HW4R — Randomized algorithms & Random structures

MPRI 1.24 — Wed. Dec. 16, 2015 - **Due on Wed. Jan. 6, 2016**

You are asked to complete the exercise marked with a [★] and to send me your solutions at:
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(or drop it in my mail box at the 4th floor of Sophie Germain) on **Wed. Jan. 6, 2016**.

■ **Exercise 1 (Using fingerprints to check matrix multiplication).** [★]
The best known algorithms for multiplying two $n \times n$-matrices take time $O(n^\alpha)$ for some $\alpha > 2$. However, there is a simple randomized algorithm for verifying matrix products with high probability in $O(n^2)$ time.

Suppose I claim that $AB = C$ where A, B, and C are integer-valued $n \times n$-matrices. You can confirm this by applying both sides of this equation to a random vector v, and checking that $ABv = Cv$.

► ***Question* 1.1)** *Describe how to check if $ABv = Cv$ in $O(n^2)$ time, i.e. without computing the $n \times n$-matrix product AB.*

We desire to improve furthermore the checking procedure by using fingerprints. Assume that $AB \neq C$ and let $\alpha = \max_{i,j}\{|A_{ij}|, |B_{ij}|, |C_{ij}|\}$ bound the maximum coefficient in A, B and C.

► ***Question* 1.2)** *Explain how to choose a prime number $p = O(\log(n\alpha) \log\log(n\alpha))$ such that with probability at least $\frac{9}{10}$, we have $AB - C \neq 0 \mod p$.*
▷ Hint. *Use that the kth prime number p_k verifies $0.91k \ln k < p_k < 1.7k \ln k$, as proved by Felgner in 1990.*

Assume now that we are lucky and $AB - C \neq 0 \mod p$.

► ***Question* 1.3)** *Show that if v is chosen uniformly at random from $\{0, 1, \ldots, p-1\}^n$, then the probability that $ABv = Cv \mod p$ holds is at most $1/p$.*

■ **Exercise 2 (FPT algorithm for spotting k disjoint triangles).** [★]
Given $G = (V, E)$ an undirected graph ($n = |V|$ and $m = |E|$) and k an integer, we are looking for k vertex-disjoint triangles in G. Note that this problem is NP-complete when k is part of the input. We are looking for an algorithm of time complexity $O(f(k)n^a m^b)$ where the exponents a and b are constant, independent of k. Such an algorithm is called FPT for Fixed Parameter Tractable, which means that the complexity is a fixed-degree polynomial in the size of the input for any fixed value of the parameter k. Consider the following randomized algorithm:

Algorithm 1 FPT randomized algorithm for k disjoint triangles

- Choose independently for each vertex u a color $c_u \in \{1, \ldots, 3k\}$ uniformly at random.
- **return** "Yes" if there is a *colorful* solution, i.e. a set of k triangles whose $3k$ vertices use exactly once each color; **return** "I don't know" otherwise.

► ***Question* 2.1)** *Show that if G contains k disjoint triangles, then the probability that the algorithm answer "Yes" is at least e^{-3k}.*
▷ Hint. *use that : $k! \geqslant (k/e)^k$ for all k.*

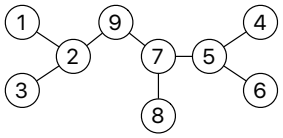
► ***Question* 2.2)** *How many times should you run this algorithm to improve success probability to $1/2$?*

In order to check whether a colorful solution exists, we propose to try all permutations π on $\{1, \dots, 3k\}$ and check if there is are triangles of colors $(\pi_1, \pi_2, \pi_3), \dots, (\pi_{3k-2}, \pi_{3k-1}, \pi_{3k})$.

► ***Question* 2.3)** *Describe an algorithm that decides if such a collection of triangles exists. (Explicit the exact data structure you are using). What is the overall expected time complexity in k, n and m, of the algorithm that uses this method to return k disjoint triangles with probability at least $1/2$ if they exists in G? What is the time complexity if k is fixed?*

■ **Exercise 3 (Binomial laws composition).** Show that when the laws considered are independent, $\mathsf{Bin}(\mathsf{Bin}(n, p), q) \sim \mathsf{Bin}(n, pq)$ for all $n \in \mathbb{N}$ and $p, q \in [0, 1]$.

■ **Exercise 4 (Prüfer sequence).** Consider a (unrooted) tree with n nodes labelled from 1 to n. The Prüfer sequence encoding this tree is obtained by removing vertices from the tree until only two vertices remain. Specifically, at step i, remove the leaf with the smallest label and set the ith element of the Prüfer sequence to the label of this leaf's neighbor. For instance, the Prüfer sequence for the tree bellow is $2, 2, 5, 5, 7, 7, 7$.



► ***Question* 4.1)** *Show that every labelled unrooted tree corresponds a unique sequence in $\{1, \dots, n\}^{n-2}$ and reciprocally.*
▷ Hint. *First, show how to compute the degree of every node from a given Prüfer sequence.*

► ***Question* 4.2)** *Conclude that there are n^{n-2} labelled unrooted trees with n nodes.*

► ***Question* 4.3)** *More precisely, show that there are*

$$\binom{n-2}{d_1-1,\, d_2-1,\, \dots,\, d_n-1} = \frac{(n-2)!}{(d_1-1)!(d_2-1)!\cdots(d_n-1)!}$$

labelled unrooted trees with n nodes whose ith vertex has degree d_i.