Characterization of relations accepted by two-way transducers

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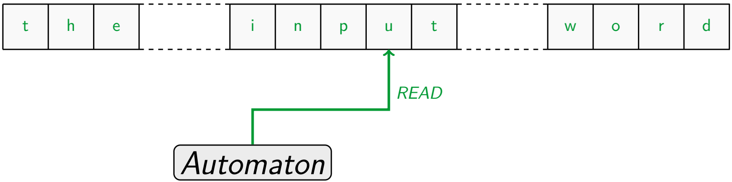
Séminaire Automate

1-way automaton over Σ

A

(Q, q_-, F, δ)

transition set: $Q \times \Sigma \times Q$



2-way automaton over Σ

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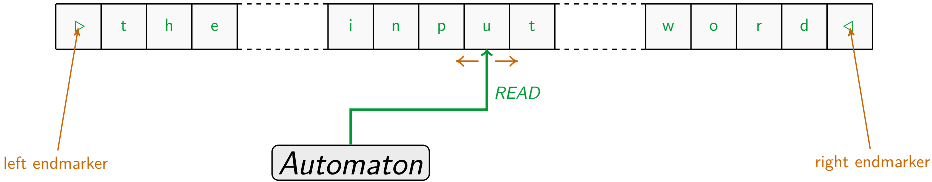
(Q, q_{-}, F, δ)

transition set: $Q \times \Sigma_{\triangleright,\triangleleft} \times \{-1, 0, 1\} \times Q$

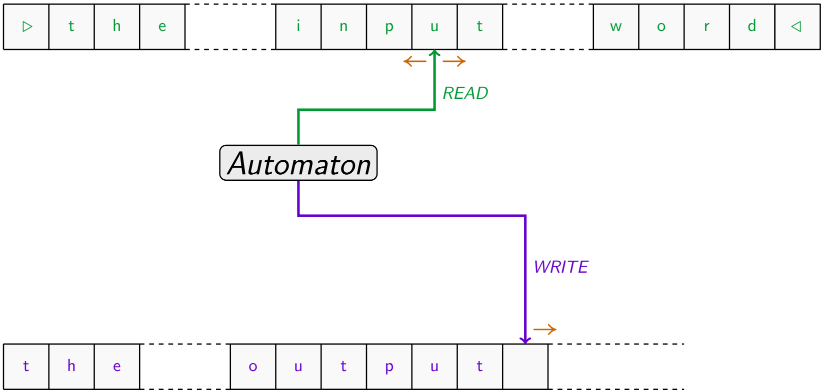


2-way transducer over Σ, Γ

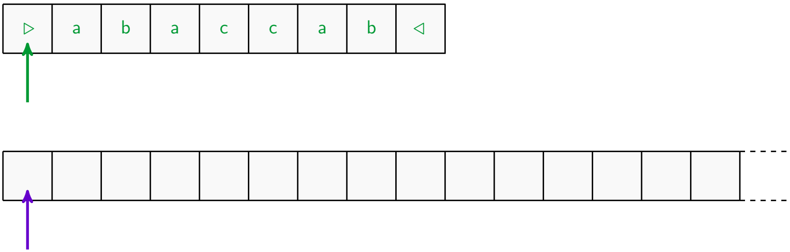
$$(A, \phi)$$

(Q, q_-, F, δ) ← A; ϕ → production function: $\delta \to \mathrm{RAT}(\Gamma^*)$

δ → transition set: $Q \times \Sigma_{\triangleright,\triangleleft} \times \{-1, 0, 1\} \times Q$



A simple example: $\textsc{Square} = \{(w, ww) \mid w \in \Sigma^*\}$



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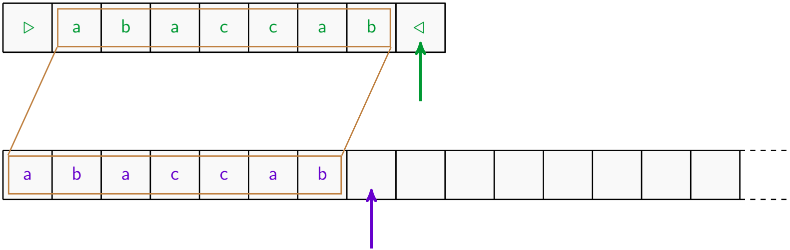

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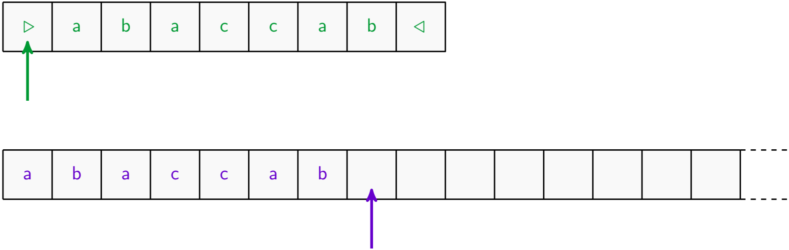

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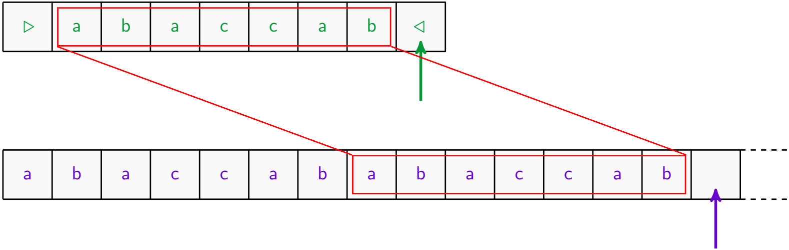

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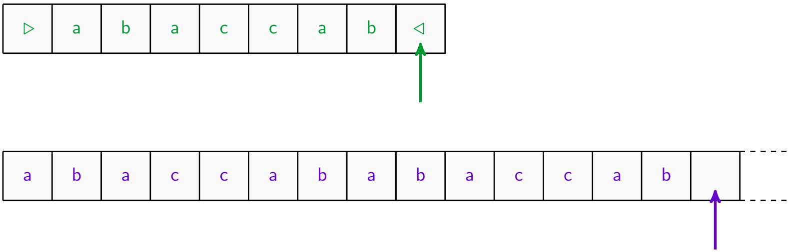

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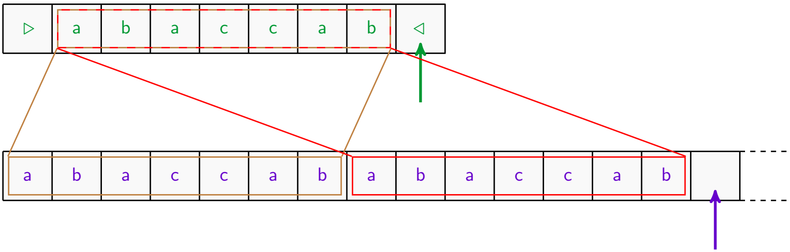

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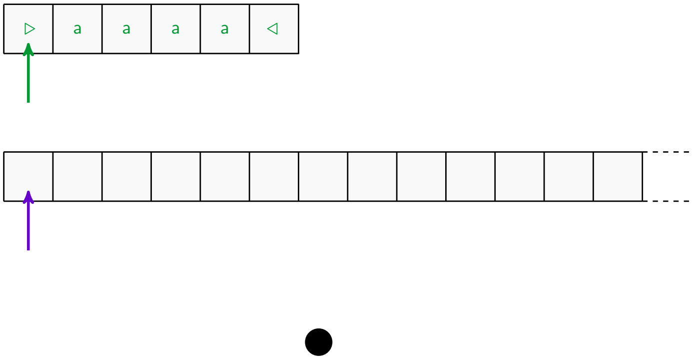
Another example: $\textsc{uMult} = \{(a^n, a^{kn}) \mid k, n \in \mathbb{N}\}$



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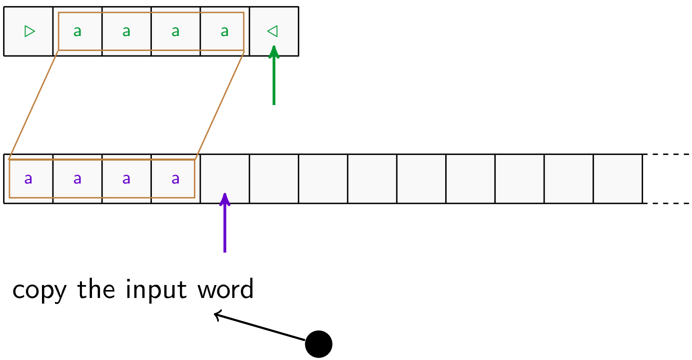

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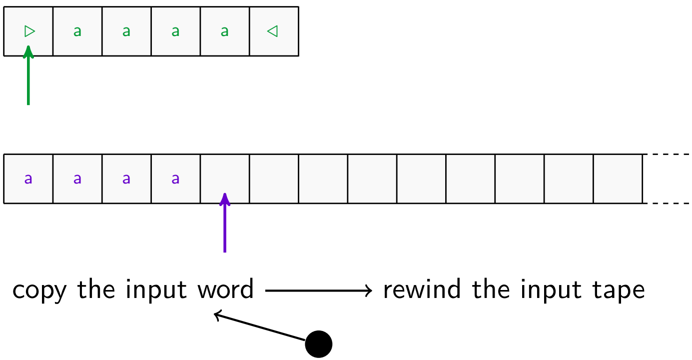

