From Well to Better, the Space of Ideals

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On the one hand, the ideals of a well quasi-order (wqo) naturally form a compact topological space into which the wqo embeds. On the other hand, Nash-Williams' barriers are given a uniform structure by embedding them into the Cantor space.

We prove that every map from a barrier into a wqo restricts on a barrier to a uniformly continuous map, and therefore extends to a continuous map from a countable closed subset of the Cantor space into the space of ideals of the wqo. We then prove that, by shrinking further, any such continuous map admits a canonical form with regard to the points whose image is not isolated.

As a consequence, we obtain a simple proof of a result on better quasi-orders (bqo); namely, a wqo whose set of non principal ideals is bqo is actually bqo.

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Contents

1 Introduction

A *quasi-order* (qo) is a transitive and reflexive relation. A quasi-order without any infinite antichain nor infinite descending chain is called a *well quasi-order* (wqo). For historical references as well as for a gentle and synthetic introduction to wqo theory we refer the reader to [Kru72]. The notion of wqo admits of several different definitions as reviewed in Proposition 2.5. In particular, using Ramsey's Theorem, the above forbidden pattern definition is equivalent to the positive condition that every sequence admits a monotone subsequence.

The notion of *better quasi-order* (bqo) was introduced by Nash-Williams in [NW65]. As its name indicates it is a stronger notion than well quasi-order. The combinatorial definition of better quasi-order relies on a generalisation of the Ramsey's Theorem to transfinite dimension: the notion of *barrier*. It generalises the positive definition of wqo given above in the sense that it requires every sequence of sequences – or 2-sequence – to admit a "monotone" sub-2-sequence, every sequence of sequences of sequences – or 3-sequence – to admit a "monotone"' sub-3-sequence, so on and so forth in the transfinite.

These α-sequences are just maps from a barrier to some set. They are sometimes called *arrays*, but for the purpose of this introduction we call them

supersequences. One aim of this paper is to show that these objects deserve this name since they share significant properties with usual sequences.

A crucial property for a sequence in a metric space is the Cauchy condition. In order to generalise the notion of being Cauchy to supersequences, observe that a sequence $(x_n)_{n\in\omega}$ in a metric space satisfies the Cauchy condition iff the mapping $\omega \to X$, $n \mapsto x_n$ is uniformly continuous, when ω is identified with a subspace of the Cantor space 2^ω via $n \mapsto 0^n 10^\omega$.

As observed notably in [AT05, Tod10], barriers can naturally be seen as subsets of the Cantor space. Viewing every barrier as a uniform subspace of 2^ω, we say that a supersequence in a uniform space is Cauchy when it is uniformly continuous. We then show the following theorem, which generalises the usual sequential compactness for zero dimensional metric spaces.

Theorem 1.1. *Every supersequence in a zero dimensional compact metric space has a Cauchy sub-supersequence.*

This combinatorial result should be compared with Erdös-Rado Theorem ([ER50]) and Pudlak-Rödl Theorem ([PR82]) as a Ramsey theorem for partitions into infinitely many classes. Note also that this result subsumes Nash-Williams' Theorem.

Given X is a complete metric space, every Cauchy sequence $f : \omega \to X$ converges, and thus extends to a continuous map $\overline{f} : \overline{\omega} \to X$, where $\overline{\omega}$ is the one point compactification of ω. Similarly for a Cauchy supersequence: any uniformly continuous map $f : B \to X$ from a barrier into a complete metric space X continuously extends to the completion $\overline{B}$ of the barrier, which coincides with the topological closure of the barrier inside the Cantor space, to yield a continuous map $\overline{f} : \overline{B} \to X$.

Theorem 1.1 has a nice corollary in the context of wqo theory. Indeed Pouzet and Sauer show in [PS06] that the set $\mathrm{I}(Q)$ of ideals of a wqo Q is naturally a compact topological partially ordered space in which Q embeds densely. Without much more work we get

Theorem 1.2. *Every supersequence $f : B \to Q$ in a wqo Q admits a Cauchy sub-supersequence f' from B' a sub-barrier of B to Q, which therefore extends to a continuous map $\overline{f'} : \overline{B'} \to \mathrm{I}(Q)$ into the ideals of Q.*

We then turn to the study of continuous extension of Cauchy supersequences. In full generality, we will be concerned with continuous maps from the topological closure of a barrier into some topological space.

Recall that a point x in a topological space is called *isolated* if the singleton $\{x\}$ is open, and *limit* otherwise. The following simple fact exhibits a property

of converging sequences that can always be achieved by going to a subsequence: If $(x_n)_{n\in\omega}$ is a sequence converging in a topological space $\mathcal{X}$ to some point x, then there is a subsequence $(x_n)_{n\in N}$ such that

1. if x is isolated, then $(x_n)_{n\in N}$ is constant equal to x;
2. if x is limit, then

   either x_n is isolated for all $n \in N$;

   or x_n is limit for all $n \in N$.

This generalises to supersequences as follows

Theorem 1.3. *Let $\overline{f} : \overline{B} \to \mathcal{X}$ be a continuous extension of a supersequence f in a topological space $\mathcal{X}$. Then there exists a sub-supersequence f' of f from a sub-barrier B' of B to $\mathcal{X}$ such that*

either *$\overline{f'} : \overline{B'} \to X$ is constant and equal to an isolated point;*

or *$\{s \in \overline{B'} \mid f(s)$ is limit$\} = \overline{C}$ for some barrier C.*

As observed by Pouzet and Sauer in [PS06] the limit points of the compact space $\mathrm{I}(Q)$ of ideals of a wqo Q are exactly the non principal ideals. Combining Theorems 1.2 and 1.3 we obtain that

Corollary 1.4. *Any supersequence in a wqo*

either *admits a constant sub-supersequence,*

or *yields a supersequence into the non principal ideals of Q.*

This allows us to give a proof of the following result, conjectured by Pouzet in his thesis [Pou78]. Pouzet and Sauer advanced a proof of this statement in [PS06], but this proof contains a gap, as clearly revealed by Alberto Marcone and acknowledged by Pouzet and Sauer.

Theorem 1.5. *If $(Q, \leq_Q)$ is wqo and the po of non principal ideals of Q, $(\mathrm{I}^*(Q), \subseteq)$ is a bqo, then $(Q, \leq_Q)$ itself is bqo.*

Theorem 1.5 can be very useful when one decomposes a qo into a sum of bqo. Proving that the index set of the sum is bqo implies indeed that the whole class is. The following example drawn from [Car13] shows that it can be at the same time very easy to prove that some index set is a wqo, and not so easy to prove that it is bqo without the help of Theorem 1.5.

Example 1.6. Consider the following qo on ω:

$$n \leq^\bullet m \text{ iff } n = m \text{ or } 2n < m$$

Proving that $(\omega, \leq^\bullet)$ is wqo is not difficult, then since there is a single non-principal ideal – the whole qo – Theorem 1.5 implies immediately that $(\omega, \leq^\bullet)$ is bqo.

Organisation of the paper. Section 2 is devoted to some classical background. It can be skipped by the reader well versed in the subject, even though it contains many notations, definitions and conventions.

In Section 3 we study the barriers as uniform spaces. A basic knowledge of the classical material on uniform spaces is assumed. We give details on the particular setting we need, namely uniform subspaces of compact and 0-dimensional spaces. This section ends with the proof of Theorem 1.1.

Pouzet and Sauer's way to topologise the space of ideals of a wqo is explained at the beginning of Section 4. We then show that Theorem 1.1 applies in this context.

Ideas of Sections 3 and 4 are exemplified on Rado's typical counterexample. Section 5 continues with the proof of Theorem 1.3. Finally, after unearthing a shrewd trick first used by R. Rado in [Rad54], we give a proof of Theorem 1.5.

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2 Preliminaries

2.1 Well Quasi-Orders

Definitions 2.1.

- A *quasi-order* (qo) is a set Q equipped with a reflexive and transitive binary relation denoted $\leq_Q$.
- An antisymmetric qo is a *partial order* (po).
- Every qo has an associated strict relation denoted $<_Q$ defined by $p <_Q q$ iff $p \leq_Q q$ and $q \not\leq_Q p$.

Remark 2.2. If Q is a po then the strict relation $<_Q$ is just $\leq_Q \setminus \Delta_Q$, where Δ_Q stands for the diagonal in Q^2. This is far from being true in any qo, since for instance the total relation Q^2 on Q is a qo. However, every qo Q can be turned into a po, *its associated po*, by quotienting by the equivalence relation $p \leq_Q q$ and $p \geq_Q q$.

In what follows Q stands for a qo, and we will write $\leq$ instead of $\leq_Q$ as long as the context remains clear.

Definitions 2.3.

