Finite versus infinite: an intricate shift

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Directed graphs

- A *directed graph* is a pair $\mathbf{D} = (X, D)$ where D is an irreflexive binary relation on X.
- A *homomorphism* from (X, D) to (X', D') is a map $h : X \to X'$ such that $x \, D \, y$ implies $h(x) \, D' \, h(y)$.
- A *coloring* of a directed graph (X, D) is a map $c : X \to Y$ such that xDx' implies $c(x) \neq c(x')$.
- The *chromatic number* of $\mathbf{D} = (X, D)$, $\chi(\mathbf{D})$, is the smallest cardinality of a set Y s.t. there exists a coloring $c : X \to Y$.

Borel chromatic number

- If X is a Polish space, the *Borel chromatic number*, $\chi_B(\mathbf{D})$, of $\mathbf{D} = (X, D)$ is the smallest cardinality of a Polish space Y such that there exists a Borel coloring $c : X \to Y$.
- Write $(X, D) \preceq (X', D')$, $(\preceq_c, \preceq_B)$ if there exists a (continuous, Borel) homomorphism from (X, D) to (X', D').

Remark:

1. $\chi_B(\mathbf{D}) \in \{1, 2, 3, \ldots, \aleph_0, 2^{\aleph_0}\}$.
2. if $(X, D) \preceq_B (X', D')$ then $\chi_B(X, D) \leqslant \chi_B(X', D')$.

Theorem (Kechris–Solecki–Todorcevic, 96)

There is a graph $\mathbf{G}_0$ on 2^ω s.t. for every analytic graph $\mathbf{G} = (X, G)$ on a Polish space X, exactly one of the following holds:

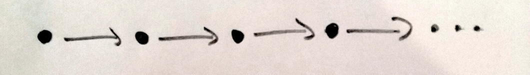
1. $\chi_B(\mathbf{G}) \leqslant \aleph_0$,
2. $\mathbf{G}_0 \preceq_c \mathbf{G}$ *(and therefore $\chi_B(\mathbf{G}) = 2^{\aleph_0}$).*

See the survey on *Descriptive graph combinatorics* by Marks and Kechris.

Graphs generated by a function

For any function $f : X \to X$, let (X, f) denote the directed graph whose arrows are given by: $x \; D_f \; y \leftrightarrow x \neq y$ and $f(x) = y$.

Example: For $\mathrm{s} : \omega \to \omega$, $n \mapsto n + 1$,

$(\omega, \mathrm{s}) =$ 

Theorem (Kechris–Solecki–Todorcevic, Miller)

Let $f : X \to X$ be a Borel function with no fixed point. Then the following are equivalent:

1. $\chi_B(X, f) \leqslant 3$,
2. $\chi_B(X, f)$ *is finite,*
3. *there exists a Borel subset B of X such that*

$$\forall x \in X \; (\exists m \in \omega \; f^m(x) \in B \text{ and } \exists n \in \omega \; f^n(x) \notin B).$$

Finite vs Infinite: The shift graph

Let $[\omega]^\infty$ be the space of infinite subsets of ω. As a subspace of 2^ω it is Polish and homeomorphic to ω^ω. The *shift map* is defined by

$$\begin{aligned} \mathsf{S} : [\omega]^\infty &\longmapsto [\omega]^\infty \\ X &\longrightarrow X \setminus \{\min X\} \end{aligned}$$

The *Shift Graph* is the directed graph $\mathcal{G}_\mathsf{S} = ([\omega]^\infty, \mathsf{S})$.

- As $\mathcal{G}_\mathsf{S}$ is acyclic, we have $\chi(\mathcal{G}_\mathsf{S}) = 2$.
- The Galvin–Prikry theorem: for every finite Borel coloring of $[\omega]^\infty$ there exists an infinite $X \subseteq \omega$ such that $[X]^\infty$ is monochromatic. In particular X and $\mathsf{S}(X)$ have same color. Hence $\chi_B(\mathcal{G}_\mathsf{S})$ is infinite. But $c : [\omega]^\infty \to \omega$, $X \mapsto \min X$ is a continuous coloring, so

$$\chi_B(\mathcal{G}_\mathsf{S}) = \omega.$$

Finite vs infinite

(Kechris–Solecki–Todorcevic, 1996) Is the following true?

If X is a Polish space and $f : X \to X$ is a Borel function, then exactly one of the following holds:

1. The Borel chromatic number of (X, f) is finite;
2. $\mathcal{G}_{\mathsf{S}} \preceq_c (X, f)$.

The answer is negative.

There exists a Polish space X and a continuous function $f : X \to X$ such that $\chi_B(X, f) = \aleph_0$ and there is no Borel homomorphism from $\mathcal{G}_{\mathsf{S}}$ to (X, f).

- However no specific example is known.
- This result follows from a representation theorem for $\mathbf{\Sigma}^1_2$ sets.
- Actually no basis result at all (Todorčević-Vidnyánszky).

