Colorful well-foundedness

Yann Pequignot

University of California, Los Angeles

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Part I
Well-quasi-orders and Better-quasi-orders

Well-quasi-orders

A **quasi-order** (qo) is a set Q together with
a *reflexive* and *transitive* binary relation $\leqslant$.

Definition

A **well-quasi-order** (wqo) is a qo that satisfies one of the following equivalent conditions.

1. Q is well-founded and has no infinite antichain;
2. there exists no bad sequence, i.e. no $(q_n)_n$ s.t.

$$\forall m, n \in \omega \quad m < n \to q_m \not\leqslant q_n.$$

3. $\mathcal{P}(Q)$ is well-founded, under:

$$X \leqslant Y \quad \longleftrightarrow \quad \forall x \in X \; \exists y \in Y \; x \leqslant y.$$

Well-quasi-orders

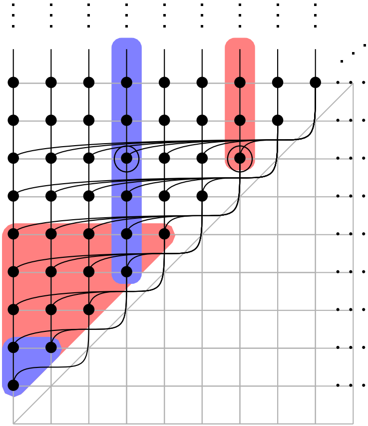
Examples of wqos

- Finite quasi-orders
- Well-orders
- If P and Q are wqo, then $P \times Q$ is wqo.
- (Higman 52') If P is wqo, then $P^{<\omega}$ is wqo under

$$(p_i)_{i<n} \leqslant (q_j)_{j<m} \quad \longleftrightarrow \quad \begin{array}{l} \exists f : n \to m \text{ strictly increasing} \\ \text{s.t. } p_i \leqslant q_{f(i)} \text{ for all } i < n \end{array}$$

- (Laver 71') Countable linear orders under embeddability.
- (Robertson-Seymour, 500 pages, 1983-2004) The finite undirected graphs under the minor relation.

A wqo Q such that $\mathcal{P}(Q)$ is not wqo



Rado's poset $\mathcal{R}$

Richard Rado, 1954.

$\mathcal{R}$ is defined on $[\omega]^2$ by:

$$\{m_0, m_1\} \leqslant \{n_0, n_1\}$$

iff

$$\begin{cases} m_0 = n_0 \text{ and } m_1 \leqslant n_1, \text{ or} \\ m_0 < m_1 < n_0 < n_1 \end{cases}$$

Better quasi-orders

Fix a quasi-order Q and treat the element of Q as *atoms*, namely they have no elements but they are different from the empty set. We define by transfinite recursion:

$$\begin{aligned} Q_0^* &= Q \\ Q_{\alpha+1}^* &= \mathcal{P}^*(Q_\alpha^*) \quad \text{(the non-empty subsets of } V_\alpha^*\text{)} \\ Q_\lambda^* &= \bigcup_{\alpha<\lambda} Q_\alpha^*, \quad \text{for } \lambda \text{ limit.} \end{aligned}$$

Let

$$Q^* = \bigcup_\alpha Q_\alpha^*.$$

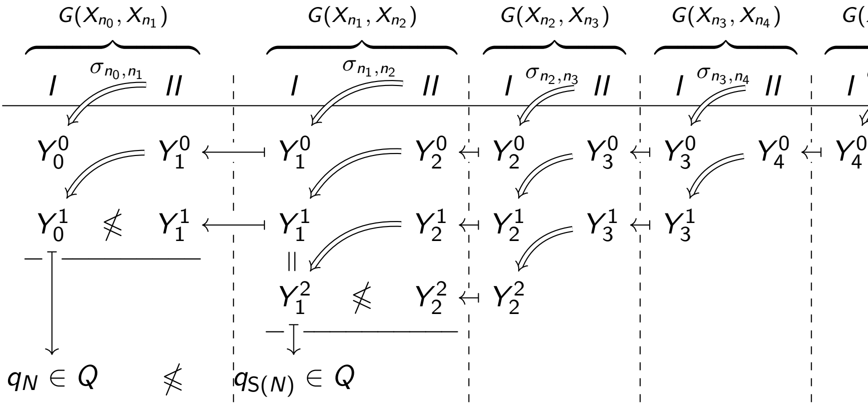
We define a quasi-order on Q^* via the existence of a winning strategy in a suitable game $G(X, Y)$.

