Duality and the equational theory of regular languages

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Abstract

On one hand, the Eilenberg variety theorem establishes a bijective correspondence between varieties of formal languages and varieties of finite monoids. On the other hand, the Reiterman theorem states that varieties of finite monoids are exactly the classes of finite monoids definable by profinite equations. Together these two theorems give a structural insight in the algebraic theory of finite automata. We explain how duality theory can account for the combination of this two theorems, as it was pointed out by (Gehrke, Grigorieff, and Pin, 2008).

The theory of formal languages is basically concerned with the description of properties of sequences, which are nothing else than sets of sequences. For any finite non empty set A of symbols called an alphabet, we define a (formal) *language* as a set of finite sequences of symbols in A. In this context, finite sequences are called words and are naturally endowed with the concatenation operation. The set of all words on the alphabet A forms the free monoid A^* for the concatenation with the empty word as the neutral element. In order to avoid to deal with such classes as the class of finite sets, we consider that a finite alphabet A is simply a positive natural number.

The specification of a language requires an unambiguous description of which word belongs to that language. One way to achieve such this is by using a machine that recognises the language. Here we are interested in the simplest model of computation: the finite state automaton. The languages on an alphabet A recognised by a finite automaton form the Boolean algebra $\mathrm{Rec}(A^*)$ of *recognisable languages*, also called *regular languages* in this case. However, the fact that A^* is a monoid leads to an alternative definition in algebraic terms. We say that a language $L \subseteq A^*$ is *recognised* by a finite monoid M if there exists a surjective monoid morphism $\varphi : A^* \to M$ and a subset P of M such that for all word $w \in A^*$

$$w \in L \quad \text{if and only if} \quad \varphi(w) \in P.$$

Then a language L is recognised by a finite state automaton if and only if it is recognised by a finite monoid. Thus, the recognisable languages on A^* are those subsets of A^* for which the membership problem reduces by a monoid morphism to the membership problem for a subset of a finite set.

1 Varieties of languages and varieties of finite monoids

The connection of the theory of formal languages with logic and especially theoretical computer science motivates the study of classes of languages. The algebraic approach allows us to describe certain classes of languages by means of classes of finite monoids.

Of particular interest in algebra are the classes of finite monoids closed under taking submonoids, quotient monoids, and finite direct products. We call such classes *varieties of finite monoids.*

Any variety of finite monoids $\mathbf{V}$ defines for each finite alphabet A the family of those languages on A recognised by a monoid in $\mathbf{V}$. It can be easily showed directly that the correspondence so defined is a function $\mathcal{V}$ which associate to each finite alphabet A a family of languages $\mathcal{V}(A^*)$ such that

1. for all A, $\mathcal{V}(A^*)$ is a Boolean subalgebra of $\mathrm{Rec}(A^*)$,
2. for all A, $\mathcal{V}(A^*)$ is *closed under quotienting*, that is for all $L \in \mathcal{V}(A^*)$ and for all $u \in A^*$, the languages

$$u^{-1}L = \{w \in A^* \mid uw \in L\}$$
$$Lu^{-1} = \{w \in A^* \mid wu \in L\}$$

   belong to $\mathcal{V}(A^*)$,

3. for all monoid morphism $\varphi : A^* \to B^*$, if $L \in \mathcal{V}(B^*)$ then $\varphi^{-1}(L) \in \mathcal{V}(A^*)$.

Such a function is called a *variety of languages.*

Reciprocally, given a variety of languages $\mathcal{V}$ we associate a variety of finite monoids $\mathbf{V}$ as follows. For each finite alphabet A and each language $L \in \mathcal{V}(A^*)$, we consider the *syntactic monoid* M_L of L defined as the quotient of A^* by the congruence

$$u \sim_L v \quad \text{if and only if} \quad \text{for all } x, y \in A^*(xuy \in L \leftrightarrow xvy \in L).$$

This congruence saturates L and is of finite index so that the corresponding quotient of A^* is a finite monoid recognising L. It is in fact the coarsest congruence which saturates L. We associate to the variety of languages $\mathcal{V}$ the variety of finite monoids $\mathbf{V}$ generated by the syntactic monoids of all languages of $\mathcal{V}$.