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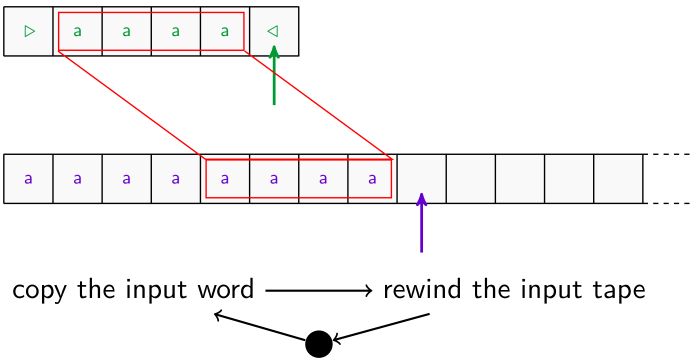

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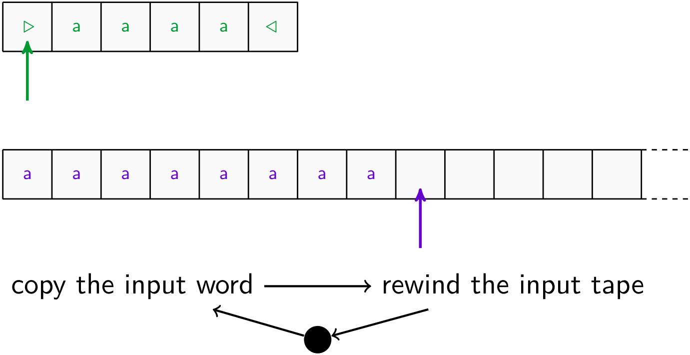

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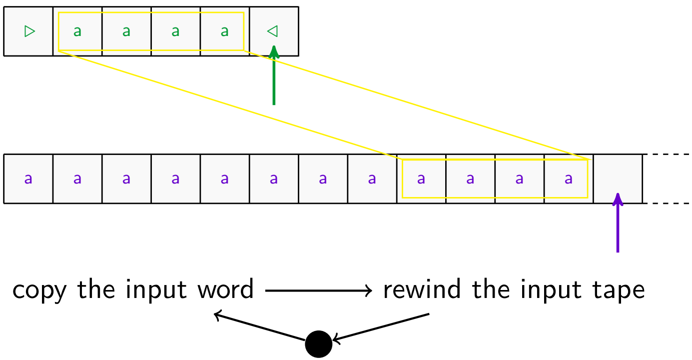

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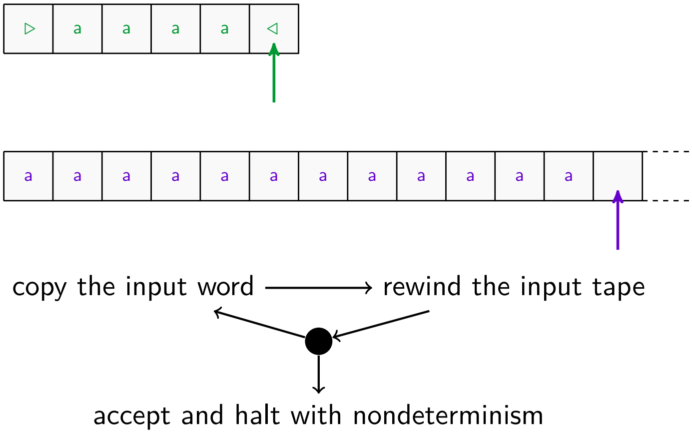

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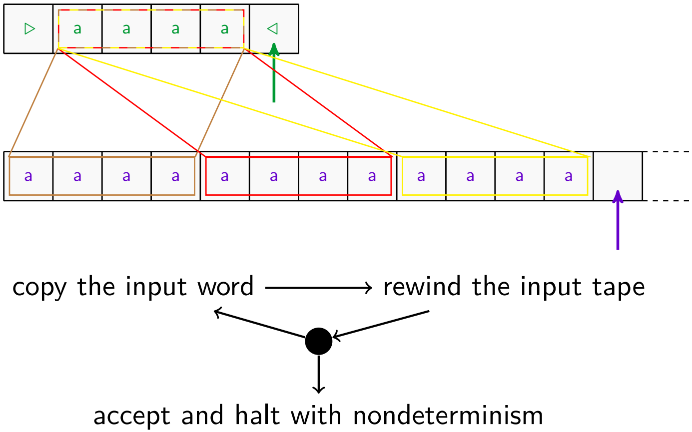

Rational operations

- Union

$$R_1 \cup R_2$$

- Componentwise concatenation

$$R_1 \cdot R_2 = \{(u_1 u_2, v_1 v_2) \mid (u_1, v_1) \in R_1 \text{ and } (u_2, v_2) \in R_2\}$$

- Kleene star

$$R^* = \{(u_1 u_2 \cdots u_k, v_1 v_2 \cdots v_k) \mid \forall i \ (u_i, v_i) \in R\}$$

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Definition ($Rat(\Sigma^* \times \Gamma^*)$)

The class of rational relations is the smallest class:

- contains finite relations
- closed under rational operations

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Theorem (Elgot, Mezei - 1965)

1-way transducers = the class of rational relations.

Hadamard operations

- Union $R_1 \cup R_2$
- H-product

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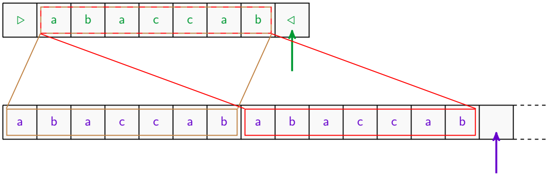
Example: $\textsc{Square} = \{(w, ww) \mid w \in \Sigma^*\} = \textsc{Id} \circledast \textsc{Id}$



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$$R^{\mathsf{H}\star} = \{(u, v_1 v_2 \cdots v_k) \mid \forall i \ (u, v_i) \in R\}$$

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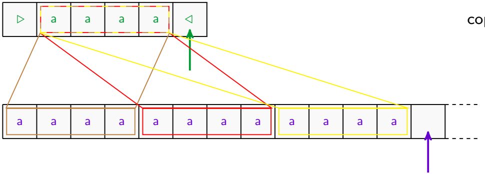

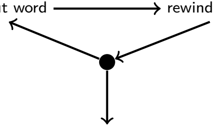

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Definition ($\textsc{Had}(\Sigma^* \times \Gamma^*)$)

The class of Hadamard relations is the smallest class:

- contains rational relations
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Hadamard relations

Proposition

two-way transducers are closed under H-operations.

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Proof

- $R_1 \cup R_2$:
  - *simulate* T_1 *or* T_2

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Proof

- $R_1 \cup R_2$:
  - *simulate* T_1 *or* T_2
- $R_1 \circledH R_2$:
  - *simulate* T_1
  - *rewind the input tape*
  - *simulate* T_2

Hadamard relations

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two-way transducers are closed under H-operations.

Proof

- $R_1 \cup R_2$:
  - *simulate T_1 or T_2*
- $R_1 \, \textcircled{\small H} \, R_2$:
  - *simulate T_1*
  - *rewind the input tape*
  - *simulate T_2*
- $R^{H\star}$:
  - *repeat an arbitrary number of times:*
    - *simulate T*
    - *rewind the input tape*
  - *reach the right endmarker and accept*

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$\textsc{Had} = $ *rotating*

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$\textsc{Had} = \textit{rotating} \subseteq \textit{two-way}$

Hadamard relations

Proposition

two-way transducers *are* *closed* *under* *H-operations*.

Proposition

Rat $\subsetneq$ $\textsc{Had}$ $=$ *rotating* $\subseteq$ *two-way*

Example

$$\textsc{uMult} = \left\{(a^n, a^{kn}) \mid k, n \in \mathbb{N}\right\} = \{(a^n, a^n) \mid n \in \mathbb{N}\}^{\mathsf{H}\star} = \textsc{uId}^{\mathsf{H}\star}$$

Main result

Theorem (Elgot, Mezei - 1965)

1-way transducers = the class of rational relations.

Main result

Theorem (This talk)

When $\Sigma = \{a\}$ *and* $\Gamma = \{a\}$:

2-way transducers = *the class of* HAD *relations.*

Main result

Theorem (This talk)

When $\Sigma = \{a\}$ and $\Gamma = \{a\}$:

2-way transducers $=$ *the class of* HAD *relations*.

Theorem This talk

With $\Sigma = \{a, \#\}$:

HAD $\subsetneq$ *two-way*

With $\Gamma = \{a, b\}$:

HAD $\subsetneq$ *two-way*

Known results on 2-way transducers

- functional = deterministic = MSO definable functions
- general incomparable MSO definable relations

[Engelfriet, Hoogeboom - 2001]

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- 1-way simulation of 2-way functional transducer:
  decidable and constructible [Filiot et al. - 2013]

Known results on 2-way transducers with unary output

When $\Gamma = \{a\}$:

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- simulation of unambiguous by 1-way [Anselmo - 1990]
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- simulation of unambiguous by 1-way [Anselmo - 1990]
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- *tropical* = 1-way [Carnino, Lombardy - 2014]

production function $\Phi : \delta \to \{a^n a^* \mid n \in \mathbb{N}\} \cup \varnothing$

rational of *period* 1

$$\Sigma = \{a\} \quad \text{and} \quad \Gamma = \{a\}$$

From 2-way transducers to HAD (unary case) [1]

Theorem

When $\Sigma = \{a\}$ *and* $\Gamma = \{a\}$*:*

$$\text{HAD} = \textit{two-way transducers}$$

Proof

- $\subseteq$*: done.*
- $\supseteq$*:* ***to do***.