Definition (Intuitive definition)

A quasi-order Q is a better-quasi-order if Q^* is well-founded.

Making sense of the definition

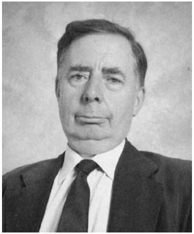
Let $(X_n)_n$ be a sequence in Q^* such that $m < n$ implies $X_n \nleq X_m$.
For every infinite subset $N = \{n_0, n_1, n_2, \ldots\}$ of ω, contemplate:



Where $S(N) = N \setminus \{\min N\}$. This defines a map $f : [\omega]^\infty \to Q$,
$f(N) = Q_N$, such that $f(N) \nleq f(S(N))$.

A working definition for better-quasi-orders

$[\omega]^\infty$ is a Polish space homeomorphic to ω^ω. Considering Q with the discrete topology, we just proved:



Proposition

If Q^* is ill-founded, then there exists a continuous map $f : [\omega]^\infty \to Q$ s.t. $f(N) \not\leqslant f(\mathsf{S}(N))$ for every N.

And in fact, this is an equivalence. So we get:

Definition (Working definition, Nash-Williams 65')

A quasi-order Q is a *better-quasi-order* (bqo) if for every continuous map $f : [\omega]^\infty \to Q$ there exists $N \in [\omega]^\infty$ such that $f(N) \leqslant f(\mathsf{S}(N))$.

Examples of better-quasi-orders

Theorem (Nash-Williams, Galvin-Prikry)

For every finite partition $[\omega]^\infty = B_0 \cup \cdots \cup B_n$ *into Borel sets, there exists an infinite* $X \subseteq \omega$ *such that* $[X]^\infty \subseteq B_i$ *for some* $i \in \{0, \ldots, n\}$.

Examples of bqos

- Finite quasi-orders
- Well-orders
- If P and Q are bqo, then $P \times Q$ is bqo.
- (Nash-Williams) If P is bqo, then P^ω is bqo under

$$(p_i)_{i\in\omega} \leqslant (q_j)_{j\in\omega} \quad \longleftrightarrow \quad \begin{array}{l} \exists f : \omega \to \omega \text{ strictly increasing} \\ \text{s.t. } p_i \leqslant q_{f(i)} \text{ for all } i \in \omega \end{array}$$

- (Laver 71') Countable linear orders under embeddability.
- *every "naturally occuring" wqo.*

Part II

Infinite vs. infinite Borel chromatic number

Directed graphs

- A *directed graph* is a pair $\mathbf{D} = (X, D)$ where D is an irreflexive binary relation on X.
- A *homomorphism* from (X, D) to (X', D') is a map $h : X \to X'$ such that $x \, D \, y$ implies $h(x) \, D' \, h(y)$.
- A *coloring* of a directed graph (X, D) is a map $c : X \to Y$ such that xDx' implies $c(x) \neq c(x')$.
- The *chromatic number* of $\mathbf{D} = (X, D)$, $\chi(\mathbf{D})$, is the smallest cardinality of a set Y s.t. there exists a coloring $c : X \to Y$.

