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COMBINATORICS ON IDEALS AND FORCING

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A natural generalization of Cohen's set of forcing conditions (the two-valued functions with domain a finite subset of ω) is the set of two-valued functions with domain an element of an ideal J on ω. The problem treated in this paper is to determine when such forcing yields a generic real of minimal degree of constructibility.

A simple decomposition argument shows that the non-maximality of J implies the non-minimality of the generic real which is obtained. In §3 and 4 we look at the case J is maximal and we show that the minimality of the generic real depends on a combinatorial property of J.

In fact the minimality result uses the notion of T-ideal and the non-minimality result that of selective ultrafilter (a notion studied in Booth [1]). These notions are generalized to the case of non-maximal ideals and shown to be equivalent in § 1. A short study of them is also made in §2 and in the appendix.

The notion of T-ideal, without any hypothesis of maximality, is used in §5 where we generalize Silver's set of forcing conditions (described in Mathias [3] p. 4). In fact Silver's forcing is related to the above in the following way: first force to get a maximal ideal, which is shown to be a T-ideal, and then force with this ideal in the above manner.

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§ 1. Combinatorics and ideals

Throughout this paper an ideal on ω will mean an ideal containing the ideal of finite subsets and a filter will mean a filter containing the filter of cofinite subsets. We use for them the letters J and F.

J and F are said to be dual if F is the set of complements of the subsets which lie in J.

We write $\mathrm{Seq}(\omega)$ for the set of finite sequences of integers, $s*t$ for the concatenation of two sequences s and t, $\mathrm{lh}(s)$ for the length of s, and (n) for the sequence of length one defined by the integer n.

We put on $\mathrm{Seq}(\omega)$ the extension ordering: s is greater than t if $\mathrm{lh}(s)$ is greater than $\mathrm{lh}(t)$ and the restriction of s to $\mathrm{lh}(t)$ is t.

Definition 1.1. i) A is a tree if A is a subset of $\mathrm{Seq}(\omega)$ and any predecessor of an element of A is in A. (So the empty sequence is in any tree.)

ii) If s is in the tree A the ramification of A at s is the set of integers n such that $s*(n)$ is in A.

iii) A function H from ω into ω is a branch of the tree A if for every k the sequence $(H(0), ..., H(k))$ is in A.

Definition 1.2. i) A is a J-tree if no ramification of A is in J.

ii) A is a strong J-tree if no finite intersection of ramifications of A is in J.

iii) H is a J-branch of the tree A if it is a branch with range not in J.

iv) J is a T-ideal if every J-tree has a J-branch.

v) J is a weak T-ideal if every strong J-tree has a J-branch.

Any T-ideal is a weak T-ideal. In case J is maximal the two notions coincide since then any J-tree is a strong J-tree.

Proposition 1.3. *The ideal of finite subsets of a T-ideal.*

Proposition 1.4. *If J is a T-ideal (resp. a weak T-ideal) and if J′ is countably generated over J then J′ is also a T-ideal (resp. a weak T-ideal).*

Proof. Let (x_n), n in ω, be a basis of J' over J: i.e. a subset of ω is in J' if and only if it is included in the union of an x_n and an element of J. We can suppose that the x_n are increasing.

Let A' be a J'-tree; define A as follows: a sequence $(n_0, \ldots, n_k)$ is in A if and only if it is an A' and ,for each j, n_j is not in x_j. Clearly A is a tree. A ramification of A is the difference of a ramification of A' with an x_n, so it is not in J and A is a J-tree. Let H be a J-branch of A; since its range has at most n points in x_n it is a J'-branch. Since A is included in A', H is a J'-branch of A'. This shows that J' is a T-ideal.

Corollary 1.5. *Every countably generated ideal is a T-ideal.*

Definition 1.6. A partition of ω is a J-partition if no finite union of elements of the partition is in the dual F of J.

Definition 1.7. J is selective if for every J-partition there exists a subset of ω which is not in J and meets each element of the partition at one point at most. Such a subset is called a selector for the given partition.

Definition 1.8. J is p-point if for every Jpartition there exists a subset of ω which is not in J and which meets each element of the partition at a finite number of points.

Proposition 1.9. *J is p-point if and only if for every decreasing sequence (x_n), n in ω, of subsets of ω which are not in J there exists a subset x, not in J, such that $x-x_n$ is finite for each n.*

Proof. It suffices to consider the partition defined by $y_0 = \omega - x_0$ and $y_{n+1} = x_n - x_{n+1}$ which is a J-partition.

Proposition 1.10. *If J is a weak T-ideal then J is a selective ideal.*

Proof. Let (x_n), n in ω, be a J-partition. Define a tree A: s is in A if and only if s meets each x_n at one point at most. A is a strong J-tree, taking a J-branch of A we get a selector for the partition which is not in J.

Definition 1.11. J is inductive if for every decreasing sequence (x_n), n in ω, of subsets of ω which are not in J, there exists a strictly increasing function H from ω into ω, with range not in J, such that $H(n+1)$ is in $x_{H(n)}$ for each n.

Proposition 1.12. *If J is a selective ideal then J is inductive.*

Proof. The following is a slight modification of a proof due to Kunen which gives the proposition in the case J is a maximal ideal (see Booth [1]).

Let (x_n), n in ω, be a decreasing sequence of subsets not in J.

As J is selective it is p-point and so (Prop. 1.9) there is an x, not in J, such that $x - x_n$ is finite for each n. Define a function g from ω into ω: $g(n)$ is the greatest element of $x - x_n$. Thus if m is greater than $g(n)$ and if m is in x then m is in x_n.

Let $g^0(0)$ be 0 and $g^{p+1}(0)$ be $g(g^p(0))$.

If a and b are such that for an integer p $a \leqslant g^p(0) \leqslant g^{p+1}(0) < b$ then b is in $x_{g^p(0)}$, and , as x_a contains $x_{g^p(0)}$, b is in x_a.

Consider the partition of ω defined by $\omega - x$ and the intersections of x with the intervals $]g^{2p}(0), g^{2p+2}(0)]$. It is a J-partition, take a selector not in J and let a_p be the point of it which is in $]g^{2p}(0), g^{2p+2}(0)]$.

Put on the set of a_p, p in ω, the following equivalence relation: a_p is equivalent to a_{p+1} if the interval $[a_p, a_{p+1}]$ is included in the interval $]g^{2p+1}(0), g^{2p+3}(0)]$. Clearly the equivalence classes have at most two elements.

These equivalence classes define with the complement of the set $\{a_p : p \text{ in } \omega\}$ a J-partition. Take a selector not in J and let $H(n)$ be its n-th point which is in an equivalence class.

As between $H(n+1)$ and $H(n)$ there is an interval $]g^p(0), g^{p+1}(0)]$, $H(n+1)$ is in $x_{H(n)}$. Hence H is the desired function.

Proposition 1.13. *If J is inductive and if (x_s), s in* Seq(ω), *is a family of subsets of ω such that no finite intersection of them is in J, there is a strictly increasing function H from ω into ω with range not in J such that $H(n)$ is in $x_{H \upharpoonright n}$ for each n.*

Proof. By the finite intersection property we can suppose that if s and t are sequences such that lh(s) is less than lh(t) and sup(s) is less than sup(t) (where sup denotes the greatest element of a sequence) then x_s is included in x_t.

