Université Paris Diderot – École doctorale de sciences mathématiques de Paris centre

Mémoire d'habilitation à diriger des recherches

Spécialité Informatique

POLYOMINOES, PERMUTOMINOES AND PERMUTATIONS

Enrica Duchi

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Chapter 1

Introduction and brief curriculum

Enumerative and bijective combinatorics

Combinatorics is nowadays a fundamental research field representing one of the main bridges between computer science and mathematics. It deals with the mathematical properties of discrete structures, as opposed to continuous ones, and structures of this kind play an important role in theoretical computer science.

Within combinatorics, *enumerative combinatorics* is more specifically dealing with the fundamental problem of counting structures in combinatorial classes with respect to a size, in an exact or in an approximate way. Apart the natural question to know how many objects there are, enumerative problems arises in many fields, ranging from the analysis of algorithms to statistical physics or bioinformatics for instance.

Sometimes it is easy to count, like in the case of permutations of $\{1, \ldots, n\}$, counted by $n!$, but in general this is not the case. Only in rare cases the answer will be a completely explicit closed formula, involving well known functions, and free from summation symbols. Whenever we do not have a simple formula for counting numbers we can hope to get one for the generating function of the combinatorial class, and this is the reason why a lot of energy has been dedicated to improve the toolbox of enumerative combinatorics: clever decomposition technics leading to functional equations for generating functions, or sophisticated methods to solve these classes of functional equations, or at least to obtain asymptotic results.

Still, sometimes these sophisticated approaches lead to surprisingly nice closed formulas or to unexpected coincidences when apparently unrelated objects reveal themselves equinumerous, then we ask for a direct combinatorial explanation of these facts: *bijective combinatorics* comes in this context.

Let us take an example: the number of spanning trees of the complete graph with nodes in $\{1, \ldots, n\}$, called Cayley trees, is n^{n-2} and it is also the number of different ways to write the cyclic permutation $(1, 2, \ldots, n)$ into a product of $n - 1$ transpositions. While the first one is a very classical result with a lot of different proofs and generalizations, the second one is less known outside the combinatorics community but it attracts the attention because of the appearance of the same numbers in *a priori* different context. Such coincidences are at the basis of bijective combinatorics. Dénes, in the 50's, looked for an explanation for the coincidence of the n^{n-2} numbers, and obtained a first bijection between doubly labelled Cayley trees and factorizations of arbitrary big cycles in transpositions, then a second bijection directly between Cayley trees and factorizations of a fixed big cycle was given by Mozkovski in the 80's. On the one side Dénes bijection makes explicit the arborescent structures of the junction operations between cycles in a product of transpositions, on the other side Mozkovski's bijections has allowed to realize that Cayley trees can be seen as planar increasing trees, that is a relevant ingredient in a lot of later works, including some of mines. In general the fact of explaining a formula through a bijection lead us to obtain a better understanding of the properties of the underlying objects.

We have already pointed out that it is not easy to find closed formulas for combinatorial classes, so that when we have a method that works, either a decomposition or some manipulations on a system of functional equations, it is natural to wonder how far this method can be generalized. In particular if we have a non standard decomposition for a combinatorial class one can wonder if it is a decomposition *ad hoc* for that class or if it can apply to other combinatorial classes. This can be done by looking for general theorems formalizing the approach, but also, and maybe more importantly in combinatorics, by looking for alternative interesting examples: while general theorems are attractive, they are only useful with good applications, which are not so easy to find...

Most of my research is set in this field of bijective and enumerative combinatorics, and is concerned with the search of good examples!

A quick overview of my research

In order to present in more details my field of research I have decided to concentrate on a coincidence of the type discussed above: the stricking similarity between classical Delest-Viennot formula for convex polyominoes and the much more recent formulas for square permutations and convex permutominoes. The later combinatorial classes and their counting formulas are much less known than polyominoes but deserve in my opinion some advertising! In a first part I will thus introduce these objects and show how some standard enumerative methods converge on their study. Then in a second part I will discuss some more advanced examples, dealing with generalizations of directed convex polyominoes and square permutations, and with a class of pattern avoiding permutations.

In the rest of this preliminary chapter I will now review some topics I have been interested in over the recent years in approximate chronological order, just to show that polyominoes, permutominoes and permutations, although they play an important role in my work, are not the only combinatorial classes I have been considering... Along the lines, I will point out those works that will appear, briefly or in details, later in the manuscript. This classification will be followed by a short curriculum vitae.

Polyominoes. My Phd thesis and some of my early papers [J9,J11,J12,J14,J15,C7,C8,C9,W2] were mainly concerned with the interplay of succession rules and object grammars in the enumerative study of convex classes of polyominoes: A *polyomino* is a finite union of unit cells in the plane, whose interior is connected. The problem of enumerating polyominoes is a famous and difficult problem. This is the reason why different sub-classes of polyominoes were introduced in the aim of pushing forward the limits of the enumerative techniques. Convexity, *i.e.* the fact that the intersection of the polyomino with any column or line is connected, is one of the most famous and useful restriction that has been studied in the literature. Many different methods successfully applied for the enumeration of convex polyominoes, like linear constructions involving generating trees, grammar-like decompositions and bijective constructions. We will talk about these methods in detail in Chapter 2 and Chapter 3.

Among the above cited results, I wish to point out the one with Simone Rinaldi [J14]: we gave a pictorial grammar-like decomposition for two subclasses of polyominoes, column-convex and directed column-convex polyominoes. While algebraic decompositions were already known for these classes (in particular through encoding in terms of context free languages), our pictorial presentation allows to convey more intuitively the algebraicity of the corresponding generating functions according to the semi-perimeter.

I have later returned to the study of polyominoes in several occasions. A few years ago, in the setting of discrete tomography, a new classification of convex polyominoes was proposed: a polyomino is said to be k**-convex** if each pair of cells can be joined by a monotonous path with at most k changes of direction. For with $k = 1, 2$, k-convex polyominoes are also called respectively *L-convex* and *Z-convex* polyominoes. The enumeration problem for L-convex polyominoes was quickly solved and, with Simone Rinaldi and Gilles Schaeffer, we studied the case of Z-convex polyominoes [C5,J6]: Altough this is a subclass of convex polyominoes, we were not able to apply the classical approaches discussed above. Our approach consisted in proposing a construction involving *hooked polyominoes* that are polyominoes in which a hook is highlighted that consists in the bottom row y under the leftmost column of the polyomino and in the bottom part of the rightmost column under y. We devised a non standard construction for this class, by inflating the polyominoes along the hook, which lead us to a system of functional equations involving Hadamard products. We showed that although the decompositions we used are quite complex, the generating function of Z-convex polyominoes according to their semi-perimeter is a rational function of the Catalan generating function. The enumeration problem for general k-convex polyominoes is still open for $k \geq 3$, but to conclude on this subject I would like to mention the paper of my PhD student Samanta Socci, which together with Adrian Boussicault and Simone Rinaldi, enumerated the class of directed k-convex polyominoes by obtaining that the generating function is rational and can be expressed using the *Fibonacci polynomials*.

More recently I came back to polyominoes by working on another subclass of these objects, the class of **permutominoes**. A permutomino is a polyomino without holes such that its boundary has exactly one segment in each row and in each column of its bounding box. A permutomino is uniquely determined by two permutations. A generalization of permutominoes, where the boundary can cross itself, is the class of **permutominides**. One of our personal contributions to the study of convex permutominoes and permutominides consists in the paper [J5], with Disanto, Pinzani and Rinaldi, were we gave bijective proofs for the enumeration of permutominides and of permutominoes, **we will briefly talk about this paper in Part I, Chapter 3, Subsection 3.4.2.** Our second contribution to the enumeration of permutominoes and permutominides is the paper with Simone Rinaldi and Samanta Socci [J3], where we introduced the concept of d-dimensional permupolygons and we proposed a path encoding of the boundary of a d-dimensional permupolygon inspired by the one given by Bousquet-Mélou and Guttmann for general polygons. This lead us to recover, in a easier way, the known results about parallelogram, directed convex, and convex permupolygons with $d = 2$, corresponding respectively to the class of parallelogram, directed convex and convex permutominides. Moreover we enumerated these classes for $d = 3$, thus generalizing to dimension 3 the enumerative results for convex subclasses of permutominoes. Indeed, by definition, d-dimensional permupolygons with $d \geq 3$ are a natural generalization of permutominoes since they can not intersect. Our characterization is new and lead us reconsider the problems of enumerating permutominoes as problems on lattice paths, and solve them via bijective techniques.

Lattice paths. As suggested by the previous example, lattice path is another recurrent source of remarkable formulas that I have been interested in. The 2^n property of **d-dimensional lattice paths with diagonal steps** that we studied with Robert Sulanke [J10] is another typical illustration: Consider on the one hand lattice paths $\mathcal{D}(n)$ in $\mathbb{Z}^d$, $d \geq 1$, starting from $(0, 0, \ldots, 0)$ and ending in $(n, n, \ldots, n)$, whose steps are the positive steps between the vertices of an hypercube, and on the other hand $\mathcal{S}(n)$ going from $(0, 0, \ldots, 0)$ to $(n, n, \ldots, n)$ and using non-zero steps $(x_1, x_2, \ldots, x_d)$, with $x_i \geq 0$. These two sets of paths are related by a remarkable identity, called the 2^n property: for all d, $\|\mathcal{S}(n)\| = 2^{n-1}\|\mathcal{D}(n)\|$. Using elementary manipulations on paths we were able to give a bijective proof of this property, and to extend it to paths in higher dimensions.

Later I have been interested in so-called **prudent walks**: these are oriented lattice self-avoiding paths in $\mathbb{Z}^2$ that are prudent, in the sense that they never even look toward a point that they already visited, *i.e.* extending the ith step of the path along its current trajectory one never intersect a vertex that has been visited by the $i - 1$ first steps. These paths can be classified in terms of k-sided prudent walks, with $k = 1, 2, 3, 4$, which are prudent walks that can respectively end on k-sides of the bounding box containing the path. In particular, 1-sided prudent walks coincide with partially directed walks, having only north, east and west unit steps. I gave in [C6] a linear recursive decomposition for 2-sided prudent walks and then a direct interpretation of the algebraicity of this class. More generally I am interested in walks on restricted regions, like in the **slitplane**, or in **cones**: while general technics are emerging that allow to classify such walks according to their step sets, there is still little combinatorial understanding of the reason why some of these objects have algebraic generating functions while other have not. Although the matter is somewhat controversial[1], I believe that the Schützenberger methodology has still its word to say on the matter: as an example, I would be happy to see a grammar-like explanation of the general occurrence of Catalan numbers in the enumeration of walks on the slitplane.

Both polyomino and lattice path enumeration problems take their origins and motivations in the physics literature. This is also the source of two others topics that I have been interested in, **Sand pile model**, and **the TASEP**. The *Sand Pile Model* is a discrete dynamical system describing pilings of n granular objects distributed on an array of columns. Together with Mantaci, Phan and Rossin [J8] we defined an extension of the Sand Pile Model by introducing the *Bidimensional Sand Pile Model*. We made a parallel between these unidimensional and bidimensional models by finding some common features and some differences. In particular we proved that, like in the unidimensional case, not all plane partitions are accessible in the *Bidimensional Sand Pile Model* but, unlikely from the unidimensional case, it is difficult to characterize the partitions that are accessible, in particular we only gave some necessary but not sufficient conditions.

The Totally Asymmetric Exclusion Process (TASEP) is a model of particles jumping along a line. This model has been largely studied in the context of out of equilibrium statistical physics, mainly after the seminal papers of the physicist B. Derrida and his coauthors. From a combinatorial point of view Catalan numbers appear in the Markov chain stationary distribution. These results have been shown by phycisists with algebraic techniques. With Gilles Schaeffer we gave the first combinatorial explanation of this appearance of Catalan numbers [J13]. Our main ingredient was the construction of a new Markov chain on a new set of configurations, consisting on particles jumping along not one but two lines. This new Markov chain satisfies two conditions: on the one hand the stationary distribution of the basic chain can be simply expressed in terms of the stationary distribution of the new chain; on the other hand the stationary distribution of the new chain is easy to understand. The first requirement is met by imposing that disregarding what happens in the second row, the new chain simulates the basic chain in the first row. The second requirement is met by adequately choosing the new two-lines configurations and the transition rules so that the new chain clearly has a uniform stationary distribution. This idea lead us to take into account other parameters like the probability for particles to jump [C10] and also the boundary conditions [C11]. Finally we extended our bijection to recover the stationary distribution of the parallel TASEP, in which particles are allowed to jump simultaneously [J7].

Sylvie Corteel and her coauthors have since then obtained many results of the same type for the PASEP which extends the TASEP in another direction, introducing the possibility for particles to jump backwards. These results have led to the discovery of a rich and new combinatorics going much further than the Catalan combinatorics that we had built on. Recently however, the physicist K. Mallick and his coauthors have obtained new and finer properties of the original TASEP: these formulas suggest that the combinatorial structure of this original problem is still richer than what is currently understood, and I believe that it could be worthwhile returning to our approach to bring a combinatorial light to K. Mallick's results and to extend them to parallel models.

Another topic in which bijections play a fundamental role is map enumeration, and in this context I have been interested in particular in **bijective proofs of Hurwitz formula**. The number of minimal transitive factorizations of a permutation in transpositions has been determined by Hurwitz in 1891, but a series of new proofs of this formula have been given in the last twenty years, with tools going from algebraic geometry to the theory of the character of the symmetric group, or by using recurrences or properties of inclusion-exclusion. With Dominique Poulalhon and Gilles Schaeffer [C2] we gave a first bijective proof of this formula. Our proof uses labelled

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[1] As beautifully explained by the work of Bousquet-Mélou and her coauthors, the fact that these generating functions are algebraic is closely related to the finiteness of a certain group of transformation of the root of the kernel of the main functional equation they satisfy, but this is not a very combinatorial explanation…

trees and minimal orientations and start from a representation of the factorizations in terms of increasing quadrangulations. Many tools have been developed in the context of the study of planar maps, leading to establish many links between a lot of families of planar maps and families of simple planar trees. Our result shows that it is possible to adapt these tools to develop a parallel theory linking labeled increasing planar maps and labeled trees. Somehow we generalize here the work of Mozkovski recalled in the introduction. Our bijective proof gave us efficient random generation algorithms for increasing quadrangulations, that can be interpreted as simple branched coverings of the sphere by itself. Later we obtained a second bijective proof of Hurwitz formula that we extended to give an interpretation of double Hurwitz numbers that have attracted in the last ten years a lot of attention from the mathematical community. Our bijective approach also allowed us to obtain new formulas linked to these numbers and to establish a particular case of a positivity conjecture of Kazarian and Zvonkine.

In parallel with my strong interest for bijective proofs, I am particularly fond of **generating trees and decomposition technics** and many of my works indeed involve exotic generation schemes.

An example of this trend is my work on **parking functions** with Cori, Guerrini and Rinaldi [J1]: we explored the links between some parking functions and the numbers of Catalan, Schroeder and Baxter. In particular we defined two new families of parking functions, one counted by Schroeder numbers and the other by Baxter numbers. These families both include the well-known class of non-decreasing parking functions, which is counted by Catalan numbers and easily represented by Dyck paths, and they both are included in the class of underdiagonal sequences, which are equinumerous with permutations. We investigated their combinatorial properties by exhibiting bijections between these two families and classes of lattice paths and discovering a link between them and some classes of pattern avoiding permutations. Moreover, we provided a quite natural generalization for each of these families and gave enumerative results by using generating trees. Another example of my interest in generating trees is my work on **square permutations**: let define the records of a permutation σ as the minima or maxima from left to right or from right to left. A point (i, j) with $j = \sigma(i)$ is an internal point of the permutation if and only if it is not a record. The number of minima from left to right is a classical statistic of random permutations that appears in many enumerative problems while the internal points often appear in problems of geometric algorithmic. A square permutation is a permutation without internal points. With Dominique Poulalhon [C4] we provided a simple linear recursive construction for square permutations that, together with Filippo Disanto, Simone Rinaldi and Gilles Schaeffer [C3], we later extended to **square permutations with a fixed number i of internal points**, thus showing that their generating function is algebraic of degree 2, more precisely it is again rational in the Catalan generating function. Our approach lead us to the enumeration of permutations according to different type of records, which answers a problem raised by Wilf years ago. **We will present this construction in Part II, Chapter 4 of this manuscript.**

More recently, together with Guerrini and Rinaldi [J2], we gave a non classical linear recursive decomposition for permutations avoiding the generalized pattern 1–23–4, which extend the popular 123-avoiding permutations. In particular our construction consists in uniquely generating a permutation of size n from one of size $n-1$ by inserting a new point but also by moving some points of the original permutation. We gave an algorithmic description of a generating tree for these permutations. The algorithm leads to a recursive bijection between 1–23–4-avoiding permutations and Dyck paths with marked valleys. It extends a known bijection between 123-avoiding permutations and Dyck paths, and makes explicit the connection between these objects that was earlier obtained by Callan through a series of non-trivial bijective steps. **We will talk about this paper in Part II, Chapter 5**.

Finally, another very recent enumerative challenge that we tackled with two different decompositions, is the enumeration of fighting fish. Together with Guerrini, Rinaldi and Schaeffer [J4], we introduced these combinatorial objects made of square tiles that form two dimensional branching surfaces. These objects have two main features: on the one hand they enjoy beautifully simple closed enumeration formulas, and on the other hand, the area of uniform random fish of size n

scales like $n^{5/4}$ as opposed to the typical $n^{3/2}$ area behavior of the staircase or direct convex polyominoes that they generalize. We enumerated the fighting fish according to their semi-perimeter by using essentially a Temperley's approach, that is a decomposition in vertical slices. Later we gave another decomposition [C1] that extends to fighting fish the classical wasp-waist decomposition on parallelogram polyominoes. **We will talk of these two decompositions in Part II, Chapter 6**.

As should appear from this overview, my research is mainly focused on bijections and enumeration technics for decomposable combinatorial structures. Admittingly the topic is quite classical but I believe it to be still highly relevant: a venerable approach, Schützenberger's methodology for structures with algebraic generating functions, is at the basis of the combinatorics of planar maps, one of the most remarkable success of combinatorics over the last decade, and even more recently bijective combinatorics underlies the advances on the study of the ASEP or of dimer configurations by Corteel *et al.* I believe that long standing open problems like the interpretation of the algebraic generating function for the hard hexagon model, or more recent ones like the interpretation of the algebraic generating function for our newly born fighting fish are still worth looking for, especially in view of the better understanding developed over the last few years of the conditions on which one and two parameter generating trees lead to algebraic series.

Curriculum vitae

Personal informations

Surname : Duchi
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Born: 30/12/73
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Personal details : Married life, 3 children (7, 10 and 12 years old).

Current employement : Assistant professor (*maître de conférence de classe normale*).
Section CNU : 27.
Personal career

- sept 2016– june 2017: parental leave (*disponibilité*).
- 2015–2016: *délégation CNRS* spent at the University of Siena.
- 2005–2015: *Maître de conférence* at LIAFA, university Paris Diderot (maternity leaves in 2006, 2008 and 2011).
- 2004–2005: Postdoc at LIAFA, *vacataire* at the University of Marne-la-Vallée.
- 2003–2004: Postdoc at CAMS (EHESS).
- 2000–2003: Phd thesis (*cotutelle*) under the direction of R. Pinzani (University of Florence) and J.M. Fédou (University of Nice Sophia-Antipolis).

Full list of publications

Preprints

- E. Duchi, *A code for square permutations and convex permutominoes,* Submitted, (2017).
- E. Duchi, D. Poulalhon, G.Schaeffer, *Bijections for simple and double Hurwitz numbers,* arXiv:1410.6521 (2014).

International Journals

- [J1] R. Cori, E. Duchi, V. Guerrini, S. Rinaldi, *Families of parking functions counted by the Schroeder and Baxter numbers,* special issue of Lattice Path Combinatorics 2015, Pomona, California, Springer Book Series "Developments of Mathematics", to appear.
- [J2] E. Duchi, V. Guerrini, S. Rinaldi, *A generating tree for permutations avoiding the pattern* $122^{+}3$*,* Fundamenta Informaticae, to appear.
- [J3] E. Duchi, S. Rinaldi, S.Socci, *Three dimensional polygons determined by permutations,* Journal of Combinatorics 9 (2018), 57-94.
- [J4] E.Duchi, V.Guerrini, S.Rinaldi, G.Schaeffer, *Fihting fish,* Journal of Physics A: Mathematical and Theoretical, 50.2. (2016).
- [J5] F. Disanto and E. Duchi and R. Pinzani and S. Rinaldi, *Polyominoes determined by permutations: enumeration via bijections,* Annals of Combinatorics 16 (2012), no. 1, 57-75.
- [J6] E. Duchi and S. Rinaldi and G. Schaeffer, *The number of Z-convex polyominoes,* Advances in Applied Mathematics 40 (2008), no. 1, 54-72.
- [J7] E. Duchi, G. Schaeffer, *A combinatorial approach to jumping particles: The parallel TASEP,* Random Structures and Algorithms 33(4) (2008), 434-451.
- [J8] E. Duchi and R. Mantaci and H.D. Phan and D. Rossin, *Bidimensional sand pile and ice pile models,* Pure Math. Appl. (PU.M.A.) 17 (2006), no. 1-2, 71-96.
- [J9] S. Brlek, E. Duchi, E. Pergola, S. Rinaldi, *On the equivalence problems for succession rules*, (developed version of the paper presented at FPSAC'02), Discrete Mathematics 298, Issues 1-3 (2005), 142-154.
- [J10] E. Duchi, R. Sulanke, *The* 2^{n-1} *factor for multi-dimensional lattice paths with diagonal steps*, Sém. Lothar. Combin. 51, (2004/05), Article B51c.
- [J11] E. Duchi, J-M Fédou, S. Rinaldi, *From object grammars to ECO-systems,* Theoretical Computer Science 314 (2004) 57-95.
- [J12] A. Del Lungo, E. Duchi, A. Frosini, S. Rinaldi, *The enumeration of convex polyominoes by using ECO method* (developed version of the paper presented at DMCS'2003), The Electronic Journal of Combinatorics 11 (2004),no. 1, R60.
- [J13] E. Duchi, G. Schaeffer, *A combinatorial approach to jumping particles*, Journal of Combinatorial Theory, Series A. (2004).
- [J14] E. Duchi, S. Rinaldi, *An object grammar for column-convex polyominoes,* Annals of Combinatorics 8 2004, no. 1.
- [J15] E. Duchi, A. Frosini , S. Rinaldi , R. Pinzani, *A note on rational succession rules*, Journal of Integer Sequences 6 (2003), Article 03.01.7.

International Conferences

- [C1] E. Duchi, V. Guerrini, S.Rinaldi and G. Schaeffer, *Fighting fish: enumerative properties*, Proceedings of 29th International Conference on Formal Power Series and Algebraic Combinatorics (London) (2017).
- [C2] E. Duchi, D. Poulalhon and G. Schaeffer, *Uniform random sampling of simple branched coverings of the sphere by itself*, 25th ACM-SIAM Symposium on Discrete Algorithms (SODA 14) , Portland (USA), (2014).
- [C3] F. Disanto, E. Duchi, S. Rinaldi, G. Schaeffer, *Permutations with few internal points. EuroComb 2011*, Budapest (Hongrie) . Acts published in Electronic Notes in Discrete Mathematics 38: 291-296 (2011)
- [C4] E. Duchi and D. Poulalhon, *On square permutations*, Fifth Colloquium on Mathematics and Computer Science, 207 222, Discrete Math. Theor. Comput. Sci. Proceedings, Nancy (2008).
- [C5] E. Duchi, S. Rinaldi, G. Schaeffer, *On the number of Z-convex polyominoes*, FPSAC'2006, Vienna (Austria).
- [C6] E. Duchi, *On some classes of Prudent Walks*, FPSAC'2005, à Taormina (Italy).
- [C7] S. Brlek, E. Duchi, E. Pergola, R. Pinzani, *Bijective construction of equivalent ECO-systems*, International Colloquium on Mathematics and Computer Science: trees, algorithms and probability (MathInfo'2002), Versailles (France).
- [C8]S. Brlek, E. Duchi, E. Pergola, S. Rinaldi, *On the equivalence problems for succession rules*, 14th International Conference on Formal Power Series and Algebraic Combinatorics, (FPSAC'02), Melbourne (Australia).
- [C9] A. Del Lungo, E. Duchi, A. Frosini, S. Rinaldi, *The enumeration of convex polyominoes by using ECO method*, International Conference Discrete Models for Complex Systems (DMCS'2003), Lyon (France). Acts published by Discrete Mathematics and Theoretical Computer Science.
- [C10] E. Duchi, G. Schaeffer, *Jumping particles I: maximal flow regime*, 16th International Conference on Formal Power Series and Algebraic Combinatorics, (FPSAC'04), Vancouver (Canada).
- [C11] E. Duchi, G. Schaeffer, *Jumping particles II: general boundary conditions*, International Colloquium on Mathematics and Computer Science: trees, algorithms and probability (MathInfo'2004), Vienna (Austria).

National and International Workshops

- [W1] E. Duchi, G. Schaeffer, *Approche combinatoire d'un modèle de particules.* Rencontres ALEA'04, Luminy (France).
- [W2] E. Duchi, E. Pergola, R. Pinzani, *Equivalence of ECO-systems*, International workshop on Combinatorics of Searching, Sorting and Coding (COSSAC'01), Ischia(Italy).

Manuscripts and Technical Reports

- [M1] E. Duchi, S. Socci, *A bijection between triangular permutations and directed convex permutominoes*, (2015).
- [M2] E. Duchi, J-M Fédou,C. Garcia, E. Pergola, *On the inversion of succession rules*, rapport intern de l'I3S, (2002).

Supervisor activities

- Co-director of the Phd thesis of Filippo Disanto (2007-2010), "cotutelle" between the University Paris 7 - Paris Diderot (E. Duchi) and the University of Siena (S. Rinaldi).
- Co-director of the Phd thesis of Samanta Socci (2013-2015), "cotutelle" between the University Paris 7 - Paris Diderot (E. Duchi) and the University of Siena (S. Rinaldi).
- Co-director with S. Rinaldi of the Master thesis (M2) of Andrea Mazzon, University of Siena (2016).

Invited talks

- *Preuve bijective de la formule d'Hurwitz à l'aide de mobile.* Journées en l'honneur de Xavier Viennot, LaBRI, 2012.
- *Fighting Fish: enumerative properties.* Séminaire Philippe Flajolet, IHP, 2017.
- *Fighting Fish: enumerative properties.* Journées de Combinatoire de Bordeaux, 2018.
- *Convex polyominoes, convex permutominoes and square permutations.* Aléa 2018, CIRM, 2018.
- *Fighting Fish: enumerative properties.* Journée Cartes, Mai 2018.

Board of examiners

- PhD thesis defense of S. Charbonier (directors Bertrand Eynard and DR François David), IPhT, University of Paris-Sud, September 2018.
- PhD thesis defense of A. Sabri (director Professor Vincent Vajnovszki), University of Bourgogne, 2015.
- PhD thesis defense of D. Battaglino (cotutelle, director Professor Simone Rinaldi, co-director Professor JeanMarc Fédou), University of Nice Sophia Antipolis and University of Siena, 2014.
- Report on the PhD thesis of F. De Carli (cotutelle, director Professor Simone Rinaldi, co-director Professor Laurent Vuillon), University of Savoie and University of Florence, 2009 (report co-signed with Dominique Rossin).

Organizing committees

- Organization of the **Alea days 2019** at CIRM, Luminy, Marseille.
- Organization at CIRM, of the opening school to the IHP trimester **Combinatorics and applications**, January 2017.
- Organization of **Gascom 2016** in La Marana, Corsica.
- Organization of the day **Combinatoire à Paris** in the context of the **Emergences** project.
- Organisation of **FPSAC 2013** in Paris.
- Organization of **Gascom 2001** in Siena, Italy.

Other activities

- Referees for journals and conferences (FPSAC, MathInfo, Electronic Journal of Combinatorics, Journal of Combinatorial Theory Series A, Journal of Integers Sequences,...)