Borel chromatic number

- If X is a Polish space, the *Borel chromatic number*, $\chi_B(\mathbf{D})$, of $\mathbf{D} = (X, D)$ is the smallest cardinality of a Polish space Y such that there exists a Borel coloring $c : X \to Y$.
- Write $(X, D) \preceq (X', D')$, $(\preceq_c, \preceq_B)$ if there exists a (continuous, Borel) homomorphism from (X, D) to (X', D').

Remark:

1. $\chi_B(\mathbf{D}) \in \{1, 2, 3, \ldots, \aleph_0, 2^{\aleph_0}\}$.
2. if $(X, D) \preceq_B (X', D')$ then $\chi_B(X, D) \leqslant \chi_B(X', D')$.

Theorem (Kechris–Solecki–Todorcevic, 96)

There is a graph $\mathbf{G}_0$ on 2^ω s.t. for every analytic graph $\mathbf{G} = (X, G)$ on a Polish space X, exactly one of the following holds:

1. $\chi_B(\mathbf{G}) \leqslant \aleph_0$,
2. $\mathbf{G}_0 \preceq_c \mathbf{G}$ *(and therefore $\chi_B(\mathbf{G}) = 2^{\aleph_0}$).*

Graphs generated by a function

For any function $f : X \to X$, let (X, f) denote the directed graph whose arrows are given by:

$$x \; D_f \; y \leftrightarrow x \neq y \text{ and } f(x) = y.$$

Remark: If X is Polish and f is Borel, then $\chi_B(X, f) \leqslant \aleph_0$.

Theorem (Kechris–Solecki–Todorcevic, Miller)

Let $f : X \to X$ be a Borel function with no fixed point. Then the following are equivalent:

1. $\chi_B(X, f) \leqslant 3$,
2. $\chi_B(X, f)$ *is finite,*
3. *there exists a Borel subset B of X such that*

$$\forall x \in X \; (\exists^\infty m \in \omega \; f^m(x) \in B \text{ and } \exists^\infty n \in \omega \; f^n(x) \notin B).$$

Finite vs Infinite: The shift graph (again :-)

Let $[\omega]^\infty$ be the space of infinite subsets of ω. As a subspace of 2^ω it is Polish and homeomorphic to ω^ω. The *shift map* is defined by

$$\begin{aligned} \mathsf{S} : [\omega]^\infty &\longmapsto [\omega]^\infty \\ X &\longrightarrow X \setminus \{\min X\} \end{aligned}$$

The *Shift Graph* is the directed graph $\mathcal{G}_\mathsf{S} = ([\omega]^\infty, \mathsf{S})$.

- As $\mathcal{G}_\mathsf{S}$ is acyclic, we have $\chi(\mathcal{G}_\mathsf{S}) = 2$ (Axiom of choice :-)
- The Galvin–Prikry theorem: for every finite Borel coloring of $[\omega]^\infty$ there exists an infinite $X \subseteq \omega$ such that $[X]^\infty$ is monochromatic. In particular X and $\mathsf{S}(X)$ have same color. Hence $\chi_B(\mathcal{G}_\mathsf{S})$ is infinite. But $c : [\omega]^\infty \to \omega$, $X \mapsto \min X$ is a continuous coloring, so

$$\chi_B(\mathcal{G}_\mathsf{S}) = \aleph_0.$$

Finite vs infinite

(Kechris–Solecki–Todorcevic, 1996) Is the following true?

If X is a Polish space and $f : X \to X$ is a Borel function, then exactly one of the following holds:

1. The Borel chromatic number of (X, f) is finite;
2. $\mathcal{G}_{\mathsf{S}} \preceq_c (X, f)$.

The answer is negative.

Finite vs infinite

Theorem (P)

There exists a closed subset C of $[\omega]^\infty$ such that

- $X \in C$ *implies* $\mathsf{S}(X) \in C$,
- *the Borel chromatic number of* (C, S) *is infinite,*
- *there is no Borel homomorphism from* $\mathcal{G}_{\mathsf{S}}$ *to* (C, S).