Let s_n be the sequence of length $n+1$ with constant value n and let y_n be x_{s_n}. Using the hypothesis that J is inductive, take a strictly increasing function H from ω into ω, with range not in J, such that $H(n+1)$ is in $y_{H(n)}$ for each n. Note that we can suppose that $H(0)$ is in x_{ϕ}.

As $(H(0), ..., H(n))$ has length $n+1$ and its sup is $H(n)$, while $s_{H(n)}$ has length $H(n)+1$ and its sup is $H(n)$, $x_{(H(0), ..., H(n))}$ contains $x_{s_{H(n)}}$ and so $H(n+1)$ is in $x_{(H(0), ..., H(n))}$. Hence $H(n)$ is in $x_{H\restriction n}$ for each n.

Proposition 1.14. *If J is an inductive ideal then J is a weak T-ideal.*

Proof. Let A be a strong J-tree.

Let s be a finite sequence of integers, if s is in A we let x_s be the ramification of A at s and if s is not in A we let x_s be ω.

We can apply 1.13 to the family (x_s), s in Seq(ω): take a function H with range not in J such that $H(n+1)$ is in $x_{H\restriction n}$ for each n. We show inductively that H is a branch of A. If $H\restriction k$ is in A then $H(k)$ is in the ramification of A at $H\restriction k$ and so $H\restriction k+1$ is in A.

As the range of H is not in J, H is a J-branch of A.

Corollary 1.15. *If J is an ideal then the following are equivalent:*
i) *J is a weak T-ideal*
ii) *J is selective*
iii) *J is inductive*

Recall the usual definition of a selective ultrafilter: an ultrafilter F is selective if for every partition of ω by elements of the dual J of F there is a selector in F.

Clearly the ultrafilter F is selective just in case its dual is.

Corollary 1.16. *An ultrafilter F is selective if and only if it its dual is a T-ideal.*

§2. Getting maximal T-ideals

Consider on 2^ω the equivalence relation of equality except on a set in J. Let $2^\omega/J$ be the quotient set. We put on it the ordering induced by the reverse inclusion ordering on 2^ω so that it becomes a boolean algebra whose zero-element is the dual of J.

We say that $2^\omega/J$ satisfies the condition of decreasing sequences (written c.d.s.) if every decreasing sequence of non-zero elements has a non-zero lower bound. Such a lower bound is called a minorant.

Definition 2.1. J satisfies the c.d.s. if $2^\omega/J$ satisfies the c.d.s. We also say that J is c.d.s.

Clearly J is c.d.s. if for every increasing sequence (x_n), n in ω, of subsets not in the dual F of J there is an x, not in F, such that $x_n - x$ is in J for each n. Passing to the complement we get:

Proposition 2.2. *J is c.d.s. if for every decreasing sequence (x_n), n in ω, of subsets not in J there is an x, not in J, such that $x - x_n$ is in J for each n.*

Using Proposition 1.9 we get:

Proposition 2.3. *If J is p-point then J is c.d.s. Hence any weak T-ideal is c.d.s.*

Remark: any maximal ideal is c.d.s.

Proposition 2.4. *J is c.d.s. if and only if every ideal countably generated over J is included in an ideal one-generated over J.*

Let M be a transitive model of ZF.

If J is an ideal in M we let c be the canonical surjection from 2^ω onto $2^\omega/J$.

If G is $2^\omega/J$-generic over M (we make no difference, when writing, between the boolean algebra $2^\omega/J$ and the set of forcing conditions obtained by deleting the zero-element) we let $J^* = c^{-1}(G)$. Clearly $M[G] = M[J^*]$.

If x and y are disjoint subsets of ω, lying in M, with union ω, then exactly one of them is in J exactly one of $c(x)$ and $c(y)$ is in G as they are complements in the boolean algebra $2^\omega/J$.

If J is c.d.s. then $2^\omega/J$ satisfies the c.d.s. and so every countable set in $M[G]$ which is included in M is in M. Hence M and $M[J^*]$ have the same subsets of ω. Thus J^* is in $M[J^*]$ a maximal ideal on ω (extending the ideal J). Also note that a countable subset of G always has a minorant in G.

Theorem 2.5. *If J is a weak T-ideal then J^* is a maximal T-ideal in $M[J^*]$ which extends the ideal J.*

Proof. If J is a weak T-ideal then J is c.d.s. (Prop. 2.3) and so J^* is a maximal ideal containing J.

Let A be a J^*-tree in $M[J^*]$; as a countable set included in M, A is in M. If s is in A let x_s be the ramification of A at s. A being a J-tree, for every s in A, $c(\omega-x_s)$ is in G. The countable family $c(\omega-x_s)$, s in A, of elements of G is bounded below an element $p = c(x)$ which is in G. The ideal $J(p)$ generated by J and x is proper and its dual contains each x_s, s in A. Let $q = c(y)$ be a minorant of p in $2^\omega/J$, and let $J(q)$ be the ideal generated by J and y. $J(q)$ is one generated over J so (Prop. 1.4) it is a weak T-ideal. A has its ramifications in the dual of $J(p)$, hence in the dual of $J(q)$ and so it is a strong $J(q)$-tree. Let H be a $J(q)$-branch of A with range z. As z is not in $J(q)$ there is a non-zero minorant r of q and $c(\omega-z)$. This condition r (weakly) forces "A has a J^* branch," for if G' is generic and contains r then $\omega-z$ is in $(J')^*$ and so H is a $(J')^*$-branch of A.

Thus we have shown that the set of conditions which force "A has a J^* branch" is dense below p. As p is in G this set meets G and so the sentence is true in $M[G]$.

Theorem 2.5 gives a way to get maximal T-ideals extending a weak T-ideal. In the following we show that we can directly within M get maximal extensions of a weak T-ideal which are T-ideals.

Lemma 2.6. *If J is a weak T-ideal and if there is a maximal ideal one-generated over J, this maximal extension is a T-ideal.*

Theorem 2.7. *Assuming the continuum hypothesis (written* CH*), if J is a weak T-ideal with no maximal one-extension then there are $2^{\aleph_1}$ maximal T-ideals extending J.*

Proof. Note that J is c.d.s. and so every ideal countably generated over J is included in a one-extension of J and hence is not maximal.

Using CH we can put well-orderings of type $\aleph_1$ on the power set of ω and on the set of trees. Fixing such orderings we can speak of "the first subset of ω such that ..." and of "the first tree such that ...".

Let f be a function from $\aleph_1$ into 2, we are going to associate to f a maximal T-ideal $J^*(f)$ extending J. $J^*(f)$ will be the union of an increasing sequence of $\aleph_1$ proper ideals, each being countably generated over J.

We define the sequence by induction. $J_0(f)$ is J. If α is limit then $J_\alpha(f)$ is the union of the $J_\beta(f)$, β less than α. Suppose $J_\alpha(f)$ is defined. Let $A_\alpha(f)$ be the first tree with all ramifications in the dual of $J_\alpha(f)$ which has not been considered earlier in the construction of the sequence. As $J_\alpha(f)$ is countably generated over J it is a weak T-ideal, so A being a strong $J_\alpha(f)$-tree has a $J_\alpha(f)$-branch. Let $x_\alpha(f)$ be the first subset of ω which is the range of a $J_\alpha(f)$-branch of $A_\alpha(f)$. The ideal $E_\alpha(f)$ generated by $J_\alpha(f)$ and $\omega - x_\alpha(f)$ is not maximal since it is countably generated over J, let $y_\alpha(f)$ be the first subset of ω which is neither in $E_\alpha(f)$ nor its dual. Let $z_\alpha(f)$ be $y_\alpha(f)$ if $f(0) = 0$ and $\omega - x_\alpha(f)$ if not. We define $J_{\alpha+1}(f)$ to be the ideal generated by $E_\alpha(f)$ and $z_\alpha(f)$.