From 2-way transducers to HAD (unary case) [1]

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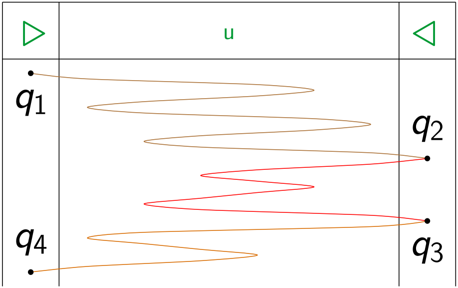
  We fix a transducer $\mathcal{T}$.

From 2-way transducers to HAD (unary case) [2]

- Consider border to border run segments;



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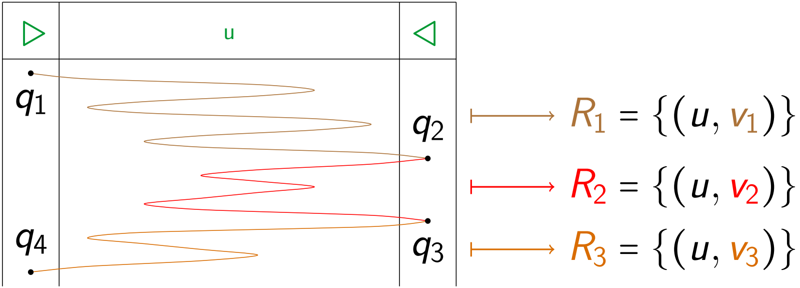

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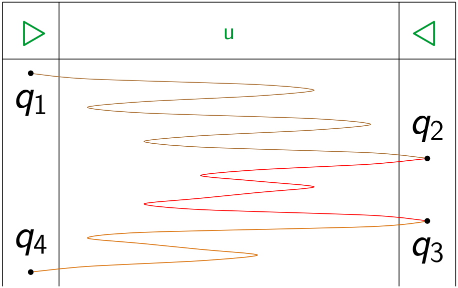

$\longmapsto R_1 = \{(u, v_1)\}$

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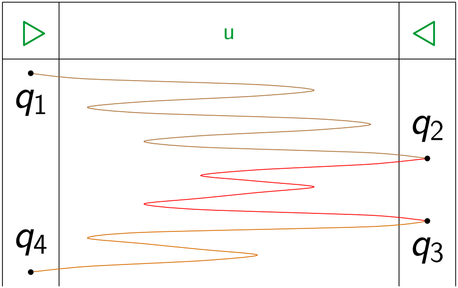

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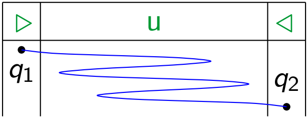
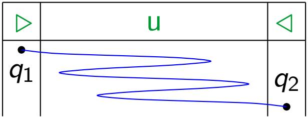
$$R_1 \circledH R_2 \circledH R_3 = \{(u, v_1 v_2 v_3)\}$$

From 2-way transducers to HAD (unary case) [3]



define a relation R_{b_i, b_j}

From 2-way transducers to HAD (unary case) [3]



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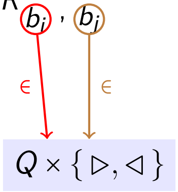

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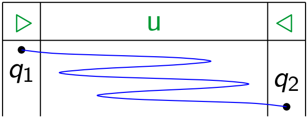

define a relation R_{b_i, b_j}

$\in$ $\in$

$Q \times \{ \triangleright, \triangleleft \}$

$2|Q|$

$$\mathrm{HIT} = \begin{pmatrix} R_{0,0} & R_{0,1} & \cdot & \cdot & \cdot & R_{0,k} \\ R_{1,0} & R_{1,1} & \cdot & \cdot & \cdot & R_{1,k} \\ \cdot & & & & & \cdot \\ \cdot & & & R_{i,j} & & \cdot \\ \cdot & & & & & \cdot \\ R_{k,0} & R_{k,1} & \cdot & \cdot & \cdot & R_{k,k} \end{pmatrix}$$

$2|Q|$

From 2-way transducers to HAD (unary case) [4]

$\langle \text{HAD}, \cup, \textcircled{H}, \text{H}\star \rangle$ is a Conway semiring.

From 2-way transducers to $\textsc{Had}$ (unary case) [4]

$\langle \textsc{Had}, \cup, Ⓗ, \textsc{h}\star \rangle$ is a Conway semiring.

Look at the successive power of the matrix HIT: HIT^k

...that is, the compositions of k border to border runs...

From 2-way transducers to $\textsc{Had}$ (unary case) [4]

$\langle \textsc{Had}, \cup, ⓗ, {}_{\mathsf{H}}\star \rangle$ is a Conway semiring.

Look at the star of the matrix HIT: $\mathrm{HIT}^{\mathsf{H}\star}$

... that is, the behavior of $\mathcal{T}$.

Remark

The relation accepted by $\mathcal{T}$ is a union of entries of $\mathrm{HIT}^{H\star}$.

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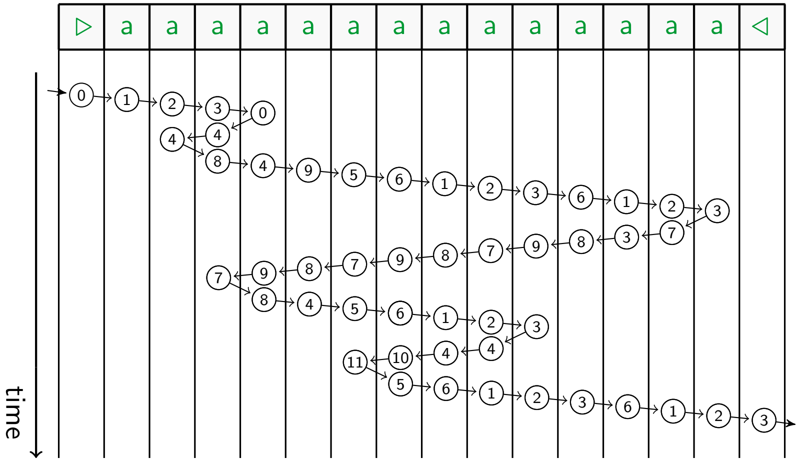
entries of $\mathrm{HIT} \in \textsc{Had}$ $\Longrightarrow$ entries of $\mathrm{HIT}^{\mathsf{H}\star} \in \textsc{Had}$

From 2-way transducers to HAD (unary case) [5]



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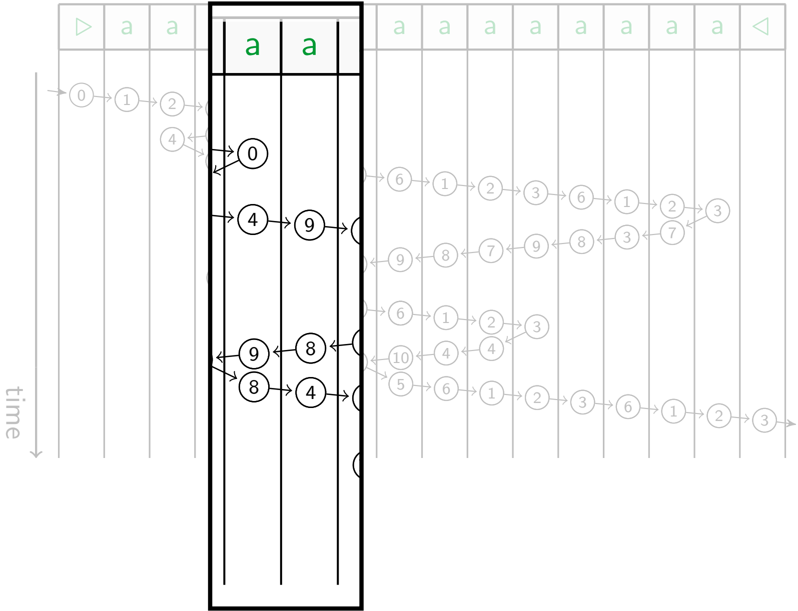

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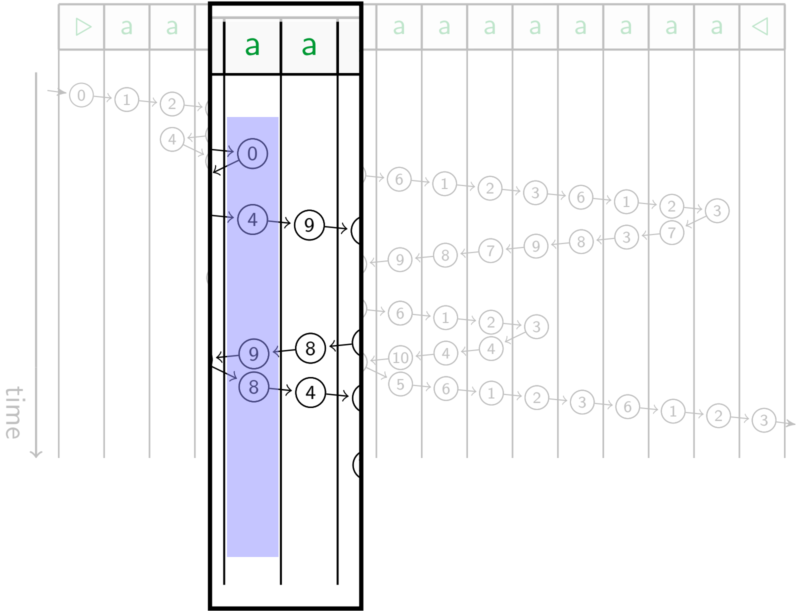

