An algebraic characterization of unary 2-way transducers

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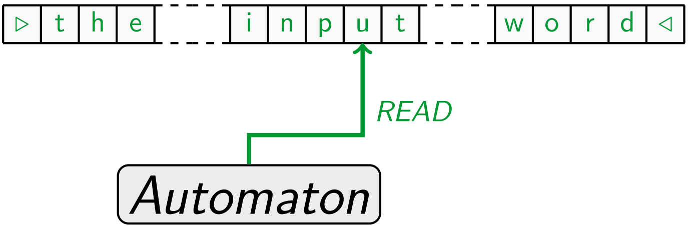
August 25, 2014

2-way automaton over Σ

A

(Q, q_-, F, δ)

transition set: $\subset Q \times \Sigma_{\triangleright,\triangleleft} \times \{-1, 0, 1\} \times Q$



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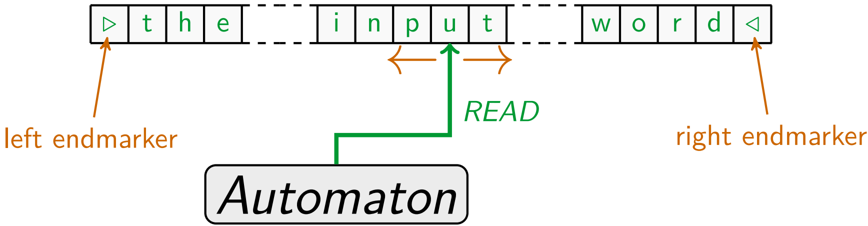
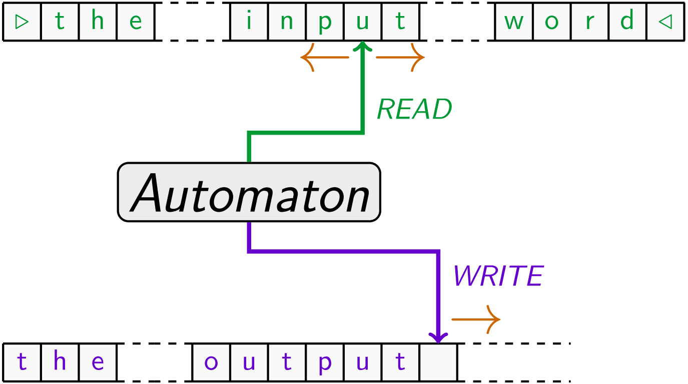


2-way transducer over Σ, Γ

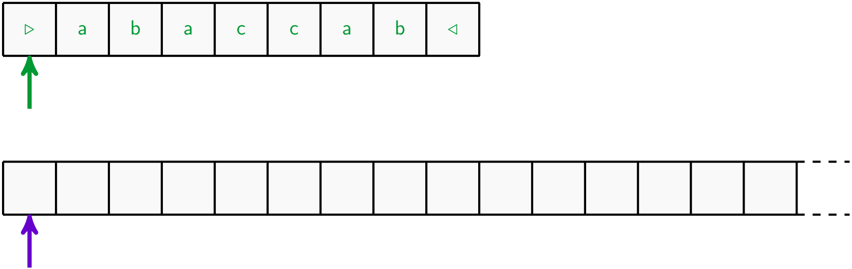
$$(A, \phi)$$

$\left(Q, q_-, F, \delta\right) \longleftarrow A$

$\phi \longrightarrow$ production function: $\delta \to Rat(\Gamma^*)$



A simple example: $SQUARE = \{(w, ww) \mid w \in \Sigma^*\}$



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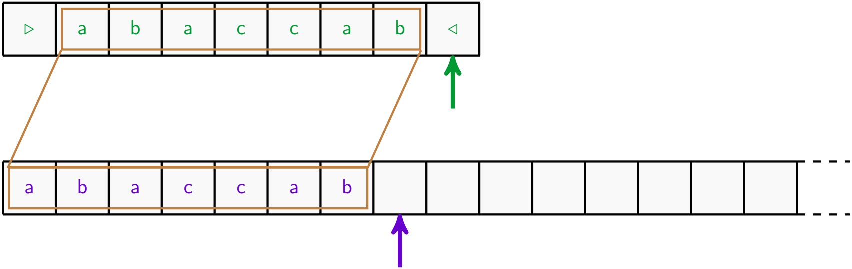

