0]\!] \geqq [\![(\forall z \in TC(x))(z \in S(\mathcal{F}))]\!] .$$

We claim that $[\![x_0 \in HS(\mathcal{F})]\!] = [\![x_0 \in S(\mathcal{F})]\!]$. Let $[\![x_0 \in S(\mathcal{F})]\!] = \operatorname{Sup}\{d$: $[\![x_0 = u]\!] = d$ for some $u \in S(\mathcal{F})\}$. By replacement we can restrict u to lie in some V_β.

We define in $\mathfrak{M}$ a function f with domain $S(\mathcal{F}) \cap V_\beta$: for $u \in S(\mathcal{F}) \cap V_\beta$, letting $d = [\![x_0 = u]\!]$ we set

$$f(u) = \{(\tilde{\sigma}(z), \sigma(a) \wedge \sigma(d)) : (z, a) \in x_0 \text{ and } \sigma \in \Omega(u) \cap N(\mathcal{F})\} .$$

Clearly $\Omega(f(u)) \supset \Omega(u) \cap N(\mathcal{F})$. Since $\mathcal{F}$ is normal and $u \in S(\mathcal{F})$, we obtain $f(u) \in S(\mathcal{F})$. Also, if $(z, a) \in x_0$ and $\sigma \in N(\mathcal{F})$ then $z \in HS(\mathcal{F})$ and by Lemma 5 we have $\tilde{\sigma}(z) \in HS(\mathcal{F})$. Thus $f(u) \in HS(\mathcal{F})$.

Letting $(z, a) \in x_0$ and $\sigma \in \Omega(u) \cap N(\mathcal{F})$, we have

$$\begin{aligned}
[\![\tilde{\sigma}(z) \in x_0]\!] &\geqq [\![x_0 = \tilde{\sigma}(x_0)]\!] \wedge [\![\tilde{\sigma}(z) \in \tilde{\sigma}(x_0)]\!] \\
&\geqq [\![x_0 = u]\!] \wedge [\![u = \tilde{\sigma}(u)]\!] \wedge [\![\tilde{\sigma}(u) = \tilde{\sigma}(x_0)]\!] \wedge [\![\tilde{\sigma}(z) \in \tilde{\sigma}(x_0)]\!] \\
&\geqq d \wedge \sigma(d) \wedge \sigma([\![z \in x_0]\!]) \geqq d \wedge (\sigma(d) \wedge \sigma(a)) .
\end{aligned}$$

This shows that $[\![f(u) \subset x_0]\!] \geqq d$, whence $[\![f(u) = x_0]\!] \geqq d$.

Thus,

$$\begin{aligned}
[\![x_0 \in HS(\mathcal{F})]\!] &\geqq \operatorname{Sup}\{[\![f(u) = x_0]\!] : u \in S(\mathcal{F}) \cap V_\beta\} \\
&\geqq \operatorname{Sup}\{[\![x_0 = u]\!] : u \in S(\mathcal{F}) \cap V_\beta\} \\
&\geqq [\![x_0 \in S(\mathcal{F})]\!] .
\end{aligned}$$

This proves our claim that $[\![x_0 \in S(\mathcal{F})]\!] = [\![x_0 \in HS(\mathcal{F})]\!]$.

Now $[\![x = x_0]\!] \geqq [\![(\forall z \in TC(x))(z \in S(\mathcal{F}))]\!]$, hence we have

$$\begin{aligned}
&[\![x \in HS(\mathcal{F})]\!] \wedge [\![(\forall z \in TC(x))(z \in S(\mathcal{F}))]\!] \\
&\quad = [\![x \in S(\mathcal{F})]\!] \wedge [\![(\forall z \in TC(x))(z \in S(\mathcal{F}))]\!]
\end{aligned}$$

and therefore

$$[\![x \in HS(\mathcal{F})]\!] \geqq [\![(\forall y \in TC(\{x\}))(y \in S(\mathcal{F}))]\!] .$$

The reverse inequality being trivial, this finishes the proof of Proposition 6.

We can now prove that if G is $\mathcal{B}$-generic over $\mathfrak{M}$ and if $\mathcal{F}$ is a normal filter then the symmetric submodel $\mathrm{Val}''_G\, HS(\mathcal{F})$ is of the type $(\mathrm{HOD}\, M[x])^{M[G]}$ for some $x \in M[G]$.

Let $x \in (\mathrm{OD}(M \cup \mathrm{Val}''_G\, S(\mathcal{F})))^{M[G]}$; we claim that there exists $\bar{x} \in S(\mathcal{F})$ such that $\mathrm{Val}_G(\bar{x}) = x$.

In fact let $\bar{y}_0, \cdots, \bar{y}_n \in S(\mathcal{F})$, $t \in M$, be such that for some formula E we have $M[G] \models \forall v(v = x \Leftrightarrow E(v, t, y_0, \cdots, y_n))$ where $y_i = \mathrm{Val}_G(\bar{y}_i)$ for $i \leqq n$.

Let α be such that $x \in (V_\alpha)^{M[G]}$ and set

$$\bar{x} = \{(u, [\![\exists v(E(v, \hat{t}, \bar{y}_0, \cdots, \bar{y}_n) \text{ and } u \in v)]\!]: u \in (V_\alpha)^{\mathcal{B}})\} .$$

Clearly $\mathrm{Val}_G(\bar{x}) = x$ and $\Omega(\bar{x}) \supset \Omega(\bar{y}_0) \cap \cdots \cap \Omega(\bar{y}_n) \in \mathcal{F}$.