Show $J^*(f)$ is maximal. If it is not ley y be the first subset of ω which is neither in $J^*(f)$ not in its dual. For every α in $\aleph_1$ y is neither in $E_\alpha(f)$ nor its dual, so $y_\alpha(f)$ is before y in the well-ordering of type $\aleph_1$ on the power set of ω, but the $y_\alpha(f)$ are all different and uncountably many while the rank of y is countable, hence a contradiction.

Show $J^*(f)$ is a T-ideal. It it is not let A be the first $J^*(f)$-tree with no $J^*(f)$-branch. A has its ramifications in the dual of $J^*(f)$, as they are countably many there is an α less than $\aleph_1$ such that they are in $J_\alpha(f)$. This implies that the A_β, $\alpha < \beta < \aleph_1$, are before A in the well-ordering of type $\aleph_1$ on the set of trees, but the A_β are all different and uncountably many while A is of countable rank, hence a contradiction.

If f and g are different functions from $\aleph_1$ into 2, let α be the first ordinal at which they differ. It is clear that $z_\alpha(f)$ is the complement of $z_\alpha(g)$, so $J^*(f)$ is different of $J^*(g)$.

Hence the $2^{\aleph_1}$ maximal T-ideals extending J.

Remark: the hypothesis in 2.7 that no one-extension of J is maximal cannot be dropped (using the fact that the sum of two T-ideals is a T-ideal, it suffices to consider the sum of two maximal T-ideals).

As a countably generated ideal is not maximal we have the following theorem (see Booth [1]):

Corollary 2.8. *Assuming* CH, *there are* $2^{\aleph_1}$ *maximal selective ideals extending a countably generated ideal.*

§3. Non-minimality results

Let M be a transitive model of **ZF**.

Definition 3.1. A real g is minimal over M if g is not in M and every real f in $M[g]$ is in M or reconstructs g (i.e. g is in $M[f]$).

If J is an ideal on ω belonging to M, let $C(J)$ be the set of two-valued functions defined on an element of J. We put on $C(J)$ the reverse inclusion ordering (thus $p \leqslant q$ means p extends q), to obtain a collection of conditions.

If G is $C(J)$-generic over M, G defines a real g: $g(n) = 0$ if and only if $\{(n, 0)\}$ is in G. As G is the set of restrictions of g to elements of J it is clear that $M[G] = M[g]$.

Definition 3.2. A real associated to a $C(J)$-generic over M is called a J-Cohen real over M.

Note that since J contains the finite subsets of ω a J-Cohen real over M is not in M.

Proposition 3.3. *If J is not maximal in M then a J-Cohen real over M is not minimal over M.*

Proof. Let x be non-measured by J, define $C(J)(x)$ to be the set of elements of $C(J)$ whose domains are included in x. Then the $C(J)$-forcing is the forcing over the product of $C(J)(x)$ by $C(J)(\omega-x)$ and $M[g] = M[g \restriction x][g \restriction \omega-x]$. So by the previous remark and the fact that x and $\omega-x$ are both necessarily infinite, it follows that $C(J)(x)$ and $C(J)(\omega-x)$ are of the same type as $C(J)$. Hence M is properly included in $M[g \restriction x]$ which is itself properly included in $M[g]$.

Before stating a result in the case J is maximal, we recall a general result on forcing (Krivine [2]).

Definition 3.4. If C and D are ordered sets, an increasing function T from C into D is said to be normal if its range is dense in D and for every p in C the image of C/p (the minorants of p in C) by T is dense below $T(p)$ in D.

Proposition 3.5. *If G is C-generic over M and T is a normal function from C into D, then the set* $\sup(T(G))$ *of elements of D greater than an element of $T(G)$ is D-generic over M and G is $T^{-1}(\sup(T(G)))$-generic over $M[\sup(T(G))]$.*

On the two-valued functions on a set x we can define the equivalence relation of equality modulo a finite set. We denote the set of equivalence classes by $2^x/\text{fin}$.

AC′ is the axiom asserting the existence of a set of representatives for $2^\omega/\text{fin}$.

Theorem 3.6. *If M satisfies* AC′ *and if J is a maximal ideal on ω whose dual is not selective then a J-Cohen real over M is not minimal over M.*

Proof. First note that given a set of representatives for $2^\omega/\text{fin}$ we get one canonically for $2^x/\text{fin}$ if x is an infinite subset of ω. If x is finite then $2^x/\text{fin}$ has one element and we can take the zero-function as a representative.

In M let (x_n), n in ω, be a partition of ω in elements of J such that if x meets each x_n in at most one point then x is in J.

By AC′ and the preceding remark we can get a family $(h_{i,n})$ of representatives for the elements of the union of the $2^{x_n}/\text{fin}$, n in ω.

With this family we define a two-valued function L on the union of the 2^{x_n}, n in ω, as follows: $L(h) = 0$ if and only if, h being in 2^{x_n} and $h_{i,n}$ being its representative, h differs from $h_{i,n}$ on a finite odd number of points.

Let K be the maximal ideal defined as follows: a subset u of ω is in K if the union of the x_n, n in u, is in J. Define a function T from $C(J)$ into $C(K)$: n is in the domain of $T(p)$ if the domain of p contains x_n and then $T(p)(n) = L(p \upharpoonright x_n)$ where $p \upharpoonright x$ means the restriction of p to x. T is clearly surjective and increasing. Let us show it is a normal function. If, in $C(K)$, d is an extension of $T(p)$, p in $C(J)$, then for each n in the domain of d either n is in the domain of $T(p)$ and so p is defined on the whole of x_n and hence for every extension q of p we have $L(q \restriction x_n) = L(p \restriction x_n) = d(n)$, or n is not in the domain of $T(p)$ and so p is not defined on the whole of x_n and there is an extension p_n of p to x_n such that $L(p_n \restriction x_n) = d(n)$. Take q to be the union of p and the p_n, n in the domain of d and not in the domain of $T(p)$; q is an extension of p, q is in $C(J)$ by the definition of K, and $T(q)$ extends d. Thus T is normal.

Now let g be J-Cohen over M, associated to the $C(J)$-generic G. Applying 3.5 we see that $T(G)$ is $C(K)$-generic over M; its associated real is f: $f(n) = 0$ if and only if $L(g \restriction x_n) = 0$. Thus f is a real in $M[g]$ not in M. Moreover G is E-generic over $M[f]$ where $E = T^{-1}(T(G))$ is the set of p in $C(J)$ such that if the domain of p contains x_n then $L(p \restriction x_n) = f(n)$.

Let X be the set of p in E that are incompatible with an element of G. Let us show that X is dense in E. Given p in E, by the hypothesis on the partition (x_n), n in ω, there is an n such that in x_n at least two points are not in the domain of p, say a and b. Let q be the extension of p to the point a such that $q(a)$ is different from $g(a)$. This q is always in E since q is not defined at b and so $T(q) = T(p)$.

X being dense in E, G can not be in $M[f]$ for then X would be in $M[f]$ $(=M[T(G)])$ and so would meet G which is impossible. Hence f does not reconstruct g and this shows the non-minimality of g over M.

§4. Minimality results

Let DC′ be the axiom of dependent choices restricted to sets of cardinality less than that of the continuum.