- In charge, together with Laurent Viennot, of the monitoring for PhD students belonging to the Algorithmic pole of IRIF (2017-2018).
- Organization, together with Matthieu Josuat-Vergès, of the weekly seminar of the Combinatorics team of IRIF (2017-2018).
- Organization of the weekly seminar of the Combinatorics team of LIAFA (2014-2015).
- Organization of a 5 weeks session on the theme “Paths” in the context of research weekly meeting organized by the Combinatorics team in LIAFA, 2013.
- Member of *conseil de laboratoire* at LIAFA (2012-2015).
- In charge of the bibliography part of the Combinatorics team for report AERES (2012) of LIAFA.

Part I

Convex polyominoes, convex permutominoes and square permutations: results and methods

The first part of this manuscript presents a parallel between three different families of combinatorial objects, the families of *polyominoes*, *permutominoes*, and *permutations*. We are interested in particular in certain subclasses of these objects, that are defined by similar convexity constraints and that enjoy strickingly similar closed formulas for their enumeration. We take advantage of the parallel between these three families to revisit some classical and less classical enumerative methods. This part of the manuscript has therefore to be interpreted as a review on the main enumerative results over these three families, drawn up in order to introduce the main methods for their enumeration, and to emphasize the links between these three classes. It is divided in two chapters: in the first one we introduce the three families of objects and give the main enumerative results and point out their similaries; in the second chapter we discuss the proof methods and point to the more advanced applications of Part II.

Chapter 2

Polyominoes, permutations and permutominoes

The chapter mainly articulates in three sections, each of which introduces a family of objects and certain of its convex subclasses, that are the ones we are interested in. We first deal with *polyominoes* and with *permutations*, that are classical combinatorial structures, and we then discuss with more details of the hybrid case of *permutominoes*, that is a particular class of polyominoes arising from an algebraic context and closely related to permutations. In particular, since the class of permutominoes is a relatively young family of objects, we get into this class a little bit more than the others, by explaining the algebraic context in which permutominoes have been introduced and their original form as a particular subclass of *permutations diagrams*. Then we restrict to the subclasses of convex polyominoes, square permutations, and convex permutominoes. Enumerative results for these subclasses according to a specific parameter, corresponding to the semiperimeter for convex polyominoes and permutominoes, and the number of points for square permutations, are briefly discussed in each section and followed by a subsection containing bibiliographical notes, concentrating mainly on the enumeration of convex subclasses according to the above parameters, since otherwise the subject would be too wide for our purpose. Indeed, our main motivation is to give a parallel between the families of convex polyominoes, convex permutominoes, and square permutations, that shares similar shapes and nice formulas according to the above parameters. A particular effort is made to emphasize the links between these three classes by alternating subsections with enumerative results with subsections in which we try to explicit the connections of a class with another. The chapter ends with a table contaning the main formulas, pointing out the striking similarities between these three families.

2.1 Convex polyominoes

2.1.1 Polyominoes, polyominoes without holes, perimeter

Definition 1. A *polyomino* is a finite union P of unit squares (called *cells*) with vertices on the lattice $\mathbb{Z}^2$ and whose interior is connected.

Polyominoes are defined up to translations, their name is usually attributed to Golomb [63]. In Figure 2.1 (a) we do not have a polyomino because the interior of the set of cells is disconnected, while in Figure 2.1 (b), (c), (d) we have three polyominoes. The length of the boundary of a polyomino is called the *perimeter* of the polyomino. Since it consists in one or more loops it is made of an even number of steps and we measure the size of polyominoes by their *semi-perimeter* rather than their perimeter. A polyomino is said to be *without holes* if its boundary consists of one single simple loop. Polyominoes in Figure 2.1 (b), (c) have one hole while the one in Figure 2.1 (d) has no holes.

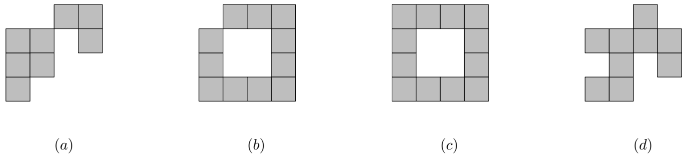

Figure 2.1: (*a*) A union of cells which is not a polyomino; (*b*), (*c*) polyominoes with one hole; (*d*) a polyomino without holes;

Polyominoes have been extensively studied in the literature and their enumeration is a notoriously intractable problem. Convexity restrictions have been introduced on polyominoes in order to obtain enumerative results, giving rise to beautiful closed formulas... In the following subsection we are going to focus on the enumeration of convex subclasses of polyominoes according to the semi-perimeter. Enumeration of more general subclasses with respect to semi-perimeter or other parameters like the *area*, (*i.e.* the number of cells) has been dealt with in the literature with more or less success. The reader will find nice surveys on polyominoes and exact enumeration results in the work by Bousquet-Mélou [18] and in the Chapter [21] by Bousquet-Mélou and Brak.

2.1.2 Convex polyominoes and their number, directed convex and parallel polyominoes

Definition 2. A polyomino is said to be:

- *vertically-convex* or *column-convex* (resp. *horizontally-convex* or *row-convex*) if all its columns (resp. rows) are connected (see Figure 2.2 (a), (b));
- *convex*, if it is both horizontally and vertically-convex (see Figure 2.2 (c)).

It seems that the study of convex polyominoes was suggested by Knuth to Klarner and Rivest [69].

We let $\mathcal{P}_n$ denote the class of convex polyominoes of semi-perimeter $n+1$ and $p_n = \|\mathcal{P}_n\|$ denote its cardinality. Then we have:

Proposition 1 (Delest and Viennot[34])**.** The number p_n of convex polyominoes of semi-perimeter $n+1$ (sequence A005436 in OEIS) is

$$p_n = (2n+5)4^{n-3} - 4(2n-5)\binom{2n-6}{n-3}, \quad n \geq 3; \quad p_1 = 1, \quad p_2 = 2. \tag{2.1}$$

We are also interested by the following classical subclasses:

Definition 3. A convex polyomino P is said to be

- *directed convex* if each of its cells can be reached from the south-west corner of P by a path remaining in P and using only north and east unit steps (see Figure 2.2 (d)).
- *parallelogram* if each of its cells can be reached both from the south-west corner using only north and east steps, and from the north-east corner using only south and west steps. Equivalently, if its boundary can be decomposed in two paths, the upper and the lower paths, which are made of north and east unit steps and meet only at their starting and final points (see Figure 2.2 (e)).

Proposition 2. We let $\mathcal{P}_n^{\triangleright}$ denote the class of directed convex polyominoes of semi-perimeter $n+1$ and $p_n^{\triangleright} = \|\mathcal{P}_n^{\triangleright}\|$ denote its cardinality. Then the number of directed convex polyominoes of semi-perimeter $n+1$, $n \geq 1$, is

$$p_n^{\triangleright} = \binom{2n-2}{n-1}, \tag{2.2}$$

corresponding to the central binomial coefficients (sequence A000984 in OEIS).

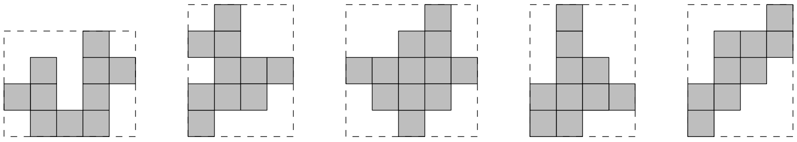
(a) (b) (c) (d) (e)

Figure 2.2: (*a*) a column-convex polyomino and its bounding box; (*b*) a row-convex polyomino; (*c*) a convex polyomino; (*d*) a directed-convex polyomino; (*e*) a parallelogram polyomino.

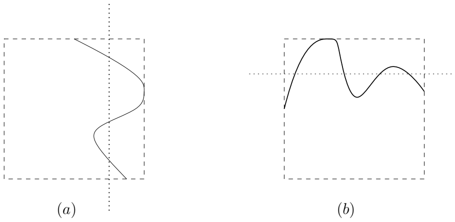
(a) (b)

Figure 2.3: (*a*) A curve that is not convex and its intersection with a vertical line. (*b*) A curve that is not convex and its intersection with a horizontal line.

Proposition 3. We let $\mathcal{P}^{/\!\!/}{}_n$ denote the class of parallelogram polyominoes of semi-perimeter $n+1$ and $p^{/\!\!/}_n = \|\mathcal{P}^{/\!\!/}{}_n\|$ denote its cardinality. Then the number of parallelogram polyominoes of semi-perimeter $n+1$, $n \geq 1$ is

$$p^{/\!\!/}_n = \frac{1}{n+1}\binom{2n}{n}, \tag{2.3}$$

corresponding to the well known sequence of Catalan numbers (sequence A000108 in OEIS).

Definition 4. Given a polyomino P we refer to the minimal rectangle containing P as its *bounding box.*

Following Bousquet-Mélou and Guttmann in [22], the convexity constraint for polyominoes without holes can be thought as a property of the self-avoiding polygon which forms the boundary of the polyomino rather than a property of the set of cells forming the interior of the object.

Proposition 4. A polyomino is convex if and only if its boundary consists of a simple loop whose length is equal to the perimeter of its bounding box.

Moreover it is directed convex if and only if it is convex and it contains the bottom left corner of its bounding box, and it is parallelogram if and only if it is convex and it contains both the bottom left and top right corners of its bounding box.

Proof. Indeed the boundary of a polyomino P is strictly longer than the perimeter of its bounding box if and only if there is a vertical or horizontal line whose intersection with the boundary of P contains at least 3 connected components. But this is immediately equivalent to the fact that P is non convex, as shown by Figure 2.3. □

2.1.3 Bibliographical notes on convex polyomino enumeration

The original proof of Formula (2.1) is due to Delest and Viennot [34] and it is based on multiple encodings of polyominoes by using words of algebraic languages. A simpler grammar-like decomposition for convex polyominoes was given by Mireille Bousquet-Mélou in [16], where she obtained a smaller system of equations that she used for the enumeration with respect to the area but also to recover Delest and Viennot's results for semi-perimeter. Later, Bousquet-Mélou and Guttmann

[22] used an encoding by contour words (expressed in terms of polygons) to explain the form of the formula as a difference. Finally, we [32] gave an alternative proof using the ECO method and the kernel method.

Concerning parallelogram polyominoes, they were first counted by Levine in [73], then by Pólya in [78]. However the proof by Pólya was never been published and Flajolet's supplied a proof in [58] by using festoons, ordered pairs of lattice paths having common start and end points. Directed convex polyominoes were introduced by Chang and Lin [74], in a paper where they provided a refined enumeration for convex polyomiones according to the number of rows and columns. While these are the first proofs concerning enumeration of parallelogram and directed convex polyominoes according to the semi-perimeter, many other proofs can be found in literature for these classes, like for example linear recursive approaches, or grammar-like decomposition, or nice bijective proofs involving especially parallelogram polyominoes. For more details we refer to the vast literature on the subject, in particular to [75] and [17].

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2.2 Square permutations

2.2.1 Square permutations and their number

The class of *square permutations* was first introduced by Mansour and Severini [77] under the name of *4 faces grid polygons*, with a definition based on upper and lower unimodality. Let π be a permutation of length n, we denote by $u(\pi)$ the longest upper unimodal sublist of π starting from $\pi(1)$ and by $\sigma(\pi)$ the subsequence of π made of $\pi(1)$, the elements not in $u(\pi)$ and $\pi(n)$. For example, let us take $\pi = 3\,7\,8\,1\,6\,5\,4\,2\,10\,9$ then $u(\pi) = 3\,7\,8\,10\,9$ and $\sigma(\pi) = 3\,1\,6\,5\,4\,2\,9$.

Definition 5. A permutation π is *square* if the sequence $\sigma(\pi)$ is lower unimodal.

The main counting result of Mansour and Severini in [77] is then:

Proposition 5 (Mansour and Severini [77])**.** We let $\mathcal{S}_n$ denote the class of square permutations with length n and $s_n = \|\mathcal{S}_n\|$ denote its cardinality. Then we have:

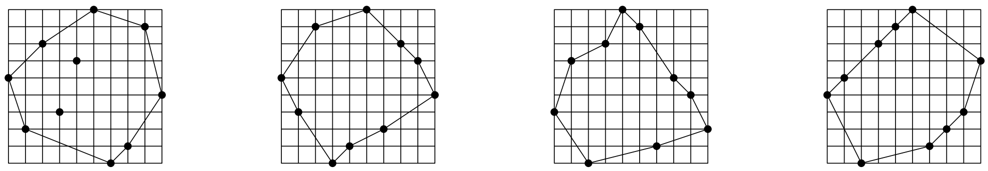
$$s_n = (n+2)2^{2n-5} - 4(2n-5)\binom{2n-6}{n-3}, \quad n \geq 3; \quad s_1 = 1, \quad s_2 = 2. \tag{2.4}$$

We give now a more pictorial characterization which we believe better explains the terminology, and which is based on the more classical notion of *records* of permutations. For this purpose, recall that a *left-right maximum* (resp. *right-left maximum)* of a given permutation π is an entry $\pi(j)$ such that $\pi(j) > \pi(i)$ for each $i < j$ (resp. for each $i > j$) and a *left-right minimum* (resp. *right-left minimum)* is an entry $\pi(j)$ such that $\pi(j) < \pi(i)$ for each $i < j$ (resp. for each $i > j$). The entries of a permutation π that are left-right or right-left minima or maxima are called the *records* of π.

Observe that the sequence $u(\pi)$ can be obtained by concatenating the sequence of left-right maxima and of right-left maxima of π and that $\sigma(\pi)$ is lower unimodal if and only if it consists only of left-right minima and right-left minima of π:

Proposition 6. A permutation is square if and only if all its entries are records.

Now recall that any permutation $\pi = \pi(1), \ldots, \pi(n)$ of length n can be represented on the grid $\mathbb{Z}^2$ by the set $\mathcal{G}_\pi$ of points $(i, \pi(i))$, $i = 1, \ldots, n$. The sequence of left-right minima (resp. right-left minima, left-right maxima, right-left maxima) forms the *left-lower path* (resp. *right-lower path*, *left-upper path* and *right-upper path*) of $\mathcal{G}_\pi$. A point of a permutation π is an *extremal point* if it has abscissa or ordinate 1 or n; it is a *fixed point* if its abscissa equals its ordinate; it is a *double point* if it is both on the left-lower and right-upper paths, or both on the left-upper and right-lower paths. Observe that we can uniquely associate with $\mathcal{G}_\pi$ a geometric figure, obtained by the concatenation of the four paths defined above, as illustrated by Figure 2.4. The shape of the resulting diagram explains why these permutations are called *square permutations* (see Figure 2.4 (b)).



(a) (b) (c) (d)

Figure 2.4: Four permutations: the first is not square since it has internal points, the second is square, the third is triangular, and the last one is parallel.

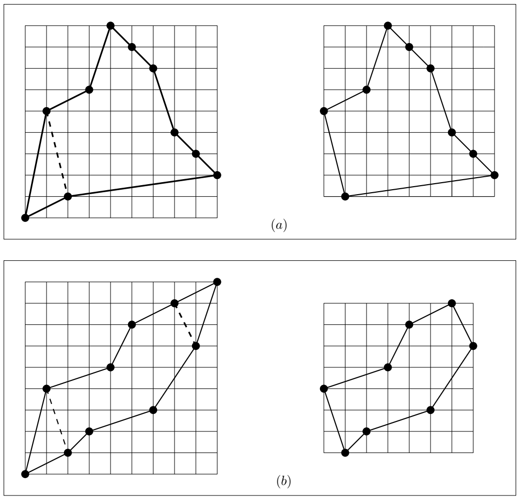


Figure 2.5: (a) A square permutation with $\sigma(1) = 1$ and the triangular one obtained by removing its first point; (b) A square permutation with $\sigma(1) = 1$ and $\sigma(n) = n$, and the parallel one obtained by removing its first and its last point.

2.2.2 Triangular and parallel permutations

In analogy with subclasses of polyominoes, it is natural to define the following subclasses of square permutations:

Definition 6. A *triangular permutation* is a square permutation such that each point belongs either to the left-upper path, the right-upper path, or the right-lower path (see Figure 2.4 (c)).

A *parallel permutation* is a square permutation such that all points belong to the left-upper or the right-lower path (see Figure 2.4 (d)).

Proposition 7. Triangular permutations of size n are in one to one correspondence with square permutations of size $n+1$ starting with a fixed point. Similarly parallel permutations of size n are in correspondence with square permutations of size $n+2$ starting and ending with a fixed point (see Figure 2.5).

Proposition 8. We let $\mathcal{S}_n^{\triangleright}$ denote the class of triangular permutations of length n and $s_n^{\triangleright} = \|\mathcal{S}_n^{\triangleright}\|$ denote its cardinality. Then the number of triangular permutations of length n, with $n \geq 1$, is

$$s_n^{\triangleright} = \binom{2n-2}{n-1}. \tag{2.5}$$

Proposition 9. We let $\mathcal{S}_n^{\sslash}$ denote the class of parallel permutations of length n and $s_n^{\sslash} = \|\mathcal{S}_n^{\sslash}\|$ denote its cardinality. Then the number of parallel permutations of length n, with $n \geq 1$, is

$$s_n^{\sslash} = \frac{1}{n+1}\binom{2n}{n}. \tag{2.6}$$

2.2.3 Square permutations vs convex polyominoes

Convex polyominoes and square permutations have similar formulas

$$p_n = s_n + 4^n, \quad n \geq 3,$$

for which there is not direct combinatorial interpretation. On the other side there is a bijective correspondence [49] between directed convex polyominoes and triangular permutations explaining the following equalities:

$$p_n^{\triangleright} = s_n^{\triangleright}, \quad \text{and} \quad p_n^{/\!/} = s_n^{/\!/},$$

it will be presented in details in Subsection 3.3.2 of Chapter 3. This correspondence gives new equidistribution results between the length of the right-upper boundary of the polyomino and the position of the maximum value of the permutation and between the number of horizontal steps in the right-upper boundary of the polyomino and the number of points in the right-upper path of the permutation. When restricted to the parallel case it recovers an already known bijection between parallelogram polyominoes and parallel permutations due to folklore, unfortunaly it seems hard to use it for the general cases of convex polyominoes and square permutations, also due to the possible combinatorial interpretations for the difference term 4^n.

2.2.4 Bibliographical notes on square permutation enumeration

Parallel, triangular, and square permutations were first introduced and enumerated by Mansour and Severini [77] using recurrences and the kernel method. Later, in [47] we recovered these results by using a simpler construction based on a linear recursive approach and proved that the four parameter generating function of square permutations according to left-right minimum, right-left minimum, left-right maximum, right-left maximum is algebraic.

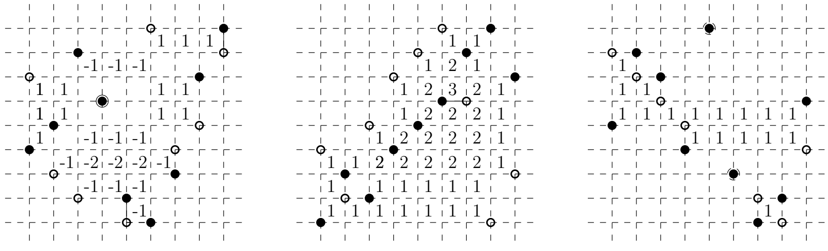
Independently Albert *et al.* [2] gave an alternative proof of the enumeration of square permutations also using the kernel method but mixed with encoding in terms of languages. Finally, as we will discuss later in Subsection 3.4.3 of Chapter 3, motivated by the similarities between Formulas (2.1)–(2.3) and (2.4)–(2.6), we gave in [41] a direct encoding approach for square permutations inspired by the method applied by Bousquet-Mélou and Guttmann in the case of convex polyominoes [22]. In particular, as a consequence of our approach, we obtained a combinatorial interpretation of the fact that Formula (2.4) is a difference of two simple quantities. Another consequence of our encoding is a linear time random generator.

2.3 Convex permutominoes

Permutominoes were introduced in an algebaic context: In 1979, Kazhdan and Lusztig [76] defined the family of Kazhdan-Lusztig polynomials of a Coxeter group, that are polynomials with integer coefficients indexed by pairs of elements of the group. They are related to Schubert varieties and also play an important role in representation theory. In order to prove the existence of these polynomials, Kazhdan and Lusztig [76] introduced the class of R-polynomials, whose computation has been shown by Dier in [53] to be related to the enumeration of some increasing paths in the Bruhat graph of the Coxeter group. In [65], Incitti computed the Kazhdan-Lusztig R-polynomials in the symmetric group $\mathcal{S}$ by giving an algorithm to generate all the increasing paths between two permutations in the Bruhat graph of $\mathcal{S}$. In order to provide such an algorithm he used *permutation diagrams* that are some diagrams associated with a pair of permutations introduced by Kassel *et al.* in [68]. As we are going to explain, the definition of these diagrams naturally leads to the definition of convex permutominoes.

2.3.1 Permutation diagrams and permutominoes

Let us start by recalling the definition of permutation diagrams reformulated by Incitti in [65]: Given a permutation $\sigma \in \mathcal{S}_n$ and $(h, k) \in \{1, \ldots, n\} \times \{1, \ldots, n\}$, we define $\sigma[h, k] = \|\{i \in$



(a) (b) (c)

Figure 2.6: (a) The permutation diagram of $(4\,5\,8\,6\,2\,1\,3\,7\,9, 7\,3\,2\,6\,1\,9\,4\,5\,8)$; (b) the permutation diagram of $(1\,3\,2\,4\,5\,6\,8\,9\,7, 4\,2\,5\,7\,8\,9\,6\,1\,3)$ having only positive multiplicity; (c) the permutation diagram of $(5\,8\,7\,4\,9\,3\,1\,2\,6, 8\,7\,6\,5\,9\,3\,2\,1\,4)$ having multiplicity in $\{0, 1\}$.

$\{1, \ldots, n\} : i \leq h, \sigma(i) \geq k\}\|$, that is $\sigma[h, k]$ is the number of points of σ lying above $y = k$ and on the left of $x = h$.

Moreover, given two permutations $\sigma_1, \sigma_2 \in \mathcal{S}_n$ and a pair of integers $(h, k) \in \{1, \ldots, n\} \times \{1, \ldots, n\}$, the *multiplicity* of (h, k) in (σ_1, σ_2) is the following:

$$(\sigma_1, \sigma_2)[h, k] = \sigma_2[h, k] - \sigma_1[h, k]$$

We consider the multiplicity extended to every $(h, k) \in \mathbb{R} \times \mathbb{R}$, so that we can talk of *multiplicity of* (σ_1, σ_2).

Definition 7. Let (σ_1, σ_2), with $\sigma_1, \sigma_2 \in \mathcal{S}_n$, the *permutation diagram* of (σ_1, σ_2) is given by the diagram of σ_1, the diagram of σ_2, and the multiplicity of (σ_1, σ_2).

This diagram can be represented by using black and white points respectively for permutations σ_1 and σ_2 and by writing inside each unit cell the common multiplicity of all the points of the cell (indeed points (h, k) that are inside the same unit cell have the same multiplicity, except for those in the lower and right border of the cell).

In Figure 2.6 there are three examples of permutation diagrams, where the cells with multiplicity 0 are represented by empty cells.

We are going to introduce a different way to compute the multiplicity $(\sigma_1, \sigma_2)[h, k]$, using the boundary of the pair of permutations:

Definition 8. Given a pair of permutations (σ_1, σ_2), we consider the vertical line segments joining two points $(i, \sigma_1(i))$ and $(i, \sigma_2(i))$ for $i = 1, \ldots, n$ with $\sigma_1(i) \neq \sigma_2(i)$. Observe that there are two types of vertical line segments that we represent respectively by $\underset{\bullet}{\overset{\circ}{|}}$ or $\underset{\circ}{\overset{\bullet}{|}}$ depending if $\sigma_1(i) < \sigma_2(i)$ or not. Similarly we consider horizontal line segments joining two points $(\sigma_1^{-1}(\ell), \ell)$ and $(\sigma_2^{-1}(\ell), \ell)$, for all $\ell \in \{1, \ldots, n\}$ with $\sigma_1^{-1}(\ell) \neq \sigma_2^{-1}(\ell)$, that we represent respectively by $\bullet\!\!-\!\!\circ$ or $\circ\!\!-\!\!\bullet$ depending if $\sigma_1^{-1}(\ell) < \sigma_2^{-1}(\ell)$ or not. Moreover we call *fixed points* of (σ_1, σ_2) the points in which the two permutations coincide. The *boundary* of (σ_1, σ_2) consists of these vertical and horizontal line segments and their fixed points.

Lemma 1. Given a pair of permutations (σ_1, σ_2), then

1. the unit cell C lying on the left and at the bottom of the leftmost vertical line segment of the boundary has multiplicity $m = 0$ (see Figure 2.7 (a)).

2. two adjacent cells C_1 and C_2 that are not separated by a horizontal or vertical line segment of the boundary have the same multiplicity (see Figure 2.7 (b), (c), (d), (e)).

3. two adjacent cells that are separated by a horizontal line segment of the boundary (denoted C_1 and C_2 from bottom to top), have respectively multiplicity m and $m + 1$ or m and $m - 1$ depending if the segment is $\bullet\!\!-\!\!\circ$ or $\circ\!\!-\!\!\bullet$ (see Figure 2.8 (a), (b)).

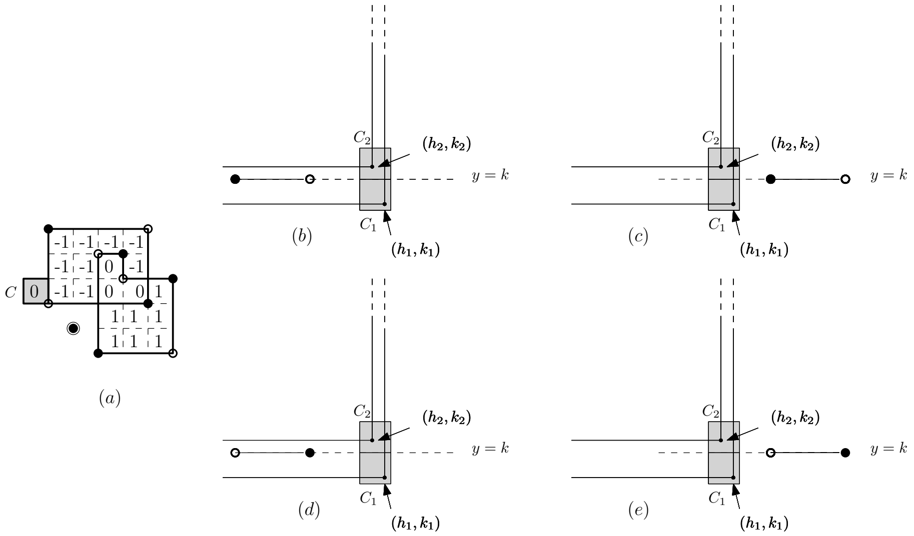


Figure 2.7: (a) The permutation diagram of $(6\,2\,1\,5\,3\,4, 3\,2\,5\,4\,6\,1)$ and its boundary. The unit cell C is in grey; (b) Two adjacent cells C_1 C_2 with a ●—○ line segment on their left; (c) Two adjacent cells C_1 C_2 with a ●—○ line segment on their right; (d) Two adjacent cells C_1 C_2 with a ○—● line segment on their left; (e) Two adjacent cells C_1 C_2 with a ○—● line segment on their right.

4. two adjacent cells that are separated by a vertical line segment of the boundary (denoted C_1 and C_2 from left to right), have respectively multiplicity m and $m-1$ or m and $m+1$ depending if the segment is $\overset{\bullet}{\underset{\circ}{|}}$ or $\overset{\circ}{\underset{\bullet}{|}}$ (see Figure 2.8 (c), (d)).

Proof. Let us take (h_1, k_1) and (h_2, k_2), respectively belonging to C_1 and C_2 (with exception of the lower and right border), then $(\sigma_1, \sigma_2)[h_1, k_1]$ and $(\sigma_1, \sigma_2)[h_2, k_2]$ denote respectively the multiplicity of C_1 and C_2. We prove the lemma point by point:

1. since the unit cell C has no permutation points on its left, its multiplicity is 0.