- Say Q is *well founded* if there is no infinite $<$-decreasing sequence in Q.
- We say two elements p and q of Q are *incomparable*, when both $p \not\leq q$ and $q \not\leq p$ hold. In this case we write $p \perp q$. A set of pairwise incomparable elements in Q is called an *antichain*.
- A sequence $f : \omega \to Q$ is called *good* if there exists $m < n < \omega$ such that $f(m) \leq f(n)$, otherwise f is called *bad*.
- A sequence $f : \omega \to Q$ such that for all $m, n \in \omega$ the relation $m \leq n$ implies $f(m) \leq f(n)$ is said to be *perfect*.
- A subset D of Q is a *downset* if $q \in D$ and $p \leq q$ implies $p \in D$. For any $S \subseteq Q$, we write $\downarrow S$ for the downset generated by S in Q, *i.e.* the set $\{q \in Q \mid \exists p \in S \; q \leq p\}$. We also write $\downarrow p$ for $\downarrow\{p\}$.
- We give the dual meaning to *upset*, $\uparrow S$ and $\uparrow q$.
- We denote by $\mathrm{Down}(Q)$ (resp. $\mathrm{Up}(Q)$) the po of *downsets* (resp. *upsets*) of Q equipped with inclusion.

- A downset $I \in \mathrm{Down}(Q)$ is called an *ideal* if it is non empty and directed, i.e for all $p, q \in I$ then there exists $r \in I$ with $p \leq r$ and $q \leq r$. For any $q \in Q$ the set $\downarrow q$ is a particular example of an ideal called a *principal* ideal.

- We denote by $\mathrm{I}(Q)$ the po of ideals of Q equipped with inclusion. We let $\mathrm{I}^*(Q)$ be the po of non principal ideals of Q.

- The *cofinality* of an ideal $I \in \mathrm{I}(Q)$, denoted $\mathrm{cof}(I)$, is the least cardinal λ such that there exists $B \subseteq I$ with $|B| = \lambda$ and $I = \downarrow B$. The principal ideals are the ideals of cofinality 1. There are no ideals of cofinality k for $2 \leq k < \omega$.

- We denote respectively by $\mathrm{I}_{\leq\omega}(Q)$ and $\mathrm{I}_\omega(Q)$ the po of ideals of Q which have countable cofinality and the po of ideals with cofinality ω. Observe that $\mathrm{I}_{\leq\omega}(Q) = \{\downarrow q \mid q \in Q\} \cup \mathrm{I}_\omega(Q)$ and that $\mathrm{I}_\omega(Q) \subseteq \mathrm{I}^*(Q)$.

- An upset $U \in \mathrm{Up}(Q)$ is said to admit a finite basis or to be finitely generated if there exists a finite $F \subseteq Q$ such that $U = \uparrow F$. We write $\mathrm{Up}_{<\infty}(Q)$ for the po of finitely generated upsets with reverse inclusion.

- We say that Q has *the finite basis property* if every upset of Q admits a finite basis, *i.e.* if $\mathrm{Up}(Q) = \mathrm{Up}_{<\infty}(Q)$.

We turn the power-set of Q, denoted $\mathcal{P}(Q)$, into a qo by letting $X \leq Y$ exactly when $\downarrow X \subseteq \downarrow Y$. Then the po associated to $\mathcal{P}(Q)$ is $\mathrm{Down}(Q)$.

Definition 2.4. Finally, we say that Q is a *well-quasi-order* (wqo) when one of the equivalent conditions of the next proposition is fulfilled.

Proposition 2.5. *Let Q be a qo. Then the following assertions are equivalent.*

(i) *Q is well founded and has no infinite antichain,*

(ii) *there is no bad sequence $f : \omega \to Q$,*

(iii) *every sequence $f : \omega \to Q$ admits a perfect subsequence,*

(iv) *Q has the finite basis property,*

(v) *$\mathcal{P}(Q)$ is well founded,*

(vi) *$\mathrm{Down}(Q)$ is well founded,*

(vii) *$\mathrm{Up}(Q)$ is well founded under reverse inclusion,*

(viii) $\mathrm{Up}_{<\infty}(Q)$ *is well founded under reverse inclusion.*

Proof. Folklore. □

Remark 2.6. In Proposition 2.5, we cannot replace $\mathrm{Down}(Q)$ by $\mathrm{I}(Q)$. Consider for example the po A consisting of a countable set partially ordered by equality. We have $\mathrm{I}(A) \cong A$ so, in particular, even though A is not wqo, $\mathrm{I}(A)$ is well founded. Nonetheless, if Q is wqo then $\mathrm{I}(Q)$ is well founded.

Observe also the following

Fact 2.7. *For an infinite wqo Q, we have* $\mathrm{Card}(\mathrm{I}(Q)) \leq \mathrm{Card}(Q)$. *If Q is a po, the equality holds.*

2.2 Families of finite sets of natural numbers

Notations.

- We let ω be the set of natural numbers. We use the set theoretic definition $n = \{0, \dots, n-1\}$. Given an infinite subset X of ω and a natural number k, we denote by $[X]^k$ the set of subsets of X of cardinality k, and by $[X]^{<\infty}$ the set of finite subsets of X. We have that $[X]^{<\infty} = \bigcup_{k\in\omega}[X]^k$. We write $[X]^\infty$ for the set of infinite subsets of X.
- For any $X \in [\omega]^\infty$ and any $s \in [\omega]^{<\infty}$, we let $X/s = \{k \in X \mid k > \max s\}$ and we write X/n for $X/\{n\}$. For any non empty set $S \subseteq \omega$ we write ${}_*S$ for $S \setminus \{\min S\}$.
- For any $s \in [\omega]^{<\infty}$ we let $x_s \in 2^\omega$ be the characteristic function of s on ω. So for instance $x_{\{2,4\}} = 001010000\cdots$. Note that $x_\emptyset = 0^\omega$.
- We write $u \sqsubseteq v$ when u is an *initial segment*, or *prefix* of v, *i.e.* when $u = v$ or when there is $n \in v$ such that $u = \{k \in v \mid k < n\}$. Note that this definition coincides with the usual prefix relation on sequences when subsets of ω are identified with their increasing enumeration with respect to the usual order on ω.

We gather here several combinatorial operations on general families of subsets of ω.

Definitions 2.8. Given a family $F \subseteq \mathcal{P}(\omega)$ we make the following definitions.

- The *base* of F is the usual set-theoretic union, denoted by $\bigcup F$;

- for any $X \in [\omega]^\infty$, the *shrinkage* of F to X, denoted by $F|X$, is defined to be the family
$$F|X := \{s \in F \mid s \subseteq X\};$$
- For every $n \in \omega$, the *ray* of F at n is, by definition, the family
$$F_n := \{s \in [\omega/n]^{<\infty} \mid \{n\} \cup s \in F\}.$$
- For every $n \in \omega$, we will denote by $F_{\uparrow n}$ the subset of F given by
$$F_{\uparrow n} := \{s \in F \mid \{n\} \sqsubseteq s\}.$$

Observe that for all $n \in \omega$ we have the bijection

$$\begin{aligned} F_n &\longrightarrow F_{\uparrow n} \\ s &\longmapsto \{n\} \cup s \end{aligned}$$

Fact 2.9. *Let $F \subseteq \mathcal{P}(\omega)$ and $X \in [\omega]^\infty$. For every $n \in X$ we have*
$$F_n|X = (F|X)_n.$$

2.3 Nash-Williams' fronts and barriers

We use the fundamental definition first enunciated by Nash-Williams [NW65].

Definition 2.10. A family $B \subseteq [\omega]^{<\infty}$ is a called a *front* on $X \in [\omega]^\infty$ if

1. either $B = \{\emptyset\}$, or X is the domain of B,
2. B is a $\sqsubseteq$-antichain,
3. (Density) for all $X' \in [X]^\infty$ there is an $s \in B$ such that $s \sqsubset X'$.
4. If moreover B is a $\subseteq$-antichain, then B is called a *barrier* on X.

The barrier $\{\emptyset\}$ is called the *trivial barrier*.

Remark 2.11. In the literature, fronts are sometimes called *blocks* or *thin blocks*. Since in section 3 we will have another use for the term *block* we follow the terminology of [Tod10].

Facts 2.12. *Let $X \in [\omega]^\infty$. If B is a front (resp. a barrier) on X, then*

(i) *for all $Y \in [X]^\infty$, $B|Y$ is a front (resp. a barrier) on Y.*

(ii) *If B is non trivial and $n \in X$, then B_n is a front (resp. a barrier) on X/n.*

Remarks 2.13.

1. If B is a front on M and $C \subseteq B$ is a front on N then $N \subseteq M$ and $C = B|N$. Therefore the fronts contained in B are exactly the shrinkages of B.

2. If for $M \in [\omega]^\infty$ and all $m \in M$ the family $F(m)$ is a front on M/m then the family
$$F = \bigcup_{m \in M} \{\{m\} \cup s \mid s \in F(m)\}$$
is a front on M. Observe though that there exist sequences of barriers $\langle F(m) \mid m \in \omega\rangle$ such that F is not a barrier.

We now recall some important combinatorial results about barriers and fronts for later use. We refer the reader to [Tod10, AT05] for proofs. Throughout this paper we will use extensively the following fundamental theorem.