Thus

$$\mathrm{Val}''_G\, S(\mathcal{F}) = (\mathrm{OD}(M \cup \mathrm{Val}''_G\, S(\mathcal{F})))^{M[G]} .$$

Using Proposition 6 we obtain

$$\mathrm{Val}''_G\, HS(\mathcal{F}) = \{x \in M[G]: TC(\{x\}) \subset \mathrm{Val}''_G\, S(\mathcal{F})\} .$$

Therefore we have

$$(\mathrm{HOD}\,(M \cup \mathrm{Val}''_G\, HS(\mathcal{F})))^{M[G]} = \mathrm{Val}''_G\, HS(\mathcal{F}) .$$

Applying 7.4, Theorem 3 we get x such that

$$\mathrm{Val}''_G\, HS(\mathcal{F}) = (\mathrm{HOD}\, M[x])^{M[G]} .$$

We now show that if G is $\mathcal{B}$-generic over $\mathfrak{M}$ then for every $x \in M[G]$ the class $(\mathrm{HOD}\, M[x])^{M[G]}$ is a symmetric submodel of $M[G]$.

We shall suppose that x is a non-empty transitive set; let $t \in M$ be such that $\mathrm{Val}_G(t) = x$ and $[\![\hat{\varnothing} \in t \text{ and } t \text{ is transitive}]\!] = \mathbf{1}$.

Let α be a cardinal in $\mathfrak{M}$ greater than the ranks of t and $\mathcal{B}$ and set $t_1 = \{(s \mid b, \mathbf{1}): s \in (V_\alpha)^M \text{ and } b = [\![s \in t]\!]\}$ and $t' = t_1 \cup \{(t, \mathbf{1})\}$; then $[\![t_1 = t]\!] = \mathbf{1}$ and $[\![t' = TC(\{t\})]\!] = \mathbf{1}$. Thus we have

$$[\![\mathrm{HOD}\, M[t] = \mathrm{HOD}\,(M \cup t')]\!] = \mathbf{1} .$$

Let $\mathcal{F}_1$ be the filter generated by $\{\Omega(u): (u, \mathbf{1}) \in t_1\}$ and let $\mathcal{F}$ be the filter generated by $\mathcal{F}_1 \cup \{\Omega(t)\}$.

Let $\sigma \in \Omega(t)$. If $u = s \mid b$ where $s \in (V_\alpha)^M$ and $b = [\![s \in t]\!]$ then $\tilde{\sigma}(u) = \tilde{\sigma}(s) \mid \sigma(b)$ and $\sigma(b) = [\![\tilde{\sigma}(s) \in t]\!]$; hence $(\tilde{\sigma}(u), \mathbf{1}) \in t_1$ and $\Omega(\tilde{\sigma}(u)) \in \mathcal{F}_1$; therefore $\sigma \in N(\mathcal{F}_1)$.

Thus $\Omega(t) \subset N(\mathcal{F}_1)$, whence $\Omega(t) \subset N(\mathcal{F})$ and $\mathcal{F}$ is a normal filter.

Now, $\mathrm{Val}_G(t') \subset \mathrm{Val}''_G\, S(\mathcal{F})$. Since $\mathrm{Val}_G(t')$ is transitive and $\mathcal{F}$ is normal, we have, by Proposition 6:

$$\mathrm{Val}_G(t') \subset \mathrm{Val}''_G\, HS(\mathcal{F}) .$$

Thus,

$$(\mathrm{HOD}\, M[x])^{M[G]} \subset (\mathrm{HOD}\,(\mathrm{Val}''_G\, HS(\mathcal{F})))^{M[G]} = \mathrm{Val}''_G\, HS(\mathcal{F})$$

(by our proof of the other half of Theorem 3).

Also, by 7.3, Theorem 1,

$$\mathrm{Val}''_G\, S(\mathcal{F}) \subset (\mathrm{OD}(M \cup \{\mathrm{Val}_G(t)\} \cup \{\mathrm{Val}_G(u) \colon (u, \mathbf{1}) \in t_1\}))^{M[G]}$$

and so $\mathrm{Val}''_G\, S(\mathcal{F}) \subset (\mathrm{OD}\, M[x])^{M[G]}$, whence $\mathrm{Val}''_G\, HS(\mathcal{F}) \subset (\mathrm{HOD}\, M[x])^{M[G]}$. This finishes the proof of Theorem 3.

9. Inner models over which the universe is generic

9.1. The following theorem is due to Vopěnka and Hájek ([15]).

THEOREM 1. *Let $\mathfrak{N}$ be a model of* ZF, *M an inner model of $\mathfrak{N}$ and suppose that $N = M[a]$ for some $a \subset M$; then N is a generic extension of* $(\mathrm{HOD}\, M)^N$ *and there exists $b \subset M$ such that* $(\mathrm{HOD}\, M)^N = M[b]$.

Proof. Let $u \in M$ be such that $a \subset u$, and set $A = (P(P(u)))^N \cap (\mathrm{OD}\, M)^N$. We put the inclusion ordering on A. Let $X \subset A$ be in $(\mathrm{OD}\, M)^N$; then $\bigcup X$ is in $(\mathrm{OD}\, M)^N$ too; hence X has a supremum in A. Thus A is an $(\mathrm{OD}\, M)^N$-complete boolean algebra.