2. let us suppose that the two adjacent cells are one on the top of the other at ordinate k. Since C_1, C_2 are not separated by a horizontal line segment and σ_1, σ_2 are permutations, this line segment must lie at ordinate k on their left or on their right. Observe that we could also have a fixed point instead of a line segment, but this can be seen as a degenerate case in which the line segment has length 0. We then have four possible cases depending if the horizontal line segment is on their left or on their right and if it is ●—○ or ○—●. We represented these four cases in Figure 2.7, where for the point (h_1, k_1) (resp. (h_2, k_2)), we draw the horizontal and vertical lines delimiting the area on the left of h_1 (resp. h_2) and on the top of k_1 (resp. k_2). Let us suppose we are in the first case (see Figure 2.7 (b)), then

$$\begin{aligned}(\sigma_1, \sigma_2)(h_2, k_2) = \sigma_2(h_2, k_2) - \sigma_1(h_2, k_2) &= \sigma_2(h_1, k_1) - 1 - (\sigma_1(h_1, k_1) - 1)\\ &= \sigma_2(h_1, k_1) - \sigma_1(h_1, k_1)\end{aligned}$$

   The other three cases of Figure 2.7 can be proved in the same way as well as the case in which the adjacent cells are one on the right of the other.

3.,4. let us suppose $\sigma_1(h_1, k_1) = m$. If C_1, C_2 are separated at ordinate k by a ●—○ line segment (see Figure 2.8 (a)) then there is a point of σ_1 at ordinate k and on the left of C_1,C_2.

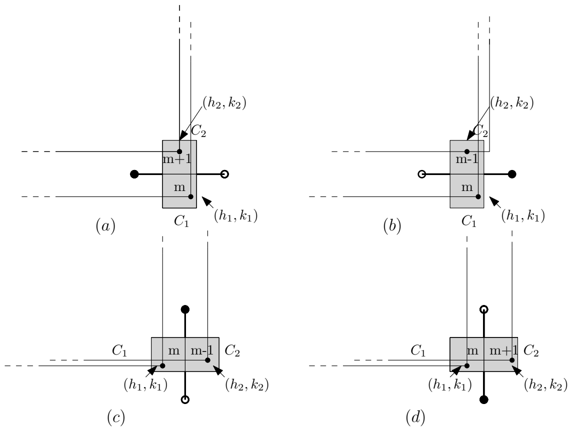


Figure 2.8: (a) two adjacent cells separated by a $\bullet\!\!-\!\!\!-\!\!\circ$ line segment; (b) two adjacent cells separated by a $\circ\!\!-\!\!\!-\!\!\bullet$ line segment; (c) two adjacent cells separated by a $\overset{\bullet}{\underset{\circ}{|}}$ line segment; (d) two adjacent cells separated by a $\overset{\circ}{\underset{\bullet}{|}}$ line segment.

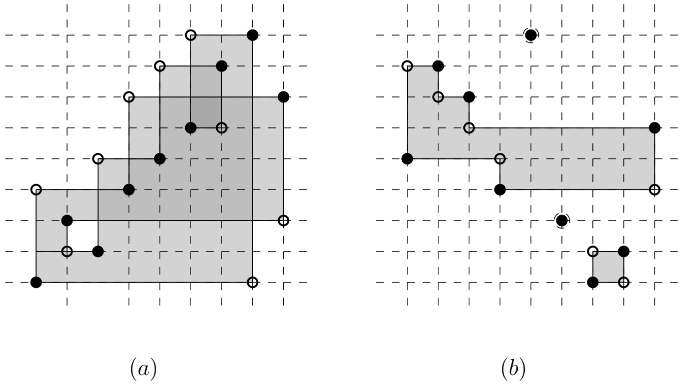


Figure 2.9: (a) Another way of representing the permutation diagram of Figure 2.6 (b); (b) Another way of representing the permutation diagram of Figure 2.6 (c).

Therefore we have the following:

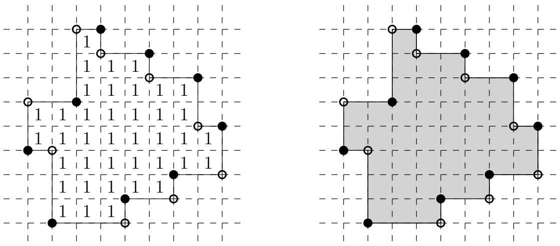
$$\begin{aligned}(\sigma_1,\sigma_2)(h_2,k_2) &= \sigma_2(h_2,k_2)-\sigma_1(h_2,k_2)=\sigma_2(h_1,k_1)-(\sigma_1(h_1,k_1)-1)\\ &= \sigma_2(h_1,k_1)-\sigma_1(h_1,k_1)+1=m+1\end{aligned}$$

The remaining cases can be proved in the same way and are represented in Figure 2.8 (b), (c), (d).

□

A permutation diagram with non negative multiplicity can be drawn using different levels of grey, the set of cells with positive multiplicity is called the *support* of the permutation diagram. See Figure 2.9 for an example.

As we have already said, Incitti [65] gave an algorithm to generate all the increasing paths between two permutations in the Bruhat graph of $\mathcal{S}$ in order to compute the Kazhdan-Lusztig R-polynomials in the symmetric group. The algorithm was given in terms of diagrams of permutations, since there is a nice combinatorial characterization of the Bruhat order relation $\leq$ in



(a) (b)

Figure 2.10: An example of permutomino and its alternative representation.

$\mathcal{S}_n$: let $\sigma_1, \sigma_2 \in \mathcal{S}_n$, then $\sigma_1 \leq \sigma_2 \iff [(\sigma_1,\sigma_2)[h,k] \geq 0 \,\text{for all}\, h,k \in \{1,\ldots,n\}]$. Then Incitti analysed the computation of the Kazhdan-Lusztig R-polynomials $\tilde{R}_{x,y}(q)$ according to different cases. In particular he computed the Kazhdan-Lusztig R-polynomials $\tilde{R}_{\sigma_1,\sigma_2}(q)$ for a particular subclass (σ_1,σ_2) of permutations, for whose diagrams he coined the term of *permutominoes*:

Definition 9. The permutation diagram of a pair $(\sigma_1,\sigma_2) \in \mathcal{S}_n$ is called a *permutomino* in the sense of Incitti if

1. $(\sigma_1,\sigma_2)[h,k] \in \{0,1\}$ for every $(h,k) \in \mathbb{R}^2$;
2. $\sigma_1(i) \neq \sigma_2(i)$ for all $i \in \{1,\ldots,n\}$;
3. the support of the diagram of (σ_1,σ_2) is connected and simply connected (that is without holes).

See Figure 2.10 for an example.

Lemma 2. The diagram of the pair of permutations (σ_1,σ_2) is a permutomino if and only if the boundary of (σ_1,σ_2) forms a single and simple loop.

Proof. $\Longrightarrow$ If the diagram of (σ_1,σ_2) is a permutomino then the border of its support coincide with its boundary. Indeed, if this was not the case, there would be a line segment of the boundary outside or inside the support, contradicting the fact (see Lemma 1 Item 3,) that the support is connected, simply connected and contains only 1. Moreover, since the support is connected and simply connected the boundary is made up of a single and simple loop.

$\Longleftarrow$ If the boundary of (σ_1,σ_2) is a single and simple loop we have, from Lemma 1, that unit cells outside the loop have multiplicity 0 whereas unit cells inside the loop have multiplicity 1. Therefore the support is connected and simply connected. Moreover there are no fixed points of (σ_1,σ_2) otherwise the boundary would not be a single loop.

□

We would like to point out that the class of permutominoes is a particular subclass of polyominoes without holes and, like polyominoes, permutominoes are not necessarily convex, in the sense that the points in their boundary are not necessarily records (see Figure 2.11 for an example.) In the next section we are going to focus on the class of convex permutominoes. To our knowledge this is the largest class of permutominoes that has been enumerated according to their size. There is no formula for general permutominoes and this is probably a difficult problem that is comparable to the enumeration of general polyominoes.

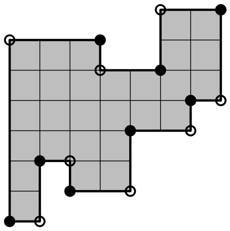


Figure 2.11: A permutomino that is not convex, with $\sigma_1 = 1\,3\,2\,7\,4\,6\,5\,8$ and $\sigma_2 = 7\,1\,3\,6\,2\,8\,5\,6$
.

2.3.2 Convex permutominoes and their number

We can reformulate Incitti's definition of permutominoes in terms of the properties of the boundary of polyominoes without holes.

Definition 10. Let P be a polyomino, then a *vertex* of P is an intersection point of two consecutive boundary edges with different directions, and a *side* of P is a segment joining two consecutive vertices.

Definition 11. Let P be a polyomino without holes and having $n-1$ rows and $n-1$ columns, $n \geq 1$; we assume without loss of generality that the south-west corner of its minimal bounding box is placed at $(1,1)$. The polyomino P is a *permutomino* if for each abscissa (resp. ordinate) between 1 and n there is exactly one vertical (resp. horizontal) side in the boundary of P with that coordinate (see Figure2.12). We denote n as the *size* of the permutomino, corresponding to the length of its (square) bounding box plus 1.

As for polyominoes convexity restrictions have been introduced on permutominoes.

Definition 12. A *convex permutomino* is a permutomino which is also a convex polyomino (see Figure 2.12 (b)).

Proposition 10 (Disanto *et al.* [39], Boldi[13])**.** We let $\mathcal{Q}_n$ denote the class of convex permutominoes of size n and $q_n = \|\mathcal{Q}_n\|$ denote its cardinality. Then, for any $n \geq 2$ we have the following:

$$q_n = (2n+4)\,4^{n-3} \,-\, (2n-3)\binom{2n-4}{n-2}. \tag{2.7}$$

Now we are going to define analogs of the subclasses introduced for convex polyominoes and square permutations:

Definition 13. A *directed convex permutomino* is a permutomino which is also a directed convex polyomino (see Figure 2.12 (c)), and a *parallelogram permutomino* is a permutomino which is also a parallelogram polyomino (see Figure 2.12 (d)).

Proposition 11 (Fanti *et al.* [55])**.** We let $\mathcal{Q}^{\nearrow}_n$ denote the class of directed-convex permutominoes of size n and $q^{\nearrow}_n = \|\mathcal{Q}^{\nearrow}_n\|$ denote its cardinality. Then for any $n \geq 2$, we have:

$$q^{\nearrow}_n = \frac{1}{2}\binom{2(n-1)}{n-1}.$$

Proposition 12 (Fanti *et al.* [55])**.** We let $\mathcal{Q}^{/\!/}_n$ denote the class of parallelogram permutominoes of size n and $q^{/\!/}_n = \|\mathcal{Q}^{/\!/}_n\|$ denote its cardinality. Then for any $n \geq 2$, we have

$$q^{/\!/}_n = \frac{1}{n}\binom{2(n-1)}{(n-1)}.$$

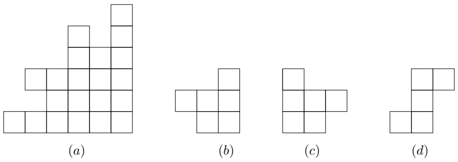


Figure 2.12: (a) a permutomino that is not convex; (b) a convex permutomino; (c) a directed convex permutomino; (d) a parallelogram permutomino;.

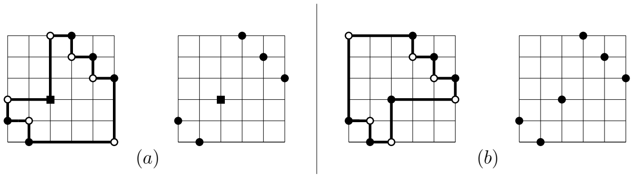


Figure 2.13: (a) A permutomino P and its associated square permutation through ϕ with a colored fixed point (represented with a little square). (b) A permutomino P and its associated square permutation through ϕ with a non-colored fixed point.

2.3.3 Convex permutominoes vs square permutation

A permutation σ of $\{1,\dots,n\}$ is *co-decomposable* if there exists two permutations π of $\{1,\dots,k\}$ and π' of $\{1,\dots,\ell\}$ with $k+\ell=n$ such that σ is the permutation $\pi(1)+\ell,\pi(2)+\ell,\dots,\pi(k)+\ell,\pi'(1),\pi'(2),\dots,\pi'(\ell)$. A square permutation is said to be *colored* if a subset of its free fixed points is colored, where a *free fixed point* is a fixed point $(i,\sigma(i)=i)$ that is not a left-right minimum or a right-left maximum. An explicit connexion between the class of convex permutominoes and the class of square permutations was obtained by Bernini *et al.* [11] who showed the following theorem:

Theorem 1. (Bernini *et al.* [11]) There is a bijection ϕ between

- convex permutominoes of size n, and
- colored square permutations of size n that are co-indecomposable.

To obtain the colored permutation associated to a convex permutomino P, one should first color the bottom vertex on the leftmost edge of P in black, and then alternatively color all other vertices of P in white or black following the boundary of P in clockwise order, so that adjacent vertices have different colors. Then $\phi(P)$ is the colored permutation consisting of the permutation represented by black points, with fixed points belonging to the upper boundary of P as colored fixed points 2.13 (a) (resp. belonging to the lower boundary of P as non-colored fixed points 2.13 (b)).

The previous theorem leads to a relation between the generating function of convex permutominoes with respect to the size and the generating function of co-indecomposable square permutations with respect to the size and number of free fixed points. However this relation does not explain the similarity between Formulas (2.4) and (2.7) nor their particular form as a difference between an asymptotically dominant rational term and a simple subdominant algebraic term. In [41] we used the previous bijection to give a common proof of the two formulas, by explaining their common form and by allowing to generalize them to take into account natural parameters extending the Narayana refinement for Catalan numbers. We will see this approach in details in Subsection 3.4.3 of Chapter 3.

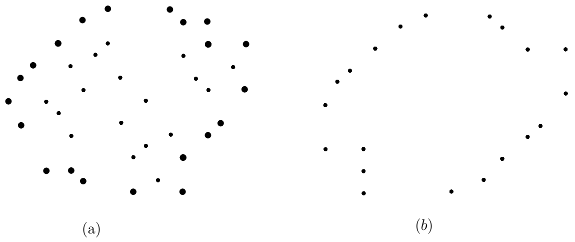


Figure 2.14: (*a*) A set of points with the records highlighted; (*b*) A set of points which is only constituted of records.

2.3.4 Bibliographical notes on convex permutomino enumeration

The problem of enumerating convex permutominoes according to their size was independently solved by Boldi *et al.* [13] and Rinaldi *et al.* [39], while the problem of counting permutominoes according to several convexity constraints was raised by Fanti *et al.* in [56], who enumerated the classes of parallelogram and directed convex permutominoes according to their size [56, 55] as well as the class of stack permutominoes. Later, in [37], we gave a nice bijective proof of (2.7), where we encoded permutominoes in terms of lattice paths. More recently we gave in [41] a direct encoding approach for convex permutominoes inspired by the method applied by Bousquet-Mélou and Guttmann in the case of convex polyominoes [22] and we obtained a combinatorial interpretation of the fact that Formula (2.4) is a difference of two simple quantities.

Another class of permutominoes, that are convex and symmetric with respect to the diagonal $y = x$, has been introduced and counted according to the size by Disanto *et al.* in [40].

2.4 Geometrical interpretations and asymptotical relations

The classes of convex polyominoes, square permutations and convex permutominoes can be defined in terms of records. Indeed definition of records has a natural geometrical interpretation which extends to any set of points in the plane (see Figure 2.14):

Definition 14. A point (i, j) in a set of points $\mathcal{G}$ is called:

- *left-right maximum* of $\mathcal{G}$ if there is no point (i', j') of $\mathcal{G}$ with $i' < i$ and $j' > j$.
- *right-left maximum* of $\mathcal{G}$ if there is no point (i', j') of $\mathcal{G}$ $i' > i$ and $j' > j$.
- *left-right minimum* of $\mathcal{G}$ if there is no point (i', j') of $\mathcal{G}$ with $i' < i$ and $j' < j$.
- *right-left minimum* of $\mathcal{G}$ if there is no point (i', j') of $\mathcal{G}$ with $i' > i$ and $j' < j$.
- an *internal point* if it belongs to none of the previous types.

We have already seen that the left-right maxima of a permutation π correspond to the left-right maxima of its diagram $\mathcal{G}_\pi$, and so on for right-left maxima, left-right minima and right-left maxima.

We can state that the first link between convex polyominoes and square permutations is that they can be both described in terms of records of a set of points in the plane:

Proposition 13.

- A polyomino (or a permutomino) is convex if and only if all the lattice points of its boundary are records (see Figure 2.15 (a)).
- A permutation is square if and only if all the points of its diagram are records (see Figure 2.15 (b)).

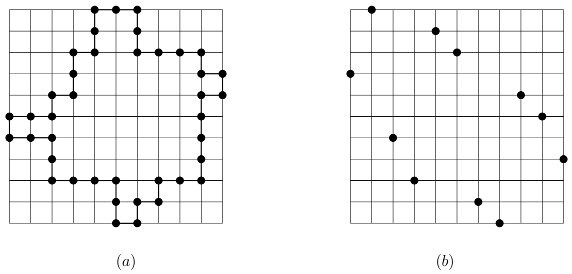


Figure 2.15: (*a*) The set of lattice points of the boundary of a convex polyomino; (*b*) The set of points of a square permutation .

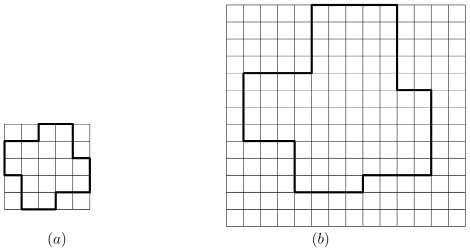


Figure 2.16: (*a*) A permutomino of size 6. (*b*) A corresponding inflated permutomino on a grid of size 13 × 14.

Convex permutominoes are closely related with big convex polyominoes having fixed number of sides. Indeed consider the class $\mathcal{C}_{M,N}(n)$ of convex polyominoes with $2n$ sides on a grid of size $M \times N$ and take uniformly at random a convex polyomino P as M, N tends to ∞. Then the probability that P is *degenerate* tends to 0, where we say that a polyomino is degenerate if we can find more than one side of P on the same horizontal line or on the same vertical line (since the number of sides of P is small with respect to the dimension of the grid). Equivalently we can say that a big convex polyomino with $2n$ vertices has the shape of an inflated big convex permutomino with $2n$ vertices.

Proposition 14. The number $c_{M,N}(n)$ of convex polyominoes with $2n$ vertices on a $M \times N$ grid satisfies

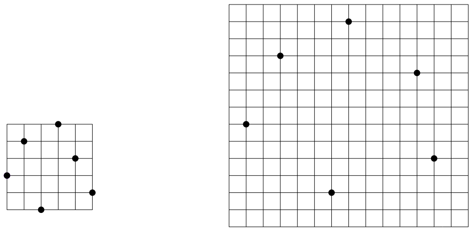
$$c_{M,N}(n) \underset{M,N\to\infty}{\sim} q_n \cdot \binom{M}{n}\binom{N}{n}.$$

Proof. Given a non degenerate convex polyomino P it can be uniquely decomposed into a convex permutomino giving its reduced shape and a selection of n of the N columns and n of the M rows that are used by the vertices (see Figure 2.16 for an example). ☐

To conclude the parallel between convex polyominoes, convex permutominoes and square permutations, we would like to point out that convex permutominoes with respect to convex polyominoes play the same role as square permutations with respect to sets of records in the plane. Indeed we can prove the following proposition, which indirectly highlights the relation between the number of square permutations and convex permutominoes.

Proposition 15. The number $E_{M,N}(n)$ of configurations of n records on a $M \times N$ grid satisfies

$$E_{M,N}(n) \underset{M,N\to\infty}{\sim} s_n \cdot \binom{M}{n}\binom{N}{n}.$$



(a) (b)

Figure 2.17: (*a*) A square permutation of length 6. (*b*) A corresponding permutation on a grid of size 13 × 14.

Proof. Given a configuration of n records on a grid $M \times N$ it can be uniquely decomposed into a square permutation giving its reduced shape and a selection of n of the N columns and n of the M rows that are used by the points (see Figure 2.17 for an example). ☐

2.5 A summary and some questions

A common pattern We have seen all along this chapter the striking similarities between the enumeration of convex polyominoes, square permutations, and convex permutominoes, as summarized in the following table:

Class \ Constraints	Parallel	Directed Convex / Triangular	Convex/Square
Polyominoes	$p^{/\!\!/}_n = \frac{1}{n+1}\binom{2n}{n}$ $n\geq1$	$p^{\triangleright}_n = \binom{2(n-1)}{n-1}$ $n\geq1$	$p_n = (2n+5)4^{n-3}-4(2n-5)\binom{2n-6}{n-3}$ $n\geq3;\quad p_1=1\quad p_2=2$
Permutations	$s^{/\!\!/}_n = \frac{1}{n+1}\binom{2n}{n}$ $n\geq1$	$s^{\triangleright}_n = \binom{2(n-1)}{n-1}$ $n\geq1$	$s_n = (2n+4)4^{n-3}-4(2n-5)\binom{2n-6}{n-3}$ $n\geq3;\quad s_1=1\quad s_2=2$
Permutominoes	$q^{/\!\!/}_n = \frac{1}{n}\binom{2(n-1)}{(n-1)}$ $n\geq1$	$q^{\triangleright}_n = \frac{1}{2}\binom{2(n-1)}{n-1}$ $n\geq1$	$q_n = (2n+4)\,4^{n-3} - (2n-3)\binom{2n-4}{n-2}$ $n\geq2$

These results can be reformulated in terms of generating functions: the common generating function of parallel structures is the classical Catalan generating function,

$$C(t) = \frac{1-2t-\sqrt{1-4t}}{2t}$$

solution of the equation

$$C(t) = t(1+C(t))^2$$

while the generating function of directed or triangular structures is essentially the generating function of central binomial coefficients:

$$B(t) = \frac{1}{1-2t\cdot(1+C(t))} = \frac{1}{\sqrt{1-4t}}$$

The generating functions $P(t)$, $S(t)$, and $Q(t)$ respectively of convex polyominoes, square permutations and convex permutominoes share a common form:

$$P(t) = \frac{t^2}{1-4t}\left(4+\frac{2t}{1-4t}\right) - \frac{4t^3}{(1-4t)^{3/2}}$$

$$S(t) = \frac{t^2}{1-4t}\left(2+\frac{2t}{1-4t}\right) - \frac{4t^3}{(1-4t)^{3/2}}$$

$$Q(t) = \frac{t^2}{1-4t}\left(2 + \frac{2t}{1-4t}\right) - \frac{t^2}{(1-4t)^{3/2}}$$

A remarkable feature of these three formulas is their common shape

$$\textbf{rational series} \qquad - \qquad \textbf{algebraic series},$$

where we recall that, a formal power series $F(t) = \sum_{n\geq 0} f_n t^n$ in the variable t with coefficients in a field A of characteristic zero is said to be *rational* if there exist two polynomials $P(t)$ and $Q(t)$ with coefficients in A such that $F(t) = P(t)/Q(t)$, and it is said to be *algebraic* if there exists a nontrivial polynomial $P(t,u)$ with coefficients in A such that $P(t, F(t)) = 0$. It is said to be *trascendental* if it is not algebraic.

Some explanations that we will give. To explain the previous similarities we will explore in Chapter 3 various possible derivations of these formulas. First we will see in Section 3.1 that the classes of convex polyominoes, square permutations, and convex permutominoes admit similar linear decompositions leading to similar functional equations for their respective generating functions: in particular these three systems of equations can be solved by the same method, that is indeed expected to lead to algebraic series. In the case of their parallelogram and directed (resp. parallel and triangular) subclasses, the occurrence of simple algebraic functions is explained combinatorially via isomorphic algebraic decompositions in Section 3.2, and the exact coincidences of the counting sequences via direct bijections in Section 3.3. In particular we will present our bijection (see [49]) between directed convex polyominoes and triangular permutations which gives new interesting equidistribution results, and discuss an explanation for the factor $\frac{1}{2}$ between the number of directed convex polyominoes and permutominoes of size n.

The general case of convex polyominoes, square permutations and convex permutominoes is harder to deal with: To obtain a bijective proof of the relation between convex polyominoes and square permutations of size n, we need primarily to identify the polyominoes corresponding to the 4^n term by which the two classes differ in the rational part of their formulas. This question is still open, and we can only give a common explanation of the general shape of the formulas of convex polyominoes and square permutations building on an approach of Bousquet-Mélou and Guttmann [22]. As opposed to this, we present in Subsection 3.4.2 of Chapter 3 a direct explanation of common rational part in formulas for square permutations and convex permutominoes [41].

More convexity? We have anticipated that the next chapter is dedicated to showing that we can explain similarities between these three classes of objects in a coherent way. Now there is a natural question arising: are there some other classes whose behavior is similar to these ones?

We know indeed that there are a lot of Catalan structures (see [82]) for which there exists a large number of bijections. Associated to these Catalan structures there are often binomial structures, like in the case of directed convex polyominoes, permutominoes and triangular permutations, for which the Catalan bijections can be adapted. From this point of view, the relation between our three families are not extremely surprising.

Convex polyominoes instead have been considered for a long time as a remarkable and singular artefact, extending the case of parallelogram and directed polyominoes but without analogs for other Catalan structures. Only in the last ten years have emerged the two analogs we discuss here, square permutations and convex permutominoes. This raises the question of what other Catalan structures have "general convex" extensions. Is it possible, for example, to find a generalization of Dyck paths and left Dyck factors to a more general class giving rise to a formula similar to the ones for our convex structures?

There are also natural generalizations of Catalan and binomial numbers, like the Fuss-Catalan numbers for k-ary trees $\frac{1}{kn+1}\binom{kn+1}{n}$, is it possible to find analogs of convex polyominoes in this context?

Refined enumeration The enumeration of parallel, directed and convex polyominoes can be refined to take into account other parameters like the width and the height, or the area of the polyomino. A natural analog of the width for square permutations is given by the sum of the number of points in the left-upper path of the permutation and the number of points in its right-upper path, while the analog for the height is given by the sum of the number of points in the left-upper path and in the left-lower path. In [47] we gave a linear recursive decomposition for square permutations which lead us to take into account the number of each type of record of the permutation and to prove the algebraicity of the corresponding generating function. However the resulting expressions are complicated so that we did not have exact formulas that would give us the enumeration of square permutations according to the number of points in the upper path and in the left path. In [41] we gave another method to count square permutations, by difference, which will be discussed in details in Subsection 3.4.3. This method allows us to enumerate square permutations according to the number of points in the upper path of the permutation and according to the number of points in its left path. Observe however that these two parameters do not exactly correspond to the width and the height in polyominoes, since they overlap in the points of the left-upper path, while the width and the height of a polyomino always sum to its size.

As we will discuss in Subsection 3.4.3 this method also applies to convex permutominoes and it can be used to obtain the enumeration according to analogs of the width or to the height. Concerning the area we do not have enumerative results for convex permutominoes, while for square permutations one can wonder what would be a proper analog of the area.

Enumeration of more general classes Bousquet-Mélou and Guttmann [22] have extended the notion of convex polyominoes to higher dimensions, in terms of convex polygons on the d-dimensional lattice. Following their steps we generalized [48] the concept of permutominoes to higher dimensions and counted three-dimensional convex permutominoes according to their size. A natural question would thus be to look for an interesting notion of square permutations in higher dimensions: the concept of multidimensional permutation already existing we should have to translate the constraint of convexity to dimension 3... In Subsection 3.3 we will see that there is a direct correspondance between parallelogram permutominoes and parallel permutations briefly consisting in replacing a point of the permutation with a corner of the permutomino. Then one can wonder if, by taking a three-dimensional convex permutomino and replacing corner with points, we always obtain a permutation and how the concept of convexity is translated in the permutation.

Another interesting question is linked to square permutation with a fixed number of points. In [38] we introduced these permutations and showed that their generating function is algebraic (we will see this in details in Part II, Chapter 4). Then one can wonder what are proper analogs of internal points for convex polyominoes and convex permutominoes.