The Variety Theorem of Eilenberg then states that

Theorem 1.1 (Eilenberg, 1974)**.** *The correspondence between varieties of finite monoids and varieties of languages described above is bijective and order preserving.*

2 Free profinite monoids and the Reiterman theorem

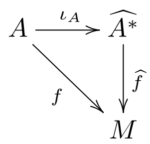
The Birkhoff theorem (Birkhoff, 1935) states that classes of (possibly infinite) monoids are closed under taking submonoids, quotients, and any products if and only if they are definable by equations. Here an equation is a formal equality between elements of a free monoid on a finite set. For example, the equation $xy = yx$ for commutative monoids can be seen as the pair (xy, yx) of words on the alphabet $\{x, y\}$. A monoid M then satisfies the equation (xy, yx) if for all monoid morphism $\varphi : \{x, y\}^* \to M$ we have $\varphi(xy) = \varphi(yx)$. Since $\{x, y\}^*$ is the free monoid on $\{x, y\}$ this simply amounts to the condition that for all interpretation $\varphi : \{x, y\} \to M$ of x and y in M the induced morphism $\varphi : \{x, y\}^* \to M$ equalises the words xy and yx. This is just another way of saying that for all x and y in M, $xy = yx$.

The Reiterman theorem is a counterpart for varieties of finite monoids of the Birkhoff theorem. However, in the case of a variety of finite monoids, the situation is somewhat modified. While classes of finite monoids defined by set of equations of the type just described are indeed varieties of finite monoids, not every variety of finite monoids is defined by a set of such equations. A more general kind of equation is needed. Actually, as it is notably observed in (Almeida, 2005; Almeida and Weil, 1995), this can be explained by the fact that in general varieties of finite monoids lack free objects. Indeed, in order to have a counterpart to the Birkhoff theorem we have to find objects which relate to finite monoids the same way as the free monoids relate to monoids.

The fact is that the free objects necessary to state the counterpart for finite monoids of Birkhoff theorem are to be taken in the category of *profinite monoids*, whose objects are compact Hausdorff topological monoids which are in addition zero dimensional, i.e. they admit a basis of clopen sets, and whose morphisms are the continuous monoid morphisms.

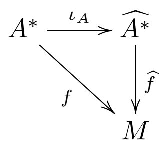
For the reader initiated to category theory, we can explain this fact as follows. We recall that by the Adjoint Functor Theorem the existence of free objects relates to the completeness of the category, i.e. the existence of all small limits (see for example (Mac Lane, 1998)). Though the (essentially) small category of finite monoids does not enjoy this property, its pro-completion provides an optimal complete category in which the category of finite monoids embeds (see (Johnstone, 1986; Grothendieck and Verdier, 1972–1973; Lambek, 1966)). The pro-completion of the category of finite monoids can be seen as the full subcategory of the category of topological monoids consisting of all projective limits of finite discrete monoids. By general topology and the Tychonoff's theorem, any such projective limit is a compact Hausdorff topological monoid which is in addition zero dimensional. In fact Numakura proved in (Numakura, 1957) that any such topological monoid is a projective limit of finite discrete monoids. Hence the pro-completion of the category of finite monoids is (equivalent to) the category of Hausdorff compact zero dimensional monoids.

For a set A, the *free profinite monoid* on A is defined by the following universal property. It is the unique profinite monoid $\widehat{A^*}$ along with a set function $\iota_A : A \to \widehat{A^*}$ such that for all set functions $f : A \to M$ into a profinite monoid M there exists a unique continuous monoid morphism $\widehat{f} : \widehat{A^*} \to M$ with the following diagram commuting.