- However no "natural" example is known.
- The proof consists of showing that the collection of closed sets as above is a true $\mathbf{\Pi}^1_2$ set, hence non empty.
- It relies on a representation theorem for $\mathbf{\Sigma}^1_2$ sets.

Actually there is no basis result at all since:

Theorem (Todorčević,Vidnyánszky)

The set of codes for closed subsets C of $[\omega]^\infty$ for which (C, S) has finite Borel chromatic number is $\mathbf{\Sigma}^1_2$-complete.

Representation of analytic sets

Let $\mathbb{G} = 2^\omega$ be the Polish space of (codes for) countable directed graphs, where $\alpha \in 2^\omega$ codes (X_α, D_α) given by

$$X_\alpha = \{n \mid \alpha(\langle n, n\rangle) = 0\}, \text{ and}$$
$$m\ D_\alpha\ n \leftrightarrow \alpha(\langle m, n\rangle) = 1 \text{ and } m, n \in X_\alpha.$$

Proposition (Folklore)

A subset A of ω^ω is $\mathbf{\Sigma}^1_1$ iff there exists a continuous function $\omega^\omega \to \mathbb{G}$, $\alpha \mapsto \mathbf{G}_\alpha$ such that

$$\alpha \in A \quad \longleftrightarrow \quad (\omega, <) \preceq \mathbf{G}_\alpha$$

Proof sketch: Let $T = \{(x\restriction_n, y\restriction_n) \mid (x, y) \in C \text{ and } n \in \omega\}$ and set $T(\alpha) = \{s \in \omega^{<\omega} \mid (\alpha\restriction_s, s) \in T\}$. We have

$$\begin{aligned}
\alpha \in A \quad &\longleftrightarrow \quad \exists \beta \in \omega^\omega\ \forall n\ (\alpha\restriction_n, \beta\restriction_n) \in T \\
&\longleftrightarrow \quad \exists \beta \in \omega^\omega\ \forall n\ \beta\restriction_n \in T(\alpha) \\
&\longleftrightarrow \quad (\omega, <) \preceq (T(\alpha), \sqsubset) = \mathbf{G}_\alpha.
\end{aligned}$$

Representation of $\mathbf{\Sigma}^1_2$ sets

Recall that a subset $P \subseteq \omega^\omega$ is $\mathbf{\Sigma}^1_2$ if there exists a closed subset C of $\omega^\omega \times \omega^\omega \times \omega^\omega$ such that

$$\alpha \in P \quad \longleftrightarrow \quad \exists \beta \in \omega^\omega \ \forall \gamma \in \omega^\omega \ (\alpha, \beta, \gamma) \notin C.$$

Theorem (Marcone, 95')

A subset $P \subseteq \omega^\omega$ is $\mathbf{\Sigma}^1_2$ iff there exists a continuous function $\omega^\omega \to \mathbb{G}$, $\alpha \mapsto \mathbf{G}_\alpha$ such that

$$\alpha \in P \quad \longleftrightarrow \quad \mathcal{G}_{\mathsf{S}} \preceq_c \mathbf{G}_\alpha.$$

Again any $\mathbf{G} \in \mathbb{G}$ is considered with the discrete topology.

A $\mathbf{\Pi}^1_2$ complete set

Corollary

$\{\mathbf{G} \in \mathbb{G} \mid \mathcal{G}_\mathsf{S} \preceq_c \mathbf{G}\}$ is a $\mathbf{\Sigma}^1_2$ non $\mathbf{\Pi}^1_2$ subset of $\mathbb{G}$.

Proof.

- It is not too hard to give a $\mathbf{\Sigma}^1_2$ definition.
- Suppose it is also $\mathbf{\Pi}^1_2$. As $\mathbf{\Pi}^1_2$ is closed under continuous preimages, the representation theorem implies that $\mathbf{\Sigma}^1_2 \subseteq \mathbf{\Pi}^1_2$. This would contradict the existence of a universal $\mathbf{\Sigma}^1_2$ set. $\square$

Definition

A countable directed graph $\mathbf{G} \in \mathbb{G}$ is **better** if $\mathcal{G}_\mathsf{S} \not\preceq_c \mathbf{G}$ when the vertex set is considered with the discrete topology.