Representation of analytic sets

Let $\mathbb{G} = 2^\omega$ be the Polish space of (codes for) countable directed graphs, where $\alpha \in 2^\omega$ codes (X_α, D_α) given by

$X_\alpha = \{n \mid \alpha(\langle n, n\rangle) = 0\}$, and $m\ D_\alpha\ n \leftrightarrow \alpha(\langle m, n\rangle) = 1$, $m, n \in X_\alpha$.

Recall that a subset $A \subseteq \omega^\omega$ is *analytic* ($\mathbf{\Sigma}^1_1$) if there exists a closed subset C of $\omega^\omega \times \omega^\omega$ s.t. $\alpha \in A \leftrightarrow \exists\beta \in \omega^\omega\ (\alpha, \beta) \in C$.

Proposition (Folklore)

A subset A of ω^ω is $\mathbf{\Sigma}^1_1$ iff there exists a continuous function $\omega^\omega \to \mathbb{G}$, $\alpha \mapsto \mathbf{G}_\alpha$ such that

$$\alpha \in A \quad \longleftrightarrow \quad (\omega, <) \preceq \mathbf{G}_\alpha$$

Proof sketch: Let $T = \{(x\restriction_n, y\restriction_n) \mid (x, y) \in C \text{ and } n \in \omega\}$ and set $T(\alpha) = \{s \in \omega^{<\omega} \mid (\alpha\restriction_s, s) \in T\}$. We have

$$\begin{aligned} \alpha \in A \quad &\longleftrightarrow \quad \exists\beta \in \omega^\omega\ \forall n\ (\alpha\restriction_n, \beta\restriction_n) \in T \\ &\longleftrightarrow \quad \exists\beta \in \omega^\omega\ \forall n\ \beta\restriction_n \in T(\alpha) \\ &\longleftrightarrow \quad (\omega, <) \preceq (T(\alpha), \sqsubset) = \mathbf{G}_\alpha. \end{aligned}$$

Representation of $\mathbf{\Sigma}^1_2$ sets

Recall that a subset $P \subseteq \omega^\omega$ is $\mathbf{\Sigma}^1_2$ if there exists a closed subset C of $\omega^\omega \times \omega^\omega \times \omega^\omega$ such that

$$\alpha \in P \quad \longleftrightarrow \quad \exists \beta \in \omega^\omega \; \forall \gamma \in \omega^\omega \; (\alpha, \beta, \gamma) \notin C.$$

Theorem (Marcone, 1995)

A subset $P \subseteq \omega^\omega$ is $\mathbf{\Sigma}^1_2$ iff there exists a continuous function $\omega^\omega \to \mathbb{G}$, $\alpha \mapsto \mathbf{G}_\alpha$ such that

$$\alpha \in P \quad \longleftrightarrow \quad \mathcal{G}_\mathsf{S} \preceq_c \mathbf{G}_\alpha.$$

Here a countable graph $\mathbf{G} \in \mathbb{G}$ is considered with the discrete topology. Notice that

$$(\omega, <) \preceq \mathbf{G} \text{ implies } \mathcal{G}_\mathsf{S} \preceq_c \mathbf{G}.$$

A $\mathbf{\Pi}^1_2$ complete set

Corollary

$\{\mathbf{G} \in \mathbb{G} \mid \mathcal{G}_S \preceq_c \mathbf{G}\}$ is a $\mathbf{\Sigma}^1_2$ non $\mathbf{\Pi}^1_2$ subset of $\mathbb{G}$.

Proof.

It is not hard to give a $\mathbf{\Sigma}^1_2$ definition. Suppose it is also $\mathbf{\Pi}^1_2$. As $\mathbf{\Pi}^1_2$ is closed under continuous preimages, the representation theorem implies that $\mathbf{\Sigma}^1_2 \subseteq \mathbf{\Pi}^1_2$. This would contradict the existence of a universal $\mathbf{\Sigma}^1_2$ set. □

Definition

A countable directed graph $\mathbf{G} \in \mathbb{G}$ is **better** if $\mathcal{G}_S \not\preceq_c \mathbf{G}$ when the vertex set is considered with the discrete topology.

The set

$$\mathrm{BG} = \{\mathbf{G} \in \mathbb{G} \mid \mathcal{G}_S \not\preceq_c \mathbf{G}\}$$

of better graphs is a $\mathbf{\Pi}^1_2$-complete set.

Shift on rays of a countable directed graph

Let $\mathbf{G} = (X, D)$ be directed graph on $X \subseteq \omega$.
The *Ray Graph* of $\mathbf{G}$ is the graph $(\vec{\mathbf{G}}, \mathsf{S})$ where:

$$\vec{\mathbf{G}} = \{(n_i)_{i\in\omega} \in \omega^\omega \mid \forall i\ n_i\ D\ n_{i+1}\} \qquad \text{(closed subset of } \omega^\omega)$$

and $\mathsf{S} : \vec{\mathbf{G}} \to \vec{\mathbf{G}}$ is the shift map given by $\mathsf{S}((n_i)_{i\in\omega}) = (n_{i+1})_{i\in\omega}$.

