ON DYSON’S CRANK CONJECTURE AND THE UNIFORM
ASYMPTOTIC BEHAVIOR OF CERTAIN INVERSE THETA
FUNCTIONS

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Abstract. In this paper we prove a longstanding conjecture by Freeman Dyson
concerning the limiting shape of the crank generating function. We fit this function
in a more general family of inverse theta functions which play a key role in physics.

1. Introduction and statement of results

Dyson’s crank was introduced to explain Ramanujan’s famous partition congru-
ences with modulus 5, 7, and 11. Denoting for \( n \in \mathbb{N} \) by \( p(n) \) the number of integer
partitions of \( n \), Ramanujan [20] proved that for \( n \geq 0 \)
\[
p(5n + 4) \equiv 0 \pmod{5}, \quad p(7n + 5) \equiv 0 \pmod{7}, \quad p(11n + 6) \equiv 0 \pmod{11}.
\]
A key ingredient of his proof is the modularity of the partition generating function
\[
\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q; q)_\infty} = \frac{q^{1/24}}{\eta(\tau)},
\]
where for \( j \in \mathbb{N}_0 \cup \{\infty\} \) we set \( (a)_j = (a; q)_j := \prod_{\ell=0}^{j-1}(1 - aq^\ell) \), \( q := e^{2\pi i \tau} \), and
\( \eta(\tau) := q^{1/24} \prod_{n=1}^{\infty}(1 - q^n) \) is Dedekind’s \( \eta \)-function, a modular form of weight \( \frac{1}{2} \).

Ramanujan’s proof however gives little combinatorial insight into why the above
congruences hold. In order to provide such an explanation, Dyson [9] famously intro-
duced the rank of a partition, which is defined as its largest part minus the number
of its parts. He conjectured that the partitions of \( 5n + 4 \) (resp. \( 7n + 5 \)) form 5
(resp. 7) groups of equal size when sorted by their ranks modulo 5 (resp. 7). This

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conjecture was proven by Atkin and Swinnerton-Dyer [4]. The first author and Ono [7] showed that partitions with given rank satisfy also Ramanujan-type congruences. Dyson further postulated the existence of another statistic which he called the “crank” and which should explain all Ramanujan congruences. The crank was later found by Andrews and Garvan [1, 12]. If for a partition $\lambda$, $o(\lambda)$ denotes the number of ones in $\lambda$, and $\mu(\lambda)$ is the number of parts strictly larger than $o(\lambda)$, then the crank of $\lambda$ is defined as

$$\text{crank}(\lambda) := \begin{cases} \text{largest part of } \lambda & \text{if } o(\lambda) = 0, \\ \mu(\lambda) - o(\lambda) & \text{if } o(\lambda) > 0. \end{cases}$$

Denote by $M(m, n)$ the number of partitions of $n$ with crank $m$. Mahlburg [17] then proved that partitions with fixed crank also satisfy Ramanujan-type congruences. In this paper, we solve a longstanding conjecture by Dyson [10] concerning the limiting shape of the crank generating function.

**Conjecture 1.1** (Dyson). As $n \to \infty$ we have

$$M(m, n) \sim \frac{1}{4} \beta \text{sech}^2 \left( \frac{1}{2} \beta m \right) p(n)$$

with $\beta := \frac{\pi}{\sqrt{6}}$.

Dyson then asked the question about the precise range of $m$ in which this asymptotic holds and about the error term. In this paper, we answer all of these questions.

**Theorem 1.2.** The Dyson-Conjecture is true. To be more precise, if $|m| \leq \frac{1}{\pi \sqrt{6}} \sqrt{n \log n}$, we have as $n \to \infty$

$$M(m, n) = \frac{\beta}{4} \text{sech}^2 \left( \frac{\beta m}{2} \right) p(n) \left( 1 + O \left( \beta^{\frac{1}{2}} |m|^\frac{1}{2} \right) \right). \quad (1.1)$$

**Remarks.**

1. The much simpler situation when $m$ is fixed can be easily concluded from work of the first author with Manschot [6].
2. In fact we could replace the error by $O(\beta^{\frac{1}{2}} m \alpha(m))$ for any $\alpha(m)$ such that
   \[ \frac{\log n}{n^{1/2}} = o(\alpha(m)) \]
   for all $|m| \leq \frac{1}{\pi \sqrt{6}} \sqrt{n \log n}$ and $\beta m \alpha(m) \to 0$ as $n \to \infty$.
   Here we chose $\alpha(m) = |m|^{-\frac{1}{2}}$ to avoid complicated expressions in the proof.
3. A key ingredient of our proof is Wright’s variant of the Circle Method [21, 22]. On the other hand the asymptotic behavior of the crank moments $\sum_{m \in \mathbb{Z}} m^k N(n, m)$ is now well understood [5]. It would be interesting to...
investigate whether probabilistic methods could be applied to give an alter-
native proof of Dyson’s conjecture (see also [8]).