We now construct a complete boolean algebra in $(\mathrm{HOD}\, M)^N$ which is isomorphic to A. Let Form be the set of formulas with two free variables; we define in N an application T from the product Form $\times$ $On \times M$ onto A as follows:

$$T(F, \alpha, x) = \{z \in N \colon z \subset u \text{ and } (V_\alpha)^N \models F(z, x)\}\ .$$

By the replacement scheme there exist $\lambda \in On$ and $t \in M$ such that $T''(\text{Form} \times \lambda \times t) = A$. We let $s = \text{Form} \times \lambda \times t$ and we define a preordering $\geqq_s$ on s: for all $x, y \in s$, $x \leqq_s y$ if and only if $T(x) \subset T(y)$. From the preordered set $(s, \leqq_s)$ we get canonically an ordered set which we will denote $\mathcal{B}$. The map T from s onto A now defines canonically an isomorphism, which we again denote by T, between $\mathcal{B}$ and A. Moreover this isomorphism T is in $(\mathrm{OD}\, M)^N$; thus $\mathcal{B}$ is $(\mathrm{OD}\, M)^N$-complete, whence $\mathcal{B}$ is a complete boolean algebra in $(\mathrm{HOD}\, M)^N$.

We define a function f from u into $\mathcal{B}$:

$$f(x) = \overset{-1}{T}(\{y \in N \colon y \subset u \text{ and } x \in y\}) \qquad \text{for } x \in u\ .$$

It is clear that f is in $(\mathrm{HOD}\, M)^N$.

Let $H = \{X \in A \colon a \in X\}$; then H is an $(\mathrm{OD}\, M)^N$-complete ultrafilter on A. We set $G = \overset{-1}{T}{}''H$; then G is $\mathcal{B}$-generic over $(\mathrm{HOD}\, M)^N$. Now, for $z \in u$ we have $z \in a$ if and only if $f(z) \in G$; therefore $a \in M[G, f]$. Thus $M[a] = (\mathrm{HOD}\, M)^N[G] = M[\mathcal{B}, f][G]$ and $M[a]$ is a generic extension of $(\mathrm{HOD}\, M)^N$; by 2.12, Lemma 1 we conclude that $M[\mathcal{B}, f] = (\mathrm{HOD}\, M)^N$. Let now π be the canonical map from s onto $\mathcal{B}$. Let

$$F = \{(x, y): x \in s,\ y \in u, \text{ and } \pi(x) = f(y)\} .$$

Then $M[\mathcal{B}, f] = M[\leqq_s, F]$ and there exists $b \subset M$ which encodes $\leqq_s$ and F.

Note: Using 5.1, Theorem 1 we see that there exists a homogeneous complete boolean algebra $\mathcal{A}$ in $M[b]$ such that $M[a]$ is an $\mathcal{A}$-generic extension of $M[b]$. The following argument, due to McAloon, shows that this result can be recovered from the above proof. Let $X = \{r \subset u: M[r] = M[a]\}$, X is an element of A; set $p = \overset{-1}{T}(X)$, $p \in \mathcal{B} \cap G$. We show that $\mathcal{B}_p$ is homogeneous in $M[b]$. Let $p_0, p_1 \leqq p$, let $x_i \in T(p_i)$, set $H_i = \{Y \in A: x_i \in Y\}$ and $G_i = \overset{-1}{T}{}''H_i$ $(i = 0, 1)$. It is clear that G_0, G_1 are $\mathcal{B}$-generic over $M[b]$, $M[G_0] = M[G_1]$, $p \in G_0 \cap G_1$, $p_0 \in G_0$, $p_1 \in G_1$. Applying 3.5, Theorem 1 we get an automorphism σ of $\mathcal{B}$ such that $\sigma(p) = p$, $\sigma''G_0 = G_1$. Thus $\sigma \restriction \mathcal{B}_p$ is an automorphism of $\mathcal{B}_p$ such that $(\sigma \restriction \mathcal{B}_p)(p_0)$ and p_1 are compatible.

9.2. We shall use the following lemma in the next section:

LEMMA 1. *Let $\mathfrak{N}$ be a model of* ZF, *M an inner model of $\mathfrak{N}$, C an ordered set in N and let G be C-generic over $\mathfrak{N}$. Suppose that C is homogeneous and belongs to* $(\mathrm{OD}\,M)^N$ *then* $(\mathrm{OD}\{x\})^{N[G]} \cap N \subset (\mathrm{OD}(M \cup \{x\}))^N$ *for all $x \in N$ and therefore* $(\mathrm{HOD}\,M)^{N[G]} \subset (\mathrm{HOD}\,M)^N$.

Proof. Let $y \in (\mathrm{OD}\{x\})^{N[G]} \cap N$; then for some E and α we have $N[G] \models y$ is the unique v such that $E(v, x, \alpha)$. Thus, there exists $p \in G$ such that $p \Vdash \hat{y}$ is the unique v such that $E(v, \hat{x}, \hat{\alpha})$. Since C is homogeneous, either all $p \in C$ force that formula or none does. Thus, y is the unique set such that $(\forall p \in C) p \Vdash E(\hat{y}, \hat{x}, \hat{\alpha})$. Since C is definable in N from elements in $M \cup On$ so is the forcing relation, whence y is definable in N from elements in $M \cup \{x\}$. This shows that $(\mathrm{OD}\{x\})^{N[G]} \cap N \subset (\mathrm{OD}(M \cup \{x\}))^N$ for all $x \in N$. From this result we get

$$(\mathrm{HOD}\,M)^{N[G]} \cap N \subset (\mathrm{HOD}\,M)^N .$$

But $(\mathrm{HOD}\,N)^{N[G]} = N$; therefore $(\mathrm{HOD} M)^{N[G]} \subset N$. Thus, we have $(\mathrm{HOD} M)^{N[G]} \subset (\mathrm{HOD}\,M)^N$.