The purpose of this section is to prove the following theorem:

Theorem 4.1. *If M satisfies* DC′ *and if J is a maximal T-ideal in M, then a J-Cohen real over M is minimal over M* (see def. 3.1 and 3.2).

The proof is a direct one. We take a J-Cohen g associated to G, $C(J)$-generic over M, and a real f in $M[g]$ and we show that either f is in M or f reconstructs g.

Let $\bar{f}$ be a denotation for f in the forcing language. We make no distinction between an element x of M and its notation as an element of $M[g]$.

All the definitions that follow make use of $\bar{f}$ and the forcing relation, so they take place in M.

Definition 4.2. Two elements p and q of $C(J)$ are said to be $\bar{f}$-compatible if for *no* integer n, p forces $\bar{f}(n) = a$ and q forces $\bar{f}(n) = b$ where a and b are distinct elements of 2.

Remark 4.3. Let p' be a mimorant of p and q' one of q, if p and q are $\bar{f}$-incompatible (i.e. not $\bar{f}$-compatible) then so are p' and q'.

If n is an integer not in the domain of a condition p and if a is in 2, we write $(p, (n, a))$ for the extension of p defined where p is and at n, where its value is a.

Similarly if s is a finite sequence, with length k, of distinct integers not in the domain of p and if i is a two-valued sequence with the same length k, we write $(p, (s, i))$ for the extension of p defined where p is and at the integers occuring in s, with value $i(n)$ at $s(n)$.

Definition 4.4. An integer n is $\bar{f}$-indifferent to a condition p (written $n\,I\,p$) if n is not in the domain of p and for every extension q of p, either n is in the domain of q or $(q, (n, 0))$ and $(q, (n, 1))$ are $\bar{f}$-compatible.

Roughly speaking, n is indifferent to p if below p n is of no use to know the interpretation of $\bar{f}$.

Remark 4.5. If n is not in the domain of q and q extends p and $n\,I\,p$ then $n\,I\,q$.

Let p be a condition, two disjoint cases are possible:
either i) $(\exists q \leqslant p)(\forall r \leqslant q)\,\forall n$ not $(n\,I\,r)$
or ii) $(\forall q \leqslant p)(\exists r \leqslant q)\,\exists n(n\,I\,r)$

The following lemmas deal with the two cases. Their proofs will be given later.

Lemma 4.6. *If M satisfies* DC′, *if J is a T-ideal and p satisfies* i) *then there is an extension q of p and a strictly increasing function H from ω into ω, with range the complement of the domain of q such that for every integer k and every two-valued sequence i with length k the two conditions $(q, (H\restriction k, i), (H(k), 0))$ and $(q, (H\restriction k, i), (H(k), 1))$ are $\bar{f}$-incompatible.*

Lemma 4.7. *If M satisfies* DC′, *if J is a T-ideal and p satisfies* ii) *then there is an extension q of p which decides $\bar{f}(n)$ for each integer n.*

Proposition 4.8. *If q is as in* Lemma 4.6 *and q is in G then f reconstructs g, i.e. g is in $M[f]$.*

Proof. Define a real g' in $M[f]$: on the domain of q g' is just q, and on the range of H we define g' by the following induction.

Suppose g' is defined on $H\restriction k$ (i.e. on the integers occuring in the sequence $H\restriction k$), then $(q, (H\restriction k, g'\restriction(H\restriction k)), (H(k), 0))$ and $(q, (H\restriction k, g'\restriction(H\restriction k)), (H(k), 1))$ speak differently about $\bar{f}(n)$ for an integer n. Take the first such n and choose $g'(H(k)$ such that $(q, (H\restriction k, g'\restriction(H\restriction k)), (H(k), g'(H(k))))$ forces $\bar{f}(n) = f(n)$.

As q is in G and as the interpretation of f in $M[G]$ is just f, se see inductively that $\{(H(k), g'(H(k)))\}$ is in G. Hence $g' = g$ and so g is in $M[f]$.

Proposition 4.9. *If q is as in* Lemma 4.7 *and q is in G then f is in M.*

The proof of Theorem 4.1 is now easy. Let D be the set of conditions q as in Lemmas 4.6 and 4.7. These lemmas just show that D is dense in $C(J)$. As D is in M it meets G; applying the two preceding propositions we deduce that f is in M or that f reconstructs g.

We now turn to the proofs of Lemmas 4.6 and 4.7.

Proposition 4.10. *Let s be a sequence of distinct integers all different from the integer n and let p be a condition which has neither n nor the integers in s in its domain. If n is indifferent to no extension q of p then there exists an extension q of p which has neither n nor the integers in s in its domain and such that for every two-valued sequence i with the same length as s the two conditions $(q, (s, i), (n, 0))$ and $(q, (s, i), (n, 1))$ are $\bar{f}$-incompatible.*

Proof. Let $i_0, ..., i_t$ be the different two-valued sequences with the same length as s.

We define an increasing sequence $q_0, ..., q_t$ of extensions of p, which have neither n nor the integers of s in their domain, by the following induction.

By the hypothesis on n and p, n is not indifferent to the extension $(p, (s, i_0))$ of p, hence there is an extension q_0 of p such that $(q_0, (s, i_0), (n, 0))$ and $(q_0, (s, i_0), (n, 1))$ are $\bar{f}$-incompatible. If q_{u-1} is defined, then n is not indifferent to the extension $(q_{u-1}, (s, i_u))$ of p, so there is an extension q_u of q_{u-1} such that $(q_u, (s, i_u), (n, 0))$ and $(q_u, (s, i_u), (n, 1))$ are $\bar{f}$-incompatible.

Using remark 4 3 we see that q_t is such that for every two-valued sequence i with the same length as s the two conditions $(q_t, (s, i), (n, 0))$ and $(q_t, (s, i), (n, 1))$ are $\bar{f}$-incompatible. Hence q_t is the desired q.

Proof of Lemma 4.6. Let p_0 be an extension of p such that no extension of p_0 has an indifferent point (such a p_0 exists by the hypothesis on p).

We define by induction a J-tree A and a decreasing function Q from A into $C(J)$ such that the integers of any sequence s in A are all distinct and not in the domain of $Q(s)$.

The empty sequence is in A and its image by Q is p_0. Let s be in A, we put $s*(n)$ in A if n is different of the integers in s and not in the domain of $Q(s)$. As $Q(s)$ is an extension of p_0, it has no extension with an indifferent point. Applying 4.10 (with s, n and $Q(s)$) we see that there exists an extension q of $Q(s)$ such that for any two-valued sequence i, with the same length as s, the two conditions $(q, (s, i), (n, 0))$ and $(q, (s, i), (n, 1))$ are $\bar{f}$-incompatible. We take such a q as $Q(s*(n))$.

One can see that the construction of A can be done assuming only DC′.

It is clear that A is a J-tree. Use the hypothesis that J is a T-ideal to take a J-branch H of A.

The sequence $Q(H\restriction k)$, k in ω, is decreasing and for each k the integers in $H\restriction k$ are not in the domain of $Q(H\restriction k)$. Hence if we let q' be the union of the $Q(H\restriction k)$, k in ω, q' is a two-valued function whose domain is disjoint from the range of H. As H is a J-branch its range is not in J and q' is an element of $C(J)$.

Let q be an extension of q' with domain just the complement of the range of H. Using Remark 4.3 and the fact that q extends $Q(H\restriction k)$ for every k, it is clear that for any two-valued sequence i, with length k, the two conditions $(q, (H\restriction k, i), (H(k), 0))$ and $(q, (H\restriction k, i), (H(k), 1))$ are $\bar{f}$-incompatible. Hence q and H are the desired ones.