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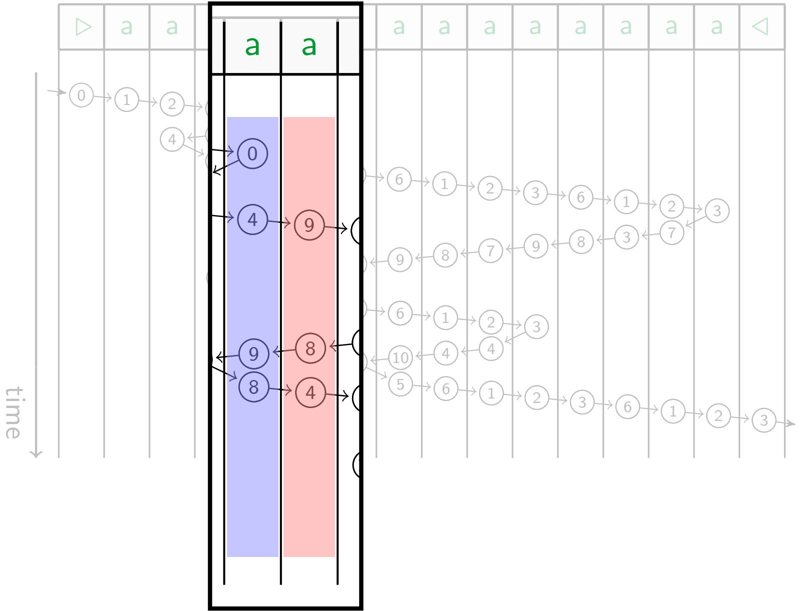

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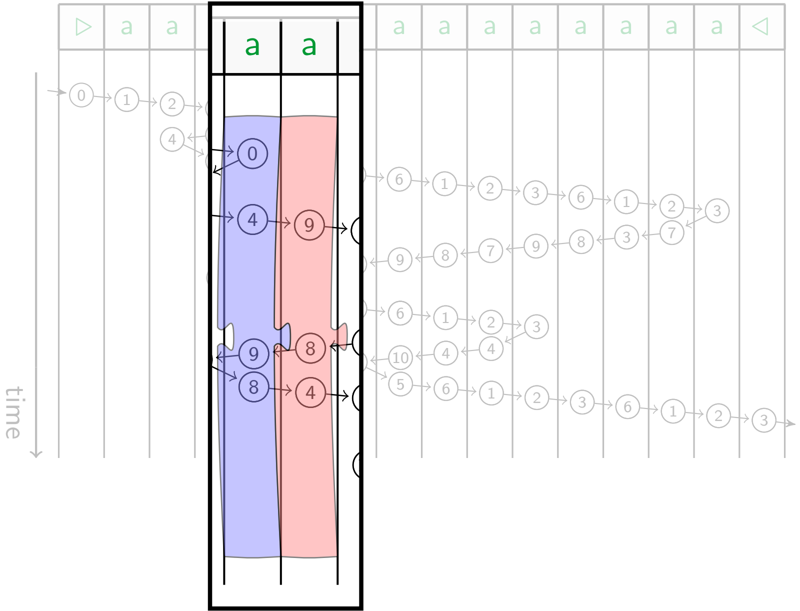

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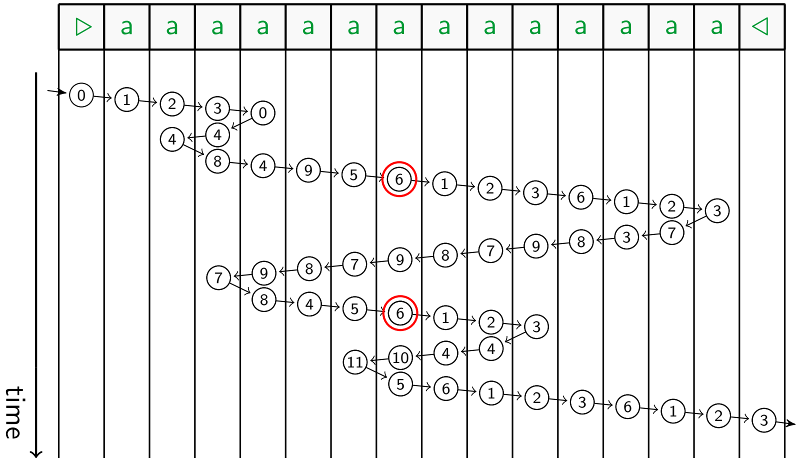

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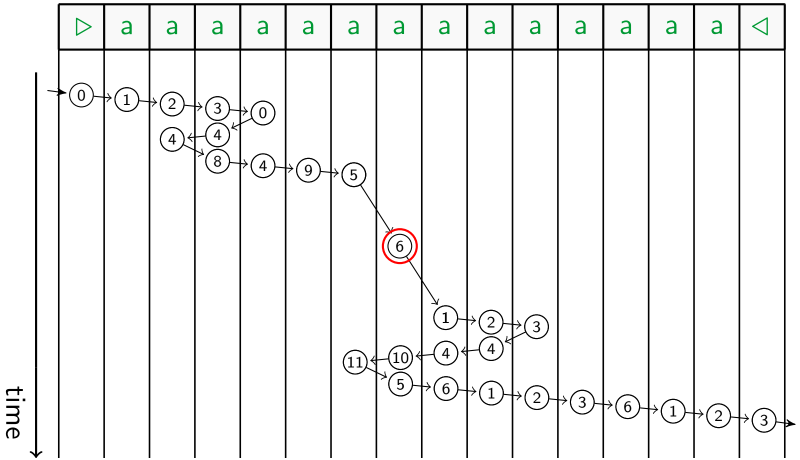

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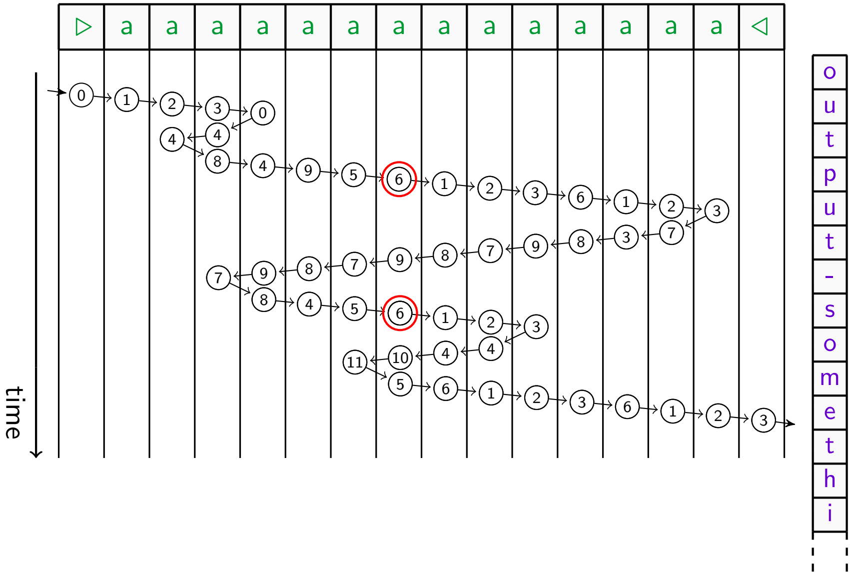

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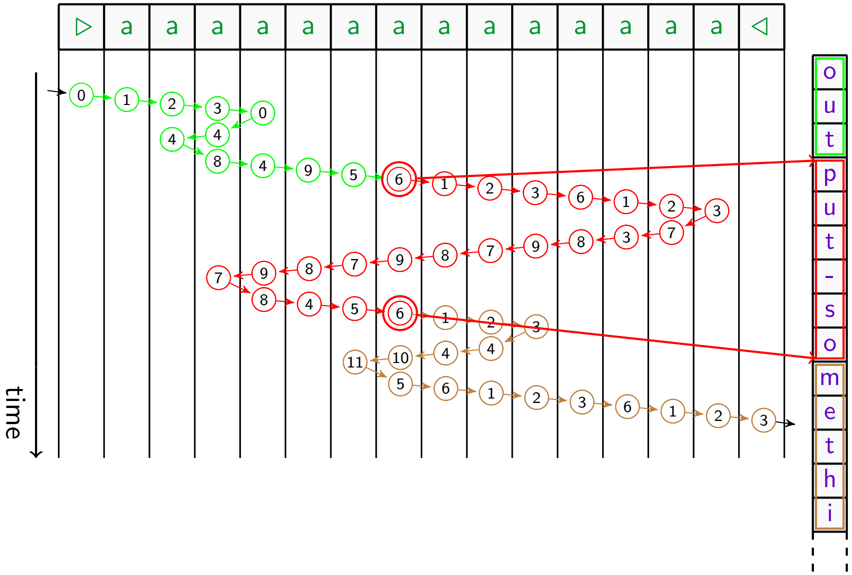