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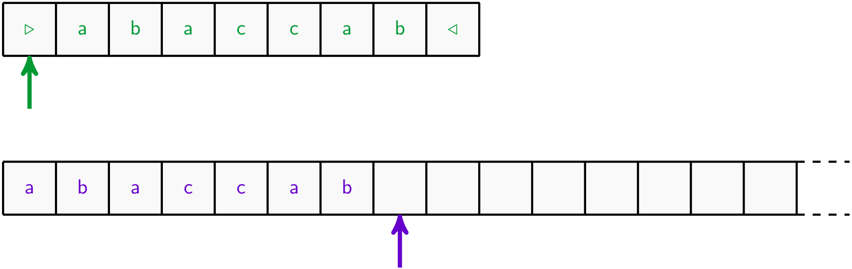

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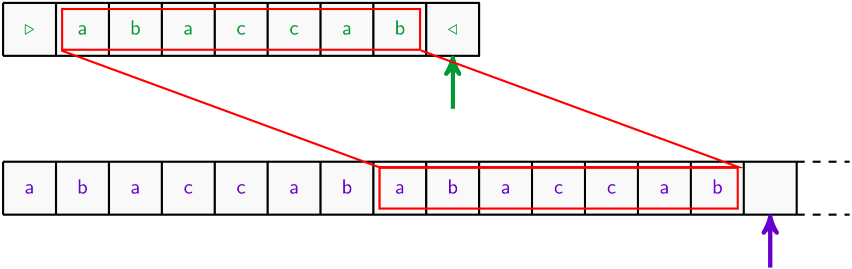

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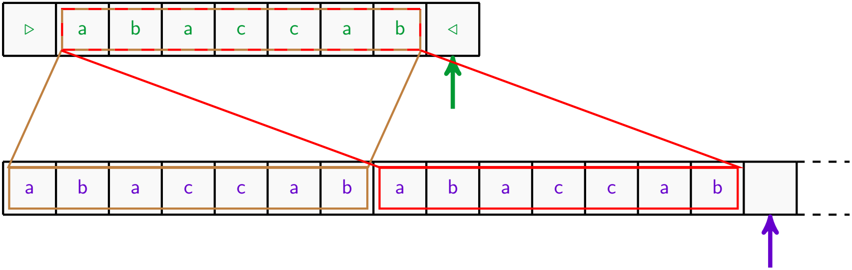

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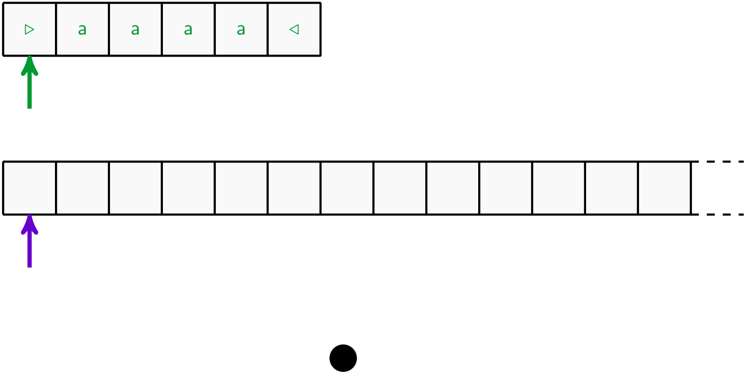
Another example: $\mathit{UnaryMult} = \{(a^n, a^{kn}) \mid k, n \in \mathbb{N}\}$



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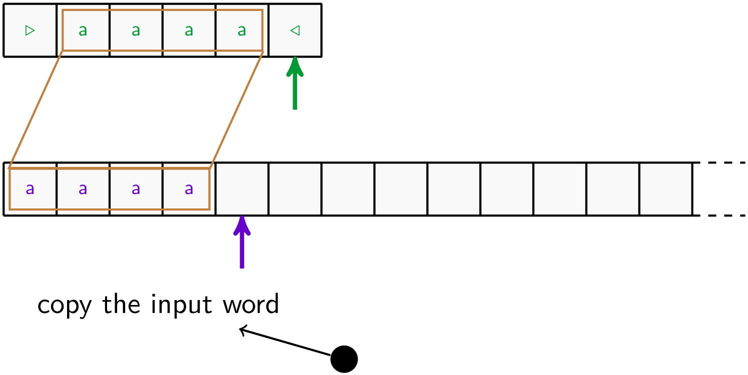