9.3. THEOREM 1. *Let $\mathfrak{N}$ be a model of* ZF, *M an inner model of $\mathfrak{N}$ and suppose that $N = M[a]$ for some $a \in N$. Then N is a quasi-generic extension of* $(\mathrm{HOD}\,M)^N$ *and there exists $b \in N$ such that $b \subset M$ and* $(\mathrm{HOD}\,M)^N = M[b]$.

Proof. It suffices to prove the theorem for countable N. Let α be the rank of a and let G be $C((V_\alpha)^N)$-generic over $\mathfrak{N}$. Then $N[G] = M[a][G] = M[r]$ for some $r \subset \omega$. Let $M' = (\mathrm{HOD}\,M)^{N[G]}$; applying 9.1, Theorem 1 we see that $N[G]$ is a generic extension of M' and that there exists $c \subset M$ such that $M' = M[c]$.

Now $C((V_\alpha)^N)$ is homogeneous and ordinal definable in $\mathfrak{N}$; applying 9.2, Lemma 1 we get $(\mathrm{HOD}\,M)^{N[G]} \subset (\mathrm{HOD}\,M)^N$; i.e., $M' \subset (\mathrm{HOD}\,M)^N$. Using 1.10, we see that $(\mathrm{HOD}\,M)^N = \bigcup \{L[x];\, x \subset M \text{ and } x \in (\mathrm{HOD}\,M)^N\}$. We apply 2.14, Theorem 3: $N[G]$ is a generic extension of $(\mathrm{HOD}\,M)^N$ and there exists $d \subset M'$ such that $(\mathrm{HOD}\,M)^N = M'[d]$.

The model $N[G]$ is a generic extension of both N and $(\mathrm{HOD}\,M)^N$, hence N is a quasi-generic extension of $(\mathrm{HOD}\,M)^N$.

Applying 1.6, Lemma 1 we see that there exists $b \subset M$ such that $(\mathrm{HOD}\,M)^N = M[b]$.

COROLLARY 2. *Let M be an inner model of $\mathfrak{N}$ such that $N = M[a]$ for some $a \in N$. Then, for every $x \in N$ there exists $y \in N$ with $y \subset M$ such that* $(\mathrm{HOD}\,M[x])^N = M[y]$ *and N is a quasi-generic extension of* $(\mathrm{HOD}\,M[x])^N$.

Proof. It suffices to replace M by $M[x]$ in Theorem 1.

COROLLARY 3. *Let M be an inner model of $\mathfrak{N}$ such that $N = M[a]$ for some $a \subset M$. If N is a generic extension of $M[x]$ then N is also a generic extension of* $(\mathrm{HOD}\,M)^{M[x]}$ *(and* $(\mathrm{HOD}\,M)^{M[x]}$ *has the form $M[y]$ with $y \subset M$).*

Proof. Applying Theorem 1 we get $y \subset M$ such that $(\mathrm{HOD}\,M)^{M[x]} = M[y]$ and $M[x]$ is a quasi-generic extension of $(\mathrm{HOD}\,M)^{M[x]}$. Now, N is a generic extension of $M[x]$ and $N = M[a]$ with $a \subset M$; therefore we can use 6.2, Corollary 6 which yields the conclusion.

THEOREM 4. *Let M be an inner model of $\mathfrak{N}$ such that $N = M[a]$ for some $a \in N$. Let X be a model of* ZF *intermediate between M and N*; i.e., $M \subset X \subset N$ *and X is transitive in N, such that* $X = (\mathrm{HOD}\,X)^N$; *then there exists $x \in X$ such that $X = M[x]$ (thus X is a class for $\mathfrak{N}$) and N is a quasi-generic extension of X (if $a \subset M$ then N is a generic extension of X). Also, the rank of x can be bounded independently of X.*

Proof. We first consider the case $a \subset M$. Set $M' = (\mathrm{HOD}\,M)^N$; by 9.1, Theorem 1, N is a generic extension of M' and $M' = M[b]$ for some $b \subset M$. The theorem is then an easy application of 7.4, Theorem 3.

We now reduce the general case to the previous one. Let α be greater than the rank of a and let G be $C((V_\alpha)^N)$-generic over $\mathfrak{N}$. By 9.2, Lemma 1 we have

$$(\mathrm{OD}\{x\})^{N[G]} \cap N \subset (\mathrm{OD}\{x\})^N \qquad \text{for all } x \in N,$$

whence

$$(\mathrm{OD}\,X)^{N[G]} \cap N \subset (\mathrm{OD}\,X)^N \quad \text{and} \quad (\mathrm{HOD}\,X)^{N[G]} \cap N \subset (\mathrm{HOD}\,X)^N = X\,.$$

But $(\mathrm{HOD}\,X)^{N[G]} \subset (\mathrm{HOD}\,N)^{N[G]} = N$; hence $(\mathrm{HOD}\,X)^{N[G]} = X$. Now $N[G] = M[a][G] = M[r]$ for some $r \subset \omega$ and so we can apply the previous case.

Remark 5. The following weakening of Theorem 4 can be proved without using 7.4, Theorem 3. Also, it is strong enough to prove 10.1, Theorem 1 (the iteration of the HOD operation).