Chapter 3

Methods

We will now prove the results of the previous chapter and take advantage of this opportunity to revisit some classical approaches for enumeration in combinatorics. In Section 3.1 we will discuss the linear recursive approach, in which each object is generated by an extention of a smaller one. This method is quite natural and versatile but often requires the introduction of extra *catalytic* parameters in order to lead to explicit functional equations for generating functions. In our cases in particular the solutions of these equations are algebraic series that satisfy simpler polynomial equations. In order to explain these algebricity results one would like to provide algebraic decompositions as discussed in Section 3.2. In Section 3.3 we will discuss the bijective approach: nice formulas call for bijections, we will see some classical and less classical bijections involving Catalan and central binomial structures. Finally, in Section 3.4 we will turn to the interpretation of the differences occuring in the explicit formulas.

3.1 The linear recursive approach

The linear recursive approach is very natural: it consists in looking for a way to grow objects from smaller to larger ones by making some local expansions, where each object should be obtained from a unique “father” so that the construction gives rise to a generating tree for the family under consideration. If the shape of the tree can be described with a simple rule there is hope for exact enumeration results...

This method was first introduced in the context of polyominoes by Temperley [84], in the form of a generation by slices. We present hereafter a formalization in terms of the generating trees and of the rules describing their shape. This formalization was given by Barcucci *et al.* in [7] and systematically discussed by Banderier *et al.* in [5]. We then illustrate the method first on the simple class of parallelogram polyominoes, and then on the more difficult case of convex permutominoes. For both of these constructions we give the rules describing the trees. Then we briefly state the rules associated with similar constructions for convex polyominoes and square permutations. We will end the section by a quick review of the alternative linear recursive approaches for the classes of polyominoes and square permutations and by a discussion on equations with one catalytic variable and the algebraicity of the corresponding generating functions.

3.1.1 Generating trees and succession rules

Let $\mathcal{O}$ be a class of combinatorial objects and let $s : \mathcal{O} \to \mathbb{N}$ be a finite size parameter on $\mathcal{O}$, *i.e.* a parameter s such that $\mathcal{O}_n = \{O \mid s(O) = n\}$ is finite. Let $\vartheta : \mathcal{O} \to 2^{\mathcal{O}}$ be an operator such that $\vartheta(\mathcal{O}_n) \subseteq 2^{\mathcal{O}_{n+1}}$. Assume that there is a unique object $O \in \mathcal{O}$ with minimum size $s(O) = m$. The operator ϑ is intended to describe how small objects produce larger ones in a unique way:

Proposition 16. If ϑ satisfies, for $n \geq m$,

- for each $O' \in \mathcal{O}_{n+1}$, there exists $O \in \mathcal{O}_n$ such that $O' \in \vartheta(O)$, and
- for every $O, O' \in \mathcal{O}_n, \vartheta(O) \cap \vartheta(O') = \emptyset$ whenever $O \neq O'$,

then the family of sets $\mathcal{F}_{n+1} = \{\vartheta(O) : O \in \mathcal{O}_n\}$ is a partition of O_{n+1}.

It means that the operator ϑ generates all the objects of the class $\mathcal{O}$ in such a way that each object $O' \in \mathcal{O}_{n+1}$ is obtained from a unique $O \in \mathcal{O}_n$.

The linear recursive construction performed by ϑ can be described by a *generating tree* $\mathcal{T}_\vartheta$, that is a rooted tree whose nodes are objects of $\mathcal{O}$. The root, placed at level 0 of the tree, is the object with minimum size parameter m. Objects having the same value of the parameter s lie at the same level, and the children of an object are those it produces through ϑ (see Chung *et al.* [29] for an early example of use of generating trees.)

Generating trees are useful for enumeration purpose when they display enough regularities to be finitely described. A particular class of such regularities are those that can be encoded by so-called *succession rules*: Let a be a positive integer called the *axiom*, $(e_i)_{i \geq 1}$ a sequence of integer called the *reproduction rule* and $q : \mathbb{N} \to \mathbb{N}$, an integer valued function called the *reproduction parameter*, then

$$\Omega = \left\{ \begin{array}{lll} (a) & & \\ (k) & \to & (e_1)(e_2)...(e_{q(k)}) \end{array} \right.$$

is the succession rule with axiom a, rule (e_i) and parameter q. A succession rule Ω induces a generating tree $\mathcal{T}_\Omega$ whose root is labeled by the axiom (a), and such that a node labeled (k) produces at the next level $q(k)$ sons labeled by $(e_1), ..., (e_{q(k)})$ respectively. Then the generating tree $\mathcal{T}_\vartheta$ can be encoded by Ω if and only if $\mathcal{T}_\vartheta$ and $\mathcal{T}_\Omega$ are isomorphic. In other terms it means that an object in $\mathcal{T}_\vartheta$ of size parameter k produces, through ϑ, $q(k)$ objects of size parameter $k+1$, that respectively produce $q(e_1)$, $q(e_2)$, $\ldots$, $q(e_{q(k)})$ objects at next level and so on.

Given a class of objects $\mathcal{O}$ generated by an operator ϑ encoded by Ω_ϑ, given its generating function $f(t) = \sum_{n \geq 0} a_n t^n$, where a_n is the number of objects of size n, we have that $f(t) = t^m \cdot f_{\Omega_\vartheta}(t) = t^m \cdot \sum_{n \geq 0} f_n t^n$, where f_n is the number of nodes at level n of the generating tree $\mathcal{T}_{\Omega_\vartheta}$.

There are many extensions of succession rules, like for example *colored* succession rules or *parametrized* succession rules, which allow to describe a large collection of growing modes for which enumeration can be performed explicitely using generating functions. In Subsection 3.1.3 we will give a colored succession rule for convex permutominoes: A colored succession rule Ω is a generalization of succession rule where there can be labels having the same value but different productions. More precisely, let $C = \{c^1, c^2, \ldots, c^\ell\}$ be a finite set, called *the set of colors* of the rule, then the labels of a colored succession rule have the form $(k)_i$, $i \in C$, and the rule is specified by a colored axiom $(a)_i$ and a set of productions which depend on the color i.

3.1.2 A generating tree for parallelogram polyominoes

We give here a first example of simple generating tree and succession rule, obtained from a linear recursive construction on the class of parallelogram polyominoes with respect to the semi-perimeter. Let P be a parallelogram polyomino. Let C_0 be its rightmost column, and let h its height. Then $\vartheta(P)$ is the set of objects produced from P by adding a column of height j, for $j = 1, \ldots, h$, on the top right of C_0, or by adding a cell on the top of C_0 (see Figure 3.1 for an example). Observe that these operations increase the semi-perimeter by one.

Proposition 17. The operator ϑ produces a generating tree for parallelogram polyominoes.

Proof. Let us check the condition of Proposition 16. Given a parallelogram polyomino P' that is not reduced to a single cell, we identify its father in the tree as follows:

- if the top level of the last column of the polyomino P' is different from the top level of the previous column (or if P' has only one column), then P' was necessarily obtained by addition of a top cell and the polyomino P obtained from P' by removing the highest cell in the last column is the only polyomino such that $P' \in \vartheta(P)$;

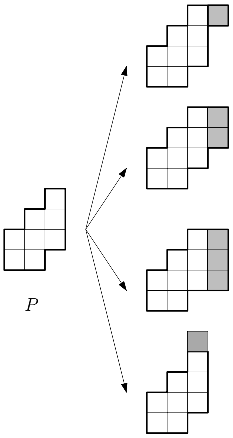


Figure 3.1: The operator ϑ produces four parallelogram polyominoes from P.

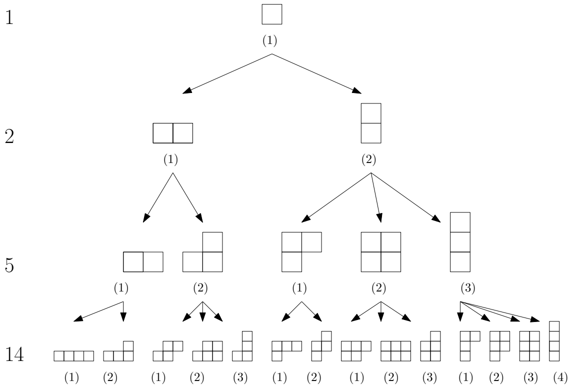


Figure 3.2: The first levels of the generating tree of ϑ and Ω_ϑ, where the labels of the succession rule Ω_ϑ are represented under each object.

- otherwise the last column has the same top level as the previous column, then the polyomino P' was necessarily obtained by addition of a last column to the polyomino P with one less column: $P' \in \vartheta(P)$. Moreover all polyominoes produced from P by addition of a last column have last columns with different height, hence they are all different.

□

Observe that, given a polyomino P with height h, the set $\vartheta(P)$ consists of $h+1$ polyominoes whose height varies from 1 to $h+1$. Moreover the initial set is reduced to the polyomino with one cell, that has height 1. The shape of the generating tree can thus be described by the following rule:

$$\Omega_\vartheta = \begin{cases} (1) \\ \\ (h) \quad \rightarrow \quad (1)\,(2)...(h)\,(h+1) \end{cases}$$

Writing the first levels of the generating tree induced by this simple succession rule and counting all the nodes at each level, the first Catalan numbers readily appear as illustrated by Figure 3.2. We are going to show that the generating function of the generating tree $\mathcal{T}_{\Omega_\vartheta}$ associated with Ω_ϑ is the Catalan one.

In order to compute the generating function $P^{\diagup\!\!\diagup}(t)$ of parallelogram polyominoes we need to introduce an additional variable y, called the *catalytic variable*, which takes into account the variations of the parameter h according to ϑ. Then $P^{\sslash}(t) = P^{\sslash}(t,1)$, with

$$P^{\sslash}(t,y) = t^2 \cdot \sum_{n\geq 0, h\geq 1} p_{n,h} t^n y^h,$$

where t^2 is due to the difference between the size of the minimal object and the level zero of the generating tree $\mathcal{T}_{\Omega_\vartheta}$ and where $p_{n,h}$ denotes the number of nodes at level n and with label h of $\mathcal{T}_{\Omega_\vartheta}$. The axiom of the succession rule Ω_ϑ implies that

$$P^{\sslash}(t,y) = t^2 y^1 + t^2 \cdot \sum_{n\geq 1, h\geq 1} p_{n,h} t^n y^h$$

since the node at level 0 contributes with y^1.

Moreover, according to Ω_ϑ, each of the $p_{n,h}$ nodes with label h and at level n produce exactly $h+1$ nodes at level $n+1$, with labels going from 1 to $h+1$. Then the term $\sum_{n\geq 1, h\geq 1} p_{n,h} t^n y^h$ in the previous equation can be obtained by summing over all the fathers in the generating tree $\mathcal{T}_{\Omega_\vartheta}$ and following the productions of the succession rule :

$$\begin{aligned}
P^{\sslash}(t,y) &= t^2 y + t^2 \cdot \sum_{n\geq 0, h\geq 1} p^{\sslash}_{n,h} t^{n+1} (y^1 + y^2 + \ldots y^{h+1}) && (3.1)\\
&= t^2 y + ty (t^2 \cdot \sum_{n\geq 0, h\geq 1} p^{\sslash}_{n,h} t^n \frac{1-y^{h+1}}{1-y}) && (3.2)\\
&= t^2 y + \frac{ty}{1-y} (P^{\sslash}(t,1) - y P^{\sslash}(t,y)) && (3.3)
\end{aligned}$$

The previous equation is a linear equation depending on one *catalytic* variable, that is a variable which takes into account the variations of the parameter, the height of the leftmost column in this case. An essential ingredient to solve it and in general, to solve equations of this form, is the so-called kernel method. Let us rewrite Equation 3.1 as follows:

$$(1 - y + ty^2) P^{\sslash}(t,y) = ty(t(1-y) + P^{\sslash}(t,1)). \qquad (3.4)$$

The coefficient of $P^{\sslash}(t,y)$ in the previous equation is called the *kernel* and the kernel method consists in coupling the variables t and y in order to cancel the left hand side of the functional equation 3.4 and obtain in the right hand side an equation for $P^{\sslash}(t,1)$. In particular, in order to set the kernel equal to 0 we take y as a solution $y(t)$ of the *kernel equation*:

$$1 - y + ty^2 = 0 \text{ or } y = 1 + ty^2.$$

If $y(t)$ can be substituted into both sides of Equation 3.4 then we obtain $P^{\sslash}(t,1)$. Here the kernel equation clearly admits a substituable power series solution

$$y_0(t) = \frac{1 - \sqrt{1-4t}}{2t} = 1 + t + 2t^2 + O(t^3)$$

which is the generating function of the well known Catalan numbers and we obtain $P^{\sslash}(t,1) = t(y_0(t) - 1)$:

Proposition 18. The generating function of parallelogram polyominoes with respect to the semi-perimeter is the series

$$P^{\sslash}(t,1) = \frac{1 - 2t - \sqrt{1-4t}}{2} = t^2 + 2t^3 + 5t^4 + O(t^5)$$

unique power series solution of the algebraic equation $P^{\sslash}(t) = (t + P^{\sslash}(t))^2$. In particular the number of parallellogram polyominoes with semi-perimeter $n+1$ is the nth Catalan number.

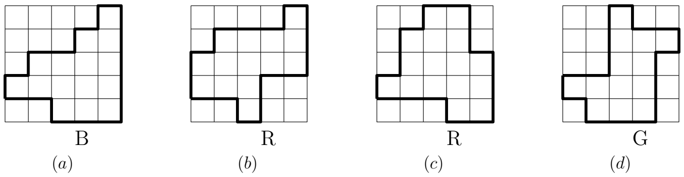


Figure 3.3: Four convex permutominos respectively belonging to B, R, R, and G.

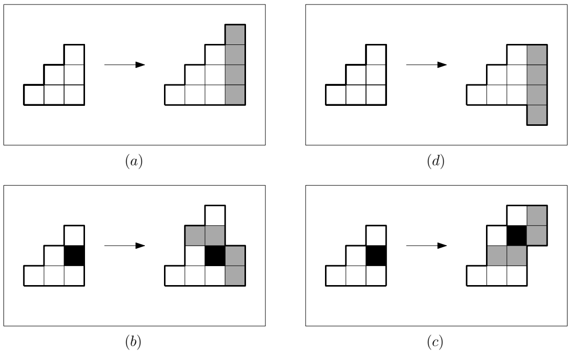


Figure 3.4: The operator ϑ acting on a convex permutomino P and a cell c_i.

3.1.3 A generating tree for convex permutominoes

Let us know illustrate how the linear recursive approach adapts to the case of convex permutominoes, following Disanto *et al.* [39]. Let P be a convex permutomino in $\mathcal{Q}_n$ and let C_0 be its rightmost column. In order to define an operator ϑ we consider the following conditions on a convex permutomino P:

- C_u: the uppermost cell of C_0 has the maximal ordinate of all cells of P;
- C_ℓ: the lowest cell of C_0 has the minimal ordinate of all cells of P.

We distinguish three different subclasses of convex permutominoes depending on whether they satisfy or not the previous conditions:

- B is the subclass satisfying both condition C_u and C_ℓ (see Figure 3.3 (a)) ;
- R is the subclass satisfying only one of the previous conditions (see Figure 3.3 (b), (c));
- G is the subclass satisfying none of the previous conditions (see Figure 3.3 (d));

Generating tree Now we are ready to define the operator ϑ introduced by Disanto *et al.*. For the sake of simplicity we choose to omit the proof that ϑ satisfies Proposition 16, as well as part of the description of the succession rule and the calculation of its associated generating function. For readers interested in more details we refer to [39]. Let $P \in \mathcal{Q}_n$ and h be the height of its righmost column C_0, and let c_i denote the i-th cell of C_0 from bottom to top, with $i = 1, \ldots, h$. Then $\vartheta(P)$ is the set of objects produced as follows:

- (a): if P satisfies condition C_u one object is produced by adding a new column of height $h+1$ whose lowest cell is on the right of the lowest cell of C_0 (see Figure 3.4 (a)).
- (b): for each cell c_i of C_0 one object is produced by adding a new row on the top and of the same length of the row containing c_i, and a column of height i ending on the right of c_i, where $i \in \{1, \ldots, h\}$ (see Figure 3.4 (b)).

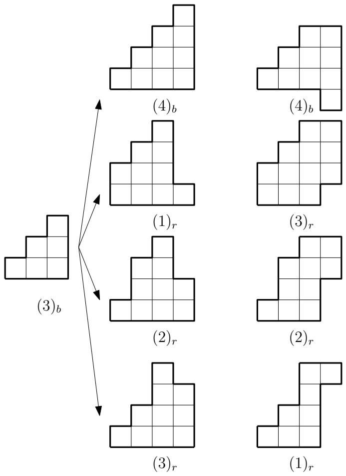


Figure 3.5: The operator ϑ acting on a convex permutomino of type B.

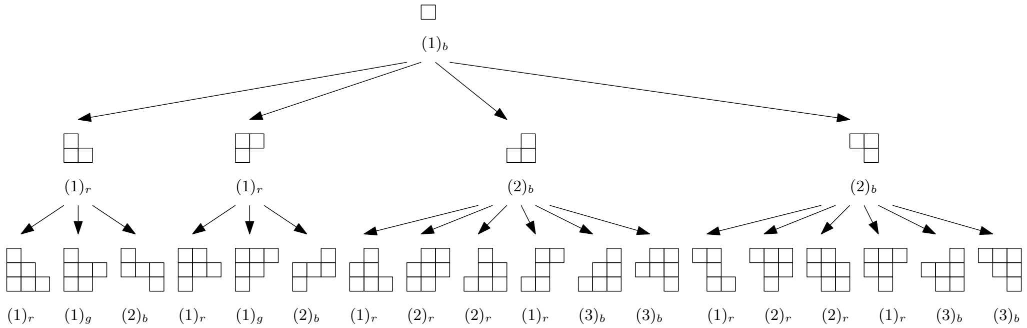


Figure 3.6: The first levels of the generating tree of ϑ.

- (c): for each cell c_i of C_0 one object is produced by adding a new row on the bottom and of the same length of the row containing c_i, and a column of height $h-i+1$ starting on the right of c_i, where $i \in \{1, \dots, h\}$ (see Figure 3.4 (c)).
- (d): if P satisfies condition C_ℓ then one object is produced by adding a new column of height $h+1$ whose upmost cell is on the right of the upmost cell of C_0 (see Figure 3.4 (d)).

Proposition 19. The operator ϑ produces a generating tree for convex permutominoes.

Proof. As in the case of parallelogram polyominoes, the strategy is to identify the father of each permutomino in the generating tree in order to prove that ϑ satisfies Proposition 16: in this case the father is simply obtained by removing the last column and a well chosen line. ☐

Succession rules We are going to analyze how the parameter h changes according to the operator ϑ. Let us suppose P of type B, then by applying case (a) (resp. (d)) of ϑ we obtain a convex permutomino of type B with parameter $h+1$, and by applying case (b) (resp. (c)) of ϑ we obtain h convex permutominoes of type R, with parameter i (resp. $h-i+1$) for $i = 1, \dots, h$ (see Figure 3.5 for an example). This can be described by the following rule:

$$(h)_b \to (1)_r^2\,(2)_r^2 \dots (h)_r^2\,(h+1)_b^2,$$

where $(k)_\ell$ stands for an object with parameter k and belonging to the class $\ell \in \{B, R, G\}$ and $(k)_\ell^j$ means label $(k)_\ell$ repeated j times. Performing a similar analysis for the other classes, we see that the succession rule associated with ϑ is the following:

$$\Omega_\vartheta = \begin{cases} (1)_b & \\ (h)_b & \rightarrow \quad (1)_r^2\,(2)_r^2 ... (h)_r^2\,(h+1)_b^2 \\ (h)_r & \rightarrow \quad (1)_r\,(1)_g\,(2)_r\,(2)_g ... (h)_r\,(h)_g\,(h+1)_r \\ (h)_g & \rightarrow \quad (1)_g^2\,(2)_g^2 ... (h)_g^2 \end{cases}$$

In Figure 3.6 we represented the first levels of the generating tree $\mathcal{T}_\vartheta$ associated with ϑ as well as the isomorphic generating tree n $\mathcal{T}_{\Omega_\vartheta}$ whose nodes are the labels of the succession rule Ω_ϑ.

From succession rule to functional equation Let $q_{n,h}$ be the number of nodes of label h at level n of $\mathcal{T}_{\Omega_\vartheta}$ and let $b_{n,h}$, $r_{n,h}$, $g_{n,h}$ be respectively the number of nodes of label h and type b, r, g at level n. We define:

$$Q(t,y) = t^2 \cdot \sum_{n\geq 0, h\geq 1} q_{n,h} t^n y^h = Q_B(t,y) + Q_R(t,y) + Q_G(t,y), \tag{3.5}$$

with $Q_B(t,y) = t^2 \cdot \sum_{n\geq 0} b_{n,h} t^n y^h$, $Q_R(t) = t^2 \cdot \sum_{n\geq 0} r_{n,h} t^n y^h$, and $Q_G(t) = t^2 \cdot \sum_{n\geq 0} g_{n,h} t^n y^h$, where the term t^2 is due to the lag between the object of minimum size 2 and the level 0 of the generating tree. Then the generating function of convex permutominos according to their size is given by $Q(t,1)$.

From the succession rule Ω_ϑ we have that

$$Q_B(t,y) = t^2 y + 2t^2 \cdot \sum_{n\geq 0, h\geq 1} b_{n,h} t^{n+1} y^{h+1} = t^2 y + 2ty Q_B(t,y)$$

and consequently

$$\begin{aligned} Q_B(t,y) &= \frac{t^2 y}{1-2ty} \qquad (3.6) \\ Q_B(t,1) &= \frac{t^2}{1-2t} \qquad (3.7) \end{aligned}$$

We know from Ω_ϑ that labels of type r are produced from labels of type b and r. Collecting the contributions we have the following equation for $Q_R(t,y)$:

$$Q_R(t,y) = \frac{2ty}{1-y}(Q_B(t,1) - Q_B(t,y)) + \frac{ty}{1-y}(Q_R(t,1) - yQ_R(t,y)),$$

which rewrites

$$Q_R(t,y)((1-y) + ty^2) = 2ty(Q_B(t,1) - Q_B(t,y)) + tyQ_R(t,1), \tag{3.8}$$

where expressions for $Q_B(t,1)$ and $Q_B(t,y)$ are already known from equations (3.6)-(3.7).

By applying the kernel method we obtain the following:

$$\begin{aligned} Q_R(t,y) &= \frac{t^2 y(2ty - 1 + \sqrt{1-4t})}{\sqrt{1-4t}(2t^2y^3 + 2ty - 3ty^2 - 1 + y)} \qquad (3.9) \\ Q_R(t,1) &= \frac{t}{\sqrt{1-4t}} - \frac{t}{1-2t}. \qquad (3.10) \end{aligned}$$

In passing we would like to point out that by taking convex permutominoes of the class B and half of the convex permutominoes of the class R we obtain the whole class of directed convex permutominoes (rotated of 180 degrees in clockwise order), this lead us to recover the generating function of directed convex permutominoes: $Q_B(t,1) + \frac{1}{2}Q_R(t,1) = \frac{t}{2}(\frac{1}{\sqrt{1-4t}} - 1)$.

Now $Q_G(t,1)$ is the last missing ingredient in order to obtain $Q(t,1)$. From succession rule Ω_ϑ and with the same reasoning as before we have

$$Q_G(t,y) = \frac{ty}{1-y}(Q_R(1,y) - Q_R(t,y)) + \frac{2ty}{1-y}(Q_G(1,y) - Q_G(t,y)),$$

and by applying the kernel method we obtain the following:

$$Q_G(t,1) = \frac{t - 7t^2 + 14t^3 - 4t^4}{(1-2t)(1-4t)^2} - \frac{t-3t^2}{(1-4t)^{\frac{3}{2}}} \tag{3.11}$$

Finally, from Equations (3.5), (3.7), (3.10), and (3.11) we have:

$$Q(t,1) = \frac{2t^2(1-3t)}{(1-4t)^2} - \frac{t^2}{(1-4t)^{\frac{3}{2}}}.$$

Extracting coefficients from the previous expression we obtain the cardinal of the set of convex permutominos of size n, with $n \geq 2$:

$$|\mathcal{Q}_n| = (2n+4)4^{n-3} - (2n-3)\binom{2n-4}{n-2}.$$

3.1.4 Two succession rules for convex polyominoes and square permutations

We would like to point out that we introduced similar constructions for convex polyominoes and square permutations. As for convex permutominoes we classified these objects into different subclasses according to their general shape. For convex polyominoes we introduced four subclasses [32] depending if the last column exceeds (partially or totally) the one preceeding it or not. The succession rule is similar to the one for convex permutominoes, with colored labels representing the different subclasses:

$$\Omega = \begin{cases} (1)_a & \\ (h)_a & \rightarrow \quad (1)_g^{h-2}\,(2)_g^{h-3}\ldots(h-2)_g \;\; (1)_r^2\,(2)_r^2\,\ldots(h-1)_r^2 \;\; (h)_b(h+1)_a \\ (h)_b & \rightarrow \quad (1)_g^{h-2}\,(2)_g^{h-3}\ldots(h-2)_g \;\; (1)_r^2\,(2)_r^2\,\ldots(h-1)_r^2 \;\; (h)_b(h+1)_a(h+1)_b \\ (h)_r & \rightarrow \quad (1)_g^{h-1}\,(2)_g^{h-2}\ldots(h-1)_g^h \;\; (1)_r\,(2)_r\,\ldots(h+1)_r \\ (h)_g & \rightarrow \quad (1)_g^h\,(2)_g^{h-1}\ldots(h)_g \end{cases}$$

In [47] we classified square permutations into four subclasses depending on the position of the first point of a permutation σ and on the position of the point occurring after the first sequence of double points in σ. We obtained the following succession rule:

$$\Omega = \begin{cases} (1)_a & \\ (1)_a & \rightarrow \quad (2)_a(1)_b \\ (h)_a & \rightarrow \quad (1)_b\,(2)_b\ldots(h)_b \;\; (2)_c\,(3)_c\,\ldots(h-1)_c \;\; (h+1)_a^2 \\ (h)_b & \rightarrow \quad (1)_b\,(2)_b\ldots(h+1)_b \;\; (1)_d\,(2)_d\,\ldots(h-1)_d \\ (h)_c & \rightarrow \quad (2)_c\,(3)_c\ldots(h+1)_c \;\; (1)_d\,(2)_d\,\ldots(h)_d \\ (h)_d & \rightarrow \quad (1)_d^2(2)_d^2\ldots(h)_d^2 \end{cases}$$

In the same paper we extended the previous rule in order to take into account the records and we showed the algebraicity of the corresponding generating function. Finally, as already discussed in the conclusions of Chapter 2 and, as we will see in details in Chapter 4, we [38] successively generalised the linear construction leading to the previous rule and proved the algebraicity of square permutations with a fixed number of internal points.

3.1.5 Other linear recursive constructions for convex polyominoes and square permutations

Succession rules are a useful but not necessary tool for passing from linear recursive constructions to generating functions, they represent an intermediate step allowing to describe in a synthetic way the growth mode of many classical classes of combinatorial objects. Beyond the previous constructions, that are translated into similar succession rules, other linear recursive constructions for the classes of convex polyominoes and for square permutations were proposed.

The first one for convex polyominoes was given by Bousquet-Mélou [17], she generated the objects by gluing a column at the beginning of the polyomino, giving rise to a generating tree with jumps due to non constant semi-perimeter variations. She then translated the construction into a linear equation with one catalytic variable and two unknowns which do not depend on this variable, such an equation can be solved by applying the kernel method with two substituable series.

The first linear construction for square permutations according to their length was given by Mansour *et al.* in [77]. They used symmetry arguments and focused on three subclasses of square permutations to obtain a system of three functional equations with two catalytic variables. After a variable substitution they simplified the system into three equations with one catalytic variable and they solved it by applying the kernel method. Later, Albert *et al.* [2], gave another linear recursive construction for square permutations (which they called *convex permutations*). Their method uses the insertion encoding of a permutation class, introduced by Albert *et al.* in [1], which provides a correspondence between permutation classes and formal languages, so that to each permutation σ is associated a word describing how σ is constructed in a step by step process. This correspondence, restricted to the case of convex permutations, gives rise to a system of six linear functional equations with one catalytic variable which they solved by applying the kernel method.