Theorem 2.14 (Nash-Williams)**.**

(i) *Let F be a front on N. For any subset S of F there exists a front $F' \subseteq F$ such that either $F' \subseteq S$ or $F' \cap S = \emptyset$.*

(ii) *Let B be a barrier on N. For any subset S of B there exists a barrier $B' \subseteq B$ such that either $B' \subseteq S$ or $B' \cap S = \emptyset$.*

Theorem 2.15. *Let F be a front on M. There exists $N \in [M]^\infty$ such that $F|N$ is a barrier on N.*

Notation. For a non empty set A, we write $A^{<\omega}$ (resp. A^ω) for the set of finite (resp. infinite) sequences of A. For $u \in A^{<\omega}$ and $x \in A^{<\omega} \cup A^\omega$, we write $u \sqsubseteq x$ (resp. $u \sqsubset x$) when x extends (resp. properly) u. We write $u^\frown x$ for the concatenation operation. For $n \in \omega$ we write $x{\restriction}n$ for the prefix of x of length n.

Definitions 2.16.

- Recall that a *tree* T on a set A is a prefix-closed subset of $A^{<\omega}$.
- A tree T on A is called *well founded* if it has no infinite branch, *i.e.* if it admits no infinite sequence $x \in A^\omega$ such that $x{\restriction}n \in T$ for all $n \in \omega$.

- A well founded tree admits a canonical *rank*. It is a strictly decreasing function ρ from T to the ordinals, defined by induction as follows.

$$\rho_T(t) = \sup\{\rho_T(s) + 1 \mid t \sqsubset s \in T\}.$$

for all $t \in T$. It is easily shown to be equivalent to

$$\rho_T(t) = \sup\{\rho_T(t^\frown(a)) + 1 \mid a \in A \wedge t^\frown(a) \in T\}.$$

The *rank* of T is by definition the ordinal $\rho_T(\emptyset)$. By convention, the rank of the empty tree is 0.

Identifying any finite subset of ω with its increasing enumeration with respect to the usual order on ω, we view any front as a subset of $\omega^{<\omega}$. For a front F we let $T(F)$ be the smallest tree on ω containing the set F, *i.e.*

$$T(F) = \{s \in \omega^{<\omega} \mid \exists t \in F \ s \sqsubseteq t\}.$$

As a direct consequence of the definition of front we have

Lemma 2.17. *For any front B on N, the tree $T(B)$ is well founded.*

Definition 2.18. Let B be a front. The tree-rank of B, denoted by $\operatorname{rk} B$, is the rank of the tree $T(B)$.

Remarks 2.19.

- The trivial barrier is the only front of null tree-rank, and for all positive integer n we have $\operatorname{rk}[\omega]^n = n$.
- Let B be a non trivial front on N and let $n \in N$. The tree $T(B_n)$ of the front B_n is naturally isomorphic to the subset

$$\{s \in T(B) \mid \{n\} \sqsubseteq s\}$$

of $T(B)$. The tree-rank of the front B is therefore related to the tree-ranks of its rays through the following formula:

$$\operatorname{rk} B = \sup\{\operatorname{rk}(B_n) + 1 \mid n \in N\}.$$

In particular, $\operatorname{rk} B_n < \operatorname{rk} B$ for all $n \in N$.

- This allows one to prove results by induction on the tree-rank by applying the induction hypothesis to the rays, following [PR82].

- As an example, Nash-Williams' Theorem 2.14 can be proved by induction on the tree-rank.

The tree $T(B)$ associated to a front B also enjoys a topological description to which we now turn.

The *Cantor space* is the product space 2^ω where 2 is the discrete two points space. A basis of clopen sets is given by the sets of the form

$$N_u = \{x \in 2^\omega \mid u \sqsubset x\}$$

for a finite sequence u in 2. For a point $x \in 2^\omega$ a neighbourhood basis is given by the sets $N_{x\restriction n}$ for $n \in \omega$.

We embed every subset of $[\omega]^{<\infty}$ into the Cantor space via $s \mapsto x_s$. By misuse of language, we sometimes identify fronts and barriers with their image inside the Cantor space. For a front B, the *closure of* B denoted $\overline{B}$ is the topological closure of the set $\{x_s \mid s \in B\}$ inside the Cantor space. We now recall some results about this closure operation on fronts. More results along this line are to be found in [Tod10, AT05].

Proposition 2.20. *Let B be a front on X. We have*

$$\overline{B} = \{x_s \in 2^\omega \mid s \in T(B)\}.$$

Proof. $\supseteq$: Let s be in $T(B) \setminus B$, so $s \sqsubset t$ holds for some $t \in B$. Let now n be in ω/s. Since B is a front there is a $u \in B$ with $u \sqsubset s \cup X/n$. If $u \sqsubseteq s$ then we have $B \ni u \sqsubset t \in B$ contradicting the fact that B is a $\sqsubset$-antichain. Hence we must have $s \sqsubset u$, we have found a $u \in B$ with $x_u \in N_{x_s\restriction n}$. It follows that $x_s \in \overline{B}$.

$\subseteq$: Conversely suppose that an element x of 2^ω belongs to $\overline{B}$. We first show that x is the characteristic function of a finite subset of X. Since $2^X = \{x_E \mid E \in \mathcal{P}(X)\}$ is closed in 2^ω and $B \subseteq 2^X$, necessarily x is the characteristic function of a subset of X. Now suppose towards a contradiction that x is the characteristic function of an infinite set $M \subseteq X$. For all finite prefix u of M, there is by definition of $\overline{B}$ some $s \in B$ such that $x_s \in N_{x\restriction \max(u)+1}$, and hence $u \sqsubseteq s$. But then M should be an infinite branch of $T(B)$, a contradiction with the well foundedness.

Hence $x = x_s$ for some $s \in [X]^{<\infty}$. It only remains to show that there exists a $t \in B$ with $s \sqsubseteq t$. By definition of the closure in 2^ω, for all $n \in \omega$ there is a $t \in B$ with $x_s\restriction n \sqsubset x_t$. For $n \in \omega/s$, $x_s\restriction n \sqsubset x_t$ means $s \sqsubseteq t$. □

Corollary 2.21. *Let B be a front on X.*

(i) *For all $M \in [X]^\infty$ we have $\overline{B|M} = \overline{B}|M$;*

(ii) *For all $n \in X$ we have $\overline{B_n} = \left(\overline{B}\right)_n$.*

Proof. (i) It is enough to prove that $\overline{B}|M \subseteq \overline{B|M}$ holds. So let $s \sqsubset t \in B$ with s a subset of M. Since B is a front there is a $u \in B$ with $u \sqsubset s \cup M/s$ and necessarily $u \in B|M$. If $u \sqsubseteq s \sqsubset t$ we have a contradiction. Hence $s \sqsubseteq u \in B|M$.

(ii) Since B_n is a front on ω/s, $\overline{B_n} = \{x_t \in 2^\omega \mid \exists u \in B_n \ t \sqsubseteq u\}$ holds. Now if $t \sqsubseteq u \in B_n$ then $\{n\} \cup t \sqsubseteq \{n\} \cup u \in B$ and thus $t \in \left(\overline{B}\right)_n$. Conversely if $x_t \in \left(\overline{B}\right)_n$ then $\{n\} \cup t \in \overline{B}$ and thus there exists $u \in B$ with $\{n\} \cup t \sqsubseteq u \in B$. Now $t \sqsubseteq_* u \in B_n$ and therefore $x_t \in \overline{B_n}$. □

2.4 Better Quasi-Orders

Definitions 2.22.

- We define the following binary relation, denoted $\lhd$, on $[\omega]^{<\infty}$. We say that $s \lhd t$ holds iff there exists $X \in [\omega]^\infty$ such that

$$s \sqsubset X \text{ and } t \sqsubset_* X.$$

- Given a family $F \subseteq [\omega]^{<\infty}$ and a set X endowed with a binary relation R, we say that a function $f : F \to X$ is *good* if there exists a pair (s,t) in F such that both $s \lhd t$ and $f(s)Rf(t)$ hold, otherwise f is said to be *bad*.

- If for all $s,t \in F$ when $s \lhd t$ holds then so does $f(s)Rf(t)$, *i.e.* if f is a relational morphism, the map f is said to be *perfect*.

Remark 2.23. When s,t belong to a barrier B, then $s \lhd t$ implies $|s| \leq |t|$.

Definition 2.24. Given a qo Q, we say that Q is a *better quasi-order* (bqo) iff any map from some barrier to Q is good.

Definition 2.25. For $B \subseteq [\omega]^{<\infty}$, let $B^2 = \{s \cup t \mid s,t \in B \wedge s \lhd t\}$. If B is a front (resp. a barrier) on N, then the family B^2 is a front (resp. a barrier) on N and for any $u \in B^2$ there exist unique $s,t \in B$ such that $s \lhd t$ and $u = s \cup t$.