THEOREM 4 *bis*. *Suppose that $N = M[a]$ for some $a \in N$ and let X be an inner model of $\mathfrak{N}$ (thus we assume that X is a class for $\mathfrak{N}$) which contains M and such that $X = (\mathrm{HOD}\,X)^N$. Then there exists $x \in X$ such that $X = M[x]$ and N is a quasi-generic extension of X.*

Proof. We consider only the case $a \subset M$. Since X is a class for $\mathfrak{N}$ we may construct $X[a]$ which is clearly the same as N. By 9.1, Theorem 1, $X[a]$ is a generic extension of $(\mathrm{HOD}\,X)^{X[a]} = X$. Let $M' = (\mathrm{HOD}\,M)^N$; then $M' = M[b]$ and N is a generic extension of M'. Applying 6.1, Theorem 1 we see that $X = M'[y]$ for some $y \in X$, whence $X = M[x]$ for some $x \in X$.

9.4. PROPOSITION 1. *Let $\mathfrak{N}$ be a countable model of* ZF *and let M be an inner model of $\mathfrak{N}$. For every ordinal α there exists a generic extension $N[H]$ of N such that, for all $y \in (V_\alpha)^N$, if N is a generic extension of $M[y]$ then*

$$(\mathrm{HOD}\,M[y])^{N[H]} = M[y]\ .$$

Proof. Set $A = \{y \in (V_\alpha)^N\colon N \text{ is a generic extension of } M[y]\}$. For $y \in A$ let $\beta(y)$ be the minimal rank of an ordered set D in $M[y]$ such that N is a D-generic extension of $M[y]$.

Set $\beta = \mathrm{Sup}\,\{\beta(y)\colon y \in A\}$ and let α be a limit ordinal greater than $\beta\cdot\omega$. Let H be $C((V_\alpha)^N)$-generic over $\mathfrak{N}$. By 4.9, Theorem 1, for every $y \in A$, there exists K which is $C((V_\alpha)^{M[y]})$-generic over $M[y]$ and such that $N[H] = M[y][K]$. Since $C((V_\alpha)^{M[y]})$ is a homogeneous ordered set in $M[y]$, we obtain $(\mathrm{HOD}\,M[y])^{N[H]} = M[y]$.

THEOREM 2. *Let $\mathfrak{N}$ be a model of* ZF *and M an inner model such that $N = M[a]$ for some $a \subset M$. For every ordinal α there exists $x \in N$ such that $x \subset M$, N is a generic extension of $M[x]$ and, for all $y \in (V_\alpha)^N$, if N is a generic extension of $M[y]$ then $M[x] \subset M[y]$.*

Proof. It suffices to prove the theorem for countable $\mathfrak{N}$.

We then apply the above proposition and let x be such that $x \subset M$ and $M[x] = (\mathrm{HOD}\,M)^{N[H]}$ (such an x exists by 9.1, Theorem 1).

THEOREM 3. *Let $\mathfrak{N}$ be a model of* ZF *and M an inner model such that $N = M[a]$ for some $a \subset M$. Let α be an ordinal and let $A \subset (V_\alpha)^N$ be such that N is a generic extension of $M[x]$ for all $x \in A$. Set $X = \bigcap\,\{M[x]\colon x \in A\}$;*

then the following conditions are equivalent:

(i) *For all* $x \in A$, X *is definable in* $M[x]$ (*with parameters*),

(ii) $(X, \varepsilon \upharpoonright X^2)$ *is a model of* ZF,

(iii) *There exists* $y \in N$ *such that* $X = M[y]$ *and* N *is a generic extension of* X.

Proof. (i) $\Rightarrow$ (ii): It is clear that X is transitive and closed under the Gödel operations. From (i) we see that X is a class and that X is almost universal. Thus, by 1.10, Lemma 1, X is an inner model.

(ii) $\Rightarrow$ (iii): It suffices to prove that implication for countable $\mathfrak{N}$. We then apply Proposition 1 to get a generic extension $N[H]$ of N such that

$$(\mathrm{HOD}\, M[x])^{N[H]} = M[x] \qquad \text{for all } x \in A .$$

Thus,

$$(\mathrm{HOD}\, X)^{N[H]} \subset \bigcap \{(\mathrm{HOD}\, M[x])^{N[H]} \colon x \in A\} \subset \bigcap \{M[x] \colon x \in A\} = X$$

and so $X = (\mathrm{HOD}\, X)^{N[H]}$.

Now with (ii) we can apply 9.3, Theorem 4: There exists y such that $X = M[y]$ and $N[H]$ is a generic extension of X; whence (iii).

Remark 4. Using 9.3, Theorem 4 *bis* one can prove Theorem 3 in case $A \in N$. This case will be enough to prove 10.1, Theorem 1.

Remark 5. Theorem 3 and Theorem 2 are still valid when we replace the condition $a \subset M$ by $a \in N$ and the word "generic" by "quasi-generic".

9.5. THEOREM 1. *Let* $\mathfrak{N}$ *be a model of* ZF, *M an inner model of* $\mathfrak{N}$ *and* $\mathfrak{B}$ *a complete boolean algebra in M such that* $N = M[G]$ *where G is* $\mathfrak{B}$*-generic over M. Let* $\langle \mathfrak{B}_\alpha \rangle_{\alpha \in \lambda}$ *be a family in M of decreasing complete subalgebras of* $\mathfrak{B}$. *There exists* $x \in N$ *such that* $M[x] = \bigcap \{M[G \cap \mathfrak{B}_\alpha] \colon \alpha \in \lambda\}$ *and N is a generic extension of* $M[x]$.