3.1.6 Some remarks about linear equation with one catalytic variable

We have seen three examples of linear recursive constructions leading to a system of linear equations depending on one catalytic variable, one of which is always independent from the others. We solved these systems by repeatedly applying the kernel method, starting from the equation which is independent from the others.

The kernel method belongs to folklore, it has been introduced around the 70's and it has been used in various combinatorial problems, as for example in Knuth's book [70] on the art of computer programming. Applications of the kernel method in probabilities date back to the paper of Fayolle et Iasnogorodski [57]. Discussions of this method can be found in the work of Banderier *et al.* [5].

In our three cases the kernel method leads to algebraic generating functions, thus raising a natural question: is it always true that the solution of a linear equation with one catalytic variable is algebraic?

But before answering these questions there is another natural question to be raised: do linear recursive constructions lead always to a system of one or more linear equations of the previous type? The answer is no, and we can find a lot of examples of this fact.

In particular, in the case of succession rules, Banderier and his co-authors [5] partially answered this question by exploring the links between the structural properties of the succession rules and the associated generating functions, by providing a partial classification of different types of rules, leading to rational, algebraic, or transcendental generating functions. They proved for example that succession rules whose productions give constant labels except for a fixed number of labels, like for example the following $(h) \rightarrow (3)^{h-3}(h+1)\,(h+2)\,(h+3)$, lead to rational generating functions. They introduced the so-called *factorial* succession rules, that are finite modifications of a rule of the type $(h) \rightarrow (1)(2)\dots(h)$, and proved that they lead to algebraic generating functions. Moreover they gave some criterion for a succession rule to give trascendental generating functions, like for example rules whose coefficients grow too fast as in the production $(h) \rightarrow (2)(3)(h+2)^{h-2}$, or whose productions are too irregular.

Moreover, Banderier *et al.* introduced a general family of so-called *factorial walks* whose generating trees can be described by factorial succession rules extending the Catalan rule we have seen for parallelogram polyominoes, and they have shown that these rules lead to functional equations that are rather generic linear equations with one catalytic variable and that these equations have algebraic solutions. Banderier *et al.* pointed out that the same results can be obtained with a detour via multivariate linear recurrences and by using the results from Bousquet-Mélou and Petkovšek in [24]. In this paper the authors give some conditions for solutions of multivariate linear recurrences to have algebraic generating functions and, among several results, they also proved that, up to some technical conditions generically satisfied by combinatorial examples, linear functional equations with one catalytic variable and a finite number of unknowns not depending on this variable lead to an algebraic generating function, giving a positive answer to our first question.

Then there is another question naturally arising: what about polynomial equations with one catalytic variable and a finite number of unknowns? In [26] Brown proved that quadratic equations with one catalytic variable give algebraic generating functions and, more generally, in [20] Bousquet-Mélou and Jehanne proved the algebraicity of the generating functions obtained from combinatorially funded polynomial equations with one catalytic variable.

For equations with more than one catalytic variable there is no such general theory, but the topic is very active, see for instance the literature on walks in the quarter plane [23] for linear equations, or the literature on coloured maps [10].

3.2 ℕ-Algebraic decompositions

All the objects that we have studied in the previous section have algebraic generating functions, however the decompositions that we have used do not explain this fact combinatorially. The *Schützenberger's methodology*, also called *DSV* [80] is an enumerative method consisting in using algebraic languages leading to functional equations that express directly the algebraicity of the original class. The idea is to estabilish a bijection between the objects and the words of an algebraic language so that the size of the objects corresponds to the length of the words of the language. If the language is generated by a non ambiguous context-free grammar, then the productions of the grammar immediately translates into a system of ℕ-algebraic equations for the generating functions (see also Schützenberger and Chomsky [28]), thus giving a satisfying explanation. The theory of *decomposable structures* (Flajolet, Salvy, and Zimmermann) [59, 60], describes recursively the objects in terms of basic operations between them. These operations are directly translated into operations between the corresponding generating functions, cutting off the passage to words. A nice presentation of this theory appears in the book of Flajolet and Sedgewick [61]. A variant is the *theory of species*, introduced by Bergeron, Labelle and Leroux [9], which also follows the philosophy of decomposable structures. Basing on ideas of Joyal [67], they define an algebra on species of structures, where the operations between the species immediately reflect on the generating functions. Finally, a convenient formalization of the approach of decomposable structures was introduced by Dutour [51] with the concept of object grammars, allowing to describe objects using very general kinds of operations. All these approaches are more or less equivalent for the purpose of explaining why a class of structures has algebraic generating function, and in the next subsections we will focus on object grammars. We will introduce this approach and we will give object grammars for parallelogram polyominoes, parallelogram permutominoes and parallel permutations. Then we will give object grammars for directed convex polyominoes, triangular permutations and directed convex permutominoes and we will end this section with a discussion around the possibility to find grammars for convex polyominoes, square permutations and convex permutominoes.

3.2.1 Introduction to object grammars

Object grammars Let us first recall some basic definitions concerning object grammars. An *object grammar* is a quadruple $\langle \mathbb{O}, \mathbb{E}, \Phi, \mathcal{A} \rangle$ where:

- $\mathbb{O} = \{\mathcal{O}^i\}_{i \in I}$ is a finite family of sets of objects (I is a finite subset of $\mathbb{N}$ of size $\|I\|$).
- $\mathbb{E} = \{E_{\mathcal{O}^i}\}_{i \in I}$ is a collection of finite subsets $\{E_{\mathcal{O}^i}\} \subset \mathcal{O}^i$, called the *terminal objects*.
- Φ is a set of *object operations* in $\mathbb{O}$, where an object operation is a mapping $\phi : \mathcal{O}^1 \times \ldots \times \mathcal{O}^k \to \mathcal{O}$, with k a positive integer and $\mathcal{O}, \mathcal{O}^1, \ldots, \mathcal{O}^k \in \mathbb{O}$.
- $\mathcal{A}$ is a fixed set of $\mathbb{O}$, called the *axiom* of the grammar.

Let $G = \langle \mathbb{O}, \mathbb{E}, \Phi, \mathcal{A} \rangle$ be an object grammar and let $\mathcal{O} \in \mathbb{O}$. A *derivation tree* for the class $\mathcal{O}$ is a finite labelled tree T satisfying the following recursive characterization:

- if T is reduced to a leaf then it is labelled by a terminal object of $\mathcal{O}$,
- if the root of T has k subtrees $T^1 \ldots T^k$, with $k \geq 1$, then each T^i is a derivation tree for a class $\mathcal{O}^i \in \mathbb{O}$, for $i \in [1, k]$, and the root of T is labelled by an operation $\phi \in \Phi$, $\phi : \mathcal{O}^1 \times \ldots \times \mathcal{O}^k \to \mathcal{O}$.

The evaluation $ev(T)$ of a derivation tree T is then recursively defined as follows:

- if T is a single node labelled e, then $ev(T) = e$,
- otherwise, if the root of T is labelled $\phi \in \Phi$ and its k subtrees are $T^1 \ldots T^k$, then $ev(T) = \phi(ev(T^1), \ldots, ev(T^k))$.

An object o is said to be generated in G, if $o \in \mathcal{A}$ and there is a derivation tree T of G such that $ev(T) = o$. The set of objects generated in G is denoted by $\mathcal{O}_G(\mathcal{A})$. The grammar is *non ambiguous* if each object admits a unique derivation tree. An *additive parameter* on the grammar is a collection of functions $s : \mathcal{O} \to \mathbb{N}$ such that for all operation $\phi : \mathcal{O}^1 \times \ldots \times \mathcal{O}^k \to \mathcal{O}$ there exists a non negative integer constant a_ϕ such that for all k-uples of objects $o_i \in \mathcal{O}^i$ with $i \in [1, k]$,

$$s(\phi(o_1, \ldots, o_k)) = a_\phi + s(o_1) + \ldots + s(o_k).$$

An object grammar is *well graded by a parameter s* if for all $n \geq 0$ the grammar generates only a finite number of objects o with $s(o) = n$. Object grammars are often graphically described and in the rest of the section we adopt this kind of representation. For further definition and examples on object grammars, we suggest the reader to see [51] or [52].

$\mathbb{N}$-algebraic series Our interest in object grammars arises from their relation to $\mathbb{N}$-algebraic series:

Definition 15. A series $F(t)$ is $\mathbb{N}$-algebraic if there etists a k-tuple of formal power series $(F_1(t), F_2(t), \ldots, F_k(t))$ such that $F = F_1(t)$ and for $1 \leq j \leq k$:

$$\begin{cases} F_1(t) &= P_1(t, F_1(t), F_2(t), \ldots, F_k(t)) \\ F_2(t) &= P_2(t, F_1(t), F_2(t), \ldots, F_k(t)) \\ \vdots &= \vdots \\ F_k(t) &= P_k(t, F_1(t), F_2(t), \ldots, F_k(t)) \end{cases}$$

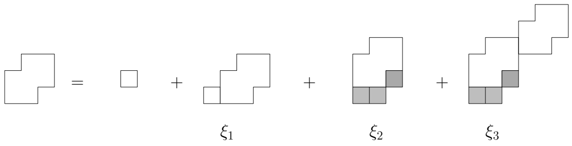


Figure 3.7: An object grammar for parallelogram polyominoes.

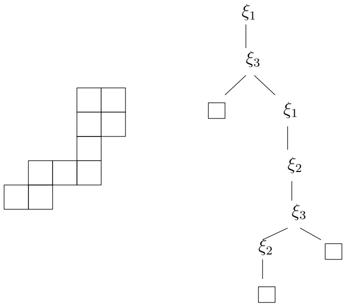


Figure 3.8: A parallelogram polyomino and its derivation tree in the associated grammar.

where each P_j, for $j = 1, \ldots, k$, is a polynomial with coefficients in $\mathbb{N}$ (non-negative integer) and such that the system is contracting (see the paper by Bousquet-Mélou [19] for more details).

Proposition 20. Let $G_{\mathcal{O}}$ be a non-ambiguous object grammar generating a class $\mathcal{O}$ of combinatorial objects and assume that the grammar is well graded by an additive size parameter s of the class. Then the generating function of class $\mathcal{O}$ with respect to the size is $\mathbb{N}$-algebraic and the rules of the grammar immediately directly translate into a system of $\mathbb{N}$-algebraic equations.

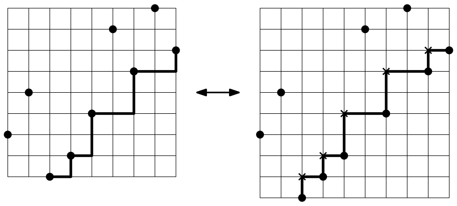
3.2.2 Grammars for Catalan objects

The example of parallelogram polyominoes The classical wasp-waist decomposition for the class $\mathcal{P}^{/\!/}$ of parallelogram polyominoes can be specified as the object grammar $G_{\mathcal{P}^{/\!/}} = \langle \{\mathcal{P}^{/\!/}\}, \{\{\Box\}\}, \{\xi_1, \xi_2, \xi_3\}, \mathcal{P}^{/\!/} \rangle$ graphically described in Figure 3.7, and meaning that a parallelogram polyomino can be:

- A unit cell.
- A parallelogram polyomino whose first column is made up of a unit cell, that is a unit cell with a parallelogram polyomino attached on its right (operation ξ_1).
- A parallelogram polyomino whose first column is made up by at least two cells and such that each column is attached to the column on its left by at least two edges, that is a raised parallelogram polyomino (operation ξ_2).
- A parallelogram polyomino whose first column is made up by at least two cells and having at least one column which is attached to the column on its left by only one edge, that is a raised parallelogram polyomino with a parallelogram polyomino attached on the right of the topmost cell of its rightmost column (operation ξ_3).

Proposition 21. The grammar $G_{\mathcal{P}^{/\!/}}$ generates the class of parallelogram polyominoes in non ambiguous way.

Proof. The proof consists in showing that there is a unique derivation tree associated with a given parallelogram polyomino P, indeed we can uniquely determine the last operation applied to P by looking at its first column:



(a)　(b)

Figure 3.9: (a) A parallel permutation σ with its staircase. (b) The “sliding” operation applied to σ. The crosses represent the positions of the points in S before applying the “sliding” operation.

- If there is only one cell in the first column then we remove it unless it is the unique cell of P (operation ξ_1^{-1}).
- Otherwise we remove a strip of cells on the bottom of P starting from the first column until we disconnect P in two polyominoes(operation ξ_3^{-1}) or we reach the last column (operation ξ_2^{-1}). See an example of derivation tree for a given parallelogram polyomino in Figure 3.8.

$\square$

Let us consider the generating function for the class of parallelogram polyominoes $P^{/\!\!/}(t) = \sum_{P\in\mathcal{P}^{/\!\!/}_n} t^{n(P)}$ according to the semi-perimeter, then we can rewrite the generating function by using the grammar $G_{\mathcal{P}^{/\!\!/}}$:

$$\begin{aligned} P^{/\!\!/}(t) &= \textstyle\sum_{P\in\mathcal{P}^{/\!\!/}_n} t^{n(P)} = t^{n(\square)} + \sum_{P_1\in\mathcal{P}^{/\!\!/}_n} t^{n(\xi_1(P_1))} + \sum_{P_1\in\mathcal{P}^{/\!\!/}_n} t^{n(\xi_2(P_1))} + \sum_{P_1,P_2\in\mathcal{P}^{/\!\!/}_n} t^{n(\xi_3(P_1,P_2))} \\ &= \textstyle t^2 + \sum_{P_1\in\mathcal{P}^{/\!\!/}_n} t^{n(P_1)+1} + \sum_{P_1\in\mathcal{P}^{/\!\!/}_n} t^{n(P_1)+1} + \sum_{P_1,P_2\in\mathcal{P}^{/\!\!/}_n} t^{n(P_1)+n(P_2)} \end{aligned}$$

and directly obtain the following polynomial equation for $P^{/\!\!/}(t)$

$$P^{/\!\!/}(t) = t^2 + 2tP^{/\!\!/}(t) + P^{/\!\!/}(t)^2,$$

showing that $P^{/\!\!/}(t)$ is $\mathbb{N}$-algebraic, and allowing to recover the explicit expression

$$P^{/\!\!/}(t) = \frac{1 - 2t - \sqrt{1-4t}}{2}$$

.

A grammar for parallel permutations In this subsection we will briefly introduce a similar grammar for parallel permutations. Given a parallel permutation σ we define a “sliding” operation on σ, corresponding to the operation of superelevation for parallelogram polyominoes: let us consider the staircase S starting with a horizontal step and taking all the non-double points of the lower path of σ (see Figure 3.9 (a)). Then the “sliding” operation consists in adding respectively a vertical and a horizontal step at the beginning and at the end of S and in adding a point under the leftmost point of S and sliding all the other lower points of the permutation horizontally along S (see Figure 3.9 (b) for an example). Equivalently, the sliding operation replaces the points at the left-upper corners of S with the points at its right-lower corners.

Now let us define a *decomposable* permutation: a permutation σ of $\{1,\ldots,n\}$ is decomposable if there exists two permutations π of $\{1,\ldots,k\}$ and π' of $\{1,\ldots,\ell\}$ with $k+\ell=n$ such that σ is the permutation $\pi_1,\pi_2\ldots\pi_k\,\pi'_1+k,\pi'_2+k\ldots\pi'_\ell+k$. In this case we say that $\sigma=\pi\oplus\pi'$. Then we have the following:

Proposition 22. The “sliding” operation is a bijection between parallel permutations of length n and indecomposable parallel permutations of length $n+1$.

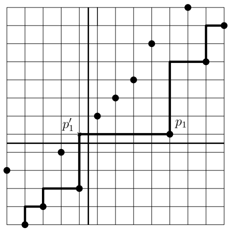


Figure 3.10: A decomposable permutation: it can not be obtained from another permutation by applying the “sliding” operation.

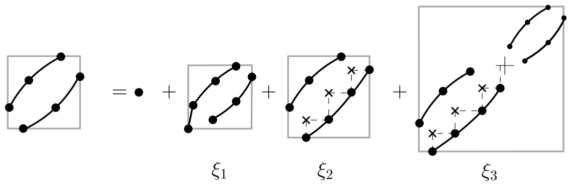


Figure 3.11: An object grammar for parallel permutations.

Sketch of proof. First we observe that the “sliding” operation transforms any parallel permutation σ of length n with k lower points into a parallel permutation σ' of length $n+1$ with $k+1$ lower points: The resulting object is indeed a permutation, since apart from the extra bottom row and right column, the same lines and columns hosting the non-double lower points of σ are used to host the lower points of σ'. In order to prove that σ' is indecomposable we observe that if σ' decomposes as $\sigma'_1 \oplus \sigma'_2$ then both factors contain at least a lower point by construction: the staircase in σ' (see Figure 3.10) then goes from the left lower decomposition quadrant to the right-upper decomposition quadrant through the left-upper quadrant. This means that the lowest point p'_1 of σ'_2 was obtained by sliding from a point p_1 placed in the left-upper decomposition quadrant in σ, *i.e.* from an upper point of σ, that gives a contradiction. Conversely given an indecomposable permutation σ' of size at least 2, we observe that its lower points are the points at the lower right angles of a staircase S whose upper right angles are all strictly below the diagonal: indeed a right-upper corner on or above the diagonal would provide a decomposition of σ'. Then removing the first point of S and sliding backward its other right-lower points yields a permutation σ. By construction this permutation σ gives σ' through the sliding operation. $\square$

Let $G_{\mathcal{S}^{/\!\!/}} = \langle \{\mathcal{S}^{/\!\!/}\}, \{\{\bullet\}\}, \{\xi_1, \xi_2, \xi_3\}, \mathcal{S}^{/\!\!/} \rangle$ be the grammar $G_{\mathcal{S}^{/\!\!/}}$ for parallel permutations graphically described in Figure 3.11, meaning that a parallel permutation can be:

- A point.
- A decomposable parallel permutation starting with a fixed point, that is obtained from a point $\oplus$ a parallel permutation (operation ξ_1).
- An indecomposable parallel permutation, that is obtained by taking a parallel permutation and by applying the sliding operation (operation ξ_2).
- A decomposable parallel permutation which does not start with a fix point, that is an indecomposable parallel permutation $\oplus$ a parallel permutation (operation ξ_3),

then we have the following:

Proposition 23. The grammar $G_{\mathcal{S}^{/\!\!/}}$ generates the class of parallel permutations in a non ambiguous way.

The proof is omitted here, as well as the derivation of the generating function since the grammar is isomorphic to that of parallelogram polyominoes.

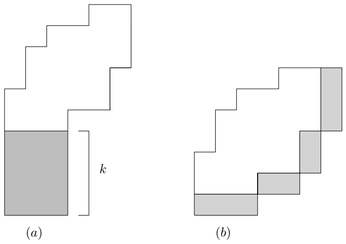


Figure 3.12: (*a*) A raised permutomino according to the "bottom" superelevation. (*b*) A raised permutomino according to the "dual" superelevation.

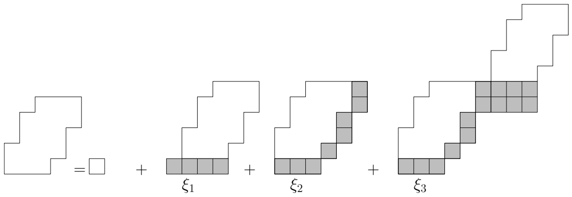


Figure 3.13: An object grammar for parallelogram permutominoes.

A grammar for parallelogram permutominoes Finally, we are going to present an object grammar for parallelogram permutominoes based mainly on two operations. The first one is the "k-bottom" superelevation. Given a permutomino P and $k \in \mathbb{N}^+$, it consists in adding k horizontal strips of unit cells under the whole bottom horizontal segment of the lower path of P (see Figure 3.12 (a)), observe that this operation does not produce a permutomino. The second operation is the "dual" superelevation, it consists in adding cells along the lower boundary of P in order to form a new lower boundary that changes direction in each column and row used by the old lower boundary (see Figure 3.12 (b)).

Let $G_{\mathcal{Q}^{/\!\!/}} = \langle \{\mathcal{Q}^{/\!\!/}\}, \{\{\square\}\}, \{\xi_1, \xi_2, \xi_3\}, \mathcal{Q}^{/\!\!/}_n \rangle$ for the class $\mathcal{Q}^{/\!\!/}$ of parallelogram permutominoes be graphically described in Figure 3.13, meaning that a parallelogram permutomino can be:

- A unit cell.
- A parallelogram permutomino such that the first column does not have any horizontal side of the boundary on its right. In other words, a parallelogram permutomino starting with a unit cell, that is a "1-bottom" raised parallelogram permutomino with a cell attached on the left and at the bottom of the first column (operation ξ_1).
- A parallelogram permutomino such that each column except the last one has at least one horizontal side of the boundary on its right, that is a "dual" raised parallelogram permutomino (operation ξ_2).
- A parallelogram permutomino with at least one column different from the first one and from the last one having no horizontal side of the boundary on its right, that is a "dual" raised parallelogram permutomino P with a "k-bottom" raised parallelogram permutomino attached on its rightmost column (operation ξ_3), where k is the height of the last column of the permutomino P.

Proposition 24. The grammar $G_{\mathcal{Q}^{/\!\!/}}$ generates the class of parallelogram permutominoes in a non ambiguous way.

The three grammars we have described for parallelogram polyominoes, parallel permutations and parallelogram permutominoes are isomorphic and they induce recursive bijections between these three classes of objects: we will give a direct description of these bijections in Section 3.3.1.

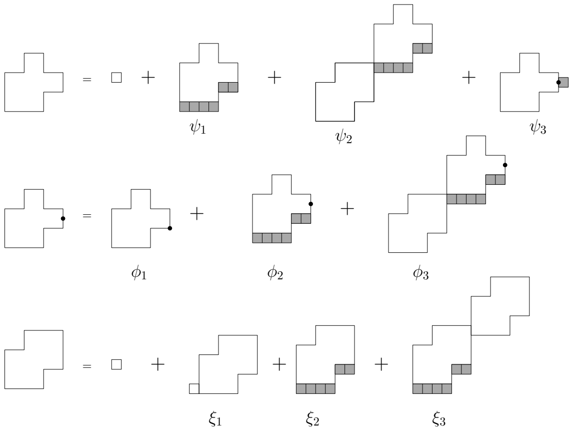


Figure 3.14: An object grammar for directed convex polyominoes.

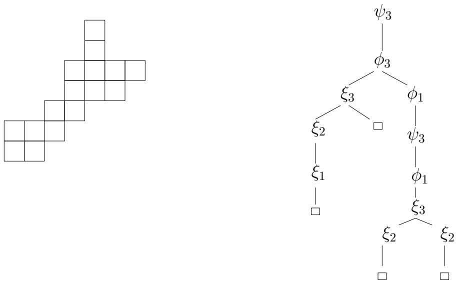


Figure 3.15: A directed convex polyomino and its derivation tree in the associated grammar.

3.2.3 Grammars for central binomial objects

A grammar for directed convex polyominoes We present now an object grammar for the class of directed convex polyominoes, introduced by Dutour in [51], the grammar is graphically represented in Figure 3.14. There are three classes involved in the grammar: the class $\mathcal{P}^{\triangleright}$ of directed convex polyominoes; the class $\mathcal{P}^{\triangleright\bullet}$ of directed convex polyominoes with a marked cell in the rightmost column; the class $\mathcal{P}^{/\!/}$ of parallelogram polyominoes. The grammar is $G_{\mathcal{P}^{\triangleright}} = \langle \{\mathcal{P}^{\triangleright}, \mathcal{P}^{\triangleright\bullet}, \mathcal{P}^{/\!/}\}, \{\{\square\}, \emptyset, \{\square\}\}, \{\psi_1, \psi_2, \psi_3, \phi_1, \phi_2, \phi_3, \xi_1, \xi_2, \xi_3\}, \mathcal{P}^{\triangleright}\rangle$, we have the following cases:

- A directed convex polyomino can be: a unit cell; a directed convex polyomino whose last column is made up by at least two cells and such that each column is attached to the column on its right by at least two edges, that is a raised directed convex polyomino (operation ψ_1); a directed convex polyomino whose last column is made up by at least two cells and having at least one column which is attached to the column on its right by only one edge, that is a parallelogram polyomino with a raised directed convex polyomino attached on the right of the topmost cell of its righmost column (operation ψ_2); a directed convex polyomino whose last column is made up by a unit cell, that is a marked directed convex polyomino with a cell on the right of the marked cell (operation ψ_3).

- A marked directed convex polyomino can be: a marked directed convex polyomino with the mark on the lowest cell of its rightmost column, that is obtained from a directed convex polyomino by putting a mark on the lowest cell of its rightmost column (operation ϕ_1); a marked directed convex polyomino whose mark is not on the lowest cell of its rightmost column and such that each column is attached to the column on its right by at least two edges, that is a marked directed convex polyomino that has been raised (operation ϕ_2); a marked directed convex polyomino whose mark is not on the lowest cell of its rightmost column and having at least one column which is attached to the column on its right by a unit cell, that is a parallelogram polyomino with a marked directed convex polyomino that has been raised and then attached on the right of the topmost cell of its righmost column (operation ϕ_3).

- For the class of parallelogram polyominoes we use the grammar $G_{\mathcal{P}^{/\!/}}$ defined in the previous subsection.

Proposition 25. The grammar $G_{\mathcal{P}^{\triangleright}}$ generates the class of directed convex polyominoes in a non ambiguous way.

Sketch of proof. The proof consists in showing that there is a unique derivation tree associated with a given directed convex polyomino P. Indeed by looking at its last column we can show that there exists a unique operation that can produce P from smaller objects:

- If P is not a single cell and the height of the last column is equal to 1, then we remove the last column and mark the cell on its left (operation ψ_3^{-1}).
- If the height of the last column of P is greater than 1 then we remove a strip of cells on the bottom of P starting from the right until we disconnect P in two polyominoes (operation ψ_2^{-1}) or we reach the end of P (operation ψ_1^{-1}).

We can apply the same reasoning to the case of marked directed convex polyominoes (see an example of derivation tree for a given directed convex polyomino in Figure 3.15). $\square$

Let us consider the generating function of parallelogram, directed convex, and marked directed convex polyominoes according to the semi-perimeter n: $P^{/\!/}(t) = \sum_{P \in \mathcal{P}^{/\!/}_n} t^{n(P)}$, $P^{\triangleright}(t) = \sum_{P \in \mathcal{P}^{\triangleright}_n} t^{n(P)}$, and $P^{\triangleright\bullet}(t) = \sum_{P \in \mathcal{P}^{\triangleright\bullet}_n} t^{n(P)}$. Then from the grammar $G_{\mathcal{P}^{\triangleright}}$ we directly obtain a $\mathbb{N}$-algebraic system of equations:

$$\begin{array}{rcl} P^{\triangleright}(t) & = & t^2 + tP^{\triangleright}(t) + P^{/\!/}(t)P^{\triangleright}(t) + tP^{\triangleright\bullet}(t) \\ P^{\triangleright\bullet}(t) & = & P^{\triangleright}(t) + tP^{\triangleright\bullet}(t) + P^{/\!/}(t)P^{\triangleright\bullet}(t) \\ P^{/\!/}(t) & = & t^2 + 2tP^{/\!/}(t) + P^{/\!/}(t)^2. \end{array} \qquad (3.12)$$

By solving the previous system we obtain the generating function for parallelogram, directed convex, and marked directed convex polyominoes.

$$\begin{array}{rcl} P^{/\!/}(t) & = & \frac{1-2t-\sqrt{1-4t}}{2} \\ P^{\triangleright\bullet}(t) & = & \frac{t}{2\sqrt{1-4t}} - \frac{t}{2} \\ P^{\triangleright}(t) & = & \frac{t^2}{\sqrt{1-4t}}. \end{array} \qquad (3.13)$$

A grammar for triangular permutations Now we are going to introduce a new and, as far as we are aware, the first object grammar for the class of triangular permutations. This grammar, represented in Figure 3.18, is a generalization of the previous one for parallel permutations. In order to describe it we define a variant of the "sliding" operation for triangular permutations. Let us consider a triangular permutation σ and let us take the staircase S' obtained starting with a horizontal step and taking all the non-double points of the lower boundary of σ and ending in the column containing the maximum of the permutation. Then the "sliding to max" (see Figure 3.16) operation consists in: adding respectively a row at the bottom of the permutation and a column on the right of the one containing the maximum; adding respectively a vertical and a horizontal step at the beginning and at the end of S'; adding a point under the leftmost point of S' and sliding horizontally all the other points along the new extended staircase.