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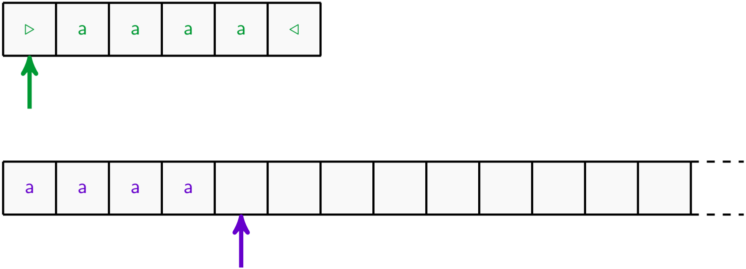
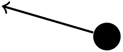


copy the input word ⟶ rewind the input tape



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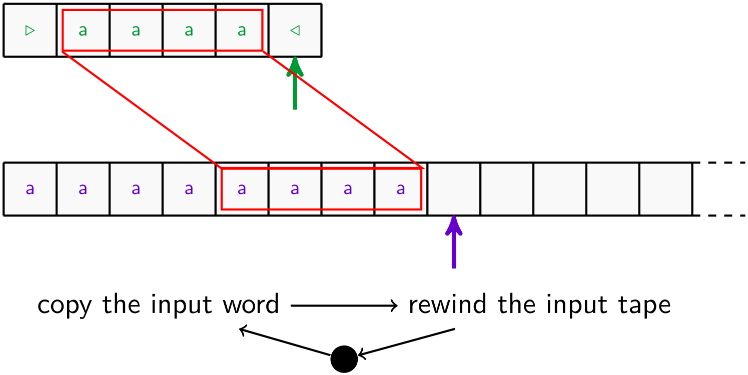

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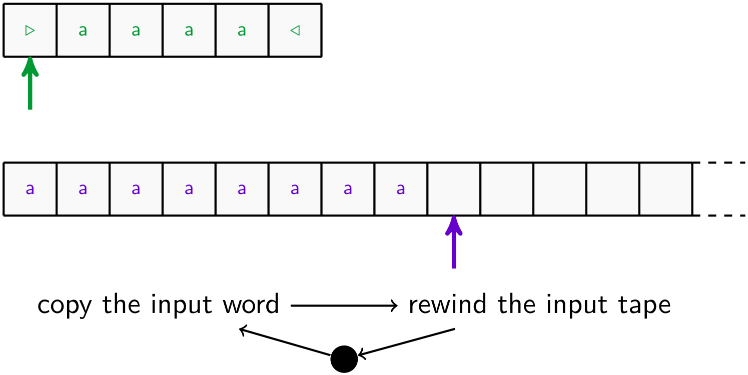

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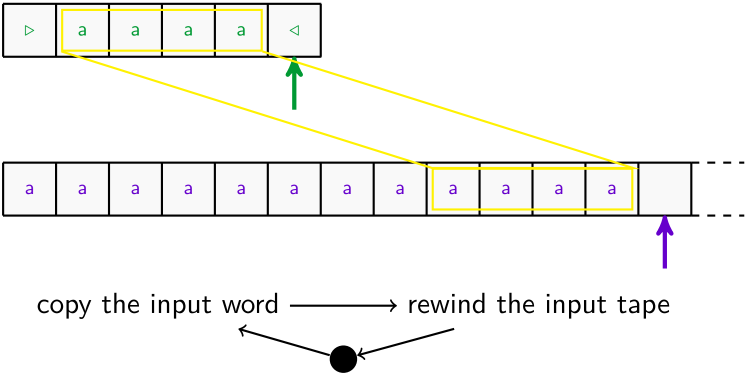

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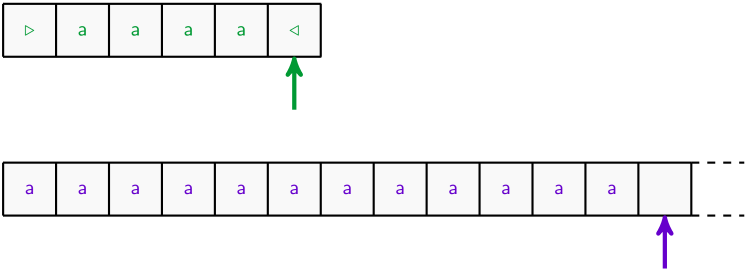