Lemma 2.26. *Let B be a front on N, R be a binary relation on a set X and let $f : B \to (X, R)$ be any function. There exists an infinite $M \subseteq N$ such that $f : B|M \to (X, R)$ is either bad or perfect.*

Proof. Consider the subset of the front B^2 given by

$$S = \{s \cup t \in B^2 \mid s, t \in B \wedge s \lhd t \wedge f(s) \leq f(t)\}.$$

By Nash-Williams' Theorem there is an infinite $M \subseteq N$ such that $B^2|M \subseteq S$ or $B^2|M \cap S = \emptyset$. Then $f : B|M \to Q$ is perfect or bad accordingly. □

3 Fronts as uniform spaces

Regarded as a topological subspace of 2^ω, any front is a discrete space. Indeed for any element s of a front F we have $N_{x_s\restriction(1+\max s)} \cap F = \{x_s\}$. However, as any compact Hausdorff space, 2^ω is really more than a topological space: it is a uniform space. Viewed as uniform subspaces of 2^ω, every non trivial front is endowed with a non discrete uniform structure.

3.1 Uniform continuity in compact 0-dimensional spaces

This subsection is devoted to a description of the uniform subspaces of compact Hausdorff zero dimensional spaces.

A compact Hausdorff space is called *zero dimensional* (0-dim) if it admits a basis of simultaneously closed and open sets, or *clopen* sets. Such a space is also called a *Boolean space* in the context of Stone duality. Any such space is a closed subset of a generalised Cantor space 2^X.

A general reference on uniform spaces is [Bou06]. Recall that any compact Hausdorff topological space admits a unique uniform structure that agrees with its topology.

Every compact Hausdorff space is thus unambiguously seen as a complete totally bounded uniform space.

The adequate framework for this paper is in fact the totally bounded "0-dimensional" uniform spaces, that is, the uniform spaces whose completion is a compact Hausdorff 0-dimensional space. These are exactly the uniform subspaces of Boolean spaces. The following notion simplifies greatly the study of these uniform subspaces:

Definition 3.1. Let S be a subset of a Boolean space X. A subset B of S is called a *block* of S (relatively to X) if there exists a clopen C of X such that $B = C \cap S$. We write $\mathrm{Blocks}(S)$ for the Boolean subalgebra of $\mathcal{P}(S)$ of blocks of S.

The uniform structure of a uniform subspace of a Boolean space X is essentially given by its blocks, see Lemma 3.6 below (see also [Bou06, §*Exercice 12*, II.38]). As a consequence, uniform continuity between such spaces admits of the following simple characterisation:

Proposition 3.2. *Let X and Y be two Boolean spaces, and let $S \subseteq X$ and $T \subseteq Y$ be endowed with the induced uniform structure. Then a function $f : S \to T$ is uniformly continuous iff for all $B \in \mathrm{Blocks}(T)$ we have $f^{-1}(B) \in \mathrm{Blocks}(S)$.*

When the condition of Proposition 3.2 is met, there exists a unique continuous map $\overline{f} : \overline{S} \to \overline{T}$ such that $\overline{f}\restriction S = f$.

Although Proposition 3.2 is folklore, we give in the rest of this subsection a serie of lemmas that lead to a proof of this fact.

Lemma 3.3. *Let X be a Boolean space. The unique compatible uniform structure on X admits*

- *as a basis the entourages of the form $U_{(C_i)} = \bigcup_i C_i \times C_i$ where (C_i) is a finite partition of X into clopen sets;*
- *as a subbasis the entourages of the form $U_C = (C \times C) \cup (X \setminus C \times X \setminus C)$ where C is a clopen set of X.*

Lemma 3.4. *Let X be a Boolean space and let F be a closed subspace of X. Then the clopen sets of F coincide with the blocks of F.*

Fact 3.5. *Let X be a Boolean space, and let $S \subseteq T \subseteq X$. Then we have* $\mathrm{Blocks}(S) = \{B \cap S \mid B \in \mathrm{Blocks}(T)\}$.

Lemma 3.6. *Let X be a Boolean space and let S be a subset of X. Then the uniform structure induced on S by X admits*

- *as a basis the entourages of the form $U_{(C_i)} = \bigcup_i C_i \times C_i$ where (C_i) is a finite partition of S into blocks;*
- *as a subbasis the entourages of the form $U_C = (C \times C) \cup (S \setminus C \times S \setminus C)$ where C is a block of S.*

Proof of Proposition 3.2. $\Rightarrow$: Suppose f is uniformly continuous and let $\widehat{f} : \overline{S} \to \overline{T}$ be its continuous extension. Then for all clopen C of Y the set $f^{-1}(C \cap T) = \widehat{f}^{-1}(C) \cap S$ is a block of S.

$\Leftarrow$: Suppose $f : S \to T$ preserves blocks by preimage. By Lemma 3.6, it is enough to show that for each block B of T the preimage of U_B by $f \times f$ is an entourage of S. In fact, $(f \times f)^{-1}(U_B) = U_{f^{-1}(B)}$. □

Fact 3.7. *Let X and Y be Boolean, $S \subseteq X$, $T \subseteq Y$, $f : S \to T$ be a function and* $\operatorname{Im} f$ *be the image of f. Then f is uniformly continuous iff $f : S \to \operatorname{Im} f$ is uniformly continuous.*

3.2 Cauchy sub-supersequences.

We give every front B the uniform structure inherited from the Cantor space through the identification

$$\begin{aligned} B &\longrightarrow 2^\omega \\ s &\longmapsto x_s \end{aligned}$$

For all front B we write $\operatorname{Blocks}(B)$ for the Boolean algebra of subsets of B given by

$$\operatorname{Blocks}(B) = \{C \cap B \mid C \text{ is a clopen of } 2^\omega\}.$$

Example 3.8. For the barrier $[\omega]^1$, the Boolean algebra $\operatorname{Blocks}([\omega]^1)$ consists in the finite or cofinite subsets of $[\omega]^1$.

Lemma 3.9. *Let B be a front on N. Then for all $n \in N$ we have the isomorphism*

$$\begin{aligned} \operatorname{Blocks}(B_n) \longrightarrow \operatorname{Blocks}(B_{\restriction n}) &= \{C \in \operatorname{Blocks}(B) \mid \forall s \in C\ \{n\} \sqsubseteq s\} \\ A &\longmapsto \{\{n\} \cup t \mid t \in A\}. \end{aligned}$$

Proof. The clopen $N_{0^{n+1}} = \{x \in 2^\omega \mid 0^{n+1} \sqsubseteq x\}$ is homeomorphic to the clopen $N_{0^n 1} = \{x \in 2^\omega \mid 0^n 1 \sqsubseteq x\}$ via $\mathcal{P}(\omega/n) \ni S \mapsto \{n\} \cup S$. This homeomorphism induces a Boolean isomorphism

$$\begin{aligned} h_n : \{C \subseteq N_{0^{n+1}} \mid C \text{ clopen in } 2^\omega\} &\longrightarrow \{D \subseteq N_{0^n 1} \mid D \text{ clopen in } 2^\omega\} \\ C &\longmapsto \{\{n\} \cup s \mid s \in C\}. \end{aligned}$$

Now if $S \in \operatorname{Blocks}(B_n)$ then there is a clopen C of 2^ω such that $C \subseteq N_{0^{n+1}}$ and $S = C \cap B_n$. Then $h_n(C) \cap B = \{\{n\} \cup s \mid s \in S\} \in \operatorname{Blocks}(B)$.

Conversely if $T \in \operatorname{Blocks}(B)$ with $T \subseteq N_{0^n 1}$ then $T = B \cap C$ for a clopen $C \subseteq N_{0^n 1}$. We have $T = h_n(C_n \cap B_n)$. □

Observe that for $S \subseteq T \subseteq [\omega]^{<\infty}$ and $N \in [\omega]^\infty$ we have

$$S|N = S \cap [N]^{<\infty} = S \cap 2^N = S \cap T \cap 2^N = S \cap T|N,$$

hence the shrinkage of S to N equals the trace of S on $T|N$.

Notation. Let $F \subseteq [\omega]^{<\infty}$. For a family $\mathcal{S} \subseteq \mathcal{P}(F)$ of subsets of F and $N \in [\omega]^\infty$ we denote by $\mathcal{S}|N$ the family

$$\{S|N \mid S \in \mathcal{S}\} = \{S \cap F|N \mid S \in \mathcal{S}\}.$$

Lemma 3.10. *Let B be a front on M and let $N \subseteq M$ be infinite. Then*

$$\mathrm{Blocks}(B)|N = \mathrm{Blocks}(B|N).$$

Proof. If $S \in \mathrm{Blocks}(B)$ then there exists C clopen in 2^ω with $S = C \cap B$. It follows that

$$S|N = S \cap B|N = S \cap B \cap B|N = C \cap B|N$$

is a block of $B|N$.

Conversely, if we have $S = C \cap B|N$ for some clopen C of 2^ω then for the block $S' = C \cap B$ of B we have $S'|N = S' \cap B|N = C \cap B \cap B|N = S$. □

Remark 3.11. Observe that for $\mathcal{F}$ a finite family of subsets of a front B on M we can find by repeated application of Nash-Williams' Theorem 2.14 an infinite $N \subseteq M$ such that for all $S \in \mathcal{F}$, $S|N = S \cap B|N$ is either empty or equal to $B|N$. In other terms, for $\mathcal{F}$ a finite family of subsets of a front B on M, there exists $N \in [M]^\infty$ such that $\mathcal{F}|N \subseteq \{\emptyset, B|N\}$.