The set

$$\mathrm{BG} = \{\mathbf{G} \in \mathbb{G} \mid \mathcal{G}_\mathsf{S} \not\preceq_c \mathbf{G}\}$$

of better graphs is a $\mathbf{\Pi}^1_2$-complete set. In particular, not $\mathbf{\Sigma}^1_2$.

Shift on rays of a countable directed graph

Let $\mathbf{G} = (X, D)$ be directed graph on $X \subseteq \omega$.
Define the *Ray Graph* of $\mathbf{G}$ as the directed graph $(\vec{\mathbf{G}}, \mathsf{S})$ where:

$$\vec{\mathbf{G}} = \{(n_i)_{i \in \omega} \in X^\omega \mid \forall i \in \omega \ n_i \ D \ n_{i+1}\}$$

and the shift map $\mathsf{S} : \vec{\mathbf{G}} \to \vec{\mathbf{G}}$ given by

$$\mathsf{S}((n_i)_{i \in \omega}) = (n_{i+1})_{i \in \omega}.$$

- If $\mathbf{G} = (\omega, <)$, then $\vec{\mathbf{G}} = [\omega]^\infty$.
- If $\mathbf{G} = (\omega, s)$, $s : n \mapsto n+1$, then $\vec{\mathbf{G}} = \{\omega \smallsetminus n \mid n \in \omega\}$.

Proposition

For every $\mathbf{G} \in \mathbb{G}$:

$$\begin{aligned} \mathcal{G}_{\mathsf{S}} \preceq_c \mathbf{G} \quad &\longleftrightarrow \quad \mathcal{G}_{\mathsf{S}} \preceq_c (\vec{\mathbf{G}}, \mathsf{S}), \\ &\longleftrightarrow \quad \mathcal{G}_{\mathsf{S}} \preceq_B (\vec{\mathbf{G}}, \mathsf{S}). \end{aligned}$$

A very *discrete* graph

Recall that $\mathrm{BG} = \{\mathbf{G} \in \mathbb{G} \mid \mathcal{G}_\mathsf{S} \not\preceq_c \mathbf{G}\}$ is $\mathbf{\Pi}^1_2$-complete.

Theorem

There exists $\mathbf{G} \in \mathbb{G}$ *such that*

$$\chi_B(\vec{\mathbf{G}}, \mathsf{S}) = \aleph_0 \quad \textit{and} \quad \mathcal{G}_\mathsf{S} \not\preceq_B (\vec{\mathbf{G}}, \mathsf{S}).$$

Sketch of the proof.

- Prove that the set $\tilde{\mathsf{F}} = \{\mathbf{G} \in \mathbb{G} \mid \chi_B(\vec{\mathbf{G}}) < \aleph_0\}$ is $\mathbf{\Sigma}^1_2$.
- Notice that $\tilde{\mathsf{F}} \subseteq \mathrm{BG}$: for if $\mathcal{G}_\mathsf{S} \preceq_c \mathbf{G}$, then $\mathcal{G}_\mathsf{S} \preceq_c \vec{\mathbf{G}}$ and so $\aleph_0 = \chi_B(\mathcal{G}_\mathsf{S}) \leqslant \chi_B(\vec{\mathbf{G}})$.
- Since BG is not $\mathbf{\Sigma}^1_2$, we cannot have $\tilde{\mathsf{F}} = \mathrm{BG}$. Hence there exists $\mathbf{G}$ with $\mathbf{G} \in \mathrm{BG}$ and $\mathbf{G} \notin \tilde{\mathsf{F}}$. Such a $\mathbf{G}$ is as desired. $\square$