Proof. Let $X = \bigcap \{M[G \cap \mathfrak{B}_\alpha] \colon \alpha \in \lambda\}$. By 2.13, Theorem 1 and 9.4, Theorem 3, it suffices to prove that X is definable in $M[G \cap \mathfrak{B}_\alpha]$ for all $\alpha \in \lambda$. Now, $\langle G \cap \mathfrak{B}_\beta \rangle_{\beta \in [\alpha, \lambda[}$ belongs to $M[G \cap \mathfrak{B}_\alpha]$; since $X = \bigcap \{M[G \cap \mathfrak{B}_\beta] \colon \beta \in [\alpha, \lambda[\}$ we see that X is definable in $M[G \cap \mathfrak{B}_\alpha]$. Hence the theorem.

Remark 2. The hypothesis $(\langle \mathfrak{B}_\alpha \rangle_{\alpha \in \lambda}) \in M$ can be weakened to

$$(\langle \mathfrak{B}_\beta \rangle_{\beta \in [\alpha, \lambda[}) \in M[G \cap \mathfrak{B}_\alpha] \qquad \text{for all } \alpha \in \lambda .$$

We note the following improvement of Theorem 1 when $\lambda = \omega$ and M satisfies AC.

THEOREM 3. *Let* $\mathfrak{M}$ *be a model of* ZFC, $\mathfrak{B}$ *a complete boolean algebra in M and* $(\mathfrak{B}_n)_{n \in \omega}$ *a decreasing family in M of complete subalgebras of* $\mathfrak{B}$. *Let*

G be $\mathcal{B}$-generic over $\mathfrak{M}$ and set $X = \bigcap\{M[G \cap \mathcal{B}_n]: n \in \omega\}$. Set $A = \{f \in X$: f is a function from ω into $\mathcal{B}$ such that $f(n) \in \mathcal{B}_n$ for all n and there exists m such that $f(m+n) \in G$ for all $n\}$; then $X = M[A]$.

Proof. By $M^{(\mathcal{B}_n)}$ we mean the class defined by the following recursion in M:

$$x \in M^{(\mathcal{B}_n)} \text{ if and only if } \forall (y, b) \in x \ (b \in \mathcal{B} \Longrightarrow b \in \mathcal{B}_n \wedge y \in M^{(\mathcal{B}_n)}) \ .$$

Clearly if $t \in M^{(\mathcal{B}_n)}$ then $\mathrm{Val}_G(t) = \mathrm{Val}_{G\cap\mathcal{B}_n}(t)$. We also note that if $t, s \in M^{(\mathcal{B}_n)}$ then $[\![t \in s]\!]^{\mathcal{B}} = [\![t \in s]\!]^{\mathcal{B}_n}$ and $[\![t = s]\!]^{\mathcal{B}} = [\![t = s]\!]^{\mathcal{B}_n}$.

Claim: For every $x \in X$ there exists in $M[A]$ a family $\bar{x}_n$, $n \in \omega$, of elements of M such that $\bar{x}_n \in M^{(\mathcal{B}_n)}$ and $\mathrm{Val}_{G\cap\mathcal{B}_n}(\bar{x}_n) = x$.

Let β be greater than the ranks of x and $\mathcal{B}$ and let $\alpha = \beta\cdot\omega$; by 2.6, Lemma 2, all terms which are to denote an element in $(V_\alpha)^{M[G]}$ can be supposed to be in $(V_\alpha)^M$. We fix in M some well-ordering of $(V_\alpha)^M$.

We define by induction a function f from ω into $\mathcal{B}$ and a sequence $\bar{x}_n$, $n \in \omega$:

$\bar{x}_0$ is the least $z \in (V_\alpha)^M \cap M^{(\mathcal{B}_0)}$ such that $\mathrm{Val}_{G\cap\mathcal{B}_0}(z) = x$,

$f(n)$ is the least $p \in \mathcal{B}_n \cap G$ such that for some $z \in (V_\alpha)^M \cap M^{(\mathcal{B}_{n+1})}$ we have $p \Vdash \bar{x}_n = z$,

$\overline{x_{n+1}}$ is the least $z \in (V_\alpha)^M \cap M^{(\mathcal{B}_{n+1})}$ such that $f(n) \Vdash \bar{x}_n = z$.

It is clear that $f \in X$; therefore $f \in A$ and hence $(\bar{x}_n)_{n\in\omega} \in M[A]$. This proves the claim.

Let $x \in X$ be such that $x \subset A$; we show that $x \in M[A]$. Applying the claim we see there exists in $M[A]$ a family $(\bar{x}_n)_{n\in\omega}$ such that $\bar{x}_n \in M^{(\mathcal{B}_n)}$ and $\mathrm{Val}_G(\bar{x}_n) = x$ for all n. Letting $f \in A$, we let $\psi(f)$ be the following condition:

"There exists a family $(\bar{y}_n)_{n\in\omega} \in M[A]$ and $h \in A$ such that for all n we have $\bar{y}_n \in M^{(\mathcal{B}_n)}$, $h(n) \Vdash ((\bar{y}_n = \overline{y_{n+1}}) \wedge (\bar{y}_n$ is a function from $\hat{\omega}$ into $\hat{\mathcal{B}}$ such that $\bar{y}_n(\hat{i}) = \widehat{f(i)}$ for all $i \leqq \hat{n}))$ and $\{(n, [\![\bar{y}_n \in \bar{x}_n]\!]^{\mathcal{B}}): n \in \omega\} \in A$."

We prove that $f \in x$ if and only if $\psi(f)$.