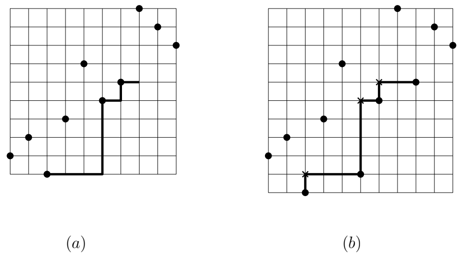


Figure 3.16: (a) A triangular permutation σ with its staircase S'. (b) The "sliding to max" operation applied to σ. The crosses represent the positions of the points in S before applying the "sliding to max" operation.

Proposition 26. The "sliding to max" operation is a bijection between triangular permutations of length n and indecomposable triangular permutations of length $n+1$ and such that the maximum is followed by a point on the lower path.

As for the grammar of directed convex polyominoes, there are three classes involved in this grammar: the class $\mathcal{S}^{\triangleright}$ of triangular permutations; the class $\mathcal{S}^{\triangleright\bullet}$ of triangular permutations with a marked vertical line right after the maximum, or right after one of the points of the sequence of consecutive lower points following the maximum; the class $\mathcal{S}^{/\!/}$ of parallel permutations.

Before giving the grammar we define another operation, called the "insert max and slide" operation. Let us consider a marked triangular permutation σ and let us consider the column C_0 on the left of the mark. Then we have three possibilities:

- the column C_0 is the last column of σ then we insert a new max on the right of σ.
- the column C_0 contains a point of the lower path of the permutation. In this case we consider the staircase S'' obtained starting with a horizontal step in this point and ending with a vertical step in a column C_1 that is on the right of C_0 and taking all the points on the right of C_0 and belonging to the lower boundary followed by the points in the upper boundary in counterclockwise order. Then we insert a new row on the top of the permutation and a new column on the right of C_0 and we add a new max in the column C_0 and also a sequence of horizontal steps at the end of S'' reaching the new inserted column on the right of C_0. Finally we make slide the point that was in column C_0 to the right along S'' so that all the other points in S'' slide too. Observe that this operation will produce a triangular permutation with an upper point different from the max minus one immediately on the right of its maximum.
- the column C_0 contains the maximum of the permutation, then we insert a new maximum on the left hand side of the current one: this operation produces a triangular permutation with the second maximum immediately on the right of the maximum.

The grammar is $G_{\mathcal{S}^{\triangleright}} = \langle \{\mathcal{S}^{\triangleright}, \mathcal{S}^{\triangleright\bullet}, \mathcal{S}^{/\!/}\}, \{\{\bullet\}, \emptyset, \{\bullet\}\}, \{\psi_1, \psi_2, \psi_3, \phi_1, \phi_2, \phi_3, \xi_1, \xi_2, \xi_3\}, \mathcal{S}^{\triangleright}\rangle$, where the operations are described by the following case analysis:

- A triangular permutation can be: a point; an indecomposable triangular permutation whose maximum is followed by a point in the lower path, that is a triangular permutation to which we have applied the "sliding to max" operation, (operation ψ_1); a decomposable triangular permutation whose maximum is followed by a point in the lower path, that is a parallel permutation $\oplus$ a triangular permutation to which we have applied the "sliding to max" operation (operation ψ_2); a triangular permutation whose max is not followed by a point on its lower path, that is a marked triangular permutation to which we have applied the "insert max and slide" operation (operation ψ_3).

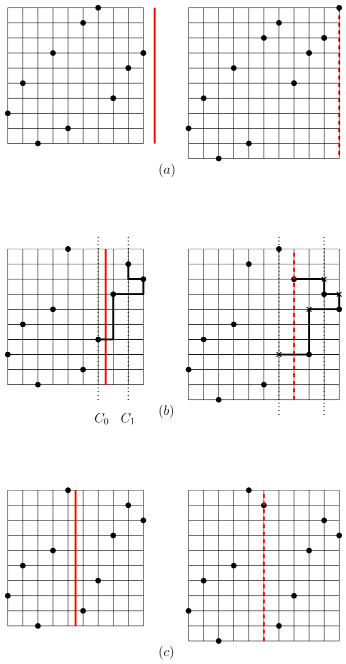


Figure 3.17: The three cases of "insert max and slide" operation: on the left we have marked triangular permutations, where the mark is represented with a red vertical line, while on the right we have the corresponding permutations after applying the "insert max and slide" operation. Observe that the permutations on the right are not marked but nevertheless we choose to represent the original mark with a dashed vertical line.

- A marked triangular permutation can be: a marked triangular permutation with a mark immediately on the right of its maximal point, that is a triangular permutation to which we have added a mark on the right of the maximum (operation ϕ_1); a marked indecomposable triangular permutation whose maximum is followed by a point in the lower path, that is a marked triangular permutation to which we have applied the "sliding to max" operation (operation ϕ_2); a marked decomposable triangular permutation whose maximum is followed by a point in the lower path, that is a parallel permutation $\oplus$ a marked triangular permutation to which we have applied the "sliding to max" operation (operation ϕ_3).
- For the class of parallel permutations we have the grammar $G_{\mathcal{S}^{/\!/}}$ previously defined.

Proposition 27. The grammar $G_{\mathcal{S}^{\triangleright}}$ uniquely generates the class of triangular permutations.

We conclude by observing that the grammar $G_{\mathcal{S}^{\triangleright}}$ is isomorphic to the grammar $G_{\mathcal{P}^{\triangleright}}$ and thus yields the same generating functions.

A grammar for directed convex permutominoes As for triangular permutation we are going to introduce a new and, as far as we are aware, the first object grammar for the class of directed convex permutominoes. But before we need to introduce four operations on this class. The first one is the "k-rightmost" extension. Given a permutomino $P \in \mathcal{Q}^{\triangleright}$ and $k \in \mathbb{N}^+$, it consists in adding k vertical strips of unit cells along the whole rightmost column of P (see Figure 3.19 (a)), observe that this operation does not produce a permutomino. The second operation is the "dual C" superelevation, it consists in considering a special column C of P and in applying the "dual" superelevation defined for the grammar of parallelogram permutominoes to the zone of the permutomino ending with C (see Figure 3.19 (b)). The third one is the "insert and remove" operation, it is applied to a directed convex permutomino whose topmost boundary side is of length at least 2 and with a marked column C' between the ones ending in the topmost

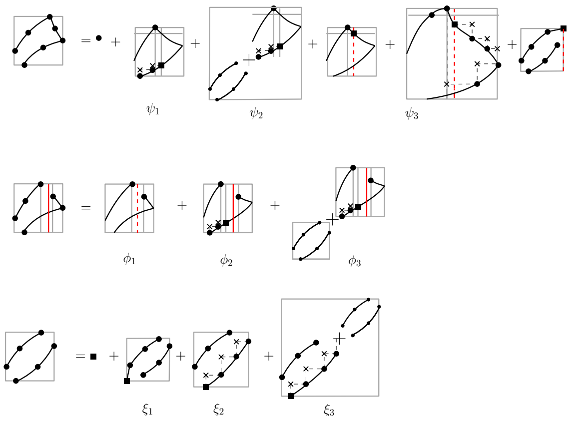


Figure 3.18: A grammar for triangular permutations.

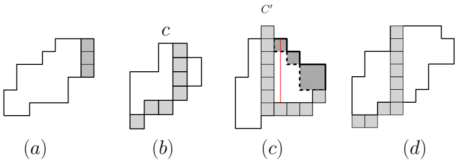


Figure 3.19: Four operations on directed convex permutominoes

side except the leftmost one (see Figure 3.19 (c)): we apply the dual operation from right to left in the zone on the right of C' and we add a cell on the top of the new inserted column on the left of C' (the cells that have been added are represented in grey). Observe that in order to obtain a permutomino we have to remove cells along the right-upper boundary of this new object in order to form a new right-upper boundary that changes direction in each column and row used by P (we remove the darkgrey zone, the new part of the boundary is represented in dashed black). The fourth one is the "dual left" superelevation, it is applied to a directed convex permutomino whose topmost boundary side is of length at least 2 and with a marked column C' between the ones ending in the topmost side except the leftmost one. It consists in applying the "dual" superelevation on the left of the lefmost column ending in the topmost boundary side of P, and in adding some additional cells in order to increase the length of the topmost boundary side by one (see Figure 3.19 (d)).

Given a permutomino P, let us denote by C_0 the rightmost column among the columns ending in the topmost boundary side. In particular if P is a parallelogram permutomino then C_0 is the rightmost column of P.

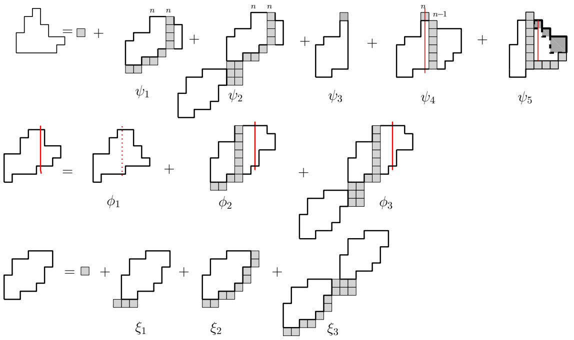


Figure 3.20: A grammar for directed convex permutominoes.

Figure 3.20 presents a grammar for the class $\mathcal{Q}^{\sslash}$ of directed convex permutominoes, $G_{\mathcal{Q}^{\triangleright}} = \langle\{\mathcal{Q}^{\triangleright}\}, \{\{\square\}, \emptyset, \{\square\}\}, \{\psi_1, \psi_2, \psi_3, \phi_1, \phi_2, \phi_3, \xi_1, \xi_2, \xi_3\}, \mathcal{Q}^{\triangleright}_n\rangle$, where the operations are described by the following case analysis:

- A directed convex permutomino can be: a unit cell; a directed convex permutomino whose topmost boundary side is of length at least two and such that each column on the left of C_0 has at least one horizontal side of the boundary on its right, that is a "dual C_0" raised directed convex permutomino (operation ψ_1); a directed convex permutomino whose topmost boundary side is of length at least two and such that at least one column on the left of C_0 has no horizontal side of the boundary on its right, that is a parallelogram permutomino to which we apply the k-rightmost extension and to which we attach a "dual C_0" raised directed convex permutomino whose bottom row is of length k (operation ψ_2); a directed convex permutomino whose topmost boundary side is of length one and which is parallelogram, that is a parallelogram permutomino to which we add a cell on the top of its rightmost column (operation ψ_3); a directed convex permutomino whose topmost boundary side is of length one, which is not parallelogram, and such that the gap between its top boundary edge and the top boundary edge of the column on its right is one, that is a permutomino obtained from a marked directed convex permutomino P with the mark on the leftmost column C_1 among the highest ones to which we add a cell on the top of C_1 and a column on the right and of the same length of C_1 (operation ψ_4); a directed convex permutomino whose topmost boundary side is of length one, which is not parallelogram, and such that the gap between its top boundary edge and the top boundary edge of the column on its right is greater than one, that is a marked directed convex permutomino P with a mark which is not on the leftmost column among the highest ones, to which we apply the "insert and remove" operation (operation ψ_5).
- A marked directed convex permutomino can be: a marked directed convex permutomino whose leftmost column between the ones ending in the topmost boundary side is marked, that is a directed convex permutomino to which we add a mark in the leftmost column among the ones ending in the topmost boundary side (operation ϕ_1); a directed convex permutomino with a marked column C_1 between the ones ending in the topmost boundary side which is not the leftmost one and such that each column on the left of C_1 has at least one horizontal side of the boundary on its right, that is a directed convex permutomino with a marked column C_1 and to which we apply the "dual left" superelevation (operation ϕ_2); a directed convex permutomino with a marked column C_1 between the ones ending in the topmost boundary side which is not the leftmost one and such that at least one column on the left of C_1 has no horizontal side of the boundary on its right, that is a parallelogram permutomino to which we apply the k-rightmost extension and to which we attach a "dual left" raised directed convex permutomino with a marked column C_1 and whose bottom row is of length k (operation ϕ_3).
- For parallelogram permutominoes we have the grammar $G_{\mathcal{Q}^{\sslash}}$ defined before.

Proposition 28. The grammar $G_{\mathcal{Q}^{\triangleright}}$ generates the class of directed convex permutominoes in a non ambiguous way.

$$\begin{aligned}
Q^{\triangleright}(t) &= t + tQ^{\triangleright}(t) + tQ^{\sslash}(t)Q^{\triangleright}(t) + tQ^{\sslash}(t) + tQ^{\triangleright\bullet}(t) \\
Q^{\triangleright\bullet}(t) &= Q^{\triangleright}(t) + tQ^{\triangleright\bullet}(t) + tQ^{\sslash}(t)Q^{\triangleright\bullet}(t) \qquad (3.14) \\
Q^{\sslash}(t) &= t + 2tQ^{\sslash}(t) + tQ^{\sslash}(t)^2.
\end{aligned}$$

The system is again linear in $Q^{\triangleright}(t)$ and $Q^{\triangleright\bullet}(t)$ and in the Catalan generating function:

$$Q^{\triangleright}(t) = \frac{Q^{\sslash}(t)}{1 - Q^{\sslash}(t)} = \frac{1}{2\sqrt{1-4t}} - \frac{1}{2}, \qquad Q^{\triangleright\bullet}(t) = (1 + Q^{\sslash}(t))\frac{Q^{\sslash}(t)}{1 - Q^{\sslash}(t)} = \frac{1}{2t\sqrt{1-4t}} - \frac{1}{2t} + \frac{1}{\sqrt{1-4t}}$$

3.2.4 Some remarks about object grammars

We would like to point out that, unlike what we have seen for parallelogram and directed subclasses, there are no known grammar-like decompositions for convex polyominoes, square permutations and convex permutominoes, but only decompositions involving object grammars and difference between two or more classes: this is natural in view of the difference occuring in the counting formula. However to the best of our knowledge it is not proven that no $\mathbb{N}$-algebraic system of equations exists for the generating function of these objects, although for some combinatorial classes it is possible to exclude the existence of grammar-like decompositions by comparing the asymptotics of the counting sequence to general classification results like the following:

Theorem 2 (Banderier et Drmota [6]). Let $F(t)$ be a $\mathbb{N}$-algebraic generating function and let ρ be its radius of convergence. Then $F(t)$ is singular at ρ, with a singular expansion of the form:

$$F(t) \overset{s}{=} C(1-\frac{t}{\rho})^{\gamma} + o((1-\frac{t}{\rho})^{\gamma})$$

where $\gamma = \frac{k}{2^d}$ for $k \in \{1\} \bigcup \{-1,-2,-3,\dots,\}$ and $d \in \{1,2,3,\dots\}$.

The first order asymptotic of the number of convex polyominoes of size n is compatible with the previous theorem and it does not allow to conclude.

One way to eliminate the possibility to have a $\mathbb{N}$-algebraic decomposition would be to refine the above theorem to give a full asymptotic expansion. We conjecture indeed that the second order in the asymptotic of convex polyominoes is not compatible with $\mathbb{N}$-algebraic series.

We have seen that the object grammars of the previous examples translate directly into systems of $\mathbb{N}$-algebraic equations. In general they thus give a more direct intuition of the algebraicity of the result compared to linear recursive decompositions, which involve a dependance in a catalytic variable and the application of the kernel method. On the other hand, while linear decompositions are usually relatively straightforward since we grow objects little by little, algebraic decompositions often require more efforts of imagination.

An interesting open question concerning these two approches is to relate them systematically. In [42] we have shown that solutions of $\mathbb{N}$-algebraic equations can be naturally generated linearly, more precisely it is possible to give linear constructions according to additive parameters for the class of derivation trees associated with an object grammar.

The inverse question is to decide whether it is possible to obtain systematically an algebraic decomposition for a class $\mathcal{O}$ starting from a linear recursive construction for $\mathcal{O}$. One way to formalize the question is to ask for conditions under which succession rules with algebraic generating function can be translated into $\mathbb{N}$-algebraic systems of equations. A possible approach is to associate to each path `p` starting from the root of the generating tree T_{Ω} of a succession rule Ω and ending in a node at level n, a word `w` made of the labels of the nodes of `p`, and to look for a context free grammar generating these words: In [32] we applied this approach to the succession rule Ω for convex polyominoes of Subsection 3.1.4, first to rederive an $\mathbb{N}$-algebraic decomposition for parallel and directed polyominoes, and then to obtain an algebraic decomposition involving a difference of the type discussed in Section 3.4 for the full class of convex polyominoes.

3.3 Direct bijections

The bijective approach consists in finding a bijective mapping between two combinatorial classes which preserves the size, thus proving that they are counted by the same numbers. On one hand, looking for a bijection is particularly tempting when a new class of combinatorial structures is conjectured to be counted by the same numbers as a class for which the enumeration is already known. On the other hand this approach is also useful to explain known nice formulas in a simpler manner and to get a better understanding of the relation between involved classes. In our case, the bijective approach will provide us with a useful ingredient, the horizontal/vertical code of a

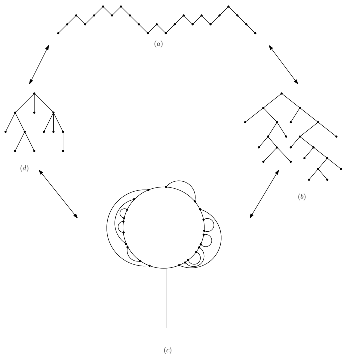


Figure 3.21: Bijections involving some classical Catalan structures: (a) a Dyck Path of length 2×11; (b) a complete binary tree with 11 internal nodes; (c) a flower representation of an arch diagram of size 2×11; (d) a plane tree with $11 + 1$ vertices.

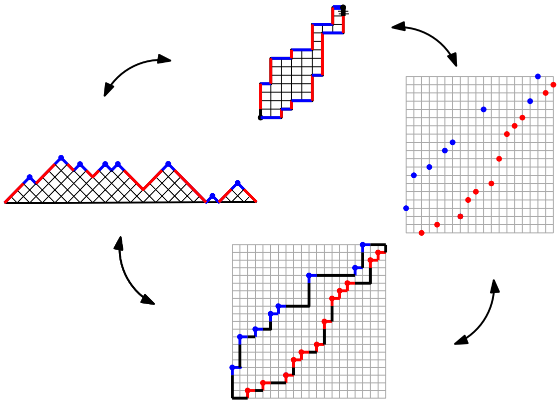


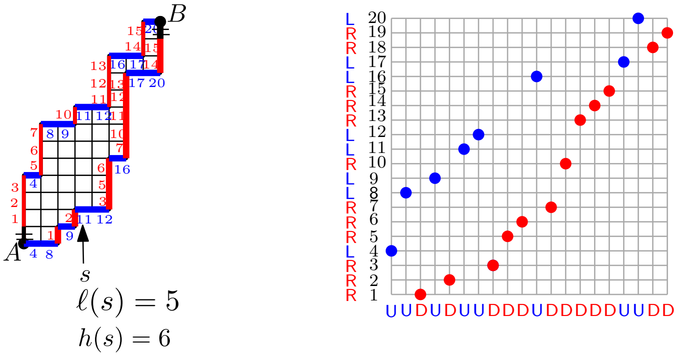
Figure 3.22: A Catalan garden involving parallelogram polyominoes, parallel permutations, parallelogram polyominoes and Dyck paths.

permutation, which we will reuse in the next section to deal with square permutations in general. It will also allow us to understand the $\frac{1}{2}$ factor between directed convex polyominoes and triangular permutations on the one hand, and directed convex permutominoes on the other hand.

3.3.1 Bijections and the Catalan garden

The most classical structures in enumerative combinatorics are the ones counted by Catalan's number, which are more than two hundred, listed by Stanley in [82], and for which there exists a large number of bijections. We illustrate in Figure 3.21 four well known Catalan structures, Dyck paths, planar trees, binary trees, and arch diagrams, related by classical bijections, see [82] for more details. These structures, together with the bijections, were refered to by Viennot as the *Catalan garden*.

The aim of part of this section is to indicate how parallelogram polyominoes, parallel permutations and parallelogram permutominoes enter in the garden. In Figure 3.22 there is a overview of an alternative Catalan garden containing these three classes and Dyck paths, that we are going to describe in details. Apart from the bijection between parallelogram permutominoes and Dyck path, which appears in [55], the presentation of these bijections is ours, although equivalent



(a) (b)

Figure 3.23: (a) A parallelogram polyomino P; (b) its associated parallel permutation $\psi(P) =$ 4 8 1 9 2 11 12 3 5 6 16 7 10 13 14 15 17 20 18 19.

bijections most certainly appear in the huge litterature on the topic ....

A bijection between parallelogram polyominoes and parallel permutations Given a parallelogram polyomino P with semi-perimeter n, we construct its image $\psi(P)$ by assigning labels to the steps of its lower boundary and reading them from left to right, except the last step (see Figure 3.23):

- We label each (blue) horizontal step s of the lower path of P with the integer $\ell(s) + h(s)$, where $h(s)$ denote the height of the column above s in P, and $\ell(s)$ the number of steps before s on the lower path of P. Equivalently, $\ell(s) + h(s)$ is the number of steps before s', the horizontal step on top of the column above s. Observe therefore that the labels of blue steps form an increasing sequence from left to right.
- We label each (red) vertical step from bottom to top in increasing order using the integers between 1 and $n - 1$ that have not been used at Step 1.
- Finally we let $\sigma(i)$ be the label of the i-th step in the lower path of P for $i = 1, \ldots, n - 1$, and take $\psi(P)$ to be the resulting permutation σ.

Conversely the points of a parallel permutation σ define two words v and w (represented on the left and on the bottom of the permutation in Figure 3.23):

- the ith letter of v is L or R depending if the point in the ith row of σ belongs to the left-upper path of σ or not;
- the ith letter of w is U or D depending if the point in the ith column of σ belongs to the left-upper path of σ or not.

Then let $\phi(\sigma)$ be the polyomino whose upper and lower boundaries are respectively encoded from left to right by v (L for horizontal steps, R for vertical ones) and by w (U for horizontal and D for vertical steps).

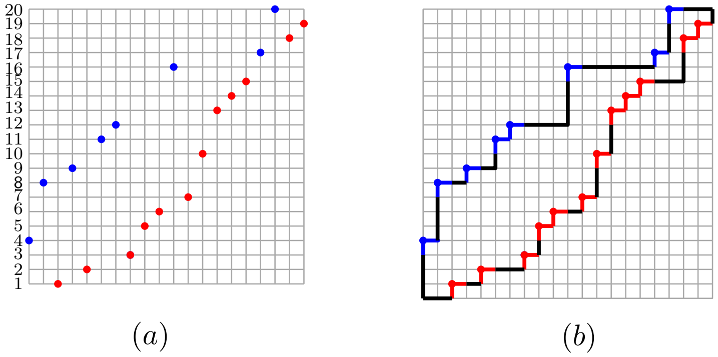


Figure 3.24: (a) A parallel permutation P; (b) its associated parallelogram permutomino.

Proposition 29. The map ψ and ϕ are inverse bijections between parallelogram polyominoes of semi-perimeter n and parallel permutations of length $n-1$, $n \geq 2$.

Sketch of proof. As already observed, the blue labels are increasing from left to right by construction and so are the red labels, which are all different from the blue ones. Hence we obtain a permutation of size $n-1$ which can be decomposed into two increasing sequence, *ie* a parallel permutation.

Observe then that if we label the colored steps of the upper boundary with integers 1,. . . , $n-1$ from left to right, then the label of the ith horizontal step of the lower boundary is the label of the ith horizontal step of the upper boundary, while the label of the ith vertical step of the lower boundary is the label of the ith colored vertical step of the upper boundary. This implies that the red points cannot be upper left points in the corresponding permutation. More precisely the red points are right-lower points which are not left-upper points of σ.

□

We will give later in Subsection 3.4.2 a generalization of the encoding by v and w which will lead us to recover the enumeration of square permutations according to their length and to explain the difference between the rational part and the algebraic part in the formula.

A bijection between parallel permutations and parallelogram permutominoes Given a parallel permutation the bijection ψ consists in putting a north-west corner in each point of the parallel permutation as in Figure 3.24, then there is a unique way to complete the permutomino starting from these corners and the fact that the permutation is parallel easily implies that we obtain a parallelogram permutomino. The inverse bijection consists in replacing each north-west corner with a point of the permutation.

Then we have the following proposition:

Proposition 30. The map ψ is a bijection between parallel permutations of length n and parallelogram permutominoes of size $n+1$, $n \geq 1$.

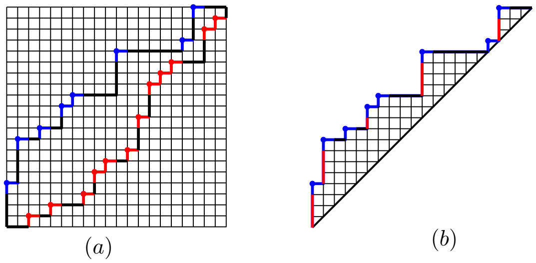


Figure 3.25: (a) A parallelogram permutomino; (b) its associated Dyck path.

A bijection between parallelogram permutominoes and Dyck paths Let us consider a parallelogram permutomino P, since it is parallel it is described by two paths, the upper and lower boundary, meeting each other only at their beginning and at their end points s and t, both lying on the main diagonal.

Lemma 3. The upper path never crosses the main diagonal, and symmetrically, neither does the lower path.

As a consequence of the previous lemma there is a trivial bijection ψ between parallelogram permutominoes and Dyck paths, consisting in taking the upper path of the permutomino P as in Figure 3.25. On the other hand, given a Dyck path, it corresponds to the upper path of a parallelogram permutomino and there is a unique way to recover its lower path by taking a turn in each free line or column on the way back.

Proposition 31. The application ψ that maps a parallelogram permutomino to its upper path is a bijection between parallelogram permutominoes of size n, $n \geq 2$, and Dyck paths of size $n-1$.

A bijection between Dyck paths and parallelogram polyominoes We give here a maybe less classical bijection ψ between Dyck paths of semi-perimeter n and parallelogram polyominoes of semi-perimeter $n+1$, arising by consecutive applications of the previous bijections, that can be described in a direct way as follows (see Figure 3.26). Let D be a Dyck path, then

- We color the peaks in blue and we color the remaining steps in red.
- Starting from the first point A we build a new path p_1 by selecting the red vertical steps and the blue horizontal steps of the Dyck path and by adding a vertical step at the beginning of this path.
- Starting from the first point A we build a new path p_2 by selecting the blue vertical steps and the red horizontal steps of the Dyck path, by reflecting the resulting path with respect to the main diagonal and by adding a vertical step at the end of it.
- We obtain the parallelogram polyomino by putting together these two paths.

The inverse bijection consists in taking the upper and lower path of a parallelogram polyomino, by removing respectively the first step and the last step of these two paths, by reflecting the second path with respect to the main diagonal and by separating in each path the vertical and the horizontal steps in order to recover the corresponding Dyck path.

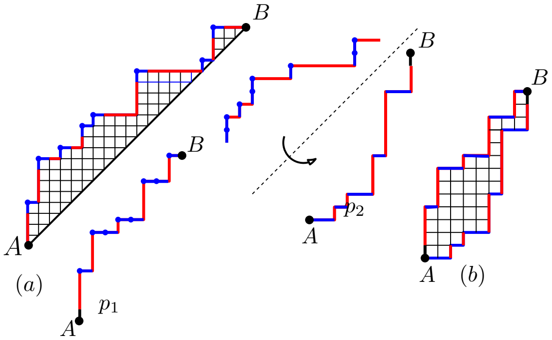


Figure 3.26: (a) A Dyck path; (b) Its associated polyomino.