Part III
Ordering functions

Ordering functions

- One way to understand objects consists of ordering them.
- For sets $A, B \subseteq \omega^\omega$, continuous reducibility (Wadge qo):

$$A \leqslant_W B \quad \longleftrightarrow \quad \begin{array}{l} \exists f : \omega^\omega \to \omega^\omega \text{ continuous such that} \\ \forall x \in \omega^\omega (x \in A \leftrightarrow f(x) \in B). \end{array}$$

- For equivalence relations E, F on ω^ω, Borel reducibility:

$$E \leqslant_B F \quad \longleftrightarrow \quad \begin{array}{l} \exists f : \omega^\omega \to \omega^\omega \text{ Borel such that} \\ \forall x, y \in \omega^\omega (x \, E \, y \leftrightarrow f(x) \, F \, f(y)). \end{array}$$

- What about functions?

All spaces considered are Polish zero-dimensional spaces, denoted by variables X, Y,...

Continuous reducibility on functions

Definition (Hertling-Weihrauch, Carroy)

Say that $f : X \to Y$ *reduces to* $g : X' \to Y'$ if there are $\sigma : X \to X'$ continuous and $\tau : \mathrm{im}(g \circ \sigma) \to Y$ continuous such that $f = \tau \circ g \circ \sigma$.

$$
\begin{tikzcd}
X' \arrow[r, "g"] \arrow[dr, phantom, "\rotatebox{-90}{$\leq$}"] & Y' \arrow[d, "\tau"] \\
X \arrow[u, "\sigma"] \arrow[r, "f"'] & Y
\end{tikzcd}
$$

Theorem (Carroy, 2012)

Continuous reducibility is a well-order on continuous functions with compact domains.

Conjecture (Carroy)

Continuous reducibility is a wqo on continuous functions.

Topological embeddability on functions

Definition

Say that $f : X \to Y$ *embeds into* $g : X' \to Y'$ if there are embeddings $\sigma : X \to X'$ and $\tau : \operatorname{im} f \to Y'$ such that $\tau \circ f = g \circ \sigma$.

$$\begin{tikzcd} X' \arrow[r, "g"] & Y' \\ X \arrow[u, hook, "\sigma"] \arrow[r, "f"'] \arrow[ur, phantom, "\sqsubseteq"] & Y \arrow[u, hook, "\tau"'] \end{tikzcd}$$

- Embeddability is finer than reducibility: $f \sqsubseteq g \ \to \ f \leqslant q$.
- The projection $p : \omega^\omega \times \omega^\omega \to \omega^\omega$ is a maximum for continuous functions: $f : X \to Y$ is continuous iff $f \sqsubseteq p$.
- The two discontinuous functions

$$\begin{aligned} d_0 : \omega + 1 &\longrightarrow 2 \\ \omega &\longmapsto 0 \\ n &\longmapsto 1 \end{aligned} \qquad\qquad \begin{aligned} d_1 : \omega + 1 &\longrightarrow \omega \\ \omega &\longmapsto 0 \\ n &\longmapsto n+1 \end{aligned}$$

form a 2-element *basis* for discontinuous functions: $f : X \to Y$ is discontinuous iff $d_0 \sqsubseteq f$ or $d_1 \sqsubseteq f$.

Topological embeddability on functions, continued.

Theorem

The following classes admits a minimum under embeddability:

1. *(Solecki, 98') The class of Baire class* 1 *functions that are not* σ*-continuous.*
2. *(Zapletal, 04') The Borel functions that are not* σ*-continuous.*
3. *(Carroy-Miller, 17') The class of Baire class* 1 *functions that are not* F_σ*-to-one.*

Theorem (Carroy-Miller, 17')

The following classes admits a finite basis under embeddability:

1. *The Borel functions that are not in the first Baire class.*
2. *The Borel functions that are not* σ*-continuous with closed witnesses.*

Conjecture, $\alpha > 1$**:**

The Borel functions that are not Baire class α admit a finite basis.