Suppose $f \in x$; by the claim there exists a family $(\bar{y}_n)_{n\in\omega}$ in $M[A]$ such that $\bar{y}_n \in M^{(\mathcal{B}_n)}$ and $\mathrm{Val}_{G\cap\mathcal{B}_n}(\bar{y}_n) = f$. It is now easy to define in X a function h such that $h(n) \in G \cap \mathcal{B}_n$ and $h(n) \Vdash (\bar{y}_n = \overline{y_{n+1}}) \wedge (\bar{y}_n$ is a function from $\hat{\omega}$ into $\hat{\mathcal{B}}$ such that $\bar{y}_n(\hat{i}) = \widehat{f(i)}$ for all $i \leqq \hat{n})$; by the definition of A we then have $h \in A$, whence $\psi(f)$.

Suppose $\psi(f)$. Then for some m we have $h(m+n) \in G$ for all n. Thus $\mathrm{Val}_G(\overline{y_{n+1}}) = f$ for all n. Since $\{(n, [\![\bar{y}_n \in \bar{x}_n]\!]): n \in \omega\} \in A$ we see that for some m we have $[\![\overline{y_{m+n}} \in \overline{x_{m+n}}]\!] \in G$, whence $f \in x$.

From the equivalence $f \in x$ if and only if $\psi(f)$ we get $x \in M[A]$.

Let Seq (A) be the set of finite sequences of elements of A. There exists a canonical surjective map from Seq $(A) \times M$ onto $M[A]$.

An easy modification of the above proof shows that if $x \in X$ is included in Seq $(A) \times M$ then $x \in M[A]$. From this we deduce that if $x \in X$ is included in $M[A]$ then $x \in M[A]$, whence $X = M[A]$. This finishes the proof of Theorem 3.

10. Iterated HOD submodels

10.1. THEOREM 1. *Let $\mathfrak{N}$ be a model of* ZF *and let M be an inner model such that $N = M[a]$ for some $a \in N$. Then there exists a family X_α, $\alpha \in N$, of inner models of $\mathfrak{N}$ such that $X_0 = N$, $X_{\alpha+1} = (\mathrm{HOD}\, M)^{X_\alpha}$ for all α and $X_\alpha = \bigcap \{X_\beta \colon \beta < \alpha\}$ for limit ordinals α. Moreover for every α there exists x such that $X_\alpha = M[x]$ and if α is a successor ordinal then we can suppose $x \subset M$. Also, X_1 is a generic extension of X_α for all $\alpha \geqq 1$, and if $a \subset M$ then N is a generic extension of X_α for all α.*

Proof. By 9.3, Theorem 1, we can reduce to the case $a \subset M$.

By induction we shall construct a sequence A_α of non-empty sets such that:

(*) If $x \in A_0$ then $M[x] = M[a] = N$.

If $x, y \in A_\alpha$ then $M[x] = M[y]$ and $A_\alpha = \{z\colon z$ is of minimal rank such that $M[z] = M[x]\}$.

If $x \in A_\alpha$ and $y \in A_{\alpha+1}$ then $M[y] = (\mathrm{HOD}\, M)^{M[x]}$.

If α is a limit and $x \in A_\alpha$ then $M[x] = \bigcap \{M[y]\colon y \in A_\beta$ for some $\beta < \alpha\}$ and

(**) If $x \in A_\alpha$ then N is a generic extension of $M[x]$.

It is clear that A_α, if it does exist, is uniquely determined by (*). If α is a successor ordinal then the construction of A_α and the induction step follow from 9.3, Corollary 3.

We now suppose α is a limit ordinal.

Set $X = \bigcap \{M[y]\colon y \in A_\beta$ for some $\beta < \alpha\}$. We note that if $\beta < \gamma < \alpha$, $y \in A_\beta$ and $z \in A_\gamma$ then $M[y] \supset M[z]$. Therefore, if $\beta < \alpha$ and $y \in A_\beta$ then

$$X = \bigcap \{M[z]\colon z \in A_\gamma \text{ and } \beta < \gamma < \alpha\} .$$

But condition (*) makes the sequence $\langle A_\gamma \rangle_{\beta<\gamma<\alpha}$ definable in $M[y]$; hence X is definable in $M[y]$. Thus condition (i) of 9.4, Theorem 3 is satisfied; (iii) gives the induction step.

Remark 2. Theorem 1 can be generalized as follows:

Let Y be an inner model of $\mathfrak{N}$ which contains M. There exists a decreasing sequence Y_α, $\alpha \in On$, of inner models of $\mathfrak{N}$ such that $Y_0 = N$, $Y_{\alpha+1} = (\mathrm{HOD}\, Y)^{Y_\alpha}$ for all α, and $Y_\lambda = \bigcap \{Y_\alpha; \alpha < \lambda\}$ for limit ordinals λ. N is a quasi-generic extension of each Y_α and there exists a_α such that $Y_\alpha = M[a_\alpha]$ (even if $Y \neq M[x]$ for all $x \in N$).

10.2. The following theorem is a corollary of 10.1, Theorem 1 and the last remark of 2.5.

THEOREM 1. *There exists a formula $E(\alpha, v, w)$ of the language of set theory such that the following are provable in the theory* ZF:

(i) $\forall \alpha \exists y,\ E(\alpha, x, y),\ E(0, x, y) \Rightarrow L[x] = L[y]$.

(ii) $E(\alpha, x, y) \wedge E(\alpha + 1, x, z) \Rightarrow L[z] = (\mathrm{HOD})^{L[y]}$.

(iii) $0 \neq \lambda = \bigcup \lambda \wedge E(\lambda, x, y) \Rightarrow L[y] = \bigcap \{L[z]\colon E(\alpha, x, z)$ *for some* $\alpha < \lambda\}$.