Proposition 4.11. *Let s be a sequence of k distinct integers which are indifferent to a condition p; then for every extension q of p and two-valued sequences i and i′ with length k the two conditions* $(q, (s, i))$ *and* $(q, (s, i'))$ *are* $\bar{f}$*-compatible.*

Proof. We proceed by induction on k. The case $k = 0$ is clear. Suppose the property true for k, we show it is true for $k+1$.

Suppose that, for an integer n, $(q, (s, i))$ and $(q, (s, i'))$ decide $\bar{f}(n)$; we shall show that they make the same decision. Let r be an extension of q which has not the integers of s in its domain and such that $(r, (s\restriction k, i\restriction k), (s(k), i'(k)))$ decides $\bar{f}(n)$. As $s(k)$ is indifferent to p and r extends p, $s(k)$ is indifferent to r, and so $(r, (s, i))$ and $(r, (s\restriction k, i\restriction k), (s(k), i'(k)))$ are $\bar{f}$-compatible. The two decide $\bar{f}(n)$, so they make the same decision. Now if we put r and $(s(k), i'(k))$ together, we can apply the induction hypothesis on k to see that $(r, (s, i'))$ and

$(r, (s\restriction k, i\restriction k), (s(k), i'(k)))$ are $\bar{f}$-compatible. The two decide $\bar{f}(n)$, so they make the same decision. Hence $(r, (s, i))$ and $(r, (s, i'))$ decide $\bar{f}(n)$ in the same way, and so do $(q, (s, i))$ and $(q, (s, i'))$. This shows the induction step.

Proposition 4.12. *Let s be a sequence of distinct integers indifferent to a condition p, and let q be an extension of p such that the integers of s are in the domain of q. If q' is the condition obtained from q by deleting the integers of s from the domain, then*

a) *If q decides $\bar{f}(m)$ then q' also decides $\bar{f}(m)$.*

b) *If the integer n is indifferent to q then n is also indifferent to q'.*

Proof. a) As q' extends p, the integers of s are indifferent to q'.

Applying Prop. 4.11, we see that every extension of q' is $\bar{f}$-compatible with q, so every extension of q' which decides $\bar{f}(m)$ makes this decision as q does. Hence q' does decide $\bar{f}(m)$, and this in the same direction as q.

b) To prove that n is indifferent to q', we show that for every extension r of q' which has neither n nor the integers of s in its domain and for every two-valued sequence i with the same length as s, if $(r, (s, i), (n, 0))$ and $(r, (s, i), (n, 1))$ both decide $\bar{f}(m)$, m any integer, they do it in the same direction. Let r' be an extension of $(r, (s, q\restriction s))$ which decides $\bar{f}(m)$. As $(r, (s, q\restriction s))$ extends q, n is indifferent to r'; applying a) we can suppose that n is not in the domain of r'. Applying Prop. 4.11, we see that r' is $\bar{f}$-compatible with both $(r, (s, i), (n, 0))$ and $(r, (s, i), (n, 1)$, so that these two conditions decide $\bar{f}(m)$ in the same direction.

Proposition 4.13. *Let p be such that it is dense below p to have an indifferent point (*condition ii)*). If s is a sequence of distinct integers indifferent to p then, for every integer m, the set of n which are indifferent to an extension r of p, r deciding $\bar{f}(m)$ and the integers of s not in the domain of r, is not in J.*

Proof. Let q be an extension of p deciding $\bar{f}(m)$, using Prop. 4.12 we can suppose that the integers of s are not in the domain of q. We prove that the set X of the integers n which are indifferent to an extension r of q, the integers of s not in the domain of r, is not in J.

Suppose not. Note that the integers of s are in X. Let r be an extension of q whose domain contains X, such an r exists since X is in J. Prop. 4.12 and the fact that the integers of s are in the domain of r implies that no extension of r has an indifferent point, contradicting the hypothesis of density below p.

Proof of Lemma 4.7. We define inductively a J-tree A and a decreasing function Q from A into $C(J)$ such that if s is in A with length k the integers of s are indifferent to $Q(s)$ and $Q(s)$ decides $\bar{f}$ up to k.

The empty sequence is in A and its image by Q is the given condition p. If s is in A with length k, we put $s*(n)$ in A if n is indifferent to an extension r of $Q(s)$ which decides $\bar{f}(k)$ and which has not the integers of s in its domain, and we let $Q(s*(n))$ be such an r.

One can show that the construction of A can be done assuming only DC′.

Prop. 4.13 shows that A is a J-tree. Use the hypothesis that J is a T-ideal to take a J-branch H of A.

Let q be the union of the $Q(H \upharpoonright k)$, k in ω. As the integers of $H \upharpoonright k$ are indifferent to $Q(H \upharpoonright k)$, they are not in its domain, so the range of H is disjoint from the domain of q. Thus q is in $C(J)$, it extends p and for each k it decides $\bar{f}(k)$.

Hence the proof of Theorem 4.1 is now complete.

Remark 4.14. Theorem 4.1 can be strengthened. Let J be a maximal T-ideal in M and g a J-Cohen over M. If f is in $M[g]$ a function from ω into M then either f is in M or f reconstructs g.

Proof. Take X in M containing the range of f. Replace in definition 4.2 the condition "a, b in 2" by "a, b in X". The proof works in the same way.

Remark 4.15. The minimality result implies the nonexistence of Cohen reals (J_0-Cohen reals, where J_0 is the ideal of finite subsets of ω), hence the non-denumerability of the continuum of M in $M[g]$. In particular if M satisfies the continuum hypothesis then M and $M[g]$ have the same cardinals. In this last case we can replace in the preceding remark the condition "f is a function from ω into M" by "f is a countable set included in M" (provided that M satisfies the axiom of choice).

To prove this last statement let u be a bijection in M between an ordinal and a set which contains f, clearly $M[f] = M[u^{-1}(f)]$; if α is the order type of $u^{-1}(f)$ we can form a bijection v from α onto f such that $M[f] = M[v]$. In $M[g]$ α is countable as is f; if M and $M[g]$ have the same cardinals α is also countable in M and there is a bijection t from ω onto f such that $M[f] = M[t]$. Now it suffices to apply 4.14.

Getting Theorems 4.1, 3.6, and 1.15 together gives:

Theorem 4.16. *If M is a model of* ZFC *then a J-Cohen real over M is minimal over M if and only if J is a maximal T-ideal.*

§5. Generalized Silver's forcing

In this section, it is understood that M is a model of ZF + DC!

Recall that if J is c.d.s. (Def. 2.1) then a set which is $2^\omega/J$-generic over M is the image by c (the canonical surjection from 2^ω onto $2^\omega/J$) of a maximal ideal J^* extending J. Moreover a countable set in $M[J^*]$ which is included in M is in M.

In $M[J^*]$ we can define $C(J^*)$. Let g be a J^*-Cohen real over $M[J^*]$. Clearly a subset x of ω is in J^* if and only ig $g \upharpoonright x$ is in $M[J^*]$; but $g \upharpoonright x$ is in $M[J^*]$ if and only if it is in M, so J^* is M-definable from g and hence $M[J^*][g] = M[g]$. We say that g is obtained by double forcing from J over M.

If F is the dual of J, we let $S(J)$ be the set of two-valued functions defined on a subset of ω which is not in F, and we put on it the reverse inclusion ordering.