Dyson’s conjecture follows from a more general result concerning the coefficients $M_k(m, n)$ defined for $k \in \mathbb{N}$ by

$$C_k(\zeta; q) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M_k(m, n) \zeta^m q^n := \frac{(q)^{2-k}}{(\zeta q)_{\infty} (\zeta^{-1} q)_{\infty}}.$$

Note that $M(m, n) = M_1(m, n)$. Denoting by $p_k(n)$ the number of partitions of $n$ allowing $k$ colors, we have.

**Theorem 1.3.** For $k$ fixed and $|m| \leq \frac{1}{6 \beta_k} \log n$, we have as $n \to \infty$

$$M_k(m, n) = \frac{\beta_k}{4} \text{sech}^2 \left( \frac{\beta_k m}{2} \right) p_k(n) \left( 1 + O \left( \beta_k^2 |m|^\frac{1}{2} \right) \right),$$

with $\beta_k := \pi \sqrt{\frac{k}{6n}}$.

**Remarks.**

1. We note that for $k \geq 3$, the functions $C_k(\zeta; q)$ are well-known to be generating functions of Betti numbers of moduli spaces of Hilbert schemes on $(k - 3)$—point blow-ups of the projective plane [13] (see also [6] and references therein). The results of this paper immediately gives the limiting profile of the Betti numbers for large second Chern class of the sheaves. Recently, Hausel and Rodriguez-Villegas [16] also determined profiles of Betti numbers for other moduli spaces, in particular the Gumbel distribution was found for the Hilbert scheme of points on $\mathbb{C}^2$.

2. Note that our method of proof would allow determining further terms in the asymptotic expansion of $M_k(m, n)$.

3. Again we could replace the error by $O(\beta_k^2 \alpha_k^2(m))$ for any $\alpha_k(m)$ such that

$$\frac{\log n}{(kn)^{\frac{3}{4}}} = o(\alpha_k(m))$$

for all $|m| \leq \frac{1}{6 \beta_k} \log n$ and $\beta_k \alpha_k(m) \to 0$ as $n \to \infty$.

4. The function $C_k$ can also be represented as a so-called Lerch sum. This representation, which was a key representation in [7], is not used in this paper.

5. The special case $k = 2$ yields the birank of partitions [14].

6. We expect that our methods also apply to show an analogue of (1.1) for the rank.
This paper is organized as follows. In Section 2, we recall basic facts on modular and Jacobi forms which are the base components of $C_k$ and collect properties on Euler polynomials. In Section 3, we determine the asymptotic behavior of $C_k$. Finally, in Section 4, we use Wright’s version of the Circle Method to finish the proof of Theorem 1.3.

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2. Preliminaries

2.1. Modularity of the generating functions. A key ingredient of our asymptotic results is to employ the modularity of the functions $C_k$. To be more precise, we write (throughout $q := e^{2\pi i\tau}$, $\zeta := e^{2\pi i w}$ with $\tau \in \mathbb{H}, w \in \mathbb{C}$)

$$C_k (\zeta; q) = \frac{i \left( \zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}} \right) q^{k/2} \eta^{3-k}(\tau)}{\vartheta (w; \tau)}, \quad (2.1)$$

where

$$\eta(\tau) := q^{\frac{k}{24}} \prod_{n=1}^{\infty} (1 - q^n),$$

$$\vartheta (w; \tau) := i \zeta^{\frac{1}{2}} q^{\frac{k}{8}} \prod_{n=1}^{\infty} (1 - q^n) (1 - \zeta q^n) (1 - \zeta^{-1} q^{n-1}).$$

The function $\eta$ is a modular form, whereas $\vartheta$ is a Jacobi form. To be more precise, we have the following transformation laws (see e.g. [18]).

Lemma 2.1. We have

$$\eta \left( -\frac{1}{\tau} \right) = \sqrt{-i\pi} \eta(\tau),$$
\[ \vartheta \left( \frac{w}{\tau}; -\frac{1}{\tau} \right) = -i \sqrt{-i \tau e^{\pi i w^2/\tau}} \vartheta (w; \tau). \]

2.2. Euler polynomials. Recall that the Euler polynomials may be defined by their generating function
\[ \frac{2e^{xt}}{e^t + 1} =: \sum_{r=0}^{\infty} E_r(x) \frac{t^r}{r!}. \quad (2.2) \]

The following lemma may easily be concluded by differentiating the generating function (2.2).