For a countably infinite family we have:

Theorem 3.12. *Let B be a front on ω and let $\mathcal{S}$ be a countable family of subsets of B. For all $M \in [\omega]^\infty$ there exists $N \in [M]^\infty$ such that $\mathcal{S}|N$ consists in blocks of $B|N$, i.e. $\mathcal{S}|N \subseteq \mathrm{Blocks}(B|N)$.*

Proof. By induction on the tree-rank of B. For the trivial barrier, the theorem is trivial.

Suppose that B is a front of non zero tree-rank on ω and that the statement of the theorem holds for fronts of strictly smaller tree-rank. Let $M \in [\omega]^\infty$.

Claim. *There exists $X \in [M]^\infty$ such that for all $m \in X$ and all $S \in \mathcal{S}$ we have $(S|X)_{\uparrow m} \in \mathrm{Blocks}(B|X)$.*

Proof of the claim. For each $n \in \omega$, since B_n is a front of strictly smaller tree-rank than B, we can apply our induction hypothesis to the countable family $\mathcal{S}_n = \{S_n \mid S \in \mathcal{S}\}$ of subsets of B_n. We thus build recursively a sequence $(X_i)_{i\in\omega}$ of infinite subsets of ω with $k_i = \min(X_i)$, such that

1. $X_{i+1} \in [X_i/k_i]^\infty$;

2. for all $i \in \omega$ we have

$$\mathcal{S}_{k_i}|X_{i+1} \subseteq \mathrm{Blocks}(B_{k_i})|X_{i+1}.$$

Set $X_0 = M$, and suppose that the sequence is defined up to i. Then let $k_i = \min X_i$ and consider the front B_{k_i}. Its tree-rank is strictly smaller than the tree-rank of B. We consider the family $\mathcal{S}_{k_i}$ of subsets of B_{k_i}. By our induction hypothesis there exists $X_{i+1} \in [X_i/k_i]^\infty$ such that $\mathcal{S}_{k_i}|X_{i+1} \subseteq \mathrm{Blocks}(B_{k_i})|X_{i+1}$.

We can then set $X = \{k_i \mid i \in \omega\}$.

To see that X satisfies the Claim, let $k_i \in X$ and $S \in \mathcal{S}$. Since $X/k_i \subseteq X_{i+1}$ we have

$$(S|X)_{k_i} = S_{k_i}|X = S_{k_i}|X/k_i = S_{k_i}|X_{i+1}|X/k_i.$$

Hence by Lemma 3.10, $S_{k_i}|X_{i+1} \in \mathrm{Blocks}(B_{k_i})|X_{i+1}$ implies

$$S_{k_i}|X_{i+1}|X/k_i \in \mathrm{Blocks}(B_{k_i})|X_{i+1}|X/k_i = \mathrm{Blocks}(B_{k_i}|X/k_i).$$

Finally,

$$\{\{k_i\} \cup s \mid s \in (S|X)_{k_i}\} = (S|X)_{\uparrow k_i}$$

is a block of $B|X$ by Lemma 3.9. □

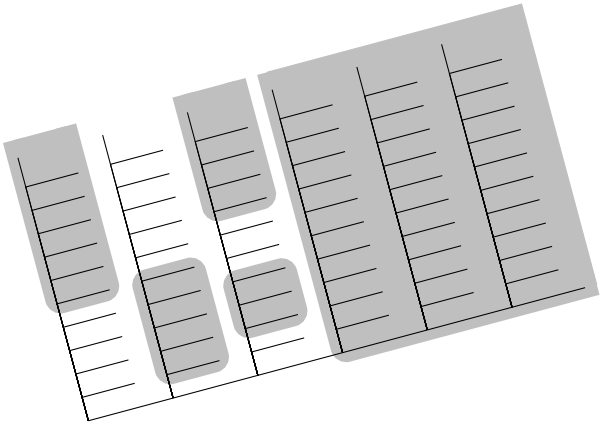

Figure 1: A block of the barrier $[\omega]^2$.

By the Claim, there is no loss of generality in assuming that B is a front on M and that for all $m \in M$ and all $S \in \mathcal{S}$ we have $S_{\uparrow m} \in \mathrm{Blocks}(B)$.

We fix an enumeration $\{S^i \mid i \in \omega\}$ of $\mathcal{S}$. By applying repeatedly Nash-Williams' Theorem we can build a sequence $N_0 \supseteq N_1 \supseteq N_2 \supseteq \cdots$ of infinite

subsets of M such that for $i \in \omega$ $N_{i+1} \subseteq N_i / \min(N_i)$ and for all $j \leq i$ the set $S^j|N_i = S^j \cap B|N_i$ is either empty or equal to $B|N_i$.

For all $i \in \omega$ set $k_i = \min N_i$ and set $N = \{k_0, k_1, \ldots\}$. We claim that for all $S \in \mathcal{S}$ we have $S|N$ is a block of $B|N$. To see this let $S^j \in \mathcal{S}$. We have $\{k_i \mid j \leq i\} \subseteq N_j$. We can partition $B|N$ as

$$B|N = \left(\bigcup_{i=0}^{j-1} B_{\uparrow k_i}|N \right) \cup B|\{k_i \mid j \leq i\}.$$

We also have

$$S^j|N = \left(\bigcup_{i=0}^{j-1} S^j_{\uparrow k_i}|N \right) \cup S^j|\{k_i \mid j \leq i\}.$$

On the one hand, $S^j_{\uparrow k_i} \in \mathrm{Blocks}(B)$ and thus $S^j_{\uparrow k_i}|N$ belongs to $\mathrm{Blocks}(B|N)$ by Lemma 3.10. On the other hand $S^j|\{k_i \mid j \leq i\}$ is either empty or equal to $B|\{k_i \mid j \geq i\}$, and thus is a block of $B|N$. Therefore $S^j|N$ is a block of $B|N$ as a finite union of blocks of $B|N$. □

We can now come to the main result of this section, which is Theorem 1.1 of the Introduction.

Theorem 3.13. *Let B be a front on some infinite subset $M \subseteq \omega$. For all $f : B \to 2^\omega$ there exists an infinite $N \subseteq M$ such that the restriction $f : B|N \to 2^\omega$ is uniformly continuous.*

Proof. Applying Theorem 3.12 to $\mathcal{S} = \{f^{-1}(C) \mid C \text{ is clopen in } 2^\omega\}$ yields a infinite N for which $f|N : B|N \to X$ satisfies for all clopen set C that

$$(f|N)^{-1}(C) = f^{-1}(C) \cap B|N = f^{-1}(C)|N \in \mathrm{Blocks}(B|N).$$

Therefore $f|N$ is uniformly continuous by Proposition 3.2. □

4 The space of ideals of a wqo

We present the space of ideals of a wqo Q as endowed with the topology induced from the generalised Cantor space 2^Q. Recall that the clopen subsets of 2^Q are finite unions of sets of the form

$$N(F, G) = \{P \subseteq Q \mid F \subseteq Q \wedge G \cap Q = \emptyset\}$$

for $F, G \in [Q]^{<\infty}$. The clopen $\subseteq$-upsets of 2^Q, *i.e.* the clopen subsets C such that $P \in C$ and $P \subseteq P'$ implies $P' \in C$, are the finite unions of sets of the form $\langle F \rangle = N(F, \emptyset)$. For $q \in Q$ we write $\langle q \rangle$ instead of $\langle \{q\} \rangle$.

4.1 The space of ideals of a wqo

The results of this subsection come from [PS06, BPZ07]. In order to keep our exposition self-contained we nonetheless provide proofs.

Let Q be any qo. We embed Q, or more precisely the po associated with Q, inside the Cantor space 2^Q via $q \mapsto {\downarrow}q$. We have $Q \subseteq \mathrm{I}(Q) \subseteq \mathrm{Down}(Q)$. We denote by $\overline{Q}$ the closure of Q in 2^Q through the above identification.

Lemma 4.1. *Let Q be a qo. The set $\mathrm{Down}(Q)$ is closed, and Q is dense in $I(Q)$ so $\overline{\mathrm{I}(Q)} = \overline{Q}$ holds.*

Proof. The set $\mathrm{Down}(Q)$ of $\leq$-downward closed subset of Q is closed in 2^Q since

$$\mathrm{Down}(Q) = \bigcap_{p \leq q} \langle q \rangle^{\complement} \cup \langle p \rangle.$$

Let I be an ideal of Q and let $N(F, G)$ be any basic neighbourhood of I in 2^Q for $F, G \in [Q]^{<\infty}$. Since I is directed and nonempty, $F \subseteq I$ implies that there exists $q \in I$ with $F \subseteq {\downarrow}q$. Since I is downward closed, $G \cap I = \emptyset$ implies $G \cap {\downarrow}q = \emptyset$. Therefore ${\downarrow}q \in N(F, G)$. Hence $I \in \overline{Q}$. □

Lemma 4.2. *If Q is a wqo, then $\mathrm{I}(Q)$ is closed in 2^Q and therefore $\overline{Q} = \mathrm{I}(Q)$.*

Proof. For $F \subseteq Q$ let $F^{\uparrow}$ denote the set of upper bounds of F, *i.e.* $F^{\uparrow} = \bigcap_{q \in F} {\uparrow}q = \{p \in Q \mid \forall q \in F \ q \leq p\}$. Since Q has the finite basis property, for all $F \in [Q]^{<\infty}$ there exists $G \in [Q]^{<\infty}$ such that $F^{\uparrow} = {\uparrow}G$. Therefore

$$\bigcup_{p \in F^{\uparrow}} \langle p \rangle = \bigcup_{r \in G} \langle r \rangle$$

is clopen as a finite union of clopen.