(iv) $(\alpha \neq \bigcup \alpha \vee (\alpha = 0 \wedge x \subset On)) \Rightarrow (\beta \geqq \alpha \wedge E(\alpha, x, y) \wedge E(\beta, x, z) \Rightarrow$ $L[y]$ *is a generic extension of* $L[z]$).

We let $(\mathrm{HOD})_\alpha^{L[x]}$ be $L[y]$ where y is such that $E(\alpha, x, y)$. Fix x; the sequence $(\mathrm{HOD})_\alpha^{L[x]}$, $\alpha \in On$, is decreasing. Moreover, either it is strictly decreasing or it is eventually constant. In case $L[x]$ is a generic extension of L, it is clear that this sequence is eventually constant.

Question: Can the sequence $(\mathrm{HOD})_\alpha^{L[x]}$, $\alpha \in On$, be strictly decreasing?

The following is an application of 9.5, Theorem 2:

PROPOSITION 2. *Let λ be a limit ordinal which has cofinality ω in $(\mathrm{HOD})_\lambda^{L[x]}$. Then there exists y which is a set of sets of ordinals such that $(\mathrm{HOD})_\lambda^{L[x]} = L[y]$.*

Proof. Let $f \in (\mathrm{HOD})_\lambda^{L[x]}$ be a function from ω into λ which is cofinal to λ and set $M = ((\mathrm{HOD})_{\lambda+1}^{L[x]})[f]$. Clearly $L[x] = M[x]$ and $(\mathrm{HOD})_\alpha^{L[x]} = (\mathrm{HOD}\, M)_\alpha^{M[x]}$ for all $\alpha \leqq \lambda$.

Let $\mathcal{B}$ be a complete boolean algebra in M and let G be $\mathcal{B}$-generic over M such that $L[x] = M[G]$. Using 3.10, Theorem 1, we have

$$(\mathrm{HOD}\, M)_\lambda^{M[x]} = \bigcap \{M[G \cap \mathcal{B}^{(\alpha+1)}]\colon \alpha \in \lambda\} = \bigcap \{M[G \cap \mathcal{B}^{(f(n)+1)}]\colon n \in \omega\}\,.$$

Since M satisfies AC we can apply 9.5, Theorem 2: there exists z which is a set of subsets of M such that $(\mathrm{HOD}\, M)_\lambda^{M[x]} = M[z]$. Now, $M = L[t]$ where t is a set of ordinals, hence there exists y which is a set of sets of ordinals such that $(\mathrm{HOD})_\lambda^{L[x]} = L[y]$.

Remark 3. It has been shown by W. Reinhart that if one works in Kelley-Morse set theory (i.e., the version of set theory with classes and the impredicative class-existence scheme) then one can define a well-ordered class

$(A, <_A)$ and a sequence of classes $\langle X_a: a \in A\rangle$ so that

(1) For each $a \in A$, X_a is a transitive model of ZF;

(2) If a is least in $(A, <_A)$, $X_a = V$;

(3) If a is successor to b in $(A, <_A)$, $X_a = (\mathrm{HOD})^{X_b}$ (and $X_a \neq X_b$);

(4) If a is a limit point of $(A, <_A)$, then $X_a = \bigcap_{b<_A a} X_b$;

(5) $(A, <_A)$ has a largest element, say z, and $X_z = (\mathrm{HOD})^{X_z}$.

This result can be established using the following proposition which serves to show that, as long as the sequence of iterated HOD's is definable, it will be composed of ZF models:

PROPOSITION 4. *Let $\mathfrak{N}$ be a model of* ZF *and let $<_A$ be a class of $\mathfrak{N}$ which is a well-ordering on the class A. Suppose that there exists a subset X of N which is a class in $(\mathfrak{N}, <_A)$ so that, letting $X_a = \{x; (a, x) \in X\}$, the sequence of classes $\langle X_a: a \in A\rangle$ satisfies conditions* (1) *to* (4) *above. Let*

$$Y = \bigcap_{a \in A} X_a = \{x: (\forall a \in A)((a, x) \in X)\} ;$$

then Y is a model of ZF (*whence, Y is an inner model of $\mathfrak{N}$*).

Proof. For each α there exists a unique map f_α from $A \cap (V_{\alpha+1} - V_\alpha)$ onto an ordinal κ_α which is an isomorphism with respect to $<_A$ and $\in$. Gluing the f_α's, we can define a bijection between A and $\sum_{\alpha \in On} \kappa_\alpha$. Thus, we can suppose that the domain A of $<_A$ is either an ordinal or the class On.

For each $a \in A$ we let $Z_a = \{(b, x); (b, x) \in X \text{ and } b \geqq_A a\}$. Since the X_b's are decreasing, and $A \subset On$, we see that $Z_a \subset X_a$, whence, by 1.4, Lemma 2, X_a satisfies ZF $(Z_a, <_A)$. Now, $Y = \{x: (\forall b \geqq_A a)((b, x) \in Z_a)\}$ and therefore Y is a class in X_a. This shows that $(Y \cap V_\alpha) \in X_a$ for all $\alpha \in On$, $a \in A$. But Y is the intersection of the X_a's and so $(Y \cap V_\alpha) \in Y$ for all $\alpha \in On$. Also, Y is a transitive class of $\mathfrak{N}$ which is closed by the eight Gödel operations. Using 1.4, Lemma 1, we conclude that Y is a model of ZF.

M. L. Harrington has informed us that combining the techniques of [2] and [6] one can construct models of ZF in which the sequence $(\mathrm{HOD})_n$, $n \in \omega$, is not definable and $(\mathrm{HOD})_\omega$ is not a model of ZF.

UNIVERSITÉ PARIS VII

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