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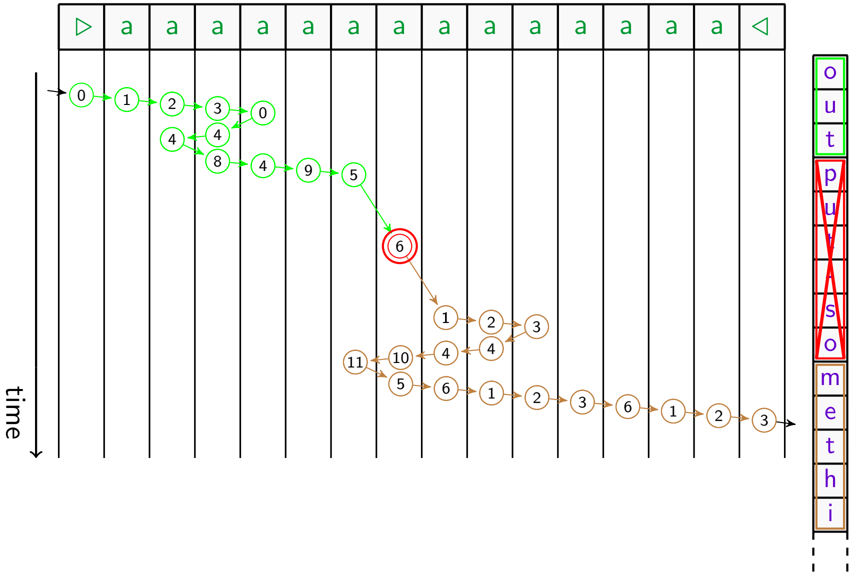

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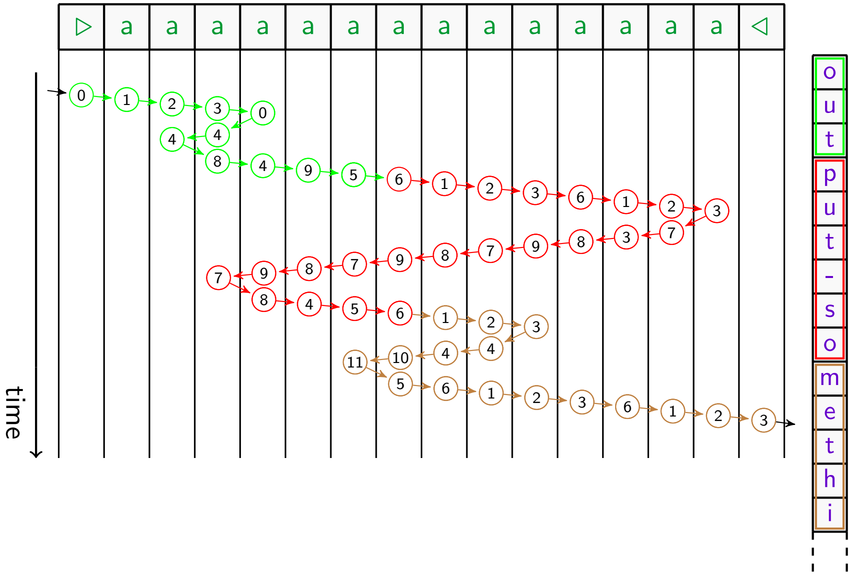

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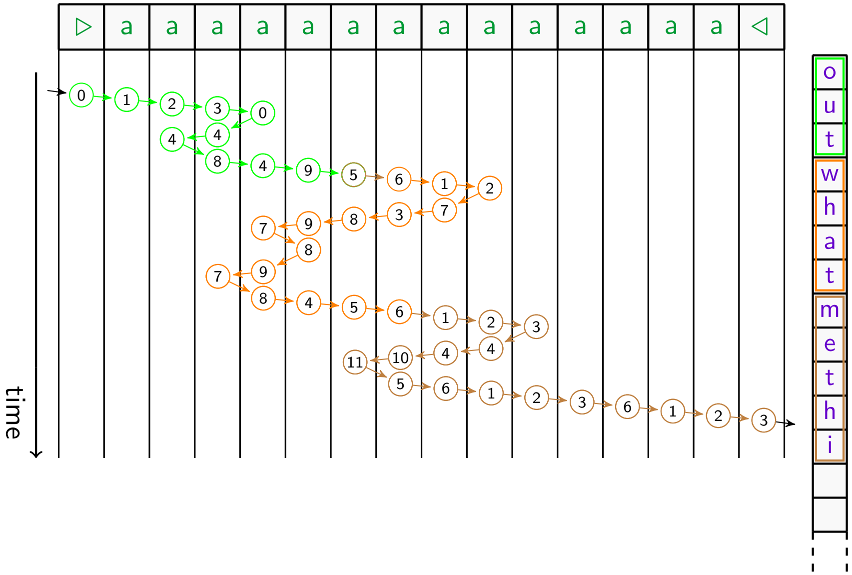

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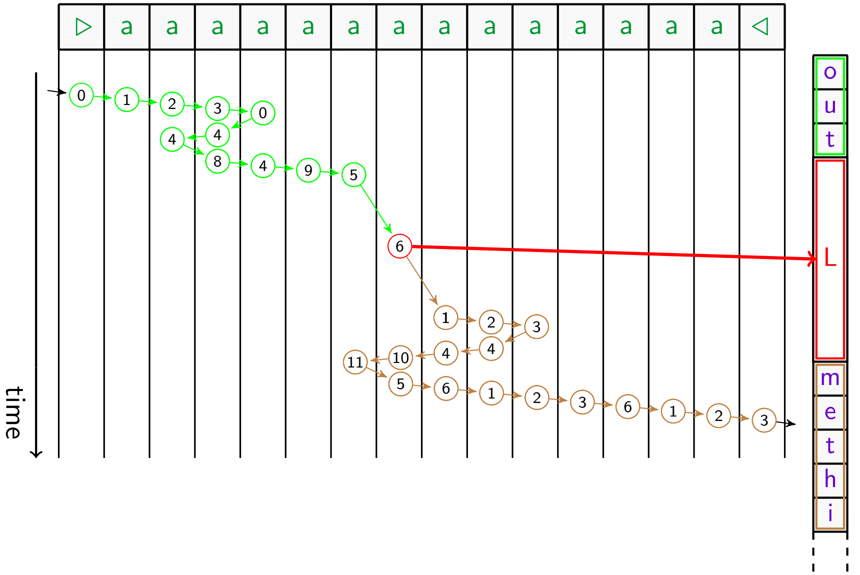

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Look at the star of the matrix HIT: $\mathrm{HIT}^{\mathsf{H}\star}$

...that is, the behavior of $\mathcal{T}$.

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The relation accepted by $\mathcal{T}$ is a union of entries of $\mathrm{HIT}^{H\star}$.

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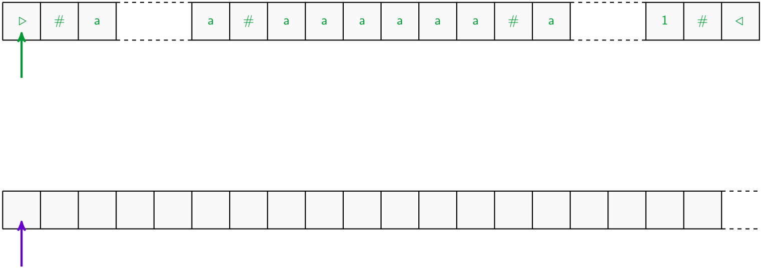
$$R = \left\{ \left(u, a^{kn}\right) \mid k, n \in \mathbb{N},\ \#a^k\# \text{ is a factor of } u \right\}$$



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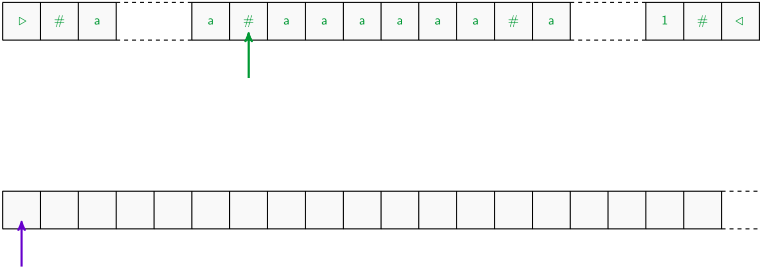

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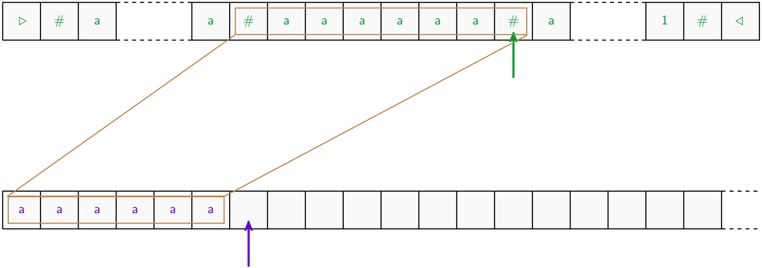

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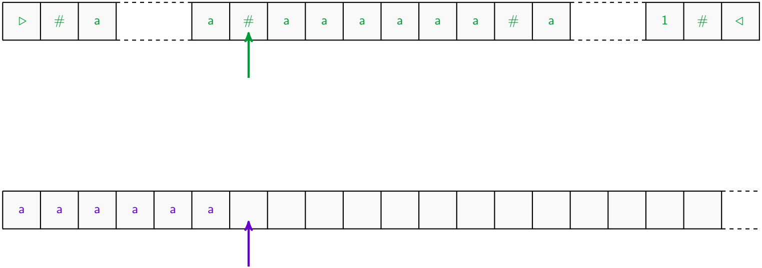