Concretely, it can be defined (see for example (Almeida and Weil, 1995)) as the limit of the following projective system. We consider the family of all finite A-generated discrete monoids, i.e. the maps $\varphi : A \to M$ with M a finite monoid such that the image of φ generates M. Then we take maps between a pair $\varphi : A \to M$, $\psi : A \to N$ of A-generated discrete monoids to be a (continuous) morphism $f : M \to N$ such that $f \circ \varphi = \psi$, in particular f must be onto. The limit of this projective system is the map $\iota_A : A \to \widehat{A^*}$ defined by the universal property described above.

In fact, by freeness of A^*, A-generated finite monoids $\varphi : A \to M$ are in bijective correspondence with monoid quotients $\varphi^* : A^* \to M$. The free profinite monoid on A is thus also the projective limit of the finite quotients of A^*. To make this idea precise, let Θ be the set of congruences of finite index on A^* partially ordered with reverse inclusion. This partial order is a directed set since the intersection of two congruences of finite index is again a congruence of finite index. The desired projective system is then the Θ indexed diagram in the category of profinite monoids defined by sending each $\theta \in \Theta$ to the corresponding finite quotient of A^*, namely A^*/θ, and when $\theta \subseteq \theta'$ letting the associated morphism $A^*/\theta \to A^*/\theta'$ be the unique morphism which commutes with the corresponding quotient maps $A^* \to A^*/\theta$ and $A^* \to A^*/\theta'$. The free profinite monoid on A is the limit in the category of profinite monoids of this projective system. Of course, the free profinite monoid on A along with the natural monoid morphism $\iota_A : A^* \to \widehat{A^*}$ is also characterised by the fact for all monoid morphisms $f : A^* \to M$ into a profinite monoid M there exists a unique continuous monoid morphism $\widehat{f} : \widehat{A^*} \to M$ with the following diagram commuting.



Remark 2.1. The free profinite monoid on a finite alphabet A can be obtained in a more pedestrian fashion. Define on A^* a metric d by letting for any words $u, v \in A^*$

$$d(u,v) = 2^{-\min\{|M| \ \mid \ M \text{ is a finite monoid which separates } u \text{ and } v\}}$$

where $|M|$ denotes the cardinality of M and where a finite monoid M separates two words u and v if there exists a morphism $\varphi : A^* \to M$ such that $\varphi(u) \neq \varphi(v)$. It can then be observed that the product on words is uniformly continuous. The free profinite monoid on A is also obtained as the metric completion of (A^*, d) with the monoid operation being the continuous extension of the product on A^*. The equivalence between the two definition can be found in (Almeida, 1994).

The points in the free profinite monoid on A are called *profinite words* and they turn out to be the right generalisation of terms in order to capture variety of finite monoids by means of equations. The satisfiability of an equation in profinite terms by a monoid is *mutatis mutandis* the same as for word equations. For a pair of profinite words $(x, y) \in \widehat{A^*} \times \widehat{A^*}$ we say that a finite monoid M satisfies the *profinite equation* (x, y) if for any continuous morphism $\varphi : \widehat{A^*} \to M$ we have $\varphi(x) = \varphi(y)$. By freeness of $\widehat{A^*}$ this is equivalent to say that for any interpretation of the variables $\varphi : A \to M$ the induced continuous morphism $\varphi : \widehat{A^*} \to M$ equalises the profinite words x and y. We can now state

Theorem 2.2 (Reiterman, 1982)**.** *A class of finite monoids is a variety of finite monoids if and only if it can be defined by profinite equations.*

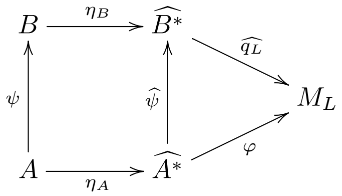
Note that the crux of the proof of this theorem is the "only if" part. Furthermore, it should be noticed that defining a variety of profinite monoid may require a set of equations between profinite words taken on unboundedly large finite alphabets.