If G is $S(J)$-generic over M, G defines a real g: $g(n) = 0$ if and only if $\{(n, 0)\}$ is in G. As G is the set of restrictions of g which are in M, it is clear that $M[G] = M[g]$.

Definition 5.1. A real associated to an $S(J)$-generic over M is called a J-Silver real over M.

Remark 5.2. If J is the ideal of finite subsets of ω, $S(J)$ is Silver's set of forcing conditions which is described in Mathias [3]. If J is maximal then $C(J) = S(J)$ and the notions of J-Cohen and J-Silver reals coincide.

Theorem 5.3. *If J is c.d.s. the double forcing from J coincides with the J-Silver forcing; i.e. a J-Silver real over M can be obtained by double forcing from J over M and conversely.*

Proof. We define a function from $S(J)$ into $2^\omega/J$: $T(p) = c(\text{domain(p)})$. T is clearly a normal function (def. 4.3).

Let g be a J-Silver real over M associated to the $S(J)$-generic G. Applying 3.5 we deduce that $T(G)$ is $2^\omega/J$-generic over M; let J^* be the maximal ideal associated to $T(G)$, then $M[T(G)] = M[J^*]$. Moreover G is $T^{-1}(T(G))$-generic over $M[J^*]$, but $T^{-1}(T(G)$ is just $C(J^*)$. Hence g is obtained by double forcing from J over M.

Conversely let g be a real obtained by double forcing: g is associated to G which is $C(J^*)$-generic over $M[J^*]$. To show that g is a J-Silver real we have to show that G is $S(J)$-generic over M; i.e that G meets each dense subset D of $S(J)$ which lies in M. As G is $C(J^*)$-generic over $M[J^*]$, it suffices to show that the intersection of D with $C(J^*)$ is dense in $C(J^*)$. Let p be an element of $C(J^*)$. If D/p is the set of minorants of p in D, D/p is dense below p in $S(J)$; so $T(D/p)$ is dense below $T(p)$ in $2^\omega/J$. As $T(p)$ is in $c(J^*)$, $T(D/p)$ meets $c(J^*)$; hence there is a q in D/p whose domain is in J^*. This shows that the intersection of D with $C(J^*)$ is dense in $C(J^*)$. Thus g is a J-Silver real over M.

Remark. Suppose that J is a c.d.s. ideal in M and that, in the double forcing, J^* is a maximal T-ideal in $M[J^*]$. Then by Theorem 4.1, the real g obtained is minimal over $M[J^*]$. In the following theorem we verify that g is in fact minimal over M.

Theorem 5.4. *Suppose that M satisfies* DC′ *and that J is a c.d.s. ideal in M. If in the double forcing the extension J^* of J is a maximal T-ideal then the real g which is obtained is minimal over M.*

Proof. Let f be a real in $M[g]$ which is not in M. Then f is not in $M[J^*]$. Reasoning in $M[J^*]$, Theorem 4.6 and the proof of 4.1 show the existence of a condition p in G (the $C(J^*)$-generic giving g) and an increasing injective function H from ω into ω, with range the complement of the domain of p, such that for every integer k and every two-valued sequence i with length k there are distinct a, b in 2 and an integer m such that $(p, (H\restriction k, i), (H(k), 0))$ forces $\bar{f}(m) = a$, and $(p, (H\restriction k, i), (H(k), 1))$ forces $\bar{f}(m) = b$. We will consider a, b, m as functions of i.

Note that p, H, a, b, m are all elements of M since J is c.d.s.

The denotation $\bar{f}$ of f in the forcing language in $M[J^*]$ associated to $C(J^*)$ can be itself denoted in the forcing language in M associated to $2^\omega/J$. Hence there is a sentence about M which means that $c(x)$ forces (forcing on $2^\omega/J$) the sentence which denotes "p forces (forcing on $C(J^*)$ in $M[J^*]$) $\bar{f}(m) = a$".

The relation between p and H is a countable conjunction of relations true in $M[J^*]$ so forced by elements of $c(J^*)$.

As J is c.d.s. there is an element of $c(J^*)$, say X, which forces simultaneously all these relations. So, for every two-valued sequence i with length k, X forces "$(p, (H \upharpoonright k, i), (H(k), 0))$ forces $\overline{f}(m(i)) = a(i)$" and X forces "$(p, (H \upharpoonright k, i), (H(k), 1))$ forces $\overline{f}(m(i)) = b(i)$".

We define a real g' in $M[f]$: on the domain of p g' is just p, and on the range of H g' is defined by the following induction.

Suppose g' is defined on $H \upharpoonright k$, then $(p, (H \upharpoonright k, g' \upharpoonright (H \upharpoonright k)), (H(k), 0))$ and $(p, (H \upharpoonright k, g' \upharpoonright (H \upharpoonright k)), (H(k), 1))$ are forced by X to decide differently $\overline{f}(m(g' \upharpoonright (H \upharpoonright k)))$. Choose $g'(H(k))$ such that $(p, (H \upharpoonright k, g' \upharpoonright (H \upharpoonright k)), (H(k), g'(H(k))))$ is forced by X to force $\overline{f}(m(g' \upharpoonright (H \upharpoonright k))) = f(m(g' \upharpoonright (H \upharpoonright k)))$.

As X is in $c(J^*)$. what is forced by X is true in $M[J^*]$. Reasonning in $M[J^*]$ and using the fact that p is in G we see inductively that $\{(H(k), g'(H(k)))\}$ is in G. Hence $g = g'$ and g is in $M[f]$.

Corollary 5.5. *If J is a weak T-ideal then a J-Silver real over M is minimal over M.*

Proof. A weak T-ideal is c.d.s. (Prop. 2.3). Using Theorems 5.3 and 2.5, which says that J^* is a T-ideal, we conclude with Theorem 5.4.

Remark 5.6. The proof of 5.5 is very indirect. In case J is a T-ideal the proof of Theorem 4.1 works to show the minimality of a J-Silver real since in no place the hypothesis of maximality is needed.

§6. Preservation and destruction of ω_1

In this section we show that the forcing with $C(J)$, J a maximal ideal on ω, collapses ω_1, just in case J is not p-point.

Theorem 6.1. *If the ground model M satisfies the continuum hypothesis* (CH) *and J is not p-point then the forcing with* $C(J)$ *collapses* ω_1.

Proof. As M satisfies CH there is a subset A of ω_1 such that

$$2^\omega \cap M = 2^\omega \cap \mathrm{L}_{\omega_1}[A]$$

The following lemma is easy:

Lemma. *If X is a coinfinite subset of* ω *and if a maps X into 2 then for each* $\alpha \in \omega_1$ *there is a map b from* $\omega - X$ *into 2 such that the union of a and b is in* $\mathrm{L}_{\beta+1}[A] - \mathrm{L}_\beta[A]$, *for a* β *greater than* α.

Now let g be a J-Cohen real over M.
Let (X_n), n in ω, be a partition of ω by elements of J such that if X meets each X_n on a finite set then X is in J.
Define a function f from ω into ω_1 as follows:
$f(n)$ = the least α such that $g \upharpoonright X_n$ is in $\mathrm{L}_\alpha[A]$.
Let p be a condition, there is an n such that $X_n \cap \mathrm{dom}(p)$ is coinfinite in X_n (if not ω-dom(p) would be in J, contradicting the definition of $C(J)$). By the lemma, given any β in ω_1, there is an extension q of p such that $q \upharpoonright X$ is in $\mathrm{L}_{\gamma+1}[A] - \mathrm{L}_\gamma[A]$ for a γ greater than β. Hence q forces that there is an n on which f is greater than β.
A density argument shows that f is then cofinal to ω_1, hence the theorem.