Proposition 32. The application ψ is a bijection between Dyck paths of length n, $n \geq 1$, and parallelogram polyominoes of semi-perimeter $n+1$.

Refinements of the previous bijections We would like to point out that the following parameters are preserved by the previous bijections: the number of (blue) horizontal steps in the upper boundary and the number of (red) vertical steps in the lower boundary of a parallelogram polyomino; the number of (blue) points in the upper path and the number of (red) points in the lower path of a parallel permutation; the number of (blue) north-west corners in the upper boundary and the number of (red) north-west corner in the lower boundary of a parallelogram permutomino; the number of (blue) north steps preceeding a peak in a Dyck path and the number of the other (red) north steps of the Dyck path.

All these structures are counted according to the various parameters by Naranaya numbers.

3.3.2 Bijections for central binomial structures

In this subsection we will present some generalizations of the previous bijections. More precisely we will present three bijections involving the classes of directed convex polyominoes, triangular permutations, and left Dyck factors with an even number of steps, that are classes counted by the central binomial coefficient. Permutominoes are temporarily left aside because directed convex permutominoes are only half as many as directed convex polyominoes and triangular permutations. The explanation of this property will only be given in the next section. In Figure 3.27 there is a overview of these new gardens that we are going to explain in details. We would like to point out that, while there are already bijections between the class of directed convex polyominoes and left Dyck factors in the literature [33], the following three bijections are ours and, as far as we are aware, there are no other known bijections between the classes of directed convex polyominoes and triangular permutations and between the classes of triangular permutations and left Dyck factors.

A bijection between directed convex polyominoes and triangular permutations For P a directed convex polyomino, let us denote by C_0 the rigthmost column ending in the topmost boundary side (simply the last column if P is parallelogram), we will denote as the *parallelogram zone* all the columns of P preceeding C_0 and as the *directed convex zone* all the columns of P following C_0 (C_0 included). Moreover, we let p_0 and p_1 denote the leftmost and rightmost point on the lower boundary of P and we let α denote the lower boundary between p_0 and p_1. We let p_2 denote the rightmost point of the topmost boundary side of P, and we let β (resp. γ) denote

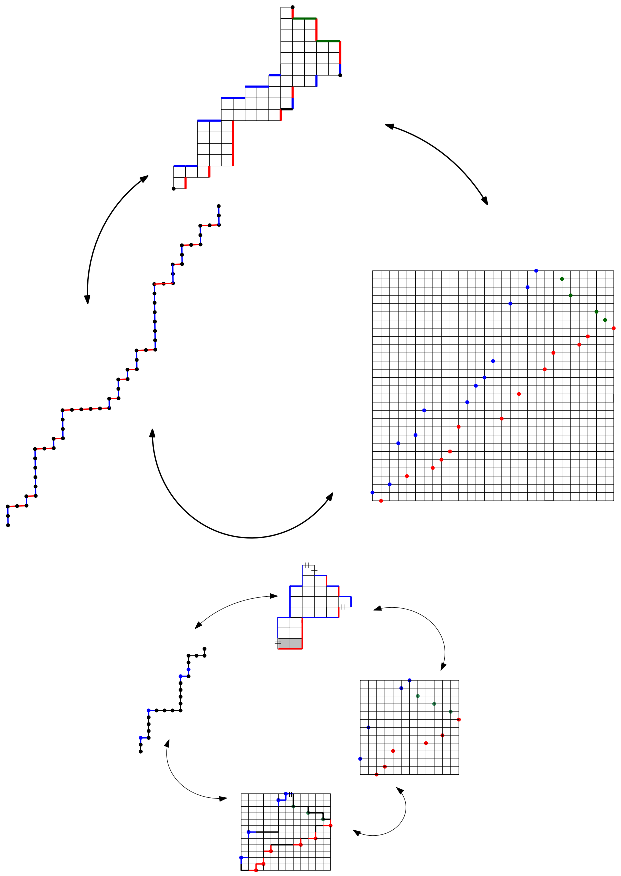


Figure 3.27: Two gardens: the first one is constituted of directed convex polyominoes, triangular permutations, and left Dyck factors with an even number of steps; the second one is constituted of directed convex polyominoes with a raised parallelogram zone, directed convex permutominoes, indecomposable triangular permutations, left Dyck factors with an odd number of steps.

the right-upper boundary between p_1 and p_2 (resp. the left upper boundary between p_0 and p_2). Moreover, we let s_0 denote the step at the bottom of C_0, h denote the height of C_0, and δ denote the boundary between s_0 and p_1.

The bijection ψ' described hereafter, which we introduced in [49], is a generalization of the bijection ψ between parallelogram polyominoes and parallel permutations. We start by giving a slightly different reformulation of ψ, consisting in moving the blue label of the upper horizontal step of the last column C_0 of a parallelogram polyomino P to its lowest of the rightmost vertical steps. This is the version of ψ that we are going to generalize. Basically the idea is the following: given a directed convex polyomino P of semi-perimeter n, we keep the same encoding of the bijection ψ in the parallelogram zone of P while we change encoding starting from C_0, by labelling in blue the vertical steps of the corners in the lower boundary after C_0 and by increasing the labels between two consecutive corners by the number of horizontal steps between them. All the remaining vertical steps are in red and represents the increasing sequence of the lower path of the permutation associated with P.

More precisely the image $\psi'(P)$ of P is obtained as follows (see Figure 3.28):

1. We color in blue and label the horizontal steps preceeding C_0 in γ: each horizontal step s gets the label $\ell(s) + h(s)$.

2. For each corner c' in δ , we color in blue the vertical step s' following c', and we label it with

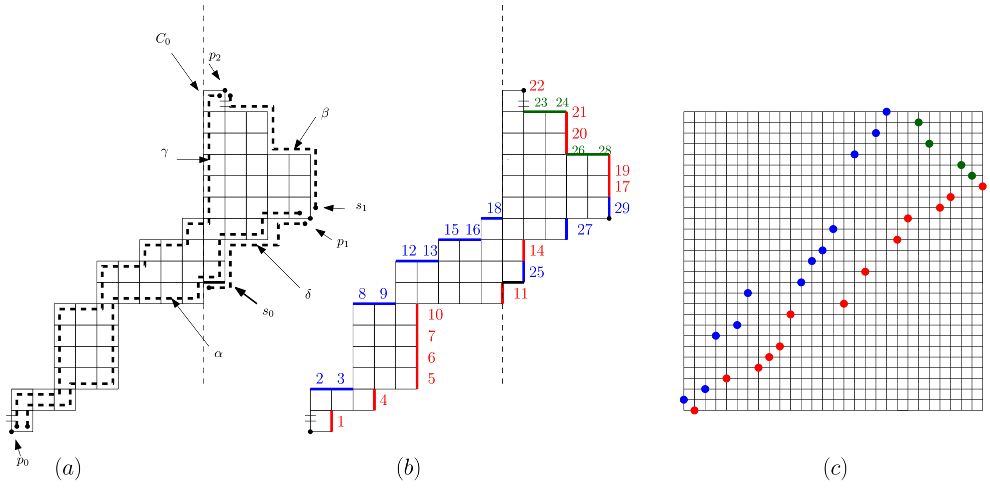


Figure 3.28: (a) A directed convex polyomino with dashed lines indicating with the paths α, β, γ and δ. (b) The colored steps and their labelling. (c) The associated triangular permutation through the bijection ψ': 2 1 3 8 4 9 12 5 6 7 10 13 15 16 18 11 25 14 27 29 17 19 28 26 20 21 24 23 22.

$\ell(s_0) + h(s_0) + x(s')$, where $x(s')$ denote the number of horizontal steps between s_0 and s' on the lower boundary.

3. We color in red the vertical steps of α and β that are not yet been colored, except for the topmost, and we label these steps first from left to right in α and then from right to left in β, using labels between 1 and $n-1$ that are not yet been used.

4. We color in green the horizontal steps of β and label them from right to left and from $n-1$ to 1 with labels that are not yet been used.

5. Finally we write the triangular permutation associated with P by reading from left to right the labels in the parallelogram zone of P as in the parallelogram case, and then, once in the directed zone, by reading from left to right the labels in δ and from right to left the labels in β.

Observe that the vertical step s_1 following p_1 is colored in blue and labeled $n-1$, where n is the semi-perimeter of P: indeed its label is $\ell(s_0) + h(s_0) + x(s_1)$, where $\ell(s_0) + h(s_0)$ equals the number of rows of P plus the number of columns on the left of C_0, and $x(s_1)$ corresponds to the number of columns on the right of C_0.

Proposition 33. The map ψ' is a bijection between directed convex polyominoes of semi-perimeter n and triangular permutations of length $n-1$. In particular, the blue, red and green labels used in the construction respectively give the points of the left-upper, left-lower and right-upper paths of the resulting triangular permutation.

Proof. We briefly sketch the proof. Let P be a directed convex polyomino with semi-perimeter n. Then $\sigma = \psi'(P)$ is a permutation of length $n-1$. Indeed observe that we labeled each vertical step on the right of each row of P and each horizontal step on the top of each column except for C_0 (see Figure 3.28). Therefore we labeled exactly $n-1$ steps, using labels between 1 and $n-1$. By construction we also know that the labels are all different: indeed the sequence of blue labels is strictly increasing, the red labels are inserted in increasing order between those who are not yet used, while the green ones are written in decreasing order between the remaining labels.

Why is the resulting permutation σ triangular? By construction we can prove that the blue sequence always remains above the red one and so does the red sequence.

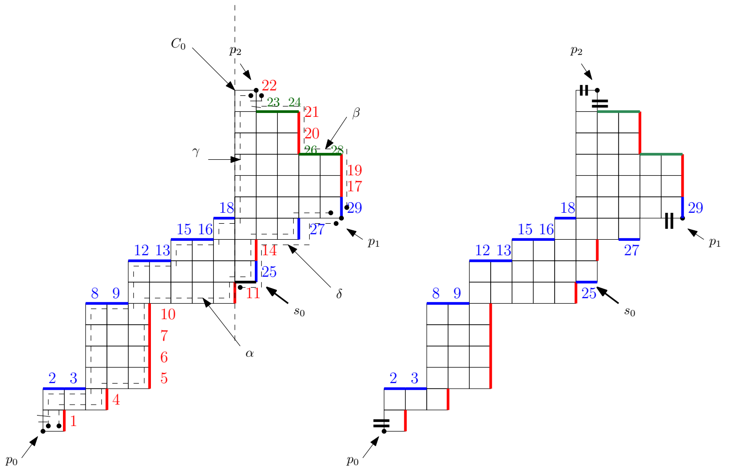


Figure 3.29: Two different representations of the same encoding.

Now, let σ be a triangular permutation of length $n-1$, we want to recover the polyomino $P = \psi'^{-1}(\sigma)$. In order to do it we first consider in σ all the points on the right of the point with ordinate $n-1$. These points determine the shape of the path β in P: points in the right-upper path give horizontal steps, while the others give vertical steps, then we add a extra horizontal step at the end of the path. The points of σ that are in the left-upper path represent either vertical or horizontal steps, depending if the difference between $n-1$ and their ordinate is less or equal to the number of points in the right-upper path of σ. Finally the points on the lower path and on the left of $n-1$ give the vertical steps on the lower boundary of σ (see Figure 3.28). $\square$

The previous theorem can be refined to obtain the following:

Proposition 34. The joint distribution of the parameters (*length* of β, *number of horizontal steps* of β) on directed convex polyominoes is the same as the joint distribution of the parameters (n *minus the position of the maximum*, *number of points in the right-upper path*) on triangular permutations.

A bijection between directed convex polyominoes and left Dyck factors with an even number of steps We are going to present a bijection between directed convex polyominoes of perimeter $2n$ and left Dyck factors with $2n-4$ steps. The bijection is based on the same encoding we used for the previous bijection, even if represented in a slightly different way. Indeed, in the context of the previous bijection we labeled the vertical step of each corner (so that the reader could easily check that there was exactly one labeled vertical step for each row) while in the context of the following bijection it will be more convenient for us to label the horizontal step of each corner rather than the vertical one, except for the last corner. Moreover, we do not need to label the red vertical steps but only to color them. In Figure 3.29 we give the encoding of the directed convex polyomino of Figure 3.28 (a) according to this new representation. Now, we want to keep this encoding but color more than $n-1$ steps: in order to obtain a left Dyck factor of length $2n-4$, we want to color $2n-4$ steps of the boundary of P. Observe that in a non-empty directed convex polyomino P there are necessarily four fixed steps: the left vertical step at the bottom of the first column of P; the horizontal step at the top of the column C_0; the right vertical step at the top of the column C_0; the horizontal step at the bottom of the last column of P. These steps do not carry any information through the bijection and they are the ones that remain uncolored. We color all the others according to the following rule (see Figure 3.30 (a)):

- We color in blue all the vertical steps in γ and all the horizontal steps in δ.

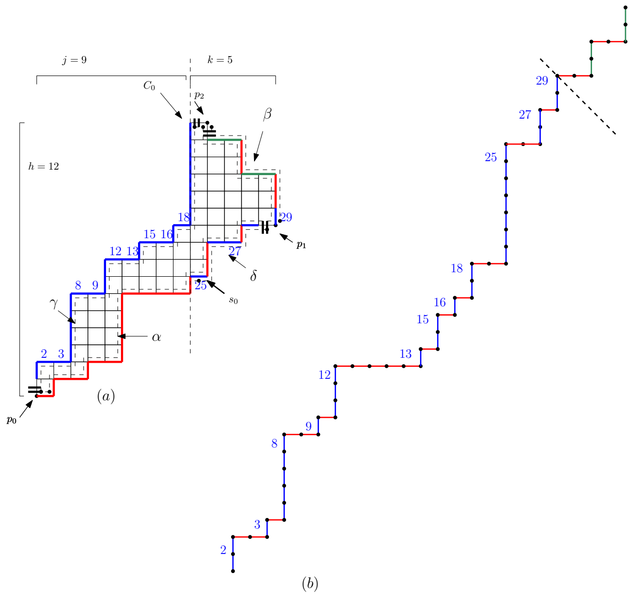


Figure 3.30: (a) A directed convex polyomino of semi-perimeter $n = 30$; (b) The associated left Dyck factor of length $2n - 4 = 56$ through the bijection ψ' .

- We color in red all the remaining steps.

Observe that the label written in each horizontal step corresponds to the number of blue steps on the left of it and in the paths γ and δ. Now, given a directed convex polyomino P with the previous encoding, we denote by ℓ_i, $i \geq 1$, the label of the i-th horizontal step from left to right and with m_i the number of red steps in the i-th column containing the i-th horizontal step and belonging to α. Then we define the map ψ' as follows: we read the labels ℓ_i from left to right and we construct the corresponding left Dyck factor by writing $\ell_i - \ell_{i-1}$ north steps, with $\ell_0 = 0$ followed by m_i east steps and so on. Finally, for each red step in β we write a south step while for each green step in β we write a north step (see Figure 3.30 (b)).

Proposition 35. The map ψ' is a bijection between directed convex polyominoes of perimeter $2n$ and left Dyck factors with $2n - 4$ steps.

Proof. By construction we coloured exactly $2n-4$ steps in P so that the length of the corresponding left Dyck factor is exactly $2n - 4$. Now, observe that the number of blue horizontal steps on the left of a column in the parallelogram zone is the same as the number of red horizontal steps, while the number of blue vertical steps is always greater or equal than the number of red vertical steps (see Figure 3.30(a)). Moreover, in the directed zone we have that the number of blue steps preceeding each column is always greater than the number of red steps. This translate in the fact that the corresponding path is a left Dyck factor. Let j be the width of the parallelogram zone of P, let k be the width of the path β, with $k \geq 1$, let h be the height of P, then the difference between the number of blue plus green steps and the number of red steps of P is given by $(h - 1 + j + 2k - 1) - (h - 2 + j) = 2k$ if P is not a parallelogram polyomino whose rightmost column is a single cell, 0 otherwise. This translate in the fact that the associated left Dyck factor ends at height $2k$ or 0. In particular, if P is parallelogram the corresponding left Dyck factor ends at height 2, since $k = 1$, except for the parallelogram polyominoes ending with a single cell, corresponding to Dyck paths.

Now, given a left Dyck factor of length $2n - 4$ we want to recover the corresponding directed convex polyomino. In order to do it we look for the point after the first $n - 1$ north steps and we cut the factor into two subpaths. The subpath on the right gives the path β in the polyomino P,

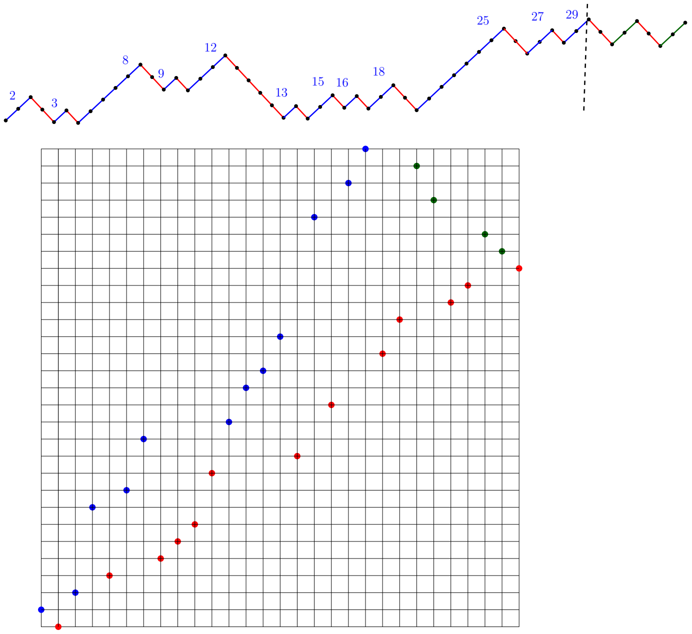


Figure 3.31: (a) A left Dyck factor of length 56; (b) The associated triangular permutation through the bijection ψ': 2 1 3 8 4 9 12 5 6 7 10 13 15 16 18 11 25 14 27 29 17 19 28 26 20 21 24 23 22. The points in green correspond to the ascending steps in the right subpath.

in particular the number of north steps into this subpath gives $k-1$. The subpath on the left gives the rest of the polyomino, in particular we can distinguish between north steps corresponding to blue steps in γ and north steps corresponding to blue steps in δ by knowing the width k of β. Indeed north steps having less than $n-k-1$ north steps on their left correspond to blue steps in γ while the others correspond to blue steps in δ. □

We would like to point out that ψ' is a generalization of the bijection between parallelogram polyominoes and Dyck paths that we presented in Subsection 3.3.1.

A bijection between left Dyck factors with an even number of steps and triangular permutations In this subsection we give a bijection between left Dyck factors of length $2n$ and triangular permutations of length $n+1$. Let us take a left Dyck factor F of length $2n$ and let us cut it into two subpaths by looking for the point after the first $n+1$ north step. Then the bijection ψ' consists in reading the peaks and the sequences of red steps in F from left to right and constructs the associated permutation σ from left to right as follows (see Figure 3.31 for an example):

- if we read a peak at height h in the left subpath of F then we insert a point in σ at height h.
- if we read an ascending (green) step in the right subpath of F then we insert a point in σ in the highest row that does not contain a point.
- if we read a sequence of red steps of length ℓ then we insert $\ell-1$ points in σ, each one in the lowest row that does not contain a point.

Finally, we insert a point in the rightmost column and in the unique empty row.

Proposition 36. The map ψ' is a bijection between left Dyck factors with $2n$ steps and triangular permutations of length $n+1$.

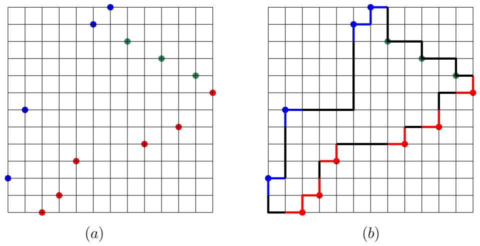


Figure 3.32: (a) An indecomposable triangular permutation 3 7 1 2 4 12 13 11 5 10 6 9 8; (b) The associated directed convex permutomino.

Proof. Let us sketch the proof. We denote by i the number of peaks and by j the number of descending steps in the left subpath of F, then we have that the length of the left subpath is $n+1+j$, while the length of the right subpath is $2n-n-1-j=n-j-1$. Now, observe that the left subpath produces exactly $i+1+j-i=j+1$ points in the permutation σ while the right subpath produces $n-j-1$ points of σ. Then, the length of the permutation σ is $n+1$. Moreover the points of σ are all different: indeed the sequence of blue points is strictly increasing, the red points are inserted in the empty rows from bottom to top, while the green ones are written in the empty rows from top to bottom. Moreover, by construction, we can prove that the blue sequence always remains above the red one and so does the green sequence.

The inverse bijection is obtained by looking respectively at the points on the left and of the right of the maximum in the triangular permutation. By reading from left to right the triangular permutation we have the following: the ordinate of the blue points on the left of the maximum gives the height of the peaks in the left Dyck factor, while $i \geq 0$ consecutive red points on the left of the maximum correspond to $i+1$ descending steps in the left Dyck factor; the sequence of red and green points on the right of the maximum describes the remaining left Dyck factor, in particular red points correspond to descending steps while green points corresponds to ascending steps. □

3.3.3 Bijections for half central binomial coefficients

Here we will finally present four bijections involving directed convex permutominoes, indecomposable triangular permutations, directed convex polyominoes whose parallelogram zone is inflated, and left Dyck factors with an odd number of steps, these classes are counted by half of the central binomial coefficients.

A bijection between indecomposable triangular permutations and directed convex permutominoes We will briefly describe a bijection between indecomposable triangular permutations of length n and directed convex permutominoes of size n. Given an indecomposable triangular permutation of size n, we define the map ψ'' as follows:

- We put a north-west corner in each point of its left-upper path.
- We put a south-east corner in each point of its lower path.

Then, starting from these corners, there is a unique way to recover the directed convex permutomino (see Figure 3.32).

Proposition 37. The map ψ'' is a bijection indecomposable triangular permutations of length n and directed convex permutominoes of size n.

Proof. Let us sketch the proof. Since we put a north-west corner in the leftmost point of the permutation and a south-east corner in its rightmost point, then the bounding box of the permutomino has size $n-1$, that is the size of the permutomino is n. By construction we can show that

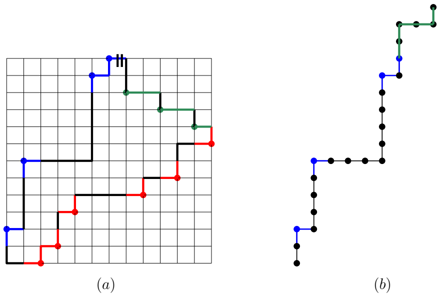


Figure 3.33: (a) A directed convex permutomino. (b) The associated left Dyck factor of odd length.

the boundary of the permutomino does not cross itself. Indeed, if it was not the case, the triangular permutation would bedecomposable. The inverse bijection can be obtained by writing a point of the permutation in each north-west corner of the left-upper boundary and in each south-east corner of the lower boundary of the permutomino. ☐

We would like to point out that this bijection, up to vertical symmetry, corresponds to a restriction of the one introduced by Bernini in [11] and discussed in Subsection 2.3.3 between convex permutominoes and colored co-indecomposable square permutations.

A bijection between directed convex permutominoes and left Dyck factors with an odd number of steps In this subsection we give a bijection between directed convex permutominoes of size n and left Dyck factors of size $2n-3$. The underlying idea is the same as in the trivial bijection between parallelogram permutominoes and Dyck paths, consisting in keeping only the upper boundary of the permutomino. However, for a directed convex permutomino we only need to keep the left-upper boundary and the length of each horizontal sequence in the right-upper boundary. More precisely, let us take a directed convex permutomino P and let us denote by ℓ_i, $i \geq 0$, the length of the i-th sequence of horizontal steps in the right-upper boundary from left to right. Then the map ψ'' consists in taking the left-upper boundary of P except for the last step and in attaching on its right a path α made up as follows: it starts with ℓ_1 north steps and alternates ℓ_{2i} north steps with $\ell_{2(i+1)}$ east steps (see Figure 3.33).

Proposition 38. The map ψ'' is a bijection between directed convex permutominoes of size n and left Dyck factors with $2n-3$ steps.

Proof. Observe that the number of vertical steps in the left-upper boundary of the permutomino P is equal to $n-1$, where n is the size of the permutomino. Let us denote by k the number of horizontal steps in the left-upper boundary of P, then we have that $k+\sum_{i\geq 0}\ell_i = n-1$, that is the length of the path associated with P through ψ'' is equal to $2(n-1)-1 = 2n-3$. Moreover, since the number of vertical steps is greater or equal (if P is parallelogram) to the number of horizontal steps in the left-upper boundary of P, and since the map ψ'' transforms some of the horizontal steps of the right-upper boundary in north steps, we obtain a left Dyck factor. In particular, its important that the subpath α starts with a sequence of vertical steps. If it was not the case and P was parallelogram, the path obtained from P by ψ'' would descend under the main diagonal.

Now, given a left Dyck factor with $2n-3$ steps, we look for the $(n+1)$-th north step from the left if it exists (if it does not exist it means that adding a last horizontal step we recover a Dyck path and we can directly reconstruct a parallelogram permutomino) and we cut the factor into two subpaths on the left of this step. Then first subpath will give the left-upper boundary of the permutomino P, while the second subpath will give the lengths of the horizontal steps in the right-upper boundary of P. We would like to point out that, as mentioned in the paper by Fanti *et al.* [55], this bijection has been suggested by Elizalde in a personal communication. ☐

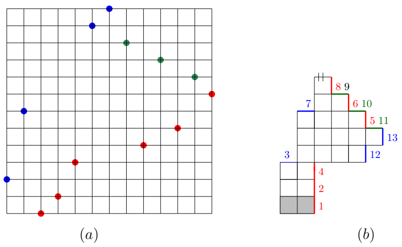


Figure 3.34: (a) An indecomposable triangular permutation 3 7 1 2 4 12 13 11 5 10 6 9 8; (b) The associated directed convex polyomino which is raised in its parallelogram zone.

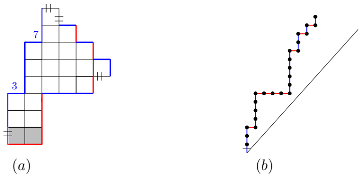


Figure 3.35: (a) A directed convex polyomino which is raised in its parallelogram zone. (b) Its corresponding left Dyck factor with odd length.

Bijections between indecomposable triangular permutations, inflated directed convex polyominoes and left Dyck factors with an odd number of steps In order to bijectively connect all the classes of our second garden and to highlight the connections with our first garden (whose classes are counted by the central binomial coefficient), we need two more bijections: one involving indecomposable triangular permutations and directed convex polyominoes that are raised in their parallelogram zone and the other one involving this class of polyominoes and left Dyck factors with odd length. The first bijection can be derived as a restriction of the bijection ψ' presented in Subsection 3.3.2, between triangular permutations and directed convex polyominoes. Indeed, it is not difficult to see that an indecomposable triangular permutation of length n corresponds to a directed convex polyomino P of semi-perimeter $n+1$ whose columns, from the first one to C_0, are raised (see Figure 3.34), in other words, each column in the parallelogram zone is attached to the one on its right with at least two edges. Finally, we can derive the second bijection from the one between directed convex polyominoes and left Dyck factors with even length, presented in Subsection 3.3.2. Indeed the condition that each column in the parallel zone of a directed convex polyomino P is attached to the next one with at least two edges, translates in the fact that the left Dyck factor never touches the main diagonal. Then we can remove the first step of the left Dyck factor, that is necessarily a north step, and obtain a left Dyck factor with odd length (see Figure 3.35).