3 The Eilenberg-Reiterman theorem

We call the Eilenberg-Reiterman theorem the combination of Eilenberg and Reiterman theorems. By this, we mean that a class of recognisable languages is a variety of languages if and only if it can be defined by profinite equations. Indeed, a variety of languages is, by Eilenberg theorem, recognised by a variety of finite monoids, which in turn is defined by profinite equations by Reiterman theorem. But how does a profinite equation directly define a variety of language?

To see this we let (x, y) be a pair of profinite words on a finite alphabet A and L be a recognisable language on a finite alphabet B. Then the language L is recognised by a finite monoid in the variety of finite monoids defined by the equation (x, y) if, by definition, the syntactic monoid M_L satisfies (x, y). This means that for all continuous morphisms $\varphi : \widehat{A^*} \to M_L$ we have $\varphi(x) = \varphi(y)$.

We can in fact be more precise, since each such map $\varphi : \widehat{A^*} \to M_L$ arises as the composition of a morphism $\widehat{\psi} : \widehat{A^*} \to \widehat{B^*}$ with the quotient map $\widehat{q_L} : \widehat{B^*} \to M_L$. Indeed for each $a \in A$, since $\widehat{q_L}$ is surjective, we can choose a $\psi(a)$ in $\widehat{q_L}^{-1}(\varphi(a))$ and let $\widehat{\psi} : \widehat{A^*} \to \widehat{B^*}$ be the continuous morphism induced by the so defined function $\psi : A \to B$.



Hence a recognisable language L on a finite alphabet B *satisfies* the profinite equation $(x, y) \in \widehat{A^*} \times \widehat{A^*}$ if for all continuous morphism $\widehat{\psi} : \widehat{A^*} \to \widehat{B^*}$ the quotient map $\widehat{q_L} : \widehat{B^*} \to M_L$ equalises $\widehat{\psi}(x)$ and $\widehat{\psi}(y)$.

The notion of satisfiablity of a profinite equation by a recognisable language thus defined we can state

Theorem 3.1 (Eilenberg-Reiterman)**.** *A class of recognisable languages is a variety of languages if and only if it can be defined by profinite equations.*

The next sections aims to explain how duality theory can account for this theorem.

4 Stone duality

Roughly speaking, Stone duality relates intimately Boolean algebras with Stone spaces. The Stone spaces are the Hausdorff compact topological spaces which are zero dimensional, i.e. which admit a basis of clopen sets. We briefly recall what this relation is about (references are notably (Stone, 1936; Johnstone, 1986; Givant and Halmos, 2009)).

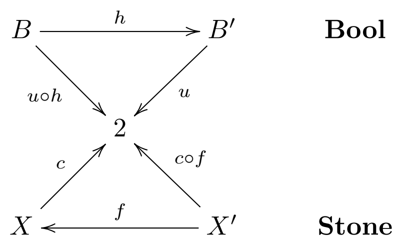
For a Boolean algebra B, we denote by X_B the set of all ultrafilters on B endowed by the topology generated by the sets of the form $\{u \in X_B \mid b \in u\}$ with $b \in B$. Equivalently, X_B can be seen as the set of all boolean homomorphisms from B to the two elements Boolean algebra 2, endowed with the topology induced by the product topology on 2^B with 2 discrete. Reciprocally, for a Stone space X we denote by $\mathrm{Clop}(X)$ the set of clopen sets of X endowed with the natural set theoretical Boolean operations. Equivalently, $\mathrm{Clop}(X)$ can be seen as the Boolean subalgebra of the power set 2^X consisting of continuous map $c : X \to 2$ with $2 = \{0, 1\}$ carrying the discrete topology. By use of the Boolean Prime Ideal Theorem, these correspondences are reciprocal up to isomorphism, so that $X_{\mathrm{Clop}(X)} = X$ and $\mathrm{Clop}(X_B) = B$. The corresponding isomorphisms are given respectively by

$$\begin{aligned} \eta : B &\to \mathrm{Clop}(X_B) \\ b &\mapsto \{u \in X_B \mid b \in u\} \end{aligned}$$

and

$$\begin{aligned} \epsilon : X &\to X_{\mathrm{Clop}(X)} \\ u &\mapsto \{a \in \mathrm{Clop}(X) \mid u \in a\}. \end{aligned}$$