Lemma 2.2. We have
\[ -\frac{1}{2} \text{sech}^2 \left( \frac{t}{2} \right) = \sum_{r=0}^{\infty} E_{2r+1}(0) \frac{t^{2r}}{(2r)!}. \]

We also require an integral representation of Euler polynomials. To be more precise, setting for \( j \in \mathbb{N}_0 \)
\[ \mathcal{E}_j := \int_{0}^{\infty} \frac{w^{2j+1}}{\sinh(\pi w)} \, dw, \quad (2.3) \]
we obtain

Lemma 2.3. We have
\[ \mathcal{E}_j = \frac{(-1)^{j+1} E_{2j+1}(0)}{2}. \]

Proof. We make the change of variables \( w \to w + \frac{i}{2} \) and then use the Residue Theorem to shift the path of integration back to the real line. Using the Binomial Theorem, we may thus write
\[ \mathcal{E}_j = -\frac{i}{2} \int_{\mathbb{R}} \frac{(w + \frac{i}{2})^{2j+1}}{\cosh(\pi w)} \, dw = -\frac{i}{2} \sum_{\ell=0}^{2j+1} \binom{2j+1}{\ell} \left( \frac{i}{2} \right)^{2j+1-\ell} \int_{\mathbb{R}} \frac{w^{\ell}}{\cosh(\pi w)} \, dw. \]

The last integral is known to equal \((-2i)^{-\ell} E_\ell\), where \( E_\ell := 2^\ell E_\ell(\frac{1}{2}) \) denotes the \( \ell \)th Euler number (see page 41 of [11]). The claim now follows using the well-known identity (see page 41 of [11])
\[ E_j(x) = \sum_{\ell=0}^{j} \binom{j}{\ell} \left( x - \frac{1}{2} \right)^{j-\ell} \frac{E_\ell}{2^\ell}. \]

\( \square \)
3. Asymptotic behavior of the function $C_k$.

Since $M_k(-m, n) = M_k(m, n)$ we from now on assume that $m \geq 0$. The goal of this section is to study the asymptotic behavior of the generating function of $M_k(m, n)$.

We define

$$C_{m,k}(q) := \sum_{n=0}^{\infty} M_k(m, n) q^n = \int_{-\frac{1}{2}}^{\frac{1}{2}} C_k(e^{2\pi i w}; q) e^{-2\pi i m w} dw$$

$$= 2 \frac{q^{\frac{1}{2}}}{\eta^k(\tau)} \int_{0}^{\frac{1}{2}} g(w; \tau)\cos(2\pi m w) dw,$$

where

$$g(w; \tau) := i \left( \zeta^\frac{1}{2} - \zeta^{-\frac{1}{2}} \right) \eta^3(\tau).$$

Here we used that $g(-w; \tau) = g(w; \tau)$. In this section we determine the asymptotic behavior of $C_{m,k}(q)$, when $q$ is near an essential singularity on the unit circle. It turns out that the dominant pole lies at $q = 1$. Throughout the rest of the paper let $\tau = \frac{i z}{2\pi}, z = \beta_k(1 + ixm^{-\frac{1}{3}})$ with $x \in \mathbb{R}$ satisfying $|x| \leq \frac{\pi m^{\frac{2}{3}}}{12\beta_k}$.

3.1. Bounds near the dominant pole. In this section we consider the range $|x| \leq 1$. We start by determining the asymptotic main term of $g$. Lemma 2.1 and the definition of $\vartheta$ and $\eta$ immediately imply.

**Lemma 3.1.** For $0 \leq w \leq 1$ we have for $|x| \leq 1$ as $n \to \infty$

$$g\left(\frac{iz}{2\pi}\right) = \frac{2\pi \sin(\pi w)}{z \sinh\left(\frac{2\pi w}{z}\right)} e^{\frac{2\pi^2 w^2}{z^2}} \left( 1 + O\left( e^{-\pi (1-w) \operatorname{Re}(\frac{1}{z})} \right) \right).$$

In view of Lemma 3.1 it is therefore natural to define

$$G_{m,1}(z) := 4\pi z \int_{0}^{\frac{1}{2}} \frac{\sin(\pi w)}{\sinh\left(\frac{2\pi w}{z}\right)} e^{\frac{2\pi^2 w^2}{z^2}} \cos(2\pi m w) dw,$$

$$G_{m,2}(z) := 2 \int_{0}^{\frac{1}{2}} \left( g\left(\frac{iz}{2\pi}\right) - \frac{2\pi \sin(\pi w)}{z \sinh\left(\frac{2\pi w}{z}\right)} e^{\frac{2\pi^2 w^2}{z^2}} \right) \cos(2\pi m w) dw.$$

Thus

$$C_{m,k}(q) = \frac{q^{\frac{1}{2}}}{\eta^k(\tau)} (G_{m,1}(z) + G_{m,2}(z)). \quad (3.1)$$
The dominant contribution comes from $G_{m,1}$.

