MPRI, Fondations mathématiques de la théorie des automates

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Avertissement : On attachera une grande importance à la clarté, à la précision et à la concision de la rédaction.

Partie 1. Étude d'un langage.

Soit $A = \{a, b\}$ et soit $L = (aa)^*(bb)^*$.

Question 1. Trouver l'automate minimal de L.

Question 2. Calculer le monoïde syntactique M de L. On donnera la liste des éléments et des relations permettant de définir M (vous devriez trouver 10 éléments, en comptant l'élément neutre).

Question 3. Donner la liste des idempotents de M.

Question 4. Déterminer la structure en $\mathcal{D}$-classes de M et dessiner les diagrammes boîtes à œufs.

Question 5. Le langage L est-il sans étoile? Justifier votre réponse.

Partie 2. Délai de synchronisation

Si K est un langage, on note $W(K)$ l'ensemble des mots qui ne sont facteurs d'aucun mot de K. Autrement dit,

$$W(K) = \{u \in A^* \mid A^*uA^* \cap K = \emptyset\}$$

Question 6. Montrer que si K est rationnel (respectivement sans étoile), $W(K)$ l'est aussi.

Soit L un langage rationnel de A^*. On dit que L a un *délai de synchronisation fini* s'il existe un entier $d \geqslant 0$ tel que pour tout $u, v, w \in A^*$ et pour tout $v \in L^d$,

$$uvw \in L^* \text{ et } v \in L^d \implies uv \in L^* \text{ et } w \in L^*.$$

Le plus petit entier d vérifiant cette condition est alors appelé le *délai de synchronisation* de L.

Question 7.

(1) Quels sont les langages de délai 0?

(2) Montrer que le langage a^*b a un délai de synchronisation (que l'on déterminera)

(3) Montrer que le langage $\{ab, ba\}$ n'a pas de délai de synchronisation fini.

Dans la suite du problème, L désigne un **langage de délai de synchronisation** d et on pose $V = W(L^{d+2}) \setminus A^*L^{d+1}A^*$.

Question 8. Soit $w \in A^*L^dA^*$. Montrer que pour tout $u_1, u_2, u_3, u_4 \in A^*$, on a

$$u_1wu_2, u_3wu_4 \in L^* \implies u_1wu_4, u_3wu_2 \in L^*$$

Question 9. Soit $v \in V$. Montrer que si v est facteur d'un mot de L^n, alors $n > d + 2$.

Question 10. En déduire que $V \subseteq W(L^*)$. En déduire que $A^*VA^* \subseteq W(L^*)$.

Question 11. Montrer que $W(L^*) = A^*VA^*$.

Question 12. Montrer que $L^dL^* = (L^dA^* \cap A^*L^d) \setminus W(L^*)$.

Question 13. En déduire que si L est un langage sans étoile de délai de synchronisation d, alors L^* est sans étoile.

Conclusion: l'étoile d'un langage sans étoile est donc parfois sans étoile...

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Warning : Clearness, accuracy and concision of the writing will be rewarded.

Part 1. Study of a language

Let $A = \{a, b\}$ and let $L = (aa)^*(bb)^*$.

Question 1. Find the minimal automaton of L.

Question 2. Compute the syntactic monoid M of L. Give the list of elements and the defining relations of M. (Hint: you should find 6 elements, including the identity).

Question 3. Give the list of all idempotents of M.

Question 4. Give the $\mathcal{D}$-class structure of M and draw the corresponding egg-box pictures.

Question 5. Is the langage L star-free? Justify your answer.

Part 2. Synchronization delay

Given a language K, $W(K)$ denotes the set of all words that are factor of no word in K. Equivalently,

$$W(K) = \{u \in A^* \mid A^*uA^* \cap K = \emptyset\}$$

Question 6. Show that if K is rational (respectively star-free), then so is $W(K)$.

Let L be a rational language of A^*. We say that L has *finite synchronisation delay* if there exists $d \geqslant 0$ such that for all $u, v, w \in A^*$ and for all $v \in L^d$,

$$uvw \in L^* \text{ and } v \in L^d \implies uv \in L^* \text{ and } w \in L^*.$$

The smallest integer d satisfying this condition is called the *synchronisation delay* of L.

Question 7.

(1) What are the languages of synchronisation delay 0?

(2) Show that the language a^*b has a synchronisation delay.

(3) Show that the language $\{ab, ba\}$ has no finite synchronisation delay.

In the remainder of the problem, L denotes a **language with synchronisation delay** d and we set $V = W(L^{d+2}) \setminus A^*L^{d+1}A^*$.

Question 8. Let $w \in A^*L^dA^*$. Show that for all $u_1, u_2, u_3, u_4 \in A^*$, one has

$$u_1wu_2, u_3wu_4 \in L^* \implies u_1wu_4, u_3wu_2 \in L^*$$

Question 9. Let $v \in V$. Show that if v is factor of a word of L^n, then $n > d + 2$.

Question 10. Deduce that $V \subseteq W(L^*)$ and that $A^*VA^* \subseteq W(L^*)$.

Question 11. Show that $W(L^*) = A^*VA^*$.

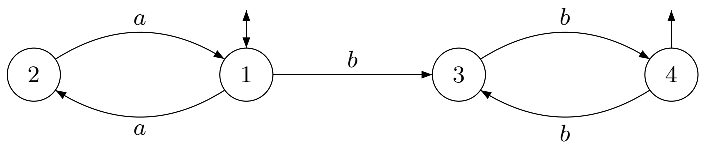
Question 12. Show that $L^dL^* = (L^dA^* \cap A^*L^d) \setminus W(L^*)$.

Question 13. Conclude that if L is a star-free language with synchronisation delay d, then L^* is star-free.

Conclusion: the star of a star-free language is sometimes star-free...