Lemma (1)

For every $\mathbf{G} \in \mathbb{G}$:

$$\mathcal{G}_{\mathsf{S}} \preceq_c \mathbf{G} \quad \longleftrightarrow \quad \mathcal{G}_{\mathsf{S}} \preceq_c \vec{\mathbf{G}}.$$

Moreover, the map $\mathbb{G} \to \mathcal{F}(\omega^\omega)$, $\mathbf{G} \mapsto \vec{\mathbf{G}}$ is $\mathbf{\Delta}^1_2$-measurable.

Where $\mathcal{F}(\omega^\omega)$ is the Effros Borel space of closed subsets of ω^ω whose Borel sets are generated by the sets of the form

$$\{F \in \mathcal{F}(\omega^\omega) \mid F \cap N_s \neq \emptyset\}$$

where $N_s = \{\alpha \in \omega^\omega \mid s \sqsubseteq \alpha\}$, $s \in \omega^{<\omega}$.

A very *discrete* graph

Recall: $\mathrm{BG} = \{\mathbf{G} \in \mathbb{G} \mid \mathcal{G}_{\mathrm{S}} \not\preceq_c \mathbf{G}\}$ is $\mathbf{\Pi}^1_2$ and not $\mathbf{\Sigma}^1_2$.

Lemma (2)

The set $\mathrm{F} = \{F \in \mathcal{F}(\omega^\omega) \mid \chi_B(F, \mathrm{S}) < \aleph_0\}$ *is* $\mathbf{\Sigma}^1_2$.

Theorem

There exists $\mathbf{G} \in \mathbb{G}$ *such that*

$$\chi_B(\vec{\mathbf{G}}, \mathrm{S}) = \aleph_0 \quad \textit{and} \quad \mathcal{G}_{\mathrm{S}} \not\preceq_c (\vec{\mathbf{G}}, \mathrm{S}).$$

Proof.

- The set $\tilde{\mathrm{F}} = \{\mathbf{G} \in \mathbb{G} \mid \chi_B(\vec{\mathbf{G}}) < \aleph_0\} = \{\mathbf{G} \in \mathbb{G} \mid \vec{\mathbf{G}} \in F\}$ is $\mathbf{\Sigma}^1_2$.
- Moreover $\tilde{\mathrm{F}} \subseteq \mathrm{BG}$: for if $\mathcal{G}_{\mathrm{S}} \preceq_c \mathbf{G}$, then by Lemma (1) $\mathcal{G}_{\mathrm{S}} \preceq_c \vec{\mathbf{G}}$ and so $\aleph_0 = \chi_B(\mathcal{G}_{\mathrm{S}}) \leqslant \chi_B(\vec{\mathbf{G}})$.
- Since BG is not $\mathbf{\Sigma}^1_2$, we cannot have $\tilde{\mathrm{F}} = \mathrm{BG}$. Hence there exists $\mathbf{G}$ with $\mathbf{G} \in \mathrm{BG}$ and $\mathbf{G} \notin \tilde{\mathrm{F}}$. Such a $\mathbf{G}$ is as desired. □

In search for a specific example

Consider the set $2^{<\omega}$ of finite binary words equipped with the subword ordering, i.e.

$$u \preccurlyeq v \quad \longleftrightarrow \quad \begin{array}{l}\text{there exists a strictly increasing map } h : |u| \to |v| \\ \text{such that for every } i < |u| \text{ we have } u(i) = v(h(i)),\end{array}$$

where $|u|$ denotes the length of $u \in 2^{<\omega}$. E.g. $01 \preccurlyeq 100100$.

- Let $\mathbf{H} = (2^{<\omega}, H)$ where $u \; H \; v \leftrightarrow u \not\preccurlyeq v$.
- Since $(2^{<\omega}, \preccurlyeq)$ is a *better-quasi-order*, so $\mathcal{G}_S \not\preceq_c (2^{<\omega}, H)$ and so $\mathcal{G}_S \not\preceq_c \vec{H}$.

Question

What is the Borel chromatic number of $\vec{\mathbf{H}}$? Is it $\aleph_0$?

Remark: there is no **continuous** 2-coloring of $\vec{\mathbf{H}}$.

One conjecture for the road

Conjecture

The Borel chromatic number of $\mathcal{G}_{\mathsf{S}}$ is *effectively* infinite: for all $n \in \omega$ and all Δ^1_1 map $c : [\omega]^\infty \to n$ there exists a Δ^1_1 point $X \in [\omega]^\infty$ with $c(X) = c(\mathsf{S}(X))$.