In order to prove the converse of the preceding theorem we need a combinatorial property of p-point ideals.
If A is any set, Seq(A) is the set of finite sequences of elements of A. We put on Seq(A) the extension ordering, $s*t$, lh(s) and (a) denote the concatenation of s and t, the length of s and the length-one sequence defined by a. $S_\omega(\omega)$ is the set of finite subsets of ω.

Definition 6.2. i) A is a p-tree if A is a non-empty subset of Seq($S_\omega(\omega)$) and any predecessor of an element of A is in A.

ii) If s is in the p-tree A the ramification of A at s is the set of elements a of $S_\omega(\omega)$ such that $s*(a)$ is in A.

iii) A function H from ω into $S_\omega(\omega)$ is called a p-branch of A if for every k the sequence $(H(0), ..., H(k))$ is in A.

Definition 6.3. i) A subset of $S_\omega(\omega)$ is called J-big if there is an X not in J such that $S_\omega(X)$ is contained in it.

ii) A is a strong J-p-tree if any finite intersection of ramifications of A is J-big.

iii) H is a J-p-branch of A if it is a branch such that the union of its range is not in J.

iv) H is a weak p-T-ideal if every strong J-p-tree has a J-p-branch.

Proposition 6.4. *J is p-point if and only if J is a weak p-T-ideal.*

The proof of this proposition is analogous to that of 1.15, we have to use the notion of p-inductive ideal:

Definition. J is p-inductive if for every decreasing sequence (X_n), n in ω, of subsets of ω not in J, there is a function H from ω into $S_\omega(\omega)$ such that i) if m is less than n then the greatest element of $H(m)$ is less than that of $H(n)$ and the cardinal of $H(m)$ is less than that of $H(n)$.

ii) the union of the range of H is not in J

iii) for each n, $H(n+1)$ is included in $X_{\mathrm{Sup}(H(n))}$.

The analogs of 1.12 and 1.13 and 1.14 hold, proving Prop. 6.4.

Theorem 6.5. *If J is a p-point maximal ideal then the forcing with $C(J)$ does not collapse ω_1.*

Proof. Let g be a J-Cohen real over M and f a function from ω into ω_1 lying in $M[g]$. Let $\bar{f}$ be a denotation of f in the forcing language.

Let p be any condition, we construct by induction a p-tree A and a decreasing function Q from A into $C(J)$ such that $Q(\emptyset) = p$ and for every s in A the domain of $Q(s)$ is disjoint of the union of $R(s)$ where $R(s)$ is the ramification of A at s.

If a is in $S_\omega(\omega)$ and is disjoint of the domain of p, we put (a) in A. If s is in A, $s = (a_0, ..., a_n)$, and $Q((a_0, ..., a_{n-1}))$ is defined, we let $u_0, ..., u_t$ be the different functions from the union of $a_0, ..., a_n$ into 2. Let $q_0, ..., q_t$ be a decreasing sequence of conditions extending $Q((a_0, ..., a_{n-1}))$ such that the domain of q_i is disjoint of the union of $a_0, ..., a_n$ and (q_i, u_i) decides $\bar{f}(n-1)$. We let $Q(s)$ be q_t and $S_\omega(\omega - (a_0 \cup ... \cup a_n \cup \text{dom}(Q(s))))$ be the ramification of A at s.

We also define $\alpha(s)$ to be the suprenum of the decisions of the q_i's. Clearly A is a J-p-tree. Let H be a J-p-branch of A. Let q be the union of the $Q(H \restriction n)$, n in ω, then q is a condition which extends p and forces f is bounded by α where α is the supremum of the $\alpha(s)$, s in A. A density argument shows that f is bounded below ω_1, hence ω_1 is preserved.

Remark. The above proof shows that if an ordinal has cofinality greater than ω in M then it still has cofinality greater than ω in $M[g]$. Hence, if CH holds in M, cardinalities and cofinalities are preserved.

Getting 4.16 and 6.5 together gives

Theorem 6.6. *Suppose M satisfies* CH *and J is a maximal ideal then*

i) *J is not p-point, C(J) collapse cardinals*

ii) J is p-point but not selective, we get a non-minimal real but we do not collapse cardinals.

iii) J is selective, we get a minimal real and cardinals are preserved.

Remark. CH implies the existence of p-point ideals which are not selective.

Appendix

The two properties of weak T-ideal and T-ideal are the same for countably generated or maximal ideals. However they do not coincide:

Proposition 1. *There is a weak T-ideal which is not a T-ideal.*

Proof. Let x_s, s in Seq(ω), be a family of disjoint subsets of ω.

We define a tree A by the following induction: if s is in A, we put $s*(m)$ in A if and only if m is in x_s.

Let J be the ideal generated by the branches of A and the finite subsets of ω. So an element of J is included, modulo a finite subset, in a finite union of branches of A.

As a branch of A takes at most one point in an x_s, we see that an element of J meets an x_s at a finite number of points. Hence each infinite subset of an x_s is not in J, showing that J is proper and that A is a J-tree.

As every branch of A is in J, J is not a T-ideal.

We now show that it is a selective ideal.

Suppose not, let (x_n), n in ω, be a J-partition whose selectors are all in J.

Note that if a set is in J then there is an infinite selector which is disjoint from it and there is an infinite subset of it which is included in a branch of A.

Using these remarks it is easy to get a family (H_n), n in ω, of distinct branches of A, each meeting infinitely many x_n.

Fix k, consider the sequences $H_n \upharpoonright k$; if infinitely many of them are different than it is easy to get an infinite selector which takes one point to each of these branches, such a selector is not in J, a contradiction; hence the $H_n \upharpoonright k$, n in ω, form a finite set.

We define by induction a strictly increasing sequence of finite sequences s_n with length k_n, n in ω, and a function f on ω such that $H_{f(n)}$ extends s_{n-1} but not s_n and infinitely many of the H_p extend s_n.

Suppose all are defined up to n. The H_p which extends s_n are infinitely many and all different, so there is a k greater than lh(s_n) such that infinitely many of them, but not all, have the same restriction to k. Choose s_{n+1} to be such a common extension and $f(n+1)$ such that $H_{f(n+1)}$ is in the "but not all".

As each H_n meets infinitely many x_m, m in ω, it is easy to construct an infinite selector which takes one point exactly in each range of $H_{f(n)}$, this point being $H_{f(n)}(k)$ for a k greater than k_n; such a selector can not be in J, a contradiction.

Definition 2. An ideal J is Ramsey if for every subset x not in J and every partition f of the pairs of elements of x in two sets there exists an homogeneous subset of x which is not in J.

Proposition 3. *If J is a weak T-ideal then J is Ramsey.*

Proof. We first assume that J is a T-ideal. The proof is just a generalization of the well-known Ramsey's theorem.

Let x and f be as in Def. 2.

We define a J-tree A inductively: $\emptyset$ is in A and the ramification of A at $\emptyset$ is x. If s is in A with length $k+1$, the ramification $x_{s \upharpoonright k}$ of A at $s \upharpoonright k$ is not in J; choose a subset x_s of $x_{s \upharpoonright k}$ such that x_s is not in J and the pairs $\{s(k), m\}$, m in x_s, have the same image by f. Put $s*(m)$ in A if and only if m is in x_s, so that x_s is the ramification of A at s.