Furthermore, this correspondence extends to morphisms in the sense that to each boolean morphism $h : B \to B'$ is associated a continuous map $X_h : X_{B'} \to X_B$ between the correspondent Stone spaces but in the reverse direction. It is simply defined by $f(u) = u \circ h$ for any $u \in X_{B'}$ seen as Boolean morphism $u : B' \to 2$. Reciprocally, to each continuous map $f : X' \to X$ between Stone spaces is associated the Boolean morphism consisting in the restriction of the preimage map to clopen sets $\mathrm{Clop}(f) : \mathrm{Clop}(X') \to \mathrm{Clop}(X)$ defined for all characteristic function c of a clopen set of X' by $h(c) = f \circ c$. We can thus view Stone duality in terms of the "double life" of the set 2 being at the same time the discrete space with two elements and the two elements Boolean algebra. In this sense, Stone duality can be depicted by the following diagram.



Stone duality states that the category **Bool** of Boolean algebras with Boolean morphisms is equivalent to the category **Stone** of Stone spaces with continuous maps where every arrow is reversed. In the setting of category theory, this amounts to $\mathbf{Bool} \cong \mathbf{Stone}^{\mathrm{op}}$ and the equivalence is given by the functors $B \mapsto X_B$, $h \mapsto X_h$ and $X \mapsto \mathrm{Clop}(X)$, $f \mapsto \mathrm{Clop}(f)$ together with the natural isomorphisms η and ϵ defined above.

In particular, Stone duality assures that monomorphisms in **Bool** correspond dually to epimorphisms in **Stone**. That is a Boolean morphism $h : A \to B$ is a Boolean embedding if and only if the dual map $X_h : X_B \to X_A$ is a quotient map.

To make this idea precise, let $A \hookrightarrow B$ be an inclusion of Boolean algebras. The dual map $X_B \twoheadrightarrow X_A$ sends each ultrafilter x on B to the ultrafilter $x \cap A$ on A. This quotient map is thus given by identifying any ultrafilters x and y on B for which

$$\text{for all } a \in A \quad (a \in x \leftrightarrow a \in y).$$

Conversely, any quotient map $q : X_B \twoheadrightarrow Y$ in Stone is given by quotienting X_B by the equivalence relation $E = \{(x, y) \in X_B \times X_B \mid q(x) = q(y)\}$. So up to isomorphism, the Boolean algebra dual to Y is the subalgebra A of B consisting of the $a \in B$ such that

$$\text{for all } (x, y) \in E \quad (a \in x \leftrightarrow a \in y).$$

Remark 4.1. In fact, for X_B the Stone dual to a Boolean algebra B, the polarity $P \subset B \times (X \times X)$ given by

$$(b, (x, y)) \in P \quad \text{if and only if} \quad b \in x \leftrightarrow b \in y$$

define an antitone Galois connection between the power sets $\mathcal{P}(B)$ and $\mathcal{P}(X_B \times X_B)$ whose Galois closed sets are respectively the subalgebras of B and the equivalence relations on X_B giving rise to Stone quotients.

5 Stone duality and recognition

We first recall that we define recognisable languages of A^* as the Boolean subalgebra of $\mathcal{P}(A^*)$ given by

$$\begin{aligned}\mathrm{Rec}(A^*) &= \left\{ \varphi^{-1}(P) \in \mathcal{P}(A^*) \;\middle|\; \begin{array}{l}\varphi : A^* \to M \text{ is a surjective morphism} \\ \text{onto a finite monoid } M \text{ and } P \in \mathcal{P}(M)\end{array} \right\} \\ &= \bigcup \left\{ \varphi^{-1}(\mathcal{P}(M)) \;\middle|\; \begin{array}{l}\varphi : A^* \to M \text{ is a surjective morphism} \\ \text{onto a finite monoid } M\end{array} \right\}\end{aligned}$$

The first explicit mention of a relation between duality theory with profinite topologies and automata theory goes back to the article (Pippenger, 1997) where we can find the following result.