Now we can see that

$$\begin{aligned} \mathrm{I}(Q) =& \{I \in \mathrm{Down}(Q) \mid \forall F \in [Q]^{<\infty} (F \subseteq I \to \exists p \in F^{\uparrow} (p \in I)\} \\ =& \mathrm{Down}(Q) \cap \bigcap_{F \in [Q]^{<\infty}} \Big(\langle F \rangle^{\complement} \cup \bigcup_{r \in F^{\uparrow}} \langle p \rangle \Big). \end{aligned}$$

is closed. □

Recall that a point x of a topological space X is isolated in X if the singleton $\{x\}$ is open. A *limit point* of a topological space X is a point that is not *isolated*, *i.e.* for every neighbourhood U of x there is a point $y \in U$ with $y \neq x$. A topological space with no isolated points is *perfect*. On the other extreme, a topological space is called *scattered* if it admits no perfect subspace.

For wqo's have the additional property:

Proposition 4.3 ([PS06]). *Let Q be wqo. The space $\mathrm{I}(Q)$ is scattered, compact and its isolated points coincide with the principal ideals.*

Proof. We first show that $\mathrm{Down}(Q)$ is scattered. Let $X \subseteq \mathrm{Down}(Q)$ be non empty. Since Q is wqo, then $\mathrm{Down}(Q)$ is well founded and there exists D which is $\subseteq$-minimal in X. Then for a finite F with $\uparrow F = Q \setminus D$ observe that $N(\emptyset, F) \cap X = \{D\}$ showing that D is isolated in X.

We now show that for $I \in \mathrm{I}(Q)$, I is isolated iff I is principal. If I is isolated, $\{I\}$ is open, and since Q is dense in $I(Q)$, $I = \downarrow q$ holds for some $q \in Q$ and I is principal. Conversely for $q \in Q$ let F be a finite basis for $Q \setminus \downarrow q$. We have $N(\{q\}, F) \cap \mathrm{I}(Q) = \{\downarrow q\}$. □

Remark 4.4. As a topological ordered space, the space $\mathrm{I}(Q)$ of ideals of a wqo is dual under Priestley duality to the bounded distributive lattice $(\mathrm{Down}(Q), \subseteq)$. From this point of view one directly gets that any order preserving map between wqos extends to a continuous order preserving map between the corresponding ideal spaces.

4.2 Extending supersequences into the ideals

We now turn to showing that any map from a front to a wqo restricts on a front to a uniformly continuous map, which therefore extends continuously to the space of ideals.

However Theorem 3.13 makes essential use of the metrisability of the codomain. In order to apply this result we need to show that if Q is wqo then for every countable subset P of Q the topological closure of P inside $\mathrm{I}(Q)$ is metrisable, in fact we show it is isomorphic to the ideal space of P.

Since the association of the topological space of ideals to any wqo is actually functorial, the following lemma should come as no surprise.

Lemma 4.5. *Let Q be wqo and $P \subseteq Q$. Then the topological ordered space $(\mathrm{I}(P), \subseteq)$ is isomorphic to the closure of P in $\mathrm{I}(Q)$.*

Proof. We first prove that the map

$$\begin{aligned} \iota : \mathrm{Down}(P) &\longrightarrow \mathrm{Down}(Q) \\ D &\longmapsto \downarrow_Q D \end{aligned}$$

is an embedding.

To see it is injective, observe that for all $D \in \mathrm{Down}(P)$, $D = P \cap \downarrow_Q D$ holds.

To see it is an order embedding, observe that monotonicity of the closure operator $D \subseteq D'$ implies $\downarrow_Q D \subseteq \downarrow_Q D'$. Conversely, if $\downarrow_Q D \subseteq \downarrow_Q (D')$, take $p \in D$ then $p \in P \cap \downarrow_Q D' = D'$.

To show it is an embedding, it is enough to prove that it is continuous, since both spaces are compact Hausdorff. For the continuity, it suffices to show that for all $q \in Q$ the set $\iota^{-1}\{E \in \mathrm{Down}(Q) \mid q \in E\}$ is clopen in $\mathrm{Down}(P)$. So let $q \in Q$. Since P is wqo as a subset of a wqo, there exists a finite $F \subseteq P$ such that $(\uparrow_Q q) \cap P = \uparrow_P F$. Now we obtain

$$\begin{aligned}\iota^{-1}\{E \in \mathrm{Down}(Q) \mid q \in E\} &= \{D \in \mathrm{Down}(P) \mid q \in \downarrow_Q D\} \\ &= \{D \in \mathrm{Down}(P) \mid F \cap D \neq \emptyset\}.\end{aligned}$$

Indeed if $q \in \downarrow_Q D$ then there is $p \in D$ with $q \leq p$. Thus $p \in (\uparrow_Q q) \cap P = \uparrow_P F$ and so there exists $r \in F$ with $r \leq p$. Hence since D is a downset we have $r \in D$.

Conversely, if there exists $r \in F \cap D$ then since $(\uparrow_Q q) \cap P = \uparrow_P F$ we have $q \leq r$. It follows that $q \in \downarrow_Q D$.

We can thus identify $\mathrm{Down}(P)$ as a subset of $\mathrm{Down}(Q)$. Under this identification we obtain that $\mathrm{I}(P)$, which is the closure of P in $\mathrm{Down}(P)$, is also the closure of P in $\mathrm{Down}(Q)$. □

Observe that if Q is a countable wqo, then $\mathrm{I}(Q)$ is a countable closed subset of the Cantor space 2^Q. Using the previous lemma, we obtain that Theorem 3.13 applies to maps from fronts into wqo, yielding Theorem 1.2 of the Introduction.

Theorem 4.6. *Let B be a front on N and Q be wqo. For every map $f : B \to Q$ there exists $M \in [N]^\infty$ such that $f|M : B|M \to Q$ is uniformly continuous, therefore extends uniquely to a continuous map $\bar{f} : \overline{B} \to \mathrm{I}(Q)$.*

Proof. Let $P = \mathrm{Im}\, f$. Then $f : B \to Q$ is uniformly continuous if and only if $f : B \to 2^Q$ is uniformly continuous iff $f : B \to \overline{P}$ is uniformly continuous. By Lemma 4.5, $\overline{P}$ is homeomorphic to $\mathrm{I}(P)$. Since P is countable $\mathrm{I}(P)$ is a subspace of the Cantor space 2^P, and we can apply Theorem 3.13. □

5 Continuous extensions into the ideals

By the previous section, any map $f : B \to Q$ from a front into a wqo restricts on a front B' to a uniformly continuous map $f' : B' \to Q$. This map then extends to a continuous map $f' : \overline{B'} \to I(Q)$. We now study such continuous map going from the closure of a front into the space of ideals of a wqo.

Here is a crucial example.

Example 5.1 (Rado's poset)**.** Let $\mathcal{R}$ be the poset $[\omega]^2$ ordered by

$$(m,n) \leq (m',n') \quad \text{iff} \quad \begin{cases} m = m' \text{ and } n \leq n', \text{ or} \\ n < m'. \end{cases}$$

We claim that $\mathrm{I}(\mathcal{R}) = \mathcal{R} \cup \{I_n \mid n \in \omega\} \cup \{\top\}$ where for all $n \in \omega$ $I_n = \bigcup_{n<k} {\downarrow}(n,k)$ and $\top = \mathcal{R}$. We have $(m,n) \leq I_k$ iff $m = k$ or $n < k$, and $a \leq \top$ for all $a \in \mathrm{I}(\mathcal{R})$. It is clear that each I_n and $\top$ are non principal ideals. We show there are no other ideals. Let I be an ideal of $\mathcal{R}$. First suppose for all $k \in \omega$ there is an $(m,n) \in I$ with $k < m$, then $I = \top$. Suppose now that there exists $m = \max\{k \mid \exists l (k,l) \in I\}$. If there is infinitely many n such that $(m,n) \in I$ then $I = I_m$. Otherwise $I = {\downarrow}(m,n)$ for $n = \max\{l \mid (m,l) \in I\}$.