Take a J-branch H of A.

For each n, $H(n)$ is such that the pairs $\{H(n), H(n+p)\}$, p in ω, have the same image by f, say $i(n)$.

The set of $H(n)$ such that $i(n) = 0$ and the set of $H(n)$ such that $i(n) = 1$ define a partition of the range of H; one of these two sets, at least, is not in J; it is the desired homogeneous set for f.

To prove the proposition with the hypothesis of weak T-ideal we have to replace A by a strong J-tree.

To do this we first note that there exists a well-ordering of Seq(ω) of order type ω which extends the non-linear inclusion ordering. The isomorphism s from ω onto Seq(ω) which is deduced from this well-ordering is constructed by blocks as follows; the first block is just formed of the empty sequence; if the n first blocks give $s(0), ..., s(k)$ then the $n+1$-st block is $s(0)*(m_0), ..., s(k)*(m_k)$ where m_i is the first integer such that $s(i)*(m_i)$ is different from $s(0), ..., s(k)$.

We now define inductively A and a decreasing sequence $(x(n))$, n in ω, of subsets of ω which are not in J.

The empty sequence $s(0)$ is in A and $x(0)$ is x. If $s(n)$ does not extend an $s(k)$, $k < n$, then we do not put $s(n)$ in A and we let $x(n)$ be $x(n-1)$.

If $s(n)$ extends an $s(k)$, $k < n$, there is a k, $k < n$, and an m such that $s(n) = s(k) * (m)$, we put $s(n)$ in A if and only if m is in $x(k)$. As $x(n-1)$ is not in J there is a subset y of it which is not in J such that all the pairs $\{m, p\}$, p in y, have the same image by f, we take such a subset as $x(n)$.

Clearly if $s(n)$ is in A, the ramification of A at $s(n)$ is $x(n)$, hence A is a strong J-tree. Taking a J-branch, we end the proof as above.

Definition 4. A strong J-partition is a J-partition which has at most one element not in J.

Definition 5. J is a weak selective ideal if for every strong J-partition there is a selector not in J.

Proposition 6. *If J is Ramsey then it is a weak selective ideal.*

Proof. Let (x_n), n in ω, be a strong J-partition. Suppose that just x_0 is not in J. On the complement of x_0, which is not in J since we have a J-partition, we define a two-valued function on the pairs: $f(\{m, n\}) = 0$ if and only if m and n are in the same element of the partition. Clearly an homogeneous set is included in an element of the partition or is a selector. As we have a strong J-partition, an homogeneous set which is not in J is a selector. Hence the proposition.

Lemma 7. J is c.d.s. if and only if for every J-partition there is a set which is not in J and meets each element of the partition on a set in J.

Proposition 8. *J is selective if and only if it is c.d.s. and weak selective.*

Proof. Use Prop. 2.3 to show one implication; transform a J-partition into a strong one, using Lemma 7, to show the other implication.

Proposition 9. *There is a Ramsey ideal which is not c.d.s.*

Proof. Let (x_n), n in ω, be a partition of ω in disjoint infinite sets. Let J be the set of subsets of ω which meet each x_n at a finite number of points.

If x is not in J then x meets an x_n on an infinite set. If f is a two-partition of the pairs of x, applying the Ramsey's theorem, there is an infinite homogeneous set included in the intersection of x and x_n. Such a set is not in J. Hence J is a Ramsey ideal. Obviously J is not c.d.s.

Proposition 10. *There is a c.d.s. ideal which is not weak selective.*

Proof. Let (x_n), n in ω, be a partition of ω in disjoint infinite subsets. Let J be the set of subsets of ω which have an infinite intersection with only a finite number of x_n.

Clearly J is an ideal which is not weak selective since (x_n), n in ω, is a strong J-partition. In fact it is not weak p-point.

Now show that J is c.d.s. Let (X_p), p in ω, be a J-partition. If the union of the X_p which are in J is not in J then this union is the desired set of Lemma 7. So we suppose that no X_p is in J. Thus each X_p meets infinitely many x_n on an infinite set, it is then easy to get a set whose intersections with the X_p are infinite subsets of different x_n. Such a set is not in J and meets the X_p on sets in J, hence it is the desired one.

Definition 11. J is a very weak T-ideal if for every x not in J there is a J-branch for every tree whose ramifications differ from x on a set in J.

Definition 12. J is weak inductive if for every decreasing sequence (x_n), n in ω, of subsets of ω which are not in J and such that $x_n - x_{n+1}$ is in J for each n, there exists a strictly increasing function H from ω into ω, with range not in J, such that $H(n+1)$ is in $x_{H(n)}$ for each n.

Definition 13. J is weak Ramsey if for every x not in J and every partition f of the pairs of elements of x into two sets, such that for each n in x either the set of m in x such that $f(\{n, m\}) = 0$ is in J or the set of m in x such that $f(\{n, m\}) = 1$ is in J, there is a homogeneous subset of x which is not in J.

Proposition 14. *If J is an ideal then the following are equivalent:*

i) *J is weak selective*
ii) *J is weak inductive*
iii) *J is a very weak T-ideal*
iv) *J is weak Ramsey*

Proof. To show that i), ii), iii) are equivalent it suffices to repeat the proofs of Prop. 1.10, 1.12, 1.13 and 1.14.

The proof that iii) implies iv) is just that given in Prop. 3 for T-ideals.

It is obvious that iv) implies ii).

Proposition 15. *There is a weak selective ideal which is not Ramsey.*

Proof. Let (x_s), s in Seq(2) (the two-valued finite sequences), be a family of infinite subsets of ω such that $x_\emptyset$ is ω and $x_{s*(0)}$ and $x_{s*(1)}$ are disjoint with union x_s for each s in Seq(2).

Define J as follows: x is in J if and only if the set of s such that the intersection of x and x_s is finite is dense in Seq(2) (w.r.t. the inclusion ordering). Clearly J is a proper ideal and the x_s are not in J.

Let y_n be the union of the $x_{s*(0)}$, s with length n.

Define a partition f of the pairs of integers: $f(\{n, n+p\}) = 0$ if and only if $n+p$ is in y_n.

A homogeneous set for f is included or disjoint, modulo a finite set, of infinitely many y_n, hence it is in J. So J is not Ramsey.

We now show that it is weak selective. Let (X_i), i in ω, be a strong J-partition. If each X_i is in J then each x_s meets infinitely many X_i since it is not in J, so it is easy to get a selector x which meets each x_s, such an x has in fact an infinite intersection with each x_s hence it is not in J. Note that if a set y is not in J then there is an s such that for every extension t of s the intersection of y and x_t is infinite, hence for every extension t of s the intersection of y and x_t is not in J. So if we consider a strong partition (X_i) whose only element not in J is X_0 there is an s such that for every extension t of s the intersection of $\omega - X_0$ and x_t is not in J and so meets infinitely many X_i, hence a selector which meets each x_t, t extending s, hence which has an infinite intersection with each x_t, t extending s, and so is not in J. Thus J is weak selective.

Corollary 16. *The following implications can not be reversed:*

i) *J is a T-ideal*

implies ii) *J is a weak T-ideal*

J is selective

J is inductive

J is Ramsey and c.d.s.

J is weak selective and c.d.s.

implies iii) *J is Ramsey*
implies iv) *J is a very weak T-ideal*
J is weak selective
J is weak inductive
J is weak Ramsey.

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