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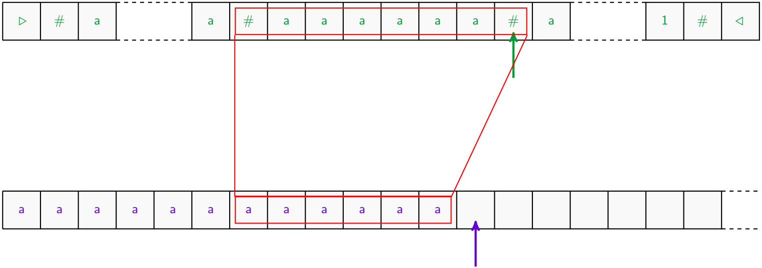

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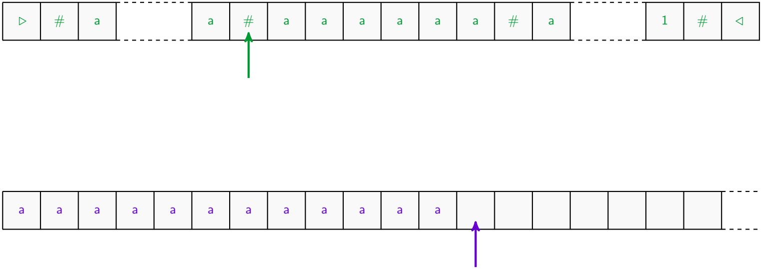

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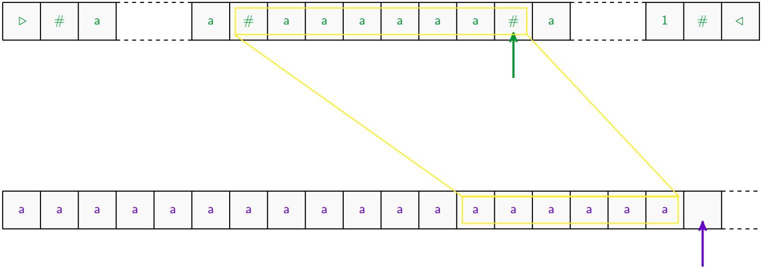

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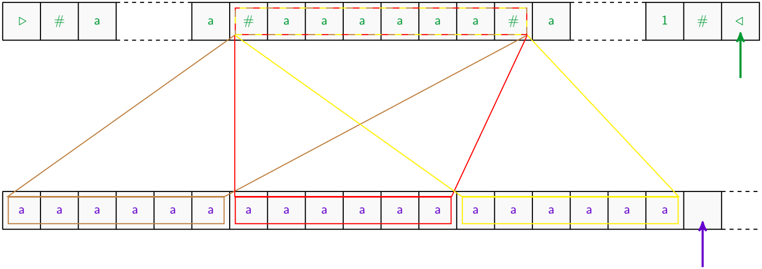

Sweeping weakens two-way transducers

Proposition

With $\Sigma = \{a, \#\}$ *and* $\Gamma = \{a\}$,

$$\textsc{Had} = \textit{sweeping} \subsetneq \textit{two-way}$$

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the family $Rat(a^*)$ is isomorphic to the rational subsets of $\mathbb{N}$

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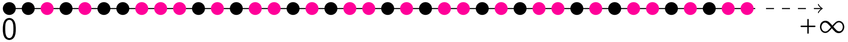

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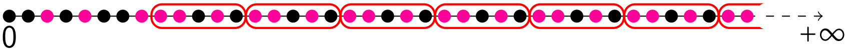

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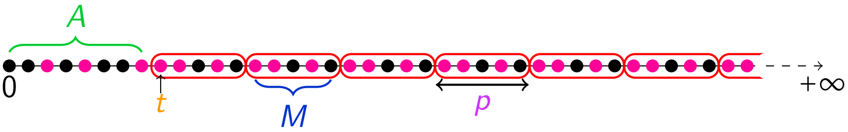

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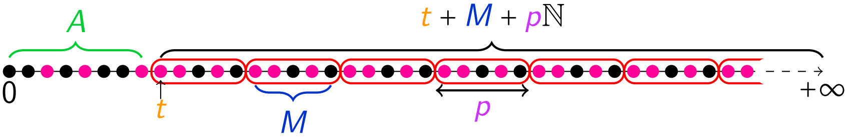

$$L = A \cup (t + M + p\mathbb{N})$$

where: $t, p \in \mathbb{N}$, $A \subseteq [\![0, t[\![$ and $M \subseteq [\![0, p[\![$

- t is *a threshold for* L
- p is *a period for* L

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$R \subseteq \Sigma^* \times \Gamma^*$. The image of $u \in \Sigma^*$ is:

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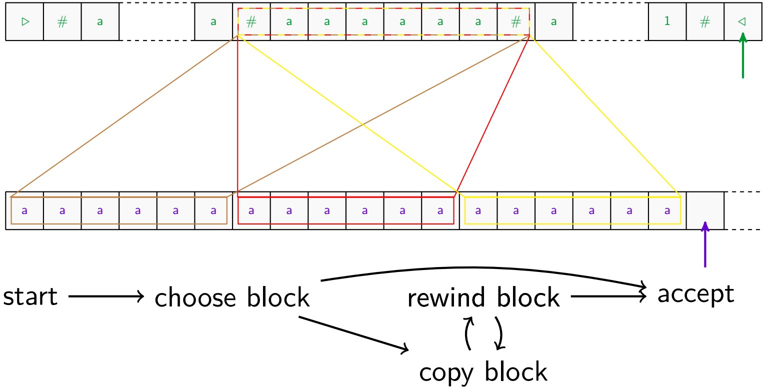

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On $\Sigma = \{a\}$ and $\Gamma = \{a, b\}$

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HAD ⊆ *two-way*

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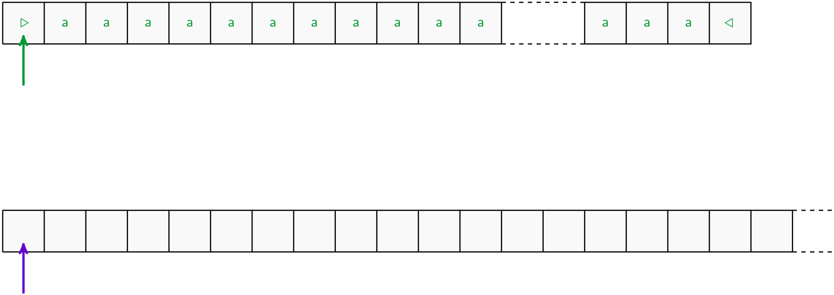

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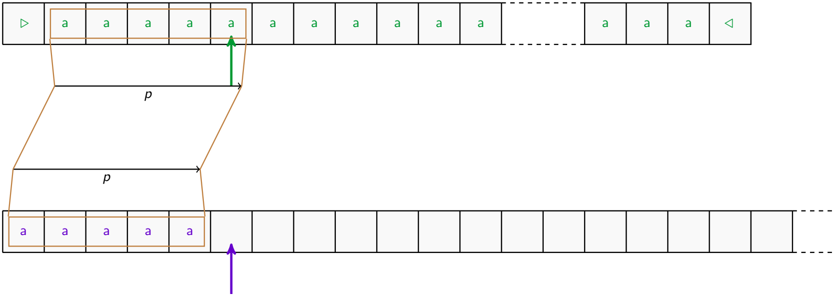

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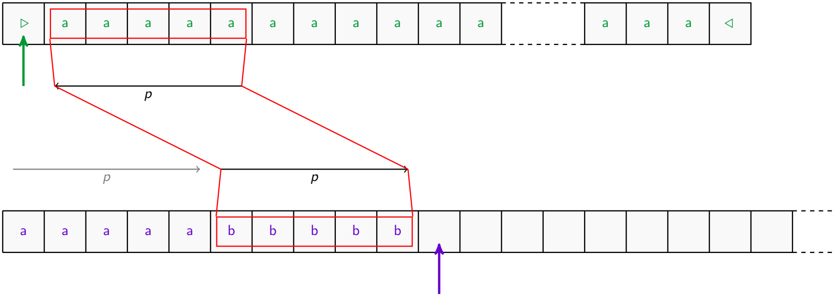

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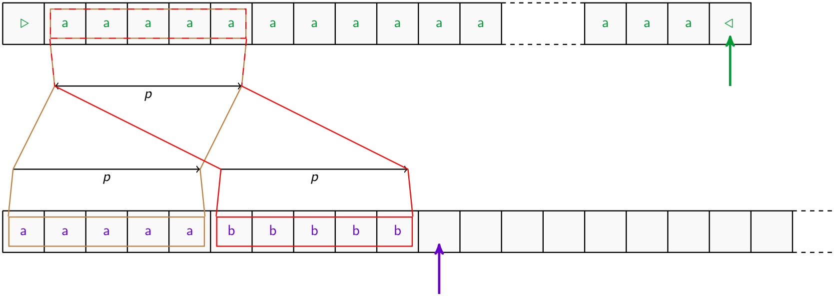

Conclusion

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general			MHAD	
input unary	RAT			~~MHAD~~
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input and ouptut unary	RAT	HAD	HAD	HAD

everything is effective. . .

Conclusion

Deterministic (= functional) case

transducer	one-way	rotating	sweeping	two-way
general			MHad	~~MHad~~
input unary			Had	
output unary		Rat		
input and ouptut unary				

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Thank you for your attention.

