HW 1 MPRI 1.24 Randomized Algorithms

Thu. Jan. 23, 2014 - **Due on Thu. Jan. 30, 2014**

You are asked to complete the exercises and send me your solutions by email at:
nicolas.schabanel@liafa.fr
(or drop it in my mail box on the 4th floor) on or before **Thu. Jan. 30, 2014 at noon**.

■ **Exercise 1 (Better randomized algorithm for Min Cut).** Consider the following recursive algorithm for Min Cut:

Procedure FastMinCut(G : an undirected multigraph)
if $n \leqslant 6$ **then**
  return an optimal min-cut obtained by exhaustive search.
else
  $t := 1 + \lceil n/\sqrt{2} \rceil$.
  Make two copies G_1 and G_2 of G.
  for $i = 1..2$ **do**
    Make $n-t$ independent random edge contractions on G_i to obtain multi-graph H_i with t vertices.
  return the best cut between FastMinCut(H_1) and FastMinCut(H_2).

► ***Question* 1.1)** *Show that its running time is* $T(n) = O(n^2 \log n)$.

► ***Question* 1.2)** *Show that it uses at most* $M(n) = O(n^2)$ *memory.*

► ***Question* 1.3)** *Show that the probability that a min cut survives* $n - t$ *random edge contractions is at least* $\frac{1}{2}$ *when* $n > 6$.

Let $P(k)$ be the minimum probability that the algorithm outputs a min cut for a multi-graph that requires k levels of recursions ($k = \Theta(\log n)$ if G has n vertices).

► ***Question* 1.4)** *Show that* $P(k) \geqslant p(k)$ *where* $p(0) = 1$ *and* $p(k+1) = p(k) - \dfrac{p(k)^2}{4}$.

Let $q(k) = \dfrac{4}{p(k)} - 1$ so that $q(0) = 3$ and $q(k+1) = q(k) + 1 + \dfrac{1}{q(k)}$.

► ***Question* 1.5)** *Show that for all* $k \geqslant 0$, $k < q(k) \leqslant k + H_{k-1} + 3$, *where* $H_k = \sum_{i=1}^{k} \frac{1}{i}$ *is the* k*th harmonic number.*

► ***Question* 1.6)** *Conclude that* FastMinCut *computes a min cut with probability* $\Omega(1/\log n)$. *Propose an algorithm which increases the success probability to* $1 - 1/n^2$. *Compare its time complexity to the best known deterministic algorithm (based on a max flow computation),* $O(mn \log(m^2/n))$.

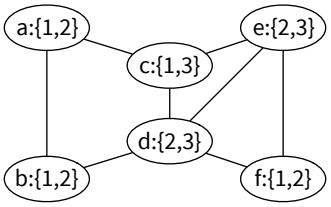
■ **Exercise 2 (Dumb algorithm for Max-SAT).** Consider the following algorithm for Max-SAT: Let τ be an arbitrary instantiation of the variables and τ' the complementary instantiation (each variable is true in τ' if and only if it is false in τ); compute the total weight of the clauses satisfied for τ and for by τ' and output the best of these two instantiations.

► ***Question* 2.1)** *Show that this algorithm is a $\frac{1}{2}$-approximation for Max-SAT, and exhibit a tight instances family.*

■ **Exercise 3 (Constraint-based 3-coloring randomized algorithm).** Given an undirected graph $G = (V, E)$ with a pair p_u of integers chosen in $\{1, 2, 3\}$ on each vertex u, the *constrained 3-coloring* problem consists in finding a valid 3-coloring α of the vertices such that the color of each vertex u is chosen in p_u, i.e. an $\alpha : V \to \{1, 2, 3\}$ such that:

$$\text{for all } uv \in E,\ \alpha_u \neq \alpha_v \quad \text{and} \quad \text{forall } u \in V, \alpha_u \in p_u.$$

► ***Question* 3.1)** *Give a solution to the following constrained 3-coloring instance:*



► ***Question* 3.2)** *Show that solving a constrained 3-coloring instance is equivalent to solve a 2-SAT instance that you are asked to describe explicitly. Give the 2-SAT instance corresponding the graph above.*

We consider the following algorithm for 3-coloring an undirected graph $G = (V, E)$.

Algorithm 1 Constraint-based 3-coloring randomized algorithm

repeat

- Choose independently for each vertex u a pair p_u of integers in $\{1, 2, 3\}$ uniformly at random.
- Solve the corresponding constrained 3-coloring using the reduction above to a 2-SAT instance.

until a 3-coloring is found

► ***Question* 3.3)** *Show that if G is 3-colorable, then this algorithm will find a 3-coloring after at most $O((\frac{3}{2})^n)$ iterations on expectation, where $n = |V|$. Give an upper bound on the expected total computation time of this algorithm if this case.*

► ***Question* 3.4)** *After how many iterations, should we stop the algorithm and declare the graph as "not 3-colorable" so as to be mistaken with probability at most $\frac{1}{n^2}$? What is the failure probability if the graph is indeed not 3-colorable?*