**Lemma 3.2.** Assume that $|x| \leq 1$ and $m \leq \frac{1}{65} \log n$. Then we have as $n \to \infty$

$$G_{m,1}(z) = \frac{z}{4} \sech^2 \left( \frac{\beta km}{2} \right) + O \left( \beta^2 k^2 \frac{m^2 \sech^2 \left( \frac{\beta km}{2} \right)}{z} \right).$$

**Proof.** Inserting the Taylor expansion of $\sin$, $\exp$, and $\cos$, we get

$$\sin(\pi w) e^{\frac{2\pi^2 w^2}{z}} \cos(2\pi mw) = \sum_{j,\nu, r \geq 0} \frac{(-1)^{j+\nu}}{(2j+1)!(2\nu)!} \pi^{2j+1}(2\pi m)^{2\nu} \left( \frac{2\pi^2}{z} \right)^r \frac{w^{2j+2\nu+2r+1}}{}.$$

This yields that

$$G_{m,1}(z) = \frac{4\pi}{z} \sum_{j,\nu, r \geq 0} \frac{(-1)^{j+\nu}}{(2j+1)!(2\nu)!} \pi^{2j+1}(2\pi m)^{2\nu} \left( \frac{2\pi^2}{z} \right)^r \mathcal{I}_{j+\nu+r}$$

where for $\ell \in \mathbb{N}_0$ we define

$$\mathcal{I}_{\ell} := \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{w^{2\ell+1}}{\sinh \left( \frac{2\pi^2 w}{z} \right)} dw.$$

We next relate $\mathcal{I}_{\ell}$ to $\mathcal{E}_{\ell}$ defined in (2.3). For this, we note that

$$\mathcal{I}_{\ell} = \int_{0}^{\infty} \frac{w^{2\ell+1}}{\sinh \left( \frac{2\pi^2 w}{z} \right)} dw - \mathcal{I}_{\ell}' \tag{3.2}$$

with

$$\mathcal{I}_{\ell}' := \int_{\frac{1}{2}}^{\infty} \frac{w^{2\ell+1}}{\sinh \left( \frac{2\pi^2 w}{z} \right)} dw \ll \int_{\frac{1}{2}}^{\infty} w^{2\ell+1} e^{-2\pi^2 w \Re \left( \frac{1}{z} \right)} dw \ll \left( \Re \left( \frac{1}{z} \right) \right)^{-2\ell^2} \Gamma \left( 2\ell + 2; \pi^2 \Re \left( \frac{1}{z} \right) \right).$$

Here $\Gamma(\alpha; x) := \int_{x}^{\infty} e^{-w} w^{\alpha-1} dw$ denotes the incomplete gamma function and throughout $g(x) \ll f(x)$ means that $g(x) = O \left( f(x) \right)$. Using that as $x \to \infty$

