MPRI, Fondations mathématiques de la théorie des automates

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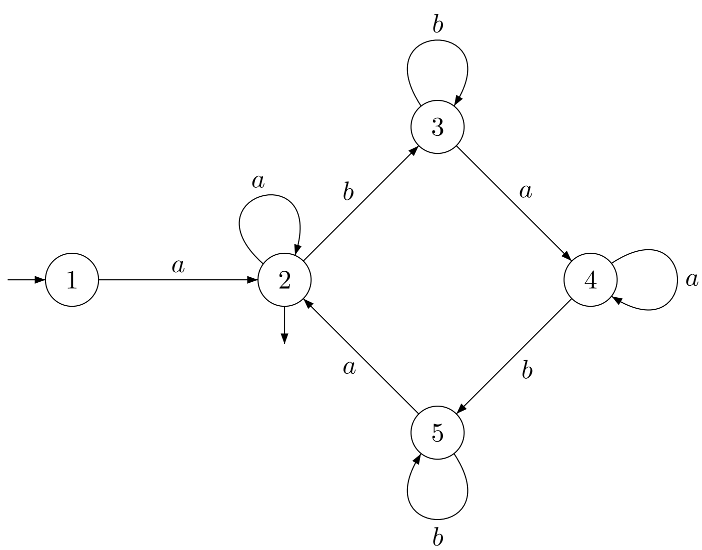
Avertissement : On attachera une grande importance à la clarté, à la précision et à la concision de la rédaction.

Partie 1. Calcul d'un monoïde syntactique.

Soit $A = \{a, b\}$. On considère l'automate $\mathcal{A}$ représenté ci-dessous:



Question 1. Donner une expression rationnelle pour le langage L reconnu par $\mathcal{A}$.

Question 2. Calculer le monoïde syntactique M de L. On donnera la liste de ses éléments (vous devriez trouver 9 éléments, en comptant l'élément neutre) et les relations permettant de le définir.

Question 3. Calculer l'image P de L dans M par le morphisme syntactique.

Question 4. Déterminer l'ensemble des idempotents de M.

Question 5. Déterminer la structure en $\mathcal{D}$-classes de M (on dessinera les diagrammes boîtes à œufs). Le monoïde M est-il commutatif? $\mathcal{R}$-trivial? $\mathcal{L}$-trivial? apériodique?

Partie 2. Opérations sur les langages.

Le *mélange* de deux langages K et L de A^* est le langage

$$K \sqcup\!\sqcup L = \{w \in A^* \mid w = u_1v_1 \cdots u_nv_n \text{ où } u_1, \ldots, u_nv_1, \ldots, v_n \text{ sont des mots de } A^* \text{ tels que } u_1 \cdots u_n \in K \text{ and } v_1 \cdots v_n \in L\}\,.$$

Par exemple

$$\{ab\} \sqcup\!\sqcup \{ba\} = \{abab, abba, baba, baab\}.$$

On admettra sans démonstration que le mélange est une opération commutative et associative, distributive par rapport à l'union.

Question 6. Donner une expression rationnelle pour le langage $(ab)^* \sqcup\!\sqcup (ab)^*$.

Question 7. Montrer que le mélange de deux langages rationnels est un langage rationnel.

Question 8. Montrer que les langages $(aab)^*$ et $(bba)^*$ sont des langages sans-étoile.

Question 9. Calculer le langage $\big((aab)^* \sqcup\!\sqcup (bba)^*\big) \cap (ab)^*$.

Question 10. Montrer que le mélange de deux langages sans-étoile n'est pas nécessairement un langage sans-étoile.

Le 2-mélange de K et L est le langage

$$K \sqcup\!\sqcup_2 L = \{u_1v_1u_2v_2 \mid u_1, v_1, u_2, v_2 \in A^*, u_1u_2 \in K, v_1v_2 \in L\}$$

Question 11. Donner une expression rationnelle pour le langage $(ab)^* \sqcup\!\sqcup_2 a^*$.

Question 12. Montrer que le 2- mélange de deux langages rationnels est un langage rationnel.