copy the input word ⟶ rewind the input tape

accept and halt with nondeterminism

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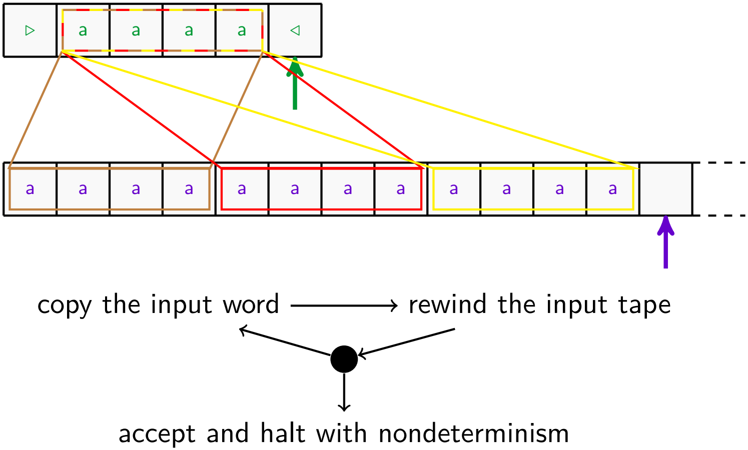

Rational operations

- Union

$$R_1 \cup R_2$$

- Componentwise concatenation

$$R_1 \cdot R_2 = \{(u_1 u_2, v_1 v_2) \mid (u_1, v_1) \in R_1 \text{ and } (u_2, v_2) \in R_2\}$$

- Kleene star

$$R^* = \{(u_1 u_2 \cdots u_k, v_1 v_2 \cdots v_k) \mid \forall i \ (u_i, v_i) \in R\}$$

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Definition ($Rat(\Sigma^* \times \Gamma^*)$)

The class of rational relations is the smallest class:

- that contains finite relations
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Theorem (Elgot, Mezei - 1965)

1-way transducers $==$ *the class of rational relations.*

Hadamard operations

- H-product

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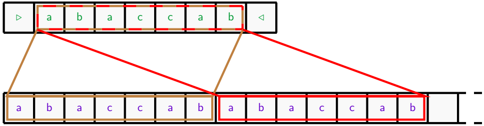

- copy the input word
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- append a copy of the input word

Hadamard operations

- H-product

$$R_1 \circledH R_2 = \{(u, v_1 v_2) \mid (u, v_1) \in R_1 \text{ and } (u, v_2) \in R_2\}$$

- H-star

$$R^{\mathrm{H}\star} = \{(u, v_1 v_2 \cdots v_k) \mid \forall i \; (u, v_i) \in R\}$$

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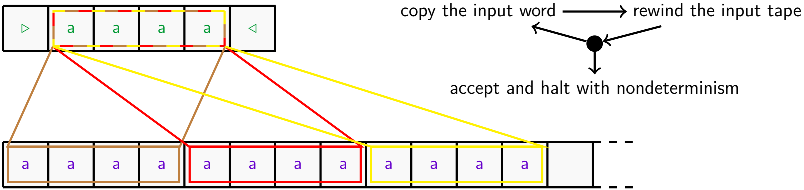

H-Rat relations

Definition

A relation R is in $H\text{-}Rat(\Sigma^* \times \Gamma^*)$ if

$$R = \bigcup_{0 \leq i \leq n} A_i \circledH B_i^{\mathsf{H}\star}$$

where for each i, A_i and B_i are rational relations.

Main result

When $\Sigma = \{a\}$ and $\Gamma = \{a\}$:

Theorem (Elgot, Mezei - 1965)

1-way transducers $=$ *the class of rational relations*.

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Theorem (*This talk*)

2-way transducers $=$ *the class of* *H-Rat relations*.

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Theorem (*This talk*)

2-way transducers $=$ *the class of* *H-Rat relations*.

Proof

- $\supseteq$: *easy*
- $\subseteq$: *difficult part*

Known results

- 2-way functional $=$ MSO definable functions

  [Engelfriet, Hoogeboom - 2001]

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When $\Gamma = \{a\}$:

- 2-way unambiguous $\longrightarrow$ 1-way

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When $\Gamma = \{a\}$:

- 2-way unambiguous $\longrightarrow$ 1-way

  [Anselmo - 1990]
- 2-way unambiguous $=$ 2-way deterministic

  [Carnino, Lombardy - 2014]

From *H-Rat* to 2-way transducers (unary case)

Property

The family of relations accepted by 2-way transducers is closed under $\cup$, $\textcircled{H}$ *and* $_H\star$.

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Proof.

- $R_1 \cup R_2$:
  - simulate T_1 or T_2

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Proof.