$$\Gamma (\ell; x) \sim x^{\ell-1} e^{-x} \tag{3.3}$$

thus yields that

$$\mathcal{I}_{\ell}' \ll \left( \Re \left( \frac{1}{z} \right) \right)^{-1} e^{-\pi^2 \Re \left( \frac{1}{z} \right)} \leq e^{-\pi^2 \Re \left( \frac{1}{z} \right)}.$$
In the first summand in (3.2) we make the change of variables $w \to zw^2/\pi$ and then shift the path of integration back to the real line by the Residue Theorem. Thus we obtain that
\[
\int_0^\infty \frac{w^{2\ell+1}}{\sinh \left( \frac{2\pi w}{z} \right)} dw = \left( \frac{z}{2\pi} \right)^{2\ell+2} E_\ell = \left( \frac{z}{2\pi} \right)^{2\ell+2} \frac{(-1)^{\ell+1} E_{2\ell+1}(0)}{2},
\]
where for the last equality we used Lemma 2.3. Thus
\[
G_{m,1}(z) = \sum_{j,\nu,r \geq 0} \frac{(-1)^{r+1}}{2^{2j+r+1}(2j+1)!(2\nu)!r!} m^{2\nu} z^{2j+2\nu+r+1}
\times \left( E_{2j+2\nu+2r+1}(0) + O \left( |z|^{-2j-2\nu-2r-2}e^{-\pi^2\Re \left( \frac{1}{z} \right)} \right) \right)
= \sum_{r=0}^\infty \frac{(mz)^{2\nu}}{(2\nu)!} \left( -\frac{z}{2} E_{2\nu+1}(0) + O \left( |z|^2 \right) \right) = \frac{z}{4} \operatorname{sech}^2 \left( \frac{mz}{2} \right) + O \left( |z|^2 \cosh(mz) \right),
\]
where for the last equality we used Lemma 2.2. To finish the proof we have to approximate $\operatorname{sech}^2 \left( \frac{mz}{2} \right)$ and $\cosh(mz)$. We have
\[
\cosh(mz) = \cosh \left( \beta_k m + i\beta_k m^3 x \right)
= \cosh(\beta_k m) \cos \left( \beta_k m^3 x \right) + i \sinh(\beta_k m) \sin \left( \beta_k m^3 x \right)
= \cosh(\beta_k m) \left( 1 + O \left( \beta_k m^3 \right) \right).
\]
This implies that
\[
\operatorname{sech} \left( \frac{mz}{2} \right) = \frac{1}{\cosh \left( \frac{mz}{2} \right)} = \frac{1}{\cosh \left( \frac{\beta_k m}{2} \right) \left( 1 + O \left( \beta_k m^3 \right) \right)},
\]
yielding
\[
\operatorname{sech}^2 \left( \frac{mz}{2} \right) = \operatorname{sech}^2 \left( \frac{\beta_k m}{2} \right) \left( 1 + O \left( \beta_k m^3 \right) \right).
\]
Thus we obtain
\[
G_{m,1}(z) = \frac{z}{4} \operatorname{sech}^2 \left( \frac{\beta_k m}{2} \right) \left( 1 + O \left( \beta_k m^3 \right) \right) + O \left( \beta_k^2 \left( 1 + \frac{1}{m^2} \right) \cosh(\beta_k m) \right)
\]
\[
\frac{z}{4} \text{sech}^2 \left( \frac{\beta_km}{2} \right) + O \left( \beta_k^2 m^\frac{3}{2} \text{sech}^2 \left( \frac{\beta_km}{2} \right) \right) + O \left( \beta_k^2 \cosh(\beta_km) \right).
\]

We may now easily finish the proof distinguishing the cases on whether \( \beta_km \) is bounded or goes to \( \infty \).

\( \Box \)

We next turn to bounding \( \mathcal{G}_{m,2} \).

**Lemma 3.3.** Assume that \( |x| \leq 1 \). Then we have as \( n \to \infty \)
\[
\mathcal{G}_{m,2}(q) \ll \frac{1}{\beta_k} e^{-\frac{\pi^2}{4\beta_k}}.
\]

**Proof.** By Lemma 3.1 we obtain that
\[
\mathcal{G}_{m,2}(z) \ll \frac{1}{|z|^2} \int_0^{\frac{\pi}{2}} \left| \frac{\sin(\pi w)}{1 - e^{-\frac{2\pi}{z} w}} e^{2\pi \text{Re}(\frac{1}{2})(w^2 + w - 2)} dw. \right|
\]

It is not hard to see that
\[
\left| \frac{\sin(\pi w)}{1 - e^{-\frac{2\pi}{z} w}} \right| \ll 1.
\]

Moreover,
\[
|z| = \beta_k \sqrt{1 + m^{-\frac{2}{3}} x^2} \gg \beta_k,
\]
\[
\text{Re} \left( \frac{1}{z} \right) \geq \frac{1}{2\beta_k}.
\]

The claim now follows, using that the maximum of \( w^2 + w - 2 \) on \([0, \frac{1}{2}]\) is obtained for \( w = \frac{1}{2} \).

\( \Box \)

Combining the above yields.

**Proposition 3.4.** Assume that \( |x| \leq 1 \). Then we have as \( n \to \infty \)
\[
\mathcal{C}_{m,k}(q) = \frac{z^{\frac{8}{7} + 1}}{4(2\pi)^7} \text{sech}^2 \left( \frac{\beta_km}{2} \right) e^{\frac{\pi}{8\tau}} + O \left( \beta_k^2 m^\frac{3}{2} \text{sech}^2 \left( \frac{\beta_km}{2} \right) e^{\pi \sqrt{\frac{\pi}{\beta_k}}} \right).
\]