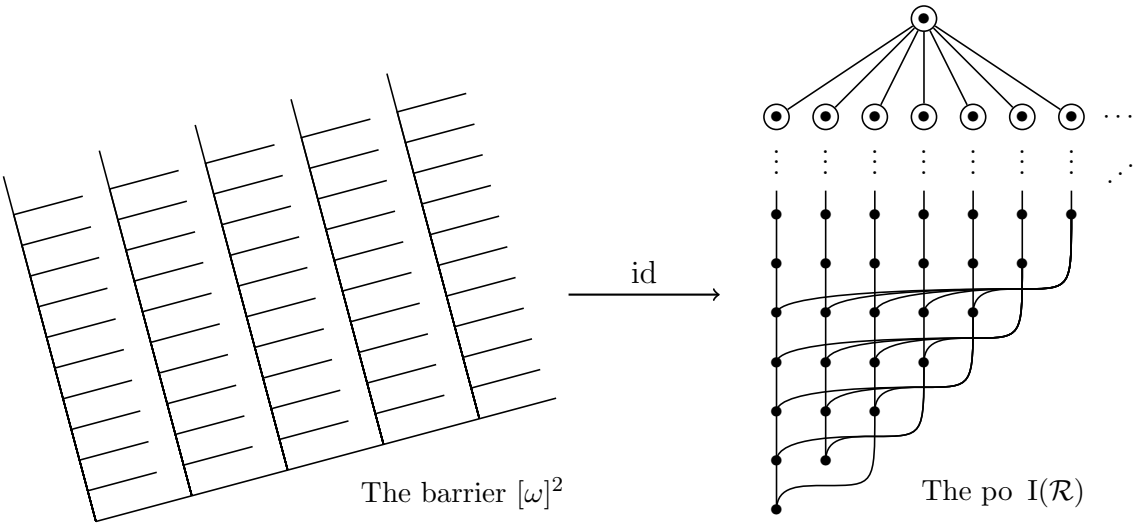

Figure 2: A bad Cauchy supersequence in Rado's poset.

It is clear that $\mathrm{I}(\mathcal{R})$ is not wqo. Now consider the supersequence $\mathrm{id} : [\omega]^2 \to \mathcal{R}$, which is the identity on the underlying sets. It is bad and one can show that it is actually uniformly continuous.

The closure of $[\omega]^2$ in 2^ω is just $[\omega]^{\leq 2} = [\omega]^2 \cup [\omega]^1 \cup \{\emptyset\}$. The continuous extension $\overline{\mathrm{id}} : [\omega]^{\leq 2} \to \mathrm{I}(\mathcal{R})$ is simply given by $\overline{\mathrm{id}}(\{m\}) = I_m$ and $\overline{\mathrm{id}}(\emptyset) = \top$. Now the restriction of $\overline{\mathrm{id}}$ to the barrier $[\omega]^1$ is a bad sequence in $\mathrm{I}^*(\mathcal{R})$ witnessing the fact that it not wqo. Hence this uniformly continuous bad supersequence on $\mathcal{R}$ yields a bad supersequence in the non principal ideals of $\mathcal{R}$. We will see that this is always the case.

5.1 Continuous extensions of supersequences

In Example 5.1, the continuous map from the closure of the barrier $[\omega]^2$ to ideals of Rado's poset enjoys interesting properties. Notably, the points of $\overline{[\omega]^2}$ whose image is a non principal ideals form the closure of a barrier, namely $[\omega]^1$. In fact, by shrinking, any continuous map enjoys such a canonical form as we shall now see.

Let $f : \overline{B} \to \mathcal{X}$ be a map from the closure of a front B on some $N \in [\omega]^\infty$ to a topological space $\mathcal{X}$. We write

$$\Lambda_f = \{s \in \overline{B} \mid f(s) \text{ is not isolated in } \mathcal{X}\}$$

for the closed set of $\overline{B}$ of those points whose image is a limit point. Observe that if g is the restriction of f to the closure of a shrinkage of B, then Λ_g is a shrinkage of Λ_f. Formally, for all $M \in [N]^\infty$ we have $\Lambda_{f\restriction(\overline{B}|M)} = \Lambda_f|M$.

Recall that by Corollary 2.21, for all $M \in [N]^\infty$ we have $\overline{B}|M = \overline{B|M}$ and $\left(\overline{B}\right)_n = \overline{B_n}$ for all $n \in N$.

Here is Theorem 1.3 from the Introduction.

Theorem 5.2. *Let B be a front on ω, $\mathcal{X}$ be a topological space and $f : \overline{B} \to \mathcal{X}$ be a continuous map. For all $N \in [\omega]^\infty$ there exists $M \in [N]^\infty$ such that*

1. *$\Lambda_f|M$ is either empty or the closure of a barrier on M.*
2. *for all $s, t \in \overline{B|M}$ if $s \notin \Lambda_f|M$ and $s \sqsubseteq t$ then $f(s) = f(t)$.*

Proof. By induction on the tree-rank of B. The theorem is obvious for the trivial barrier. We suppose it holds for all continuous maps from the closure of a front with tree-rank strictly smaller than $\alpha > 0$. Let B be a front on N with $\operatorname{rk} B = \alpha$.

Claim. *There exists $X \in [N]^\infty$ such that for all $k \in X$ the map*

$$\begin{aligned}(\overline{B|X})_k &\longrightarrow \mathcal{X}\\ s &\longmapsto f(\{k\} \cup s)\end{aligned}$$

satisfies the requirements of the theorem.

Proof of the claim. We build by induction a sequence $(X_i)_{i\in\omega}$, with k_i the minimum of X_i, such that for all $i \in \omega$ we have

1. $X_{i+1} \in [X_i/k_i]^\infty$,
2. $(\Lambda_f)_{k_i}|X_{i+1}$ is either empty or the closure of a barrier on X_{i+1};

3. for all $s,t \in \overline{B}_{k_i}|X_{i+1}$, if $\{k_i\} \cup s \notin \Lambda_f$ and $s \sqsubseteq t$ then $f(\{k_i\} \cup s) = f(\{k_i\} \cup t)$.

Set $N = X_0$, and suppose X_i is built. The family B_{k_i} is a front on X_i/k_i with $\mathrm{rk}(B_{k_i}) < \alpha$, so we can use the induction hypothesis on the continuous map $f_{k_i} : \overline{B_{k_i}} \to \mathcal{X}$ defined by $s \mapsto f(\{k_i\} \cup s)$, and we get X_{i+1}.

Setting now $X = \{k_0, k_1, \ldots\}$ we get that for all $i \in \omega$ $X/k_i \subseteq X_{i+1}$ and thus

$$\left(\Lambda_{f\restriction\overline{B|X}}\right)_{k_i} = (\Lambda_f|X)_{k_i} = (\Lambda_f)_{k_i}|X/k_i = (\Lambda_f)_{k_i}|X_{i+1}|X/k_i$$

is either empty or the closure of a barrier on X/k_i. Moreover let $s,t \in \overline{B|X}_{k_i}$ with $\{k_i\} \cup s \notin \Lambda_f$ and $s \sqsubseteq t$. Then $s,t \subseteq X_{i+1}$ and thus $f(\{k_i\} \cup s) = f(\{k_i\} \cup t)$. □

Therefore we can suppose without loss of generality that $f : \overline{B} \to \mathcal{X}$ is such that for all $n \in N$ the map

$$\begin{aligned} \overline{B}_n &\longrightarrow \mathcal{X} \\ s &\longmapsto f(\{n\} \cup s) \end{aligned}$$

satisfies the requirements of the theorem.

We now distinguish two cases

$\emptyset \notin \Lambda_f$: Since f is continuous we have $f(\emptyset) = \lim_{n\in N} f(\{n\})$ and as $f(\emptyset)$ is isolated in $\mathcal{X}$ there exists a $M \in [N]^\infty$ such that $f(\{m\}) = f(\emptyset)$ for all $m \in M$. Then for all $m \in M$ we have that $\{m\} \notin \Lambda_f$ that is $\emptyset \notin (\Lambda_f)_{\{m\}}$. Therefore $(\Lambda_f)_{\{m\}}$ is empty for all $m \in M$ and thus $\Lambda_f|M$ is empty.

$\emptyset \in \Lambda_f$: There exists $X \in [N]^\infty$ such that either for all $k \in X$ $\{k\} \notin \Lambda_f$ or for all $k \in X$ $\{k\} \in \Lambda_f$.

In the former case, we have that $\Lambda_f|X = \{\emptyset\}$ and so we can set $M = X$ which meets the conditions.