Question 13. Montrer que le 2- mélange de deux langages sans-étoile est un langage sans-étoile.

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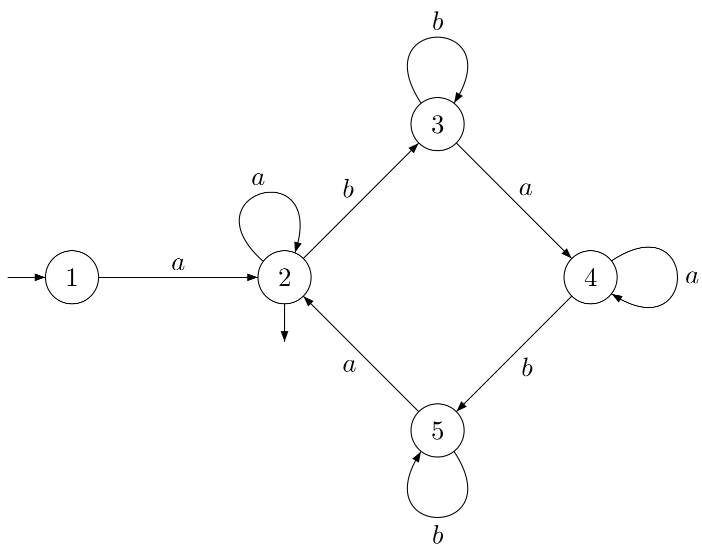
Warning : Clearness, accuracy and concision of the writing will be rewarded.

Part 1. Computation of a syntactic monoid.

Let $A = \{a, b\}$. Let $\mathcal{A}$ be the automaton represented in the picture below:



Question 1. Give a rational (= regular) expression for the language L accepted by $\mathcal{A}$.

Question 2. Compute the syntactic monoid M of L. Give the list of its elements (you should find 9 elements, including the identity) and of its defining relations.

Question 3. Compute the image of L in M under the syntactic morphism.

Question 4. Give the set of idempotents of M.

Question 5. Compute the $\mathcal{D}$-class structure of M (draw the egg-box pictures). Is the monoid M comutative? $\mathcal{R}$-trivial? $\mathcal{L}$-trivial? aperiodic?

Part 2. Operations on languages

The *shuffle* of two languages L_1 and L_2 of A is the language

$$L_1 ⧢ L_2 = \{w \in A^* \mid w = u_1 v_1 \cdots u_n v_n \text{ for some words } u_1, \ldots, u_n \\ v_1, \ldots, v_n \text{ of } A^* \text{ such that } u_1 \cdots u_n \in L_1 \text{ and } v_1 \cdots v_n \in L_2\}\,.$$

For instance

$$\{ab\} ⧢ \{ba\} = \{abab, abba, baba, baab\}.$$

One can show and we shall admit that the shuffle product defines a commutative and associative operation, which is also distributive aver union.

Question 6. Give a rational expression for the language $(ab)^* ⧢ (ab)^*$.

Question 7. Show that the shuffle of two rational languages is a rational language.

Question 8. Show that the languages $(aab)^*$ and $(bba)^*$ are star-free.

Question 9. Compute the language $\big((aab)^* ⧢ (bba)^*\big) \cap (ab)^*$.

Question 10. Show that the shuffle of two star-free languages is not necessarily star-free.

The 2-shuffle of K and L is the language

$$K ⧢_2 L = \{u_1 v_1 u_2 v_2 \mid u_1, v_1, u_2, v_2 \in A^*, u_1 u_2 \in K, v_1 v_2 \in L\}$$

Question 11. Give a rational expression for the language $(ab)^* ⧢_2 a^*$.

Question 12. Show that the 2-shuffle of two rational languages is rational.

Question 13. Show that the 2-shuffle of two star-free languages is star-free.

Corrigé

Partie 1. Calcul d'un monoïde syntactique.

Question 1. Une expression rationnelle pour L is $a(a + bb^*aa^*bb^*a)^*$.