Theorem 5.1. *The underlying topological space of the free profinite monoid on a finite set A is dual to the Boolean algebra of the recognisable subsets of A^*.*

The following proof is based on the proof given in (Gehrke, 2009) in a more general setting.

Proof. The free profinite monoid on A is the topological monoid obtained as the projective limit taken in the category of profinite monoids of the finite monoid quotients M of A^* linked with the continuous morphisms making quotient map commutes. Since forgetful functors preserves limits, the underlying Stone space of $\widehat{A^*}$ is the projective limit of the discrete spaces underlying the finite quotients of A^*. But to each quotient map $\varphi : A^* \to M$ onto a finite monoid corresponds the Boolean embedding of $\varphi^{-1} : \mathcal{P}(M) \to \mathcal{P}(A^*)$. By definition of $\mathrm{Rec}(A^*)$, this map actually embeds the powerset of M into $\mathrm{Rec}(A^*)$. Hence to the projective limit $\widehat{A^*}$ of the finite quotients M of A^* taken in **Stone** corresponds dually the inductive limit of the subalgebras $\mathcal{P}(M)$ of $\mathrm{Rec}(A^*)$ taken in **Bool**. The latter is just the direct union of the subalgebras $\mathcal{P}(M)$ of $\mathrm{Rec}(A^*)$ which is by definition $\mathrm{Rec}(A^*)$. □

The profinite words on A can thus be thought of as ultrafilters on the Boolean algebra of recognisable languages. We exploit this fact without mentioning it from now on.

On the basis of this link between profinite monoids and duality, we can already make a first step towards the Eilenberg-Reiterman theorem.

Indeed, by the previous section, and as first observed in (Pippenger, 1997), the Boolean subalgebras of $\mathrm{Rec}(A^*)$ are the duals of the Stone quotients of $\widehat{A^*}$. Hence given a Boolean algebra of recognisable languages on A, we have dually a Stone quotient $q : \widehat{A^*} \twoheadrightarrow X_B$. Setting $E = \{(x, y) \in \widehat{A^*} \times \widehat{A^*} \mid q(x) = q(y)\}$, we have seen that B can be recover as the Boolean algebra of the recognisable languages $L \in \mathrm{Rec}(A^*)$ such that

$$\text{for all } (x, y) \in E \quad (L \in x \leftrightarrow L \in y).$$

Now saying that a recognisable language $L \in \mathrm{Rec}(A^*)$ *satisfies* the profinite equation of the form $x \leftrightarrow y$ for $(x, y) \in \widehat{A^*} \times \widehat{A^*}$ if the condition $L \in x \leftrightarrow L \in y$ is satisfied, we can state the following result of (Gehrke, Grigorieff, and Pin, 2008)

Theorem 5.2. *A set of recognisable languages on A is a Boolean algebra if and only if it can be defined by a set of equations of the form $x \leftrightarrow y$ for profinite words $x, y \in \widehat{A^*}$.*

The proof that any set of equations of the form $x \leftrightarrow y$ defines a Boolean algebra can be verified in a straightforward manner.