Appendix 1

On the optimality of:

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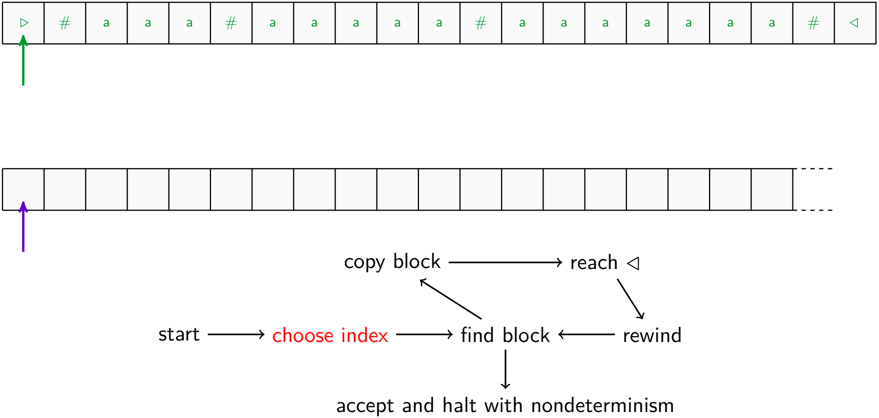
Example of Hadamard relation with polynomial period

$\Sigma = \{\#, a\}$ and $\Gamma = \{a\}$

$$R_r = \left\{ \left(\# a^{k_1} \# a^{k_2} \# \cdots \# a^{k_r} \#, a^{k_i n} \right) \mid n \in \mathbb{N} \right\}$$



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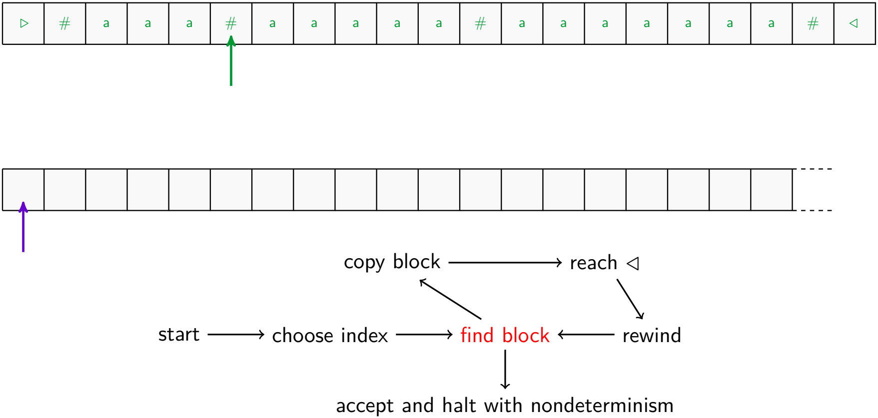

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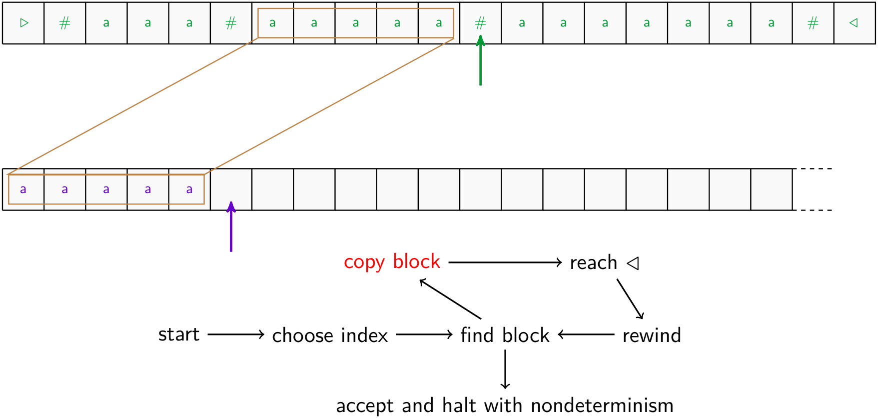

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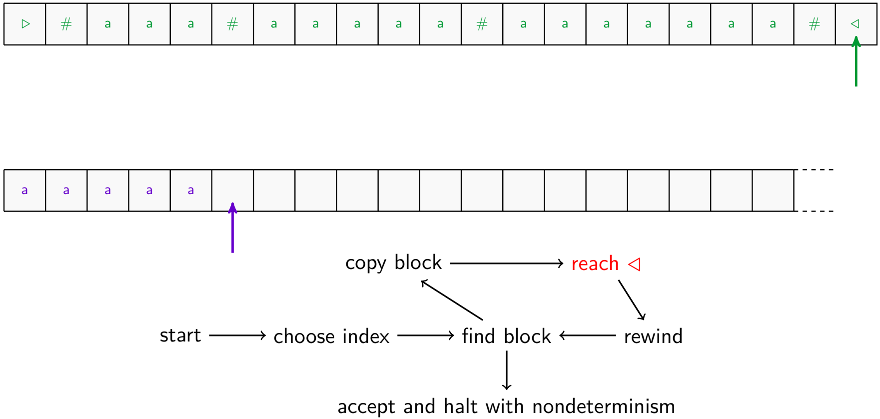

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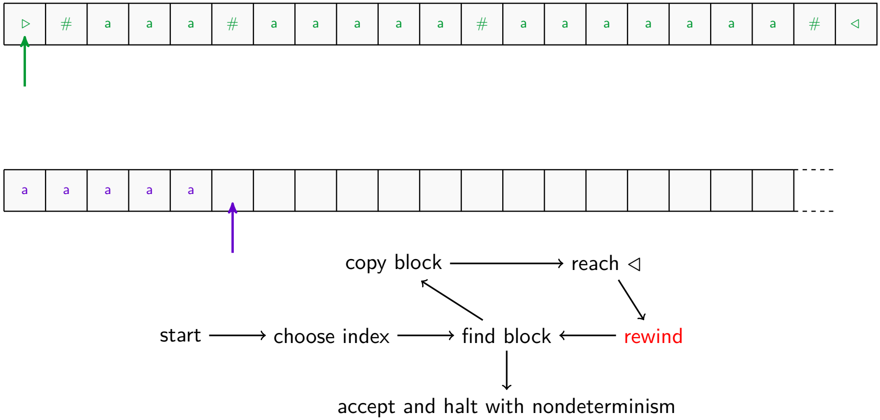

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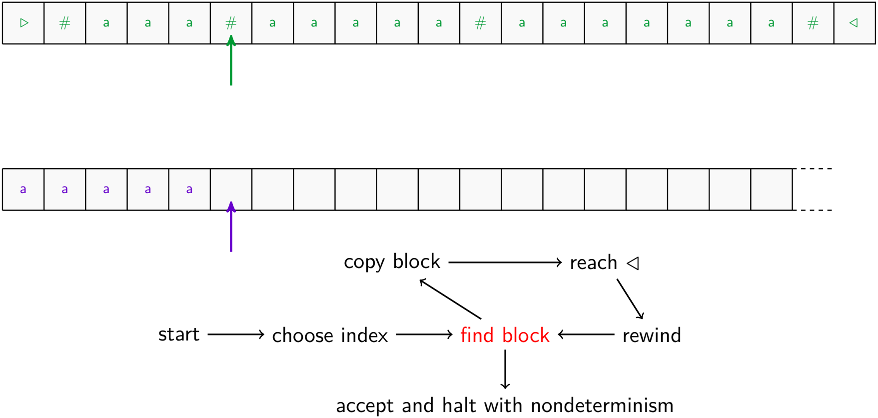
$$R_r = \left\{ \left(\# a^{k_1} \# a^{k_2} \# \cdots \# a^{k_r} \#, a^{k_i n} \right) \mid n \in \mathbb{N} \right\}$$



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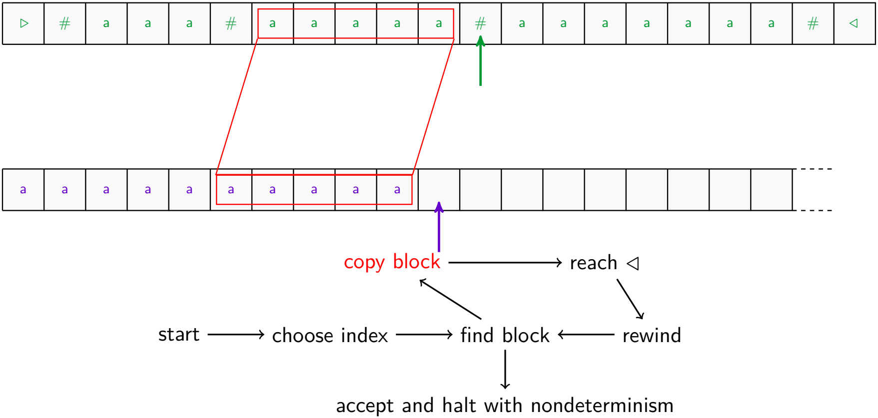