**Proof.** Recall from (3.1) that
\[
\mathcal{C}_{m,k}(q) = \frac{q^{\frac{7}{8}}}{\eta^k(\tau)} (\mathcal{G}_{m,1}(z) + \mathcal{G}_{m,2}(z)).
\]
Lemma 2.1 easily gives that
\[ \frac{q^k}{\eta^k(\tau)} = \left( \frac{z}{2\pi} \right)^{\frac{k}{2}} e^{\frac{kz^2}{8\pi}} (1 + O(\beta_k)). \]

The functions \( G_{m,1} \) and \( G_{m,2} \) are now approximated using Lemma 3.2 and Lemma 3.3, respectively. It is not hard to see that the main error term arises from approximation \( G_{m,1} \). We thus obtain
\[ C_{m,k}(q) = \frac{z^{k+1}}{4(2\pi)^{\frac{k}{2}}} e^{\frac{kz^2}{2\pi}} \text{sech}^2 \left( \frac{\beta_k m^2}{2} \right) + O \left( |z|^k \beta_k^2 m^2 \text{sech}^2 \left( \frac{\beta_k m^2}{2} \right) e^{\frac{\pi^2 k}{2} \text{Re}(\frac{1}{z})} \right). \]

The claim follows now using that
\[ |z| \ll \beta_k, \quad \text{Re} \left( \frac{1}{z} \right) \leq \frac{1}{\beta_k} = \frac{\sqrt{6n}}{\pi \sqrt{k}}. \]

**3.2. Bounds away from the dominant pole.** We next investigate the behavior of \( C_{m,k} \) away from the dominant cusp \( q = 1 \). To be more precise, we consider the range \( 1 \leq x \leq \frac{\pi m}{\beta_k} \).

**Proposition 3.5.** Assume that \( 1 \leq |x| \leq \frac{\pi m}{\beta_k} \). Then we have as \( n \to \infty \)
\[ C_{m,k}(q) \ll \sqrt{n} \exp \left( \frac{\pi \sqrt{kn}}{\sqrt{6} (1 + m^{-\frac{3}{2}})} \right). \]

**Proof.** We may bound
\[ \eta^k(\tau) C_{m,k}(q) \ll \frac{1}{|z|} \int_0^1 \left| \frac{\sin(\pi w)}{1 - e^{-\frac{4\pi^2 w}{z}}} \right| e^{2\pi^2 \text{Re}(\frac{1}{z}) w(w-1)} dw. \]

Using that \( |x| \leq \frac{\pi m}{\beta_k} \) we may easily show that
\[ \left| \frac{\sin(\pi w)}{1 - e^{-\frac{4\pi^2 w}{z}}} \right| \ll \sqrt{n}. \]
Using that the maximum of $w(w-1)$ on $[0, \frac{1}{2}]$ is obtained in $0$ gives

$$G_{m,1}(z) \ll \frac{\sqrt{n}}{|z|}.$$ 

To finish the proof, we use that by Lemma 2.1

$$\frac{1}{\eta^k(\tau)} \ll |z|^\frac{k}{2} |e^{\frac{k\tau^2}{6 \text{Re}(\frac{1}{2})}}| \ll |z| \exp \left( \frac{\pi \sqrt{kn}}{\sqrt{6} \left(1 + m^{-\frac{2}{3}}\right)} \right).$$

}\]

4. The Circle Method

In this section we use Wright's variant of the Circle Method and complete the proof of Theorem 1.3 and thus the proof of Dyson's conjecture. We start by using Cauchy's Theorem to express $M_k$ as an integral of its generating function $C_{m,k}$:

$$M_k(m,n) = \frac{1}{2\pi i} \int_C \frac{C_{m,k}(q)}{q^{n+1}} dq,$$

where the contour is the counterclockwise transversal of the circle $C := \{ q \in \mathbb{C}; |q| = e^{-\beta_k}\}$. Recall that $z = \beta_k(1 + ixm^{-\frac{1}{3}})$. Changing variables we may write

$$M_k(m,n) = \frac{\beta_k}{2\pi m^{\frac{1}{3}}} \int_{|x| \leq \frac{m^{\frac{1}{3}}}{\beta_k}} C_{m,k}(e^{-z}) e^{nx} dx.$$

We split this integral into two pieces

$$M_k(m,n) = M + E$$

with

$$M := \frac{\beta_k}{2\pi m^{\frac{1}{3}}} \int_{|x| \leq 1} C_{m,k}(e^{-z}) e^{nx} dx,$$

$$E := \frac{\beta_k}{2\pi m^{\frac{1}{3}}} \int_{1 \leq |x| \leq \frac{m^{\frac{1}{3}}}{\beta_k}} C_{m,k}(e^{-z}) e^{nx} dx.$$