6 Extended Stone duality and recognition

In the previous section, we saw that the underlying topological space of the free profinite monoid on a finite set A is the Stone dual of the Boolean algebra of recognisable languages $\mathrm{Rec}(A^*)$. An interesting question here is whether duality can account for the product on $\widehat{A^*}$. By the way duality works, the question must be answered by considering supplementary operations on the Boolean algebra of the recognisable languages. In fact $\mathrm{Rec}(A^*)$ is naturally endowed with supplementary operations. Indeed for any $L, M, N \in \mathrm{Rec}(A^*)$ we first have the lifting of the product on A^*

$$L \cdot M = \{uv \in A^* \mid u \in L \text{ and } v \in M\}$$

and most importantly the *right* and *left residuals* of N by M given by

$$M \backslash N = \{u \in A^* \mid \text{for all } v \in M,\ vu \in N\}$$
$$N/M = \{u \in A^* \mid \text{for all } v \in M,\ uv \in N\}$$

Moreover, this operations are intimately related by the preorder relation of the Boolean algebra $\mathrm{Rec}(A^*)$ which is simply the inclusion preorder. Indeed they satisfy for all $L, M, N \in \mathrm{Rec}(A^*)$ the following property

$$L \cdot M \subseteq N \quad \text{if and only if} \quad L \subseteq N/M \quad \text{if and only if} \quad M \subseteq L \backslash N \tag{1}$$

These operations turns $\mathrm{Rec}(A^*)$ into what is called a *residuated Boolean algebra*, namely a Boolean algebra B together with three binary operations $\cdot$, $\backslash$ and $/$ where $\cdot$ preserves finite joins in each coordinate and where the equation (1) is verified for all $L, M, N \in B$.

Remark 6.1. The recognisable languages of A^* are in fact closed under residuals by any subset of A^*. Indeed if L is recognised by a monoid morphism $\varphi : A^* \to M$ and $S \subseteq A^*$, then φ also recognises $S \backslash L$ and L/S since $S \backslash L = \varphi^{-1}(\varphi(S) \backslash \varphi(L))$ and $L/S = \varphi^{-1}(\varphi(L)/\varphi(S))$.

In general extended Stone duality, the supplementary operations on Boolean algebras are captured by supplementary relations on Stone spaces (see (Goldblatt, 1989)). The particular case of extended duality for residuated Boolean algebra we need here is exposed in detailed in the master thesis (Dekkers, 2008).

Roughly, to each residuated Boolean algebra $(B, \cdot, \backslash, /)$ is associated the Stone dual X_B of B with the ternary relation $R \subseteq X_B \times X_B \times X_B$ defined equivalently by

$$
\begin{aligned}
(x, y, z) \in R \quad & \text{if and only if} \quad \forall a, b \in B, (a \in x \text{ and } b \in y) \to a \cdot b \in z \\
& \text{if and only if} \quad \forall a, b \in B, (b \in y \text{ and } a \notin z) \to a/b \notin x \qquad (2) \\
& \text{if and only if} \quad \forall a, b \in B, (b \in x \text{ and } a \notin z) \to a\backslash b \notin y \qquad (3)
\end{aligned}
$$

Notice that the relation R dual to the supplementary operations on B is not functional from $X \times X$ to X in general. Nevertheless we have the following theorem whose proof can be found in (Gehrke, 2009) in a more general setting.

Theorem 6.2. *The dual space of* $(\mathrm{Rec}(A^*), \cdot, \backslash, /)$ *under extended Stone duality is the free profinite monoid* $\widehat{A^*}$*.*

We go on by characterising the Boolean subalgebras of $\mathrm{Rec}(A^*)$ which are closed under quotienting, as defined in the first part of the present paper, in terms of the residuals operations. By a *Boolean residuation ideal* of $\mathrm{Rec}(A^*)$ we mean a Boolean subalgebra B of $\mathrm{Rec}(A^*)$ such that for all $L \in B$, and for all $K \in \mathrm{Rec}(A^*)$ the languages $K\backslash L$ and L/K belong to B.