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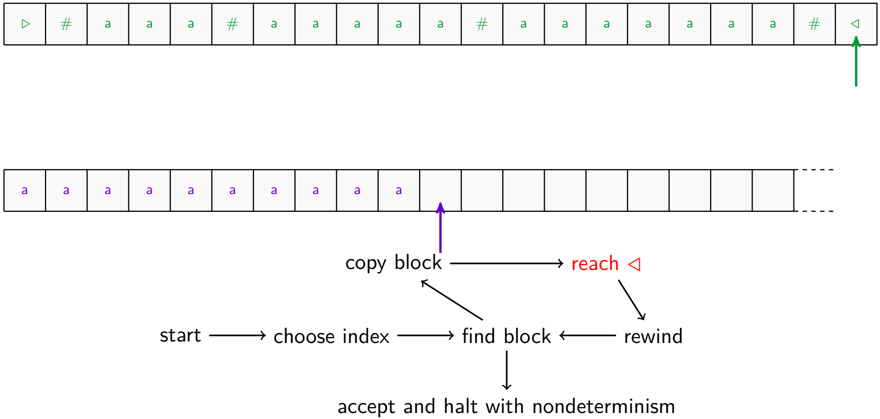

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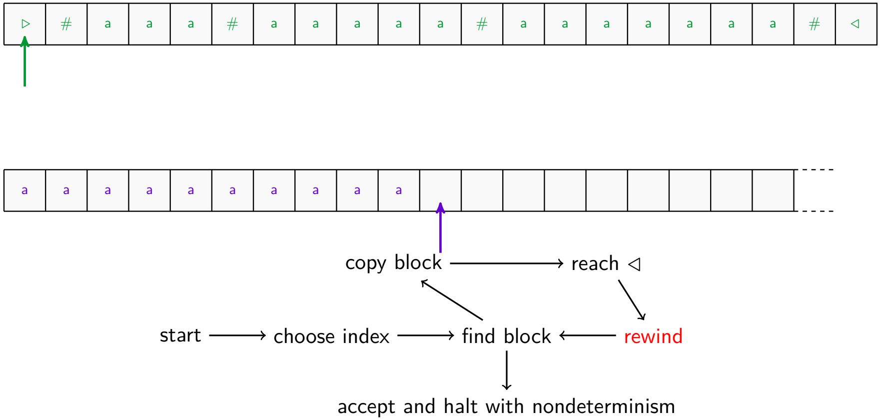

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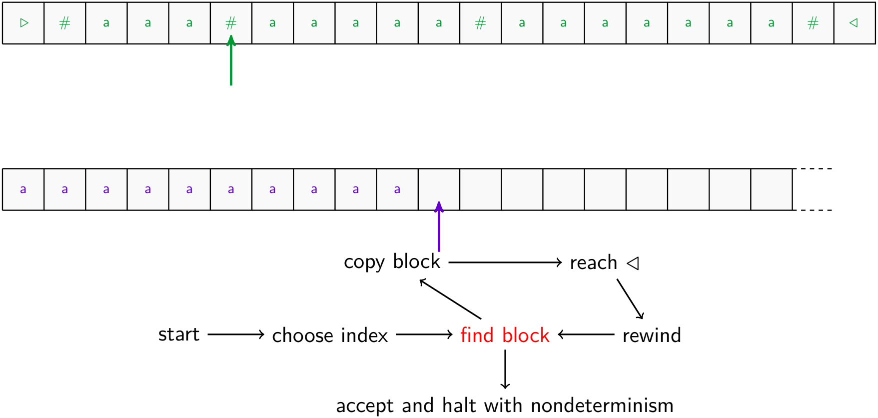

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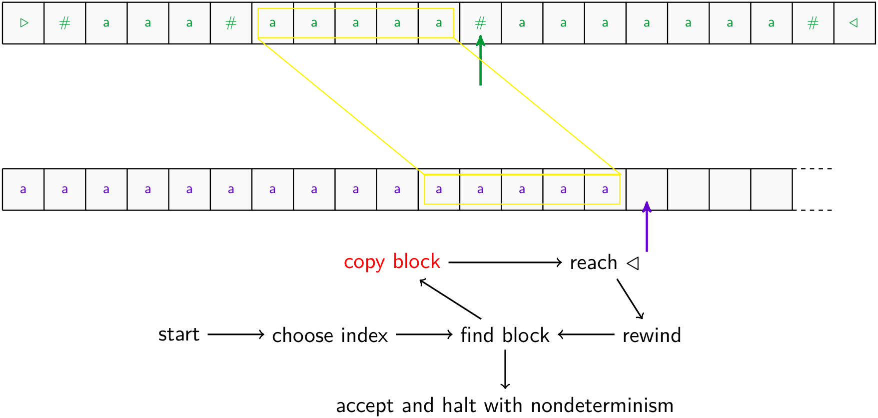

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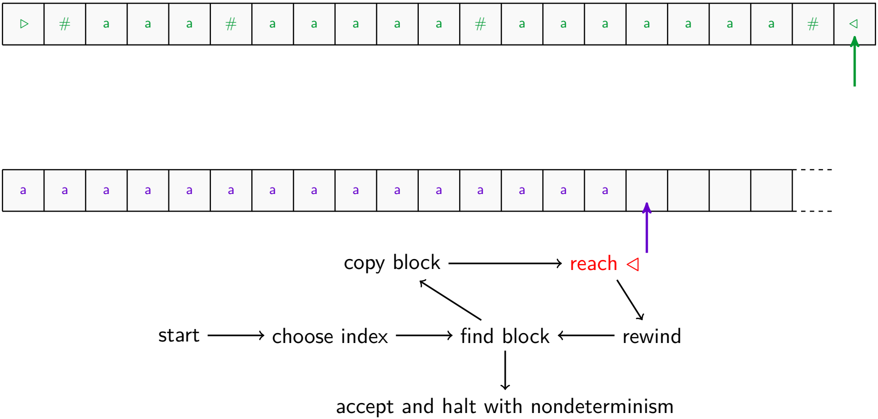

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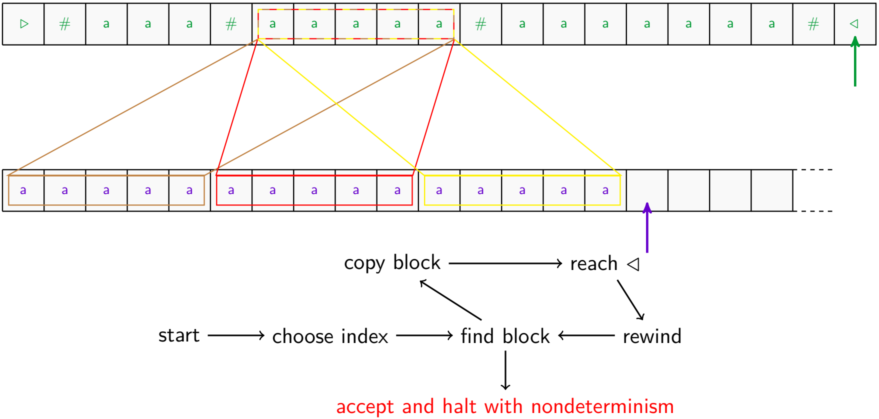

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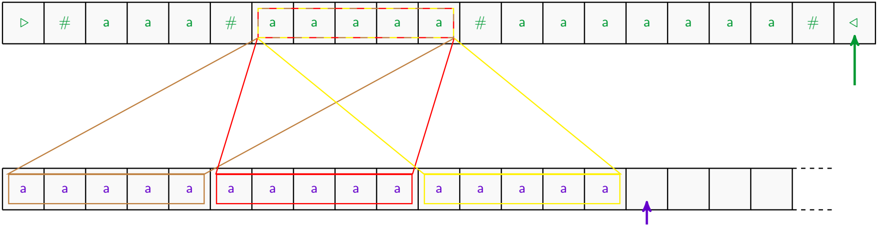

$u = \#aaa\#aaaaa\#aaaaaaa\#$ $\quad |u| = 20$

the period of $R(u)$ is $\text{lcm}(3, 5, 7) = 105$

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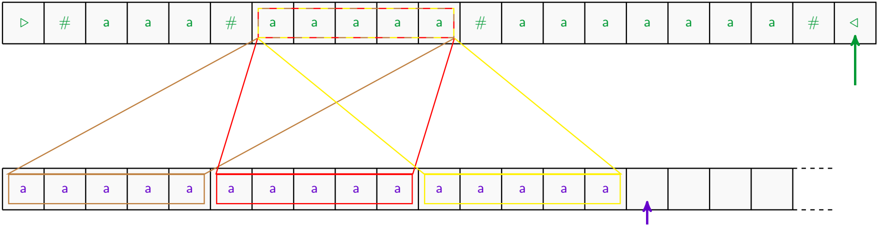

the period of $R(u)$ is in $\mathcal{O}\left(|u|^r\right)$

Appendix 2

On central loops when $\Sigma = \{a\}$ and $\Gamma = \{a\}$

Center loops when $\Sigma = \{a\}$ and $\Gamma = \{a\}$

We fix $q \in Q$.

- Consider the language :

$$L_q^\infty = \{\, \phi(\mathbf{r}) \mid \mathbf{r} \text{ is a } q\text{-central loop over some input } u\}$$

Center loops when $\Sigma = \{a\}$ and $\Gamma = \{a\}$

We fix $q \in Q$.

- Consider the subset of $\mathbb{N}$:

$$L_q^\infty = \{|\phi(\mathbf{r})| \mid \mathbf{r} \text{ is a } q\text{-central loop over some input } u\}$$

Center loops when $\Sigma = \{a\}$ and $\Gamma = \{a\}$

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- if a position is at distance $> N$ of both endmarkers, then each $\mathbf{r_i}$ may occur

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- and thus the language L_q^∞ can be produced on the output tape