- $R_1 \cup R_2$:
  - simulate T_1 or T_2
- $R_1 \textcircled{H} R_2$:
  - simulate T_1
  - rewind the input tape
  - simulate T_2

□

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- $R_1 \textcircled{H} R_2$:
  - simulate T_1
  - rewind the input tape
  - simulate T_2
- $R^{H\star}$:
  - repeat an arbitrary number of times:
    - simulate T
    - rewind the input tape
  - reach the right endmarker and accept

$\square$

From *H-Rat* to 2-way transducers (unary case)

Property

The family of relations accepted by 2-way transducers is closed under $\cup$, $\textcircled{H}$ *and* ^{H*}.

Corollary

H-Rat $\subseteq$ *accepted by 2-way transducers*

$$\left(\bigcup_{0 \leq i \leq n} A_i \textcircled{H} B_i^{H\star} \right)$$

From 2-way transducers to *H-Rat* (unary case)

A first ingredient, a preliminary result:

Lemma

With arbitrary Σ *and* $\Gamma = \{a\}$*:*

H-Rat is closed under $\cup$, $\textcircled{H}$ *and* $H\star$.

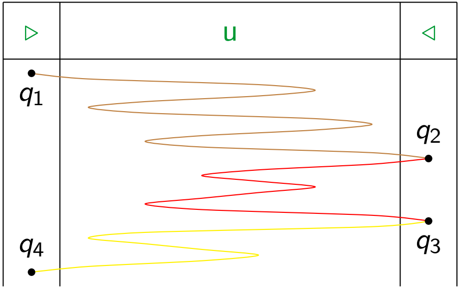
Proof.

Tedious formal computations. . . $\square$

From 2-way transducers to *H-Rat* (unary case)

We fix a transducer $\mathcal{T}$.

- Consider border to border run segments;



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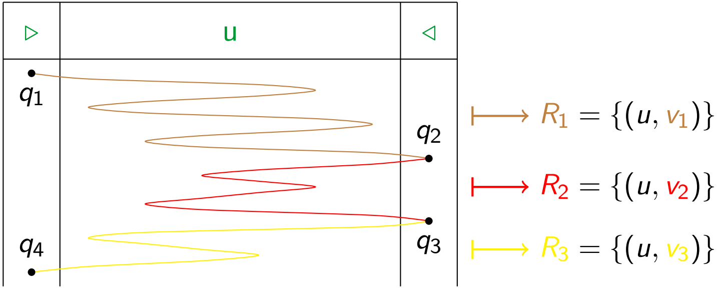

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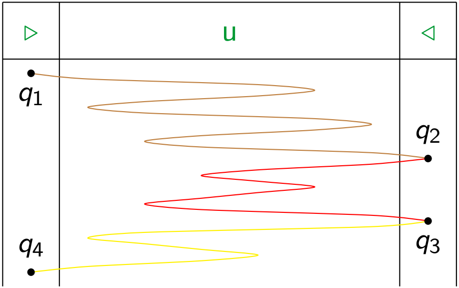

$$\longmapsto R_1 = \{(u, v_1)\}$$

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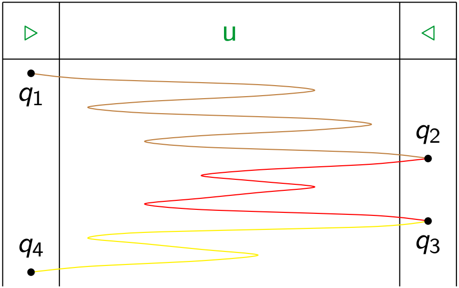

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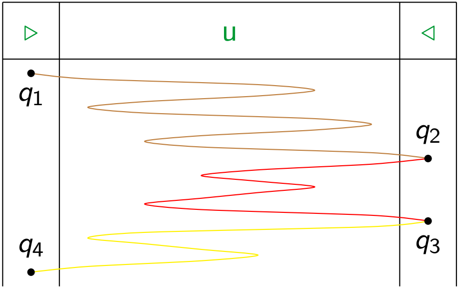
From 2-way transducers to *H-Rat* (unary case)

We fix a transducer $\mathcal{T}$.

- Consider border to border run segments;
- Compose border to border segments;
- Conclude using the closure properties of *H-Rat*.