Proposition 6.3. *The Boolean algebras of recognisable languages closed by quotients by words are exactly the Boolean residuation ideals of* $(\mathrm{Rec}(A^*), \cdot, \backslash, /)$*.*

Proof. We first observe that closure under quotients by words amounts to closure under residuals by singletons. In the other direction, we let B be a Boolean algebra of recognisable languages of A^* closed under residuals by singletons. For $L \in B$ and $K \in \mathrm{Rec}(A^*)$, we consider a monoid morphism $\varphi : A^* \to M$ onto a finite monoid which recognises L. There exists $P \subseteq M$ such that $\varphi^{-1}(P) = L$ and since M is finite we can choose a finite $K' \subseteq K$ such that $\varphi(K') = \varphi(K)$. We obtain that $K\backslash L = \varphi^{-1}(\varphi(K)\backslash P) = \varphi^{-1}(\varphi(K')\backslash P) = \varphi^{-1}(\bigcap_{v \in K'} \varphi(\{v\})\backslash P) = \bigcap_{v \in K'} \{v\}\backslash L$. Since K' is finite and B is closed under finite intersection, we have that $K\backslash L \in B$. □

From the viewpoint of extended duality, the interest for the Boolean residuation ideals I of a residuated Boolean algebra B lies in the fact that the relation defined equivalently by (2) and (3) on the dual of I behaves nicely with the relation on the dual of B. As a result of which, the Boolean residuation ideals of $\mathrm{Rec}(A^*)$ have a nice dual characterisation as the following next corollary of a result of (Dekkers, 2008) states.

Theorem 6.4. *The Boolean residuation ideals of* $\mathrm{Rec}(A^*)$ *correspond dually to the profinite monoid quotients of* $\widehat{A^*}$*. That is, a Boolean algebra of recognisable languages B embeds in* $\mathrm{Rec}(A^*)$ *as a Boolean residuation ideal if and only if the extended Stone dual of* $(B, \backslash, /)$ *is a profinite monoid quotient of* $\widehat{A^*}$*.*

Now we are able to show that Boolean algebras of recognisable languages closed under quotienting are definable by profinite equations. Indeed by Theorem 5.2, we know that any Boolean algebra B of recognisable languages on A^* is defined by a set E of equations of the form $x \leftrightarrow y$ for $x, y \in A^*$. But now if B is in addition closed under quotienting, then it is a residuation ideal of $\mathrm{Rec}(A^*)$ and thus by the previous theorem its extended dual is in fact a profinite quotient of $\widehat{A^*}$. This means that B is also defined by the congruence on $\widehat{A^*}$ generated by E. In other words, if B is closed under quotienting then it can be defined as the set of recognisable languages L on A^* satisfying for all $(x, y) \in E$ the condition

$$\forall z, z' \in \widehat{A^*} \quad zxz' \in L \leftrightarrow zyz' \in L$$

Let us say that a recognisable language L on A^* satisfies the equation of the form $x = y$ for $(x, y) \in \widehat{A^*} \times \widehat{A^*}$ if for all $z, z' \in \widehat{A^*}$ we have $zxz' \in L \leftrightarrow zyz' \in L$. This definition of satisfaction of a profinite equation by a language is consistent with the one given in the third part of the present paper.

We have thus obtained the following result of (Gehrke, Grigorieff, and Pin, 2008).

Theorem 6.5. *A set of recognisable languages on A^* is a Boolean algebra closed under quotienting if and only if it can be defined by a set of profinite equations of the form $x = y$ for $x, y \in \widehat{A^*}$.*

The previous theorem is a "local" version of the combination of Eilenberg and Reiterman theorems in the sense that a finite alphabet A is fixed. In order to account for the Eilenberg-Reiterman theorem, we have to consider sets E of equations of the form $x = y$ for profinite words $x, y \in \widehat{A^*}$ on arbitrary large finite alphabet A with the additional condition of being closed under substitution. That is, for any continuous morphism $f : \widehat{A^*} \to \widehat{B^*}$ between the free profinite monoids on finite alphabet A and B respectively,

$$\text{for all } (x, y) \in \widehat{A^*} \times \widehat{A^*}, \quad (x, y) \in E \to (f(x), f(y)) \in E.$$

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