In the following we show that $M$ contributes to the asymptotic main term whereas $E$ is part of the error term.
4.1. Approximating the main term. The goal of this section is to determine the asymptotic behavior of $M$. We show

**Proposition 4.1.** We have

$$M = \frac{\beta_k}{4} \sech^2 \left( \frac{\beta_k m}{2} \right) p_k(n) \left( 1 + O \left( \frac{m^{\frac{1}{3}}}{n^{\frac{1}{4}}} \right) \right).$$

A key step for proving this proposition is the investigation of

$$P_{s,k} := \frac{1}{2\pi i} \int_{1 - im^{-\frac{1}{3}}}^{1 + im^{-\frac{1}{3}}} v^s e^{\pi \sqrt{\frac{kn}{6}}(v + \frac{1}{v})} dv$$

for $s > 0$. These integrals may be related to Bessel functions. Denoting by $I_s$ the usual $I$-Bessel function of order $s$, we have.

**Lemma 4.2.** As $n \to \infty$

$$P_{s,k} = I_{-s-1} \left( \pi \sqrt{\frac{2kn}{3}} \right) + O \left( \exp \left( \pi \sqrt{\frac{kn}{6}} \left( 1 + \frac{1}{1 + m^{-\frac{2}{3}}} \right) \right) \right).$$

**Proof.** We use the following loop integral representation for the $I$-Bessel function [4] ($x > 0$)

$$I_\ell(2x) = \frac{1}{2\pi i} \int_\Gamma t^{-\ell-1} e^{x(t + \frac{1}{t})} dt,$$  \[4.2\]

where the contour $\Gamma$ starts in the lower half plane at $-\infty$, surrounds the origin counterclockwise and then returns to $-\infty$ in the upper half-plane. We choose for $\Gamma$ the piecewise linear path that consists of the line segments

$$\gamma_4 : \left( -\infty - \frac{i}{2m^{\frac{2}{3}}}, -1 - \frac{i}{2m^{\frac{2}{3}}} \right), \quad \gamma_3 : \left( -1 - \frac{i}{2m^{\frac{2}{3}}}, -1 - \frac{i}{m^{\frac{1}{3}}} \right),$$

$$\gamma_2 : \left( -1 - \frac{i}{m^{\frac{2}{3}}}, 1 - \frac{i}{m^{\frac{2}{3}}} \right), \quad \gamma_1 : \left( 1 - \frac{i}{m^{\frac{2}{3}}}, 1 + \frac{i}{m^{\frac{2}{3}}} \right),$$

which are then followed by the corresponding mirror images $\gamma'_2, \gamma'_3,$ and $\gamma'_4$. Note that $P_{s,k} = \int_{\gamma_1}$. Thus, to finish the proof, we have to bound the integrals along $\gamma_4, \gamma_3,$ and $\gamma_2$—the corresponding mirror images follow in the same way.
First

\[ \int_{\gamma_4} \ll \int_{-\infty}^{1} \left| \exp \left( \pi \sqrt{\frac{kn}{6}} \left( t - \frac{im^{-\frac{1}{3}}}{2} + \frac{1}{t - \frac{im^{-\frac{1}{3}}}{2}} \right) \right) \right| dt \]

\[ \ll \int_{1}^{\infty} e^{-\pi \sqrt{\frac{kn}{6}}} \left| t + \frac{im^{-\frac{1}{3}}}{2} \right|^s dt \]

\[ \ll \int_{1}^{\infty} t^s e^{-\pi \sqrt{\frac{kn}{6}}} dt \ll n^{-\frac{s+1}{2}} \Gamma \left( s + 1; \pi \sqrt{\frac{kn}{6}} \right) \ll n^{-\frac{s}{2}} e^{-\pi \sqrt{\frac{kn}{6}}}, \]

using (3.3).

Next

\[ \int_{\gamma_3} \ll m^{-\frac{1}{3}} \int_{\frac{1}{2}}^{1} \exp \left( -\pi \sqrt{\frac{kn}{6}} \left( 1 + \frac{1}{1 + m^{-\frac{2}{3}} t^2} \right) \right) \left| 1 + \frac{im^{-\frac{1}{3}} t}{2} \right|^s dt \ll e^{-\pi \sqrt{\frac{kn}{6}}}. \]

Finally

\[ \int_{\gamma_2} \ll \int_{-1}^{1} \exp \left( \pi \sqrt{\frac{kn}{6}} \left( t + \frac{t}{t^2 + m^{-\frac{2}{3}}} \right) \right) \left| t - \frac{im^{-\frac{1}{3}}}{2} \right|^s dt \]

\[ \ll \exp \left( \pi \sqrt{\frac{kn}{6}} \left( 1 + \frac{1}{1 + m^{-\frac{2}{3}}} \right) \right), \]

where we used that \( t + \frac{t}{t^2 + m^{-\frac{2}{3}}} \) obtains its maximum at \( t = 1 \). This finishes the proof. \( \square \)