Order and Chaos

For X compact, $C(X, Y)$ denotes the space of continuous functions $X \to Y$ with the topology of uniform convergence.

Proposition (Carroy, P., Vidnyánszky)

If X, Y are Polish and X is compact, then embeddability is an analytic quasi-order on $C(X, Y)$.

An analytic qo Q on a Polish space Z is analytic complete if it Borel reduces every analytic qo on any Polish space.

Theorem (Carroy, P., Vidnyánszky)

Suppose that X, Y are Polish zero-dimensional and X is compact. Then exactly one of the following holds:

1. *embeddability on $C(X, Y)$ is an analytic complete quasi-order,*
2. *embeddability on $C(X, Y)$ is a wqo. In fact, a bqo.*

Moreover 1 holds exactly when X has infinitely many non-isolated points and Y is not discrete. For instance for $C(2^\omega, 2^\omega)$.

Chaos

Let $\mathbb{G}$ denote the Polish space of (simple) undirected graphs with vertex set $\mathbb{N}$.
For $G, H \in \mathbb{G}$ let

$$G \leqslant_i H \quad \longleftrightarrow \quad \text{there is an injective homomorphism from } G \text{ to } H.$$

Theorem (Louveau-Rosendal)

The qo $\leqslant_i$ on $\mathbb{G}$ is an analytic complete quasi-order.

Theorem (Carroy, P., Vidnyánszky)

There is a continuous function $\mathbb{G} \to C(\omega^2 + 1, \omega + 1)$, $G \mapsto f^G$ that reduces $\leqslant_i$ to $\sqsubseteq$:

$$G \leqslant_i H \quad \longleftrightarrow \quad f^G \sqsubseteq f^H.$$

So embeddability on $C(\omega^2 + 1, \omega + 1)$ is an analytic complete qo.

Order

Let $\mathbb{Q}$ be the space of rationals, $(P, \leqslant_P)$ a quasi-order.
Let $P^{\mathbb{Q}}$ be the set of maps $l : \mathbb{Q} \to P$ quasi-ordered by

$$l_0 \leqslant l_1 \quad \longleftrightarrow \quad \begin{array}{l}\text{there is a topological embedding } \tau : \mathbb{Q} \to \mathbb{Q} \\ \text{such that } l_0(q) \leqslant_P l_1(\tau(q)) \text{ for all } q \in \mathbb{Q}.\end{array}$$

Theorem (van Engelen-Miller-Steel)

If P is bqo, then $P^{\mathbb{Q}}$ is bqo.

Theorem (van Engelen-Miller-Steel, Carroy)

The Polish 0-dimensional spaces with embeddability are bqo.

Proposition (Carroy, P., Vidnyánszky)

The locally constant maps are bqo under embeddability.

In search of a specific example

Consider the set $2^{<\omega}$ of finite binary words equipped with the subword ordering, i.e.

$$u \preccurlyeq v \quad \longleftrightarrow \quad \begin{array}{l}\text{there exists a strictly increasing map } h : |u| \to |v| \\ \text{such that for every } i < |u| \text{ we have } u(i) = v(h(i)),\end{array}$$

where $|u|$ denotes the length of $u \in 2^{<\omega}$. E.g. $01 \preccurlyeq 100100$.

- Let $\mathbf{H} = (2^{<\omega}, H)$ where $u\ H\ v \leftrightarrow u \not\preccurlyeq v$.
- Since $(2^{<\omega}, \preccurlyeq)$ is a *better-quasi-order*, so $\mathcal{G}_{\mathsf{S}} \not\preceq_c (2^{<\omega}, H)$ and so $\mathcal{G}_{\mathsf{S}} \not\preceq_c \vec{H}$.

Question

What is the Borel chromatic number of $\vec{\mathbf{H}}$? Is it $\aleph_0$?

Remark: there is no **continuous** 2-coloring of $\vec{\mathbf{H}}$.