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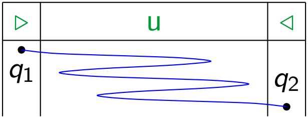
From 2-way transducers to *H-Rat* (unary case)



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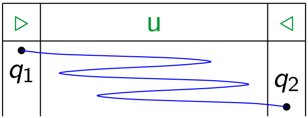

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$\in$ $\in$

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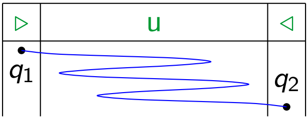

define a relation R_{b_i, b_j}

$b_i \in Q \times \{\triangleright, \triangleleft\}$, $b_j \in Q \times \{\triangleright, \triangleleft\}$

$$HIT = \left.\overbrace{\begin{pmatrix} R_{0,0} & R_{0,1} & \cdot & \cdot & \cdot & R_{0,2} \\ R_{1,0} & R_{1,1} & \cdot & \cdot & \cdot & R_{1,2} \\ \cdot & & & & & \cdot \\ \cdot & & & R_{i,j} & & \cdot \\ \cdot & & & & & \cdot \\ R_{k,0} & R_{k,1} & \cdot & \cdot & \cdot & R_{k,k} \end{pmatrix}}^{2|Q|}\right\} 2|Q|$$

From 2-way transducers to H-Rat (unary case)

Second ingredient:

The behavior of $\mathcal{T}$ is given by the matrix $HIT^{\mathrm{H}\star}$.

From 2-way transducers to *H-Rat* (unary case)

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The behavior of $\mathcal{T}$ is given by the matrix $HIT^{\mathsf{H}\star}$.

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Lemma

Each entry R_{b_1,b_2} of the matrix HIT is rational (constructible).

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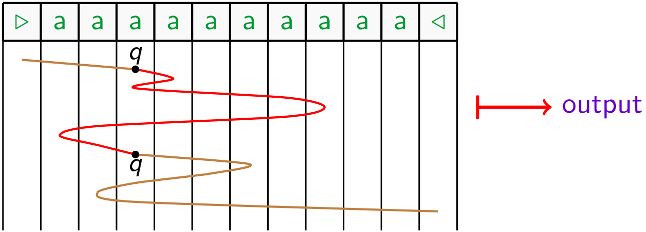

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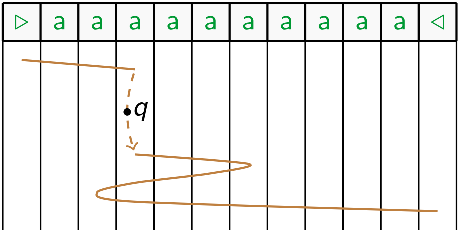

$\longmapsto$ rational output

From 2-way transducers to *H-Rat* (unary case)

Second ingredient:
The behavior of $\mathcal{T}$ is given by the matrix $HIT^{H\star}$.

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Lemma
Each entry R_{b_1,b_2} of the matrix HIT is rational (constructible).

By closure property:

Corollary
Each entry of $HIT^{H\star}$ is in H-Rat.

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Corollary
accepted by 2-way transducers $\subseteq$ H-Rat

Conclusion

Theorem

When $\Gamma = \{a\}$ and $\Sigma = \{a\}$:
2-way transducers accept exactly the H-Rat relations.

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With only $\Gamma = \{a\}$:

- 2-way $\left\{ \begin{array}{c} \text{deterministic} \\ \text{unambiguous} \\ \text{functional} \end{array} \right\}$ accept rational relations.

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With only $\Gamma = \{a\}$:

- 2-way $\left\{\begin{array}{c}\text{deterministic}\\ \text{unambiguous}\\ \text{functional}\end{array}\right\}$ accept rational relations.

- 2-way transducers are uniformizable by 1-way transducers.

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- 2-way $\left\{ \begin{array}{c} \text{deterministic} \\ \text{unambiguous} \\ \text{functional} \end{array} \right\}$ accept rational relations.

- 2-way transducers are uniformizable by 1-way transducers.

Every thing is constructible.

Conclusion

Theorem

When $\Gamma = \{a\}$ *and* $\Sigma = \{a\}$*:*
2-way transducers accept exactly the H-Rat relations.

From our construction follows:

- 2-way transducers can be made sweeping.

With only $\Gamma = \{a\}$:

- 2-way $\left\{ \begin{array}{c} \text{deterministic} \\ \text{unambiguous} \\ \text{functional} \end{array} \right\}$ accept rational relations.

- 2-way transducers are uniformizable by 1-way transducers.

Every thing is constructible.

Thank you for your attention.