We now turn to the proof of Proposition 4.1.
Proof of Proposition 4.1. Using Proposition 3.4 and making a change of variables, we obtain by Lemma 4.2

\[ M = \frac{\beta_k^{k+2}}{4(2\pi)^{\frac{k}{2}}} \sech^2 \left( \frac{\beta_k m}{2} \right) P_{k+1,k} + O \left( \beta_k^{\frac{k}{2}+3} m^{\frac{1}{2}} \sech^2 \left( \frac{\beta_k m}{2} \right) e^{\pi \sqrt{\frac{2mn}{3}}} \right) \]

\[ = \frac{\beta_k^{k+2}}{4(2\pi)^{\frac{k}{2}}} \sech^2 \left( \frac{\beta_k m}{2} \right) \left( I_{\frac{k}{2}-2} \left( \pi \sqrt{\frac{2kn}{3}} \right) \right) + O \left( \beta_k^{\frac{k}{2}+3} m^{\frac{1}{2}} \sech^2 \left( \frac{\beta_k m}{2} \right) e^{\pi \sqrt{\frac{2mn}{3}}} \right). \]

Using the Bessel function asymptotic (see (4.12.7) in [3])

\[ I_\ell(x) = \frac{e^x}{\sqrt{2\pi x}} + O \left( \frac{e^x}{x^\ell} \right) \]

yields

\[ M = \frac{\beta_k^{k+2}}{4(2\pi)^{\frac{k}{2}}} \sech^2 \left( \frac{\beta_k m}{2} \right) \left( \frac{e^{\pi \sqrt{\frac{2kn}{3}}}}{\pi \sqrt{2} \left( \frac{2kn}{3} \right)^{\frac{1}{4}}} \right) + O \left( \frac{e^{\pi \sqrt{\frac{2kn}{3}}}}{n^{\frac{1}{4}}} \right) \]

\[ + O \left( \exp \left( \pi \sqrt{\frac{kn}{6}} \left( 1 + \frac{1}{1 + m^{-\frac{1}{2}}} \right) \right) \right) + O \left( \beta_k^{\frac{k}{2}+3} m^{\frac{1}{2}} \sech^2 \left( \frac{\beta_k m}{2} \right) e^{\pi \sqrt{\frac{2mn}{3}}} \right). \]

It is not hard to see that the last error term is the dominant one. Thus

\[ M = \frac{\beta_k^{k+2}}{4(2\pi)^{\frac{k}{2}}} \sech^2 \left( \frac{\beta_k m}{2} \right) \frac{e^{\pi \sqrt{\frac{2kn}{3}}}}{\pi \sqrt{2} \left( \frac{2kn}{3} \right)^{\frac{1}{4}}} \left( 1 + O \left( m^{\frac{1}{4}} n^{-\frac{1}{4}} \right) \right). \]

Using that [16, 19]

\[ p_k(n) = 2 \left( \frac{k}{3} \right)^{\frac{1+k}{4}} (8n)^{-\frac{3+k}{4}} e^{\pi \sqrt{\frac{2kn}{3}}} \left( 1 + O \left( \frac{1}{\sqrt{n}} \right) \right) \]

now easily gives the claim. \qed
4.2. **The error arc.** We finally bound $E$ and show that it is exponentially smaller than $M$. The following proposition then immediately implies Theorem 1.3.

**Proposition 4.3.** As $n \to \infty$

$$E \ll \sqrt{n} \exp \left( \pi \sqrt{\frac{kn}{6}} \left( 1 + \frac{1}{1 + \frac{m}{3}} \right) \right).$$

**Proof.** Using Proposition 3.5, we may bound

$$E \ll \frac{\beta_k}{m^2} \int_{1 \leq x \leq \frac{\pi m}{3k}} \sqrt{n} \exp \left( \pi \sqrt{\frac{kn}{6}} \left( 1 + \frac{1}{1 + \frac{m}{3}} \right) \right) dx$$

$$\ll \sqrt{n} \exp \left( \pi \sqrt{\frac{kn}{6}} \left( 1 + \frac{1}{1 + \frac{m}{3}} \right) \right) \ll \sqrt{n} \exp \left( \pi \sqrt{\frac{kn}{6}} \left( 2 - \frac{m}{3} \right) \right).$$

\[\square\]

**References**


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