Question 2. Le monoïde syntactique M de L est donné par le tableau suivant:

	1	2	3	4	5
$*\ 1$	1	2	3	4	5
$*\ a$	2	2	4	4	2
$*\ b$	0	3	3	5	5
ab	3	3	5	5	3
ba	0	4	4	2	2
aba	4	4	2	2	4
bab	0	5	5	3	3
$*\ abab$	5	5	3	3	5
$*\ baba$	0	2	2	4	4

dans lequel les idempotents sont indiqués par une étoile. Les relations définissant M sont

$$a^2 = a \qquad b^2 = b \qquad ababa = a \qquad babab = b$$

Question 3. L'image P de L est obtenue en sélectionnant les éléments x de M tels que $1 \cdot x = 2$. Ici $P = \{a\}$.

Question 4. Idempotents:

$$E(S) = \{1, a, b, abab, baba\}$$

Question 5. La structure en $\mathcal{D}$-classes est la suivante:

^{*1}

$^*a$ aba	$^*abab$ ab
$^*baba$ ba	$^*b$ bab

Ce monoïde n'est ni commutatif, ni $\mathcal{R}$-trivial, ni $\mathcal{L}$-trivial, ni apériodique.

Partie 2. Operations on languages.

Question 6. $(ab)^* ⧢ (ab)^* = (a(ab)^*b)^*$

Question 7. One can give a proof using automata. Here is a proof based on monoids. If M is a monoid, let $\mathcal{P}(M)$ be the monoid of all subsets of M, equipped with the product defined as follows: if X and Y are subsets of M, $XY = \{xy \mid x \in X \text{ and } y \in Y\}$.

Let $\eta_1 : A^* \to M_1$ and $\eta_2 : A^* \to M_2$ be the syntactic morphisms of the languages L_1 and L_2. Let $\mu : A^* \to \mathcal{P}(M_1 \times M_2)$ be the morphism defined, for each $a \in A$, by $\mu(a) = \{(\eta_1(a), 1), (1, \eta_2(a))\}$. Let us show that μ recognises $L_1 ⧢ L_2$.

For each word $u \in A^*$, one has $\mu(u) = \{(\eta_1(u_1), \eta_1(u_2)) \mid u \in u_1 ⧢ u_2\}$. Suppose that $u \in L_1 ⧢ L_2$ and that $\mu(v) = \mu(u)$. Then there exist two words $u_1 \in L_1$ and $u_2 \in L_2$ such that $u \in u_1 ⧢ u_2$, and there exist two words $v_1, v_2 \in A^*$ such that $v \in v_1 ⧢ v_2$, $\eta_1(u_1) = \eta_1(v_1)$ and $\eta_2(u_2) = \eta_2(v_2)$. It follows that $v_1 \in L_1$ and $v_2 \in L_2$ and $v \in L_1 ⧢ L_2$. Thus μ recognizes $L_1 ⧢ L_2$.

In particular, if L_1 and L_2 are regular, then M_1 and M_2 are finite and thus $\mathcal{P}(M_1 \times M_2)$ is also finite. Therefore $L_1 ⧢ L_2$ is regular.

Question 8. It suffices to prove the result for the language $(aab)^*$ since the other language can be obtained by swapping a and b. Now $(aab)^*$ is the complement of the language

$$bA^* + A^*a + A^*\big(A^3 - (aab + aba + baa)\big)A^*$$

which is star-free.

Question 9. One gets

$$\big((aab)^* ⧢ (bba)^*\big) \cap (ab)^* = \big((ab)^3\big)^* \qquad (1)$$

Question 10. Recall that the class of star-free languages is closed under inverses of morphisms and under Boolean operations.

Let $\varphi : a^* \to \{a, b\}^*$ be the morphism defined by $\varphi(a) = ab$ and let $R = \big((ab)^3\big)^*$. Then one has $\varphi^{-1}(R) = (a^3)^*$. Since the language $(a^3)^*$ is not star-free since its syntactic monoid is the cyclic group of order 3, the language R cannot be star-free. Similarly, since $(ab)^*$ is star-free, Formula (1) shows that $(aab)^* ⧢ (bba)^*$ cannot be star-free.

If follows now from Question 8 that the shuffle of two star-free languages is not necessarily star-free.