In the latter case, for all $k \in X$ the set $(\Lambda_f|X)_k$ is the closure of a barrier $B(k)$ on X/k. The family

$$L = \bigcup_{k\in X} \{\{k\} \cup s \mid s \in B(k)\}$$

is a front on X, so by Theorem 2.15 there exists $M \in [X]^\infty$ such that $L|M$ is a barrier on M. We have that

$$\Lambda_f|M = \overline{L}|M = \overline{L|M}$$

is the closure of the barrier $L|M$ on M. This M meets the requirements. □

5.2 A proof of Pouzet's conjecture

We need a result on the convergence of sequences in the space of ideals of a wqo. Recall that for a sequence $(E_n)_{n\in\omega}$ of subsets of a set Q we have the usual relations

$$\bigcap_{n\in\omega} E_n \subseteq \bigcup_{i\in\omega}\bigcap_{j\geq i} E_j \subseteq \bigcap_{i\in\omega}\bigcup_{j\geq i} E_j \subseteq \bigcup_{n\in\omega} E_n.$$

Moreover,

Fact 5.3. *The sequence $(E_n)_{n\in\omega}$ converges to E in 2^Q if and only if*

$$\bigcup_{i\in\omega}\bigcap_{j\geq i} E_j = \bigcap_{i\in\omega}\bigcup_{j\geq i} E_j = E.$$

Proof. Suppose that $E = \bigcup_{i\in\omega}\bigcap_{j\geq i} E_j = \bigcap_{i\in\omega}\bigcup_{j\geq i} E_j$. We show that $E_n \to E$. Let $F, G \in [Q]^{<\infty}$ be such that $E \in N(F,G)$. Since we have $E = \bigcup_{i\in\omega}\bigcap_{j\geq i} E_j$ and F finite, $F \subseteq E_j$ for all sufficiently large j. Since $E = \bigcap_{i\in\omega}\bigcup_{j\geq i} E_j$ and G is finite, $G \cap E_j = \emptyset$ for all sufficiently large j. Hence $E_j \in N(F,G)$ for all sufficiently large j and therefore $E_n \to E$.

Conversely, suppose that $E_n \to E$ for some $E \subseteq Q$. If $q \in E$, then $q \in E_j$ for all sufficiently large j and thus $q \in \bigcup_{i\in\omega}\bigcap_{j\geq i} E_j$. Now if $q \notin E$ then $q \notin E_j$ for all sufficiently large j and thus $q \notin \bigcap_{i\in\omega}\bigcup_{j\geq i} E_j$. Therefore $E = \bigcup_{i\in\omega}\bigcap_{j\geq i} E_j = \bigcap_{i\in\omega}\bigcup_{j\geq i} E_j$. □

We found the following ingenious observation, inside a proof, in [Rad54].

Lemma 5.4 (Rado's trick)**.** *Let Q be a wqo and let $(f(n))_{n\in\omega} \subseteq \mathrm{Down}(Q)$. Then there exists an infinite subset N of ω such that*

$$\bigcup_{n\in N} f(n) = \bigcup_{i\in N}\bigcap_{j\in N/i} f(j).$$

Proof. Towards a contradiction suppose that for all infinite $N \subseteq \omega$ we have

$$\bigcup_{i\in N}\bigcap_{j\in N/i} f(j) \subset \bigcup_{n\in N} f(n).$$

We construct a $\subset$-descending chain $(D_i)_{i\in\omega}$ in $\mathrm{Down}(Q)$. We build recursively a sequence $(N_i)_{i\in\omega}$ of infinite subsets of ω, and set $D_i = \bigcup_{i\in N_i} f(i)$.

Set $N_0 = \omega$ and suppose we have defined $N_k \in [\omega]^\infty$. Let $n_0 \in N_k$ minimal such that there exists $q_k \in f(n_0) \setminus \bigcup_{i\in N_k}\bigcap_{j\in N_k/i} f(j)$. Then for all i in N_k

there exists $j_i \in N_k/i$ such that $q_k \notin f(j_i)$. Setting $n_{i+1} = j_{n_i}$ we obtain an infinite set $N_{k+1} = \{n_0, n_1, n_2, \ldots\}$ we have

$$q_k \in \bigcup_{j \in N_k} f(j) \setminus \bigcup_{j \in N_{k+1}} f(j).$$

The sequence $(D_k)_{k\in\omega}$ is strictly decreasing for the inclusion, contradicting the fact that Q is wqo. □

In the topological setting, Rado's trick states that the space of downsets of a wqo enjoys a much stronger property than sequential compactness as states

Proposition 5.5. *Let $(D_n)_{n\in\omega}$ be a sequence downsets of a wqo Q. Then there exists an infinite $N \subseteq \omega$ such that $(D_n)_{n\in N}$ converges to $\bigcup_{n\in N} D_n$ in 2^Q.*

We now turn to the proof of the following theorem on better quasi-orders. This is Theorem 1.5 from the introduction.

Theorem 5.6. *Let Q be wqo. If $\mathrm{I}^*(Q)$ is bqo, then Q is bqo.*

Proof. We assume that Q is wqo and that $\mathrm{I}^*(Q)$ is bqo. Let f be any map from a barrier B on ω into Q. Shrinking B, we can assume by Theorem 4.6 that $f : B \to Q$ uniquely extends to a continuous map $\bar{f} : \overline{B} \to \mathrm{I}(Q)$. By shrinking further, we can assume by Theorem 5.2 that $\Lambda = \{s \in \overline{B} \mid f(s) \in \mathrm{I}^*(Q)\}$ is either empty or the closure of a barrier C on, say, ω and that for $s, t \in \overline{B}$ with $s \notin \Lambda$ and $s \sqsubseteq t$ we have $f(s) = f(t) \in Q$. Observe that in particular for every $s \in C$, we have on the one hand that $s \notin B$, and on the other hand for all $n \in \omega/s$ we have $s \cup \{n\} \in \overline{B}$ and $f(s \cup \{n\}) \in Q$.

$\Lambda = \emptyset$: Then $f : B \to Q$ is constant and thus good.

C is trivial: Then, since Q is wqo, there is an $M \in [\omega]^\infty$ such that the restriction $\bar{f} : [M]^1 \to Q$ is perfect. Now pick any $s, t \in B|M$ with $s \lhd t$, we have $f(s) = f(\min s) \leq f(\min t) = f(t)$.

C is non trivial: Then, since $\mathrm{I}^*(Q)$ is bqo, there exists an $M \in [\omega]^\infty$ such that the restriction $\bar{f} : C|M \to \mathrm{I}^*(Q)$ is perfect. Choose any $s' \in C|M$. Since $s' \cup \{m\} \in \overline{B|M}$ for all $m \in M/s'$ and $s' = \lim_{m\in M/s'} s \cup \{m\}$, the continuity of $\bar{f}$ implies that $\bar{f}(s') = \lim_{m\in M/s'} \bar{f}(s' \cup \{m\})$. By Proposition 5.5 there is $X \in [M/s']^\infty$ with $\bar{f}(s') = \bigcup_{k\in X} {\downarrow} \bar{f}(s' \cup \{k\})$. Let $k_0 = \min X$. There exists $t' \in C|M$ with $t' \sqsubset {}_*s' \cup \{k_0\} \cup X/k_0$. Necessarily, ${}_*s' \cup \{k_0\} \sqsubseteq t'$ otherwise $t' \subset s'$ contradicting the fact that $C|M$ is a barrier. Again, by Proposition 5.5 there is $Y \in [M/t']^\infty$ such that

$f(t') = \bigcup_{l\in Y} {\downarrow} \bar{f}(t' \cup \{l\})$. Since $\bar{f} : C|M \to \mathrm{I}_\omega(Q)$ is perfect and $s' \lhd t'$, we have that

$$\bigcup_{k\in X} {\downarrow} \bar{f}(s' \cup \{k\}) = \bar{f}(s') \subseteq \bar{f}(t') = \bigcup_{l\in Y} {\downarrow} \bar{f}(t' \cup \{l\}).$$

In particular, there exists $l_0 \in Y$ such that $\bar{f}(s' \cup \{k_0\}) \leq \bar{f}(t' \cup \{l_0\})$.

Finally let $t \in B$ with $t' \cup \{l_0\} \sqsubseteq t$ and let $s \in B$ be such that $s \sqsubset s' \cup t \cup M/t$ then necessarily $s \sqsubset s' \cup t$ otherwise $B \ni t \subset s \in B$. Therefore we found $s, t \in B$ with $s \lhd t$ and since $s' \cup \{k_0\} \sqsubseteq s$ and $t' \cup \{l_0\} \sqsubseteq t$ we have

$$f(s) = \bar{f}(s' \cup \{k_0\}) \leq \bar{f}(t' \cup \{l_0\}) = f(t).$$

In each case we conclude that $f : B \to Q$ is good. It follows that Q is bqo. □

In [PS06], a stronger result is actually stated, namely: if Q is wqo and the po $\mathrm{I}_\omega(Q)$ of ideals with cofinality ω is bqo, then Q is bqo.

To see that we have actually proved this stronger statement, remember that $\mathrm{I}_{\leq\omega}(Q) \cap \mathrm{I}^*(Q) = \mathrm{I}_\omega(Q)$ and note the following simple corollary to Proposition 5.5:

Fact 5.7. *Let $f : B \to Q$ be a uniformly continuous map from a barrier into a wqo. Then its unique continuous extension $\bar{f}$ has image in $\mathrm{I}_{\leq\omega}(Q)$.*

Proof. Since B is a metrisable uniform space, every point of its completion is the limit of a sequence. Let $s \in \overline{B}$ and let $(s_n)_{n\in\omega} \subseteq B$ converge to s. Then by continuity of $\bar{f}$ we have $\bar{f}(s) = \lim_{n\in\omega} f(s_n)$. Then by Corollary 5.5 there exists a $N \in [\omega]^\infty$ such that $\bar{f}(s) = \bigcup_{n\in N} {\downarrow} f(s_n)$. Therefore $\bar{f}(s)$ has countable cofinality. □

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