Corrigé

Question 1. L'automate minimal de L est représenté ci-dessous



Question 2. Le monoïde syntactique de L est

	1	2	3	4
$*\ 1$	1	2	3	4
a	2	1	0	0
b	3	0	4	3
$*\ a^2$	1	2	0	0
ab	0	3	0	0
$*\ ba$	0	0	0	0
$*\ b^2$	4	0	3	4
a^2b	3	0	0	0
ab^2	0	4	0	0
a^2b^2	4	0	0	0

Relations:

$$a^3 = a \qquad ba = 0 \qquad b^3 = b$$

Question 3. Idempotents:

$$E(S) = \{1, a^2, b^2, 0\}$$

Question 4. $\mathcal{D}$-classes:

^{*1}

$^*a^2$	a

$^*b^2$	b

a^2b	a^2b^2
ab	ab^2

^{*0}

Question 5. The syntactic monoid of L is not aperiodic since it contains the group $\{a, a^2\}$. Therefore L is not star-free.

Part 2. Synchronization delay

Question 6. One has $W(K) = ((A^*)^{-1}K(A^*)^{-1})^c$ and both rational and star-free languages are closed under quotients and complement.

Question 7.

(1) If L has delay 0, then for all $u, w \in A^*$, $uw \in L^*$ implies $u, w \in L^*$. It follows that each letter of any word in L^* also belongs to L^*. Therefore L is a subset of A. Conversely, any subset of A is a language of synchronisation delay 0

(2) The language a^*b has delay 1. Indeed, if $v, uvw \in a^*b$, then necessarily $u \in a^*$ and $w = 1$, whence $uv, w \in L^*$

(3) The language $\{ab, ba\}$ has no finite synchronisation delay. Indeed, for any integer $d \geqslant 0$, $a(ba)^d b \in \{ab, ba\}^*$ although $b \notin \{ab, ba\}^*$, contradicting the definition for $u = a$, $v = (ba)^d$ and $w = b$.

Question 8. Let $u, v \in A^*$, $x \in L^d$ and let $w = uxv$. Suppose that u_1wu_2 and u_3wu_4 are in L^*. Since L has delay d, $u_1uxvu_2 \in L^*$ implies $u_1ux, vu_2 \in L^*$ and $u_3uvwu_4 \in L^*$ implies $u_3ux, vu_4 \in L^*$. Thus $u_1wu_4, u_3wu_2 \in L^*$ since $u_1wu_4 = (u_1ux)(vu_4)$ and $u_3wu_2 = (u_3ux)(vu_2)$.

Question 9. Let $V = W(L^{d+2}) \setminus A^*L^{d+1}A^*$. Then V is the set of all words which are not a factor of a word in L^{d+2} and which do not contain a word of L^{d+1} as a factor. We claim that $V \subseteq W(L^*)$. Otherwise, there is a word $v \in V$ which is also a factor of some word of L^*. Let n be the least integer such that v is a factor of some word in L^n. Then $uvw = x_1x_2 \cdots x_n$ for some $u, v \in A^*$ and $x_i \in L$. By the definition of V, one has $n > d + 2$.

Question 10. By the minimality of n, u is a prefix of x_1 and w is a suffix of x_n. Thus $x_2 \cdots x_{n-1}$ is a factor of v, a contradiction with the fact that v does not have a factor in L^{d+1}. This proves the claim.

It follows immediately from the definition that $A^*W(L^*)A^* = W(L^*)$. Therefore, we get $A^*VA^* \subseteq A^*W(L^*)A^* = W(L^*)$.

Question 11. Let us show the opposite inclusion. Let V' be the set of words in $W(L^*)$ without a proper factor in $W(L^*)$. It suffices to show that $V' \subset V$. Let $v \in V'$. The only thing to prove is that v does not have a factor in L^{d+1}. If $v \in A$, it is clear. Otherwise, let $v = ahb$ with $a, b \in A$ and $h \in A^*$. Suppose first that h has a factor in L^d. By the definition of V', the words ah and hb are not in $W(L^*)$ and thus there are words u_1, u_2, u_3, u_4 such that

$$u_1ahu_2, u_3hbu_4 \in L^*$$

We have seen that if h has a factor in L^d, then $u_1ahbu_4 \in L^*$, a contradiction. Let us now suppose that v has a factor in L^{d+1}. Thus $ahb = uxv$ with $u, v \in A^*$ and $x \in L^{d+1}$. Since h does not have a factor in L^d, the only possibility is that $u = v = 1$. But this contradicts the fact that $v \in W(L^*)$.

Question 12. Let $U = (L^dA^* \cap A^*L^d) \setminus W(L^*)$. We claim that $U = L^dL^*$. It is clear that L^dL^* is a subset of U. To prove the opposite inclusion, consider an element u of U. There exists $x, y \in L^d$ and $r, s \in A^*$ such that $u = xr = sy$. Since $u \notin W(L^*)$, there exist some $p, q \in A^*$ such that $puq \in L^*$. Since L has synchronization delay d, it follows from $pxrq \in L^*$ that $rq \in L^*$. Since $x \in L^*$, this implies $xrq \in L^*$. Then, from $xrq = syq \in L^*$, we conclude that $u = sy$ is in L^*, proving the claim.

Question 13. One has $L^* = 1 \cup L \cup \ldots \cup L^{d-1} \cup L^d L^*$ and $L^d L^* = \big[(L^d A^* \cap A^* L^d) \setminus W(L^*)\big]$. Moreover $W(L^*) = A^* V A^*$ and thus it suffices to verify that V is star-free. But $V = W(L^{d+2}) \setminus A^* L^{d+1} A^*$, L is star-free and thus L^{d+2} and $W(L^{d+2})$ are star-free.