3.3.4 Remarks about the bijective approach

While bijections in the Catalan garden are particularly simple the bijections for central binomial coefficients are somewhat more involved and were suprisingly difficult to find. However in the end these bijections have provided us with a better understanding for the $\frac{1}{2}$-factor between directed convex polyominoes and triangular permutations on the one hand, and directed convex permutominoes on the other hand. Indeed we have seen that the first two classes are in bijection with the class of left Dyck factors with even length while the third class is in bijection with the class of left Dyck factors with odd length. Now, given a left Dyck factor of odd length $2n-1$, we know that it can not end on the main diagonal since it has an odd number of steps. Then, starting from this path, there are two ways to obtain a left Dyck factor of even length $2n$, consisting in adding an ascending or a descending step at the end of this path. Finally, the previous bijections

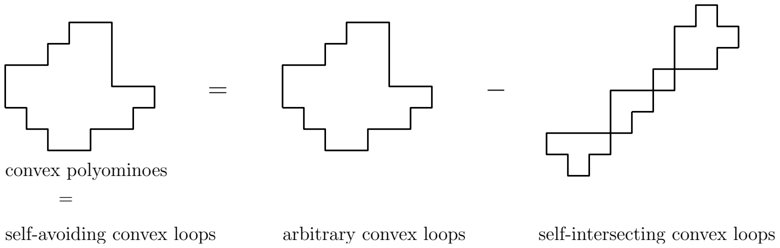


Figure 3.36: Convex polyominoes as loops that are not self intersecting.

also allowed us to understand how the parameters counting respectively the horizontal sides in the right-upper path of a direct convex polyominoe and the number of points in the left-upper path of a triangular permutation relates.

3.4 Enumeration by difference

Recall the formula for convex polyominoes, and the equivalent expression for the generating function:

$$p_n = (2n+5)4^{n-3} - 4(2n-5)\binom{2n-6}{n-3} \qquad \text{and} \qquad C(t) = \frac{t^2}{1-4t}\left(4+\frac{2t}{1-4t}\right) - \frac{4t^3}{(1-4t)^{3/2}}$$

Similarly for square permutations,

$$s_n = (2n+4)4^{n-3} - 4(2n-5)\binom{2n-6}{n-3} \qquad \text{and} \qquad S(t) = \frac{t^2}{1-4t}\left(2+\frac{2t}{1-4t}\right) - \frac{4t^3}{(1-4t)^{3/2}}$$

Finally, for convex permutominoes,

$$q_n = (2n+4)4^{n-3} - (2n-3)\binom{2n-4}{n-2} \qquad \text{and} \qquad P(t) = \frac{t^2}{1-4t}\left(2+\frac{2t}{1-4t}\right) - \frac{t^2}{(1-4t)^{3/2}}$$

In order to give a direct interpretation of the differences occuring in these formulas, one identifies supersets of combinatorial objects corresponding to the first terms and then counts the extra objects that need to be removed.

3.4.1 The enumeration of convex polyominoes by difference

The enumeration of convex polyominoes by Delest / Viennot (84) is one of the most famous success of Schützenberger methodology: due to the difference a direct $\mathbb{N}$-algebraic decomposition of the class $\mathcal{P}$ is not expected to exist, but Delest and Viennot describe the class of convex polyominoes as

$$\mathcal{P}_n \; = \mathcal{A}_n \cup \mathcal{B}_n$$

where $\mathcal{A}_n$, $\mathcal{B}_n$ and $\mathcal{A}_n \cap \mathcal{B}_n$ are encoded by algebraic langages.

This implies that

$$|\mathcal{P}_n| \; = |\mathcal{A}_n| + |\mathcal{B}_n| \; - \; |\mathcal{A}_n \cap \mathcal{B}_n|$$

or

$$P(t) = F_A(t) + F_B(t) - F_{A\cap B}(t)$$

However the generating functions $F_A(t)$ and $F_B(t)$ used by Delest and Viennot in [34] or by Bousquet-Mélou in her simplified version of the argument [17] are algebraic but not rational: this approach therefore does not really explain the difference between a rational generating function and an algebraic generating function in the generating function $\mathcal{P}(t)$ for convex polyominoes.

Instead a convincing explanation of this form was given by Bousquet-Mélou and Guttmann [22] using convex loops (see Figure 3.36). A *loop* is a sequence of unit steps on the lattice $\mathbb{Z}^2$ (denoted N, S, E, W) that form a not necessarily self avoiding walk ending at its start point. As for polyominoes we say that a loop is *convex* if its length is equal to the perimeter of its bounding box.

Then the key observation of Bousquet-Mélou and Guttmann is that a convex polyomino is a self avoiding convex loop: this allows to interpret the expression of the generating function of convex polyominos as a difference between the generating function of convex loops minus that of convex loops with at least one self intersection. Since the generating function of convex loops is rational, while self-intersecting loops admit a natural $\mathbb{N}$-algebraic decomposition, this approach fully explains the form of the formula.

Rather than give more details about their result, we adapted this approach to the case of convex permutominoes (Section 3.4.2).

But before going into the description of our approach we would like to point out that the first approach by difference was given in [13] by Boldi *et al.*, which lead them to solve the problem of the enumeration of convex permutominoes. In particular they introduced the so-called *admissible circuits* and enumerated the class of permutominoes as a difference between all the admissible circuits and the ones which do not define a permutomino. Their approach involves a bijection between the class of admissible circuits and a class of two permutations called *admissible permutations*, which lead them to enumerate the class of admissible circuits by obtaining a triple summation that they simplified by using a result from Bertoni *et al.* [12]. Finally they provided a decomposition for the class of admissible circuits which do not define a permutomino and obtained the result. However, they did not obtain the result as a difference between a rational part and an algebraic one, indeed either the class of admissible permutations and the class of admissible permutations not defining a permutomino have a difference in their formula and they are both algebraic.

Our approach is more bijective and lead us to obtain the enumeration of permutominoes as a difference between a rational part and an algebraic one.

3.4.2 Enumeration of convex permutominoes by difference

In order to adapt the strategy of Bousquet-Mélou and Guttmann to convex permutominoes, we need to identify objects that play the same role of loops for convex polyominoes: let us define a *permutominide of size* n as a loop such that each horizontal and vertical line with abscissa or ordinate in $\{1, \ldots, n\}$ contains exactly one side of the loop. A convex permutominide is *directed* if it starts from the south-west angle of its bounding box, and *parallel* if it is directed and visits the north-east angle of its bounding box.

Permutominides were introduced in [37] where two bijections are given to prove the following results:

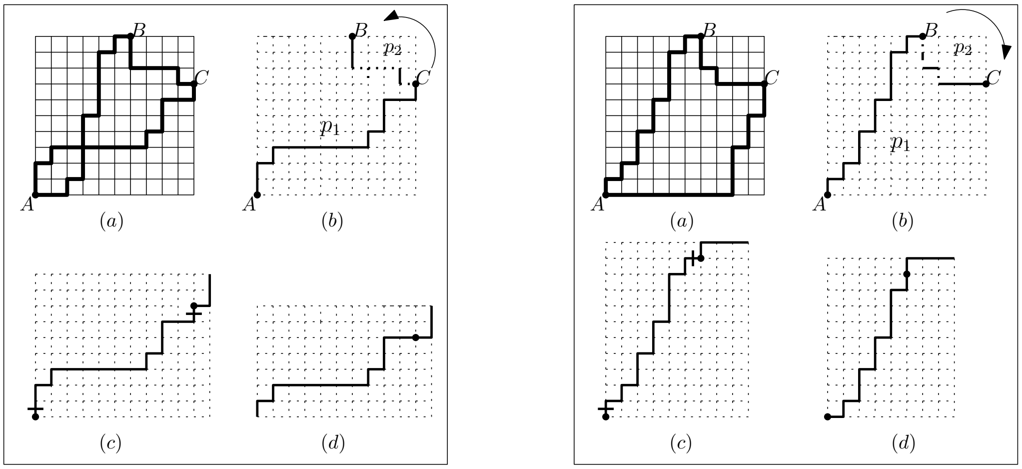
Proposition 39. The number of directed convex permutominides is $2^{2n-4} = 4^{n-2}$, and the number convex but not directed permutominides of length $2n$ with $n \geq 1$ is $(n-2) \cdot 2^{2n-5} = (2n-4)4^{n-3}$.

Proof. See paragraph below. □

Next we observe that a convex permutomino is a self avoiding convex permutominide, so that to count convex permutominoes it suffices to count convex permutominides with a self intersection:

Proposition 40. The number of convex permutominides of length $2n$ with a self intersection is equal to $2\left(\frac{2n-3}{2}\binom{2n-4}{n-2} - 2 \cdot 4^{n-3}\right)$.

Proof. See paragraph below. □



(i) (ii)

Figure 3.37: The bijection between directed convex permutominides and paths with ↑ and → steps.

As a consequence of the last two propositions the number of convex permutominoes is

$$q_n = \Big((2n-4)\cdot 4^{n-3} + 4^{n-2}\Big) - 2\left(\frac{2n-3}{2}\binom{2n-4}{n-2} - 2\cdot 4^{n-3}\right)$$
$$= (2n+4)2^{2n-4} - (2n-3)\binom{2n-4}{n-2}.$$

We observe here that this proof explains the general form of the formula as a difference between terms generated by a rational series and terms generated by an algebraic one. However the rational superset is a bit smaller than expected from the formula, resulting in an extra term in the negative term. In the next section we will see an alternative interpretation that gives directly the final formula.

A bijection for directed convex permutominides. The following proposition immediately implies the first statement of Proposition 39:

Proposition 41. There is a bijection between directed convex permutominides of size n and paths of length $2n-4$ made of north and east steps.

Let P be a directed convex permutominides, and A be the vertex in the south-west angle of its bounding box. Let B be the rightmost point on the top border of its bounding box, and C be the highest point on the right border of its bounding box (see Figures 3.37 (i)(a) and (ii)(a)). Since there is exactly one side in each horizontal and vertical line of the bounding box, the point A is incident to one vertical and one horizontal line: let us orient the loop P so that it starts along the vertical line, with a north step.

Then we define p_1 as the initial part of the path from A to B or C, depending whether

Case i. P first reaches C, after crossing the main diagonal an odd number of times: in this case $p_1 = N\bar{p}_1N$ where $\bar{p}_1$ is a north-east path with $n-1$ east steps; and we let p_2 be the path from C to B (see 3.37 (i)(b)).

Case ii. Or P first reaches point B, after crossing the main diagonal an even number of times; in this case $p_1 = N\bar{p}_1E$ where $\bar{p}_1$ is a north-east path with $n-2$ north steps, and we let p_2 be the path from B to C (see 3.37 (ii)(b)).

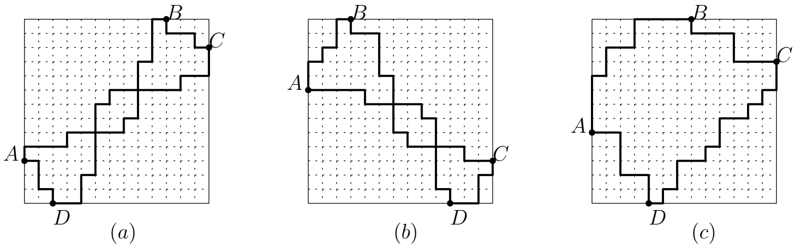


Figure 3.38: Three kinds of convex but not directed permutominides.

Then the construction of the path Q associated to P is similar in the two cases, where the notation N^i (resp. E^i) stands for i consecutive north (resp. east) steps:

Case i. Let $Q = \bar{p}_1 E^{s_1} N^{s_2} E^{s_3} \ldots$ where s_i is the length of the ith vertical side of p_2. By construction Q is a path of length $2n-4$ with at least $n-1$ east steps and (thus) less than $n-3$ north steps (see 3.37 (i)(d)).

Case ii. Let $Q = \bar{p}_1 N^{r_1} E^{r_2} N^{r_3} \ldots$ where the r_i is the length of the ith horizontal side of p_2. By construction Q is a path of length $2n-4$ with at least $n-2$ north steps and (thus) less than $n-1$ east steps (see 3.37 (ii)(d)).

To prove that the map $P \mapsto Q$ yields the announced bijection, it thus suffices to show that a permutominide P is uniquely recovered from a north east path Q of length $2n-4$. This is a consequence of the following observations:

- Given a north-east path Q of length $2n-4$, either it has at least $n-1$ east steps (Case i.) and $\bar{p}_1$ is recovered as the longuest prefix of Q with $n-1$ east steps; or it has at least $n-2$ north steps (Case ii.) and $\bar{p}_1$ is recovered as the longuest prefix of Q with $n-2$ north steps.
- The remaining part Q' of Q gives the length of the vertical (resp. horizontal) sides of p_2 for the Case i. (resp. Case ii.): this is sufficient to reconstruct p_2 as part of a permutominide P because to on its way from C to B (resp. B to C), P must take a turn at each empty vertical (resp. horizontal) line it encounters.
- Finally given p_1 and p_2 the remaining part p_3 of P can also be uniquely reconstructed: indeed on its way from B to A (resp. from C to A) in the Case i. (resp. Case ii.), the path must take a turn at each empty horizontal or vertical line it encounters.

The case of convex but not directed permutominides Observe that a convex permutomino P is non directed if and only if its lower leftmost point A is at a positive height h above the bottom of its bounding box. A refined version of the previous analysis of the case $h = 0$ of directed convex permutominides then yields the following result, which implies the second statement of Proposition 39:

Proposition 42. For each integer h with $1 \leq h \leq n-2$, there is a bijection between non directed convex permutominides of size n with lower leftmost point A at height h of its bounding box, and north-east paths of length $2n-5$.

Let P be a non directed convex permutominide of size n whose lower leftmost point A is at height h of the bounding box. Let also B be its rightmost point on the top border of its bounding box, C its highest point on the right border of its bounding box, and D its leftmost point on the bottom border of its bounding box (see Figure 3.38). Remark that B and C may coincide, while A and D are necessarily distinct since the permutominide P is assumed non directed. Due to convexity constraints P satisfies only one of the following properties:

1. the boundary of P crosses itself on the main diagonal (see Figure 3.38 (a));

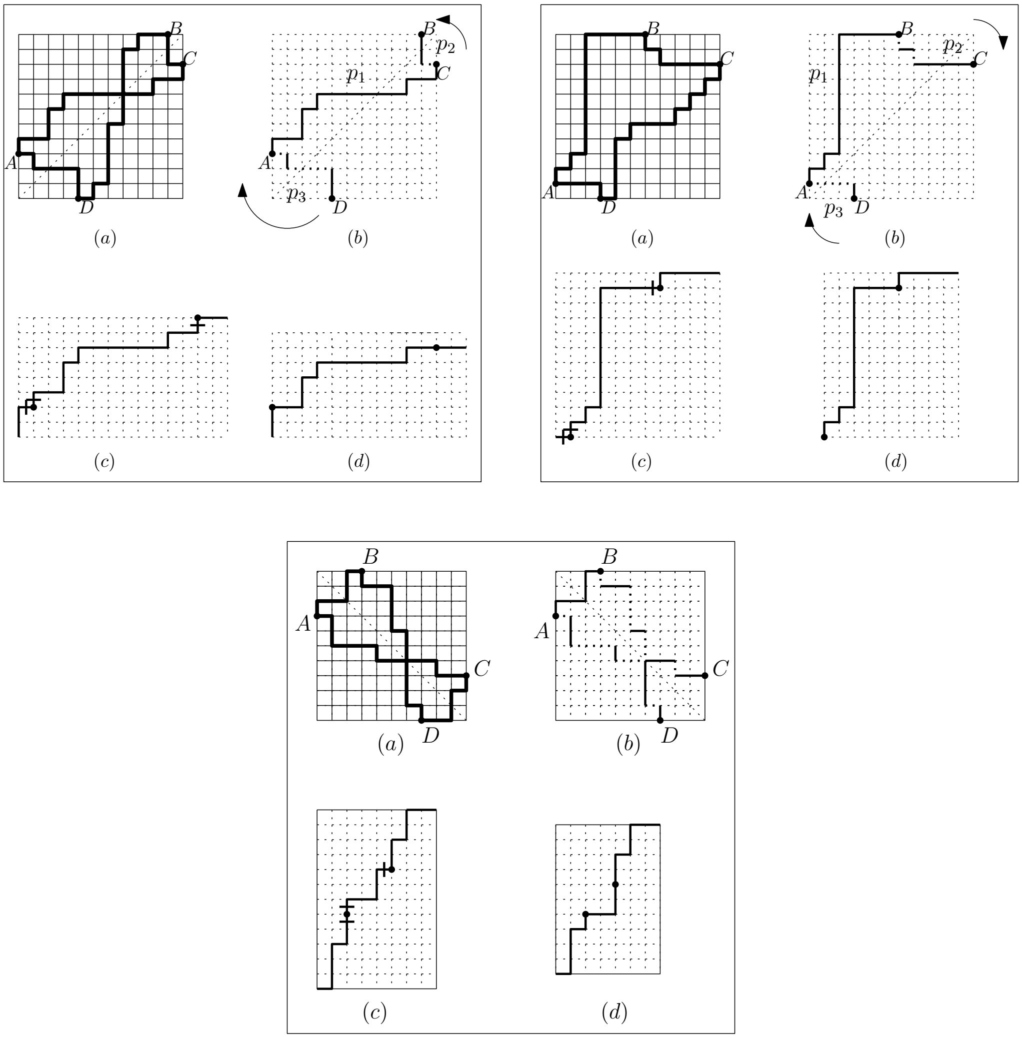


Figure 3.39: The bijection between convex but not directed permuminides crossing on the main diagonal and paths with north and east steps.

2. the boundary of P crosses itself on the anti-diagonal (see Figure 3.38 (b));
3. the boundary of P does not cross itself (see Figure 3.38 (c)), i.e. it is a self-avoiding convex permutominide, that is a convex permutomino.

These cases can be dealt with in a similar manner than for directed convex permutominides and they are only pictorially described in Figure 3.39, for more details we refer to our paper [37].

Permutominides that are not permutominoes The permutominides that are not permutominoes are those with a self intersection, as in Figures 3.38(a) and (b). Let $\overline{M}_n$ (respectively $\overline{A}_n$) denote the subclasses of permutominides with self intersections between their north-west and south-east boundaries (respectively south-west and north-east boundaries). By symmetry we have that $|\overline{M}_n| = |\overline{A}_n|$. Then the following proposition proves Proposition 40.

Proposition 43. The cardinality of $|\overline{M}_n|$ is equal to $(2n-3)\binom{2n-4}{n-2} - 2 \cdot 4^{n-3}$.

Proof. Let $P \in \overline{M}_n$, the idea is to consider the first point $I = (i, i)$ where the boundary crosses itself, $0 < i < n$. Then P uniquely decomposes into two permutominides. The lower one is

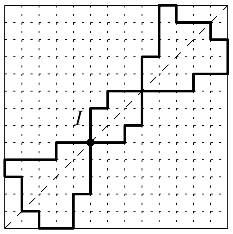


Figure 3.40: The decomposition of a convex permutominides whose boundary crosses itself on the main diagonal.

self-avoiding, hence a directed convex permutomino of size $i \geq 2$, while the second is a directed convex permutominides of size $n+1-i \geq 2$ (see Figure 3.40). Since the number of directed convex permutominoes of size n is $\binom{2n-3}{n-1}$ and, we have see in the previous subsection that the number of directed convex permutominides of size n is 4^{n-2}, we have that

$$|\overline{M}_n| = \sum_{i=2}^{n-1} \binom{2i-3}{i-1} 4^{n-i-1} = (2n-3)\binom{2n-4}{n-2} - 2 \cdot 4^{n-3}$$

□

3.4.3 Enumeration of square permutations and convex permutominoes by difference

In order to adapt the previous proofs to square permutations one would need to define an equivalent of convex loops for this class. It turns out to be more convenient to use an encoding of square permutations by their horizontal and vertical projections. This encoding, that we introduced in [41], is a generalization of the horizontal/vertical code given in Subsection 3.3.1 in the context of the bijection between parallelogram polyominoes and parallel permutations. As we shall see the set of horizontal/vertical codes is a subset of a simple family of words interpreting the rational terms in the formula for square permutations. We will start by defining this encoding for the class of square permutations, then by using the same encoding we will also rederive the formula for convex permutominoes via the bijection with the class of co-indecomposable and colored square permutations given in Subsection 2.3.3.

The rational terms in the formulas for square permutations and convex permutominoes Let $\mathcal{W} = \mathcal{A}^*$ denote the set of (bi)words on the alphabet $\mathcal{A} = \{U, D\} \times \{L, R\}$, and let $\mathcal{M}$ denote the set of marked words (w, ℓ) consisting of

- a word $w = (u_1, v_1) \cdots (u_n, v_n) \in \{(X, Y)\} \cdot \mathcal{W} \cdot \{(X, Y)\}$
- and a mark ℓ with $1 \leq \ell \leq n$ and $v_m \in \{L, Y\}$.

Let moreover

$$W(t; x, y) = \frac{1}{(1 - (1+x)(1+y)t)}$$

be the generating function of $\mathcal{W}$ with respect to the length (var. t), number of U and X (var. x) and number of L and Y (var. y). Upon dealing separately with the case $\ell = 1, n$, where the mark is on a letter Y, from the case $1 < \ell < n$, where the mark is on a letter L, we then have that the generating function of $\mathcal{M}$ with respect to the same parameters is

$$M(t; x, y) = 2 \cdot txy \cdot W(t; x, y) \cdot txy + txy \cdot W(t; x, y) \cdot t(1+x)y \cdot W(t; x, y) \cdot txy.$$

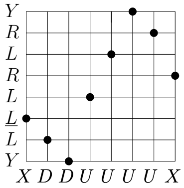


Figure 3.41: A square permutation and its code. The marked letter is underlined.

In particular $M(t, 1, 1)$ gives a combinatorial interpretation of the dominant rational term in the formula for square permutations since

$$W(t; 1, 1) = \frac{1}{1-4t} \quad \text{and} \quad M(t; 1, 1) = \frac{t^2}{1-4t}\left(2 + \frac{2t}{1-4t}\right).$$

The encoding for square permutations Given a square permutation σ, we let denote by $H(\sigma) = Xu_2 \dots u_{n-1}X$ the horizontal profile of σ, where X stands for an extremal point of σ and

$$u_i = \begin{cases} U, & \text{if } (i, \sigma(i)) \text{ is an upper point;} \\ D, & \text{otherwise} \end{cases}$$

with $i \in \{2, \dots, n-1\}$, and we let denote by $V(\sigma) = Yv_2 \dots v_{n-1}Y$ the vertical profile of σ, where Y stands for an extremal point and

$$v_i = \begin{cases} L, & \text{if } (\sigma(i), i) \text{ is a left point which is not an upper right one} \\ R, & \text{otherwise;} \end{cases}$$

with $i \in \{2, \dots, n-1\}$.

Finally let the *code* of a permutation σ be the triple $(H(\sigma), V(\sigma), \sigma(1))$ (see Figure 3.41 for an example), we are going to show that this encoding $\sigma \mapsto (H(\sigma), V(\sigma), \sigma(1))$ is indeed injective. Observe that the triple $(H(\sigma), V(\sigma), \sigma(1))$ can alternatively be viewed as a marked biword (w, ℓ) of $\mathcal{W}$ by taking $w = (X, Y)(u_2, v_2) \dots (u_{n-1}, v_{n-1})(X, Y)$ and $\ell = \sigma_1$.

A decoding algorithm for square permutations We already pointed out that the set of triples coding square permutations of size n forms a subset of $\mathcal{M}$. However this subset is not easy to describe directly. We provide an algorithmic description via the following decoding algorithm.

Let us take an element of $\mathcal{M}$ of length n, viewed as a triple (u, v, ℓ). We are going to construct a partial permutation $\sigma = \sigma(1) \dots \sigma(k)$, with $k \leq n$. The idea is to insert each point on the grid from left to right as long as a compatible decoding is possible. The rules to construct σ depend on the letters of u and v: the letter L (resp. R, U, D) means that the point of the permutation we are constructing belongs to a left path (resp. right, lower, upper path) of the corresponding permutation, while the letters X or Y means that this point is extremal. Then (see Figure 3.42):

- Write u (resp. v) along the lower edge (resp. left edge) of the grid $n \times n$ starting from the left (resp. starting from the bottom).
- Start reading the word u: u_1 is necessarily an X and it should correspond to the leftmost point of the permutation. Accordingly insert a point $\sigma(1)$ in the leftmost column and in row ℓ (the row of the marked letter of v).
- Continue reading each letter u_i of u, for $i \geq 2$:

1. If $i = n$ then insert the point $\sigma(n)$ in the unique remaining empty row and end the algorithm.
2. If $u_i = U$ and we have not yet inserted the highest point of the permutation, then take the first row j above $\sigma(i-1)$ that is labeled with L or Y and insert the point $\sigma(i) = j$.

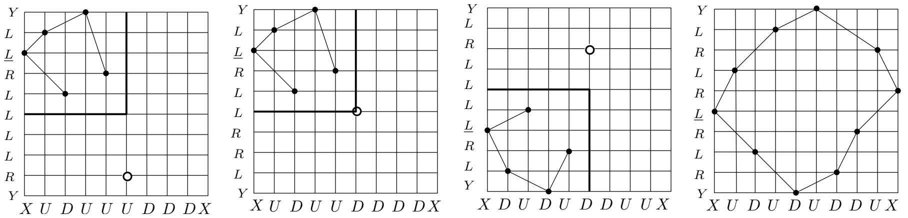


Figure 3.42: Four examples of triples (u, v, ℓ), only the fourth represents an encoding of a square permutation.

3. If $u_i = U$ and we have already inserted the highest point of the permutation, then take the first row j below $\sigma(i-1)$ that is labeled with R or Y: if $\sigma(1)\ldots\sigma(i-1)$ is contained in the grid $(i-1) \times (i-1)$ attached to the left-upper corner of the grid $n \times n$ and $j \neq i$ then the insertion is invalid (since we are constructing the right-upper path of the permutation while the rows between $i-1$ and j remain empty, that is the permutation can not be square) and the algorithm stops (see the first case in Figure 3.42, where the invalid insertion is represented with a white bullet); otherwise insert the point $\sigma(i) = j$.

4. If $u_i = D$ and we have not yet inserted the lowest point of the permutation, then take the first row j that is labeled with L or Y under $\sigma(i-1)$ and insert the point $\sigma(i) = j$: if $\sigma(1)\ldots\sigma(i-1)$ is contained in the grid $(i-1) \times (i-1)$ attached to the left-upper corner of the grid $n \times n$ and $j = i$ then the insertion is invalid (since we supposed that points belonging to the right-upper path of the permutation are coded by (U, R)) and the algorithm stops (see the second case in Figure 3.42).

5. If $u_i = D$ and we have already inserted the lowest point of the permutation, then take the first row j that is labeled with R or Y above $\sigma(i-1)$ and insert the point $\sigma(i) = j$: if $\sigma(1)\ldots\sigma(i-1)$ is contained in the grid $(i-1) \times (i-1)$ attached to the bottom-left corner of the grid $n \times n$ then the insertion is invalid (for the same reasons as before) and the algorithm stops (see the third case in Figure 3.42).

Let us suppose that the algorithm ends after reading i letters of u and v, $i \in \{1, \ldots, n\}$. Then we can state the following proposition:

Proposition 44. Given an element (w, ℓ) of $\mathcal{M}$ of size n viewed as triple (u, v, ℓ) the algorithm stops and yields:

- (in case of success) a permutation $\sigma(1)\ldots\sigma(n) \in \mathcal{S}_n$ whose code is (w, ℓ), or
- (in Cases 3 and 4 above) a triple $(\rho, (u_i, v_i), (u'', v''))$, where $\rho = \text{Standard}(\sigma(1)\ldots\sigma(i-1)) \in \mathcal{S}^{\lhd}_{i-1}$, with $2 \leq i \leq n$, is the standardization of σ on $\{1, \ldots, i-1\}$, i.e., $\rho(l) = \sigma(l) - n + i - 1$ with $l \in \{1, \ldots, i-1\}$, $(u_i, v_i) \in \{(U, L), (D, L)\}$, and (u'', v'') is a pair of words of length $n - i - 1$ where $u'' = u_{i+1}\ldots u_{n-1}X$ and $v'' = v_{i+1}\ldots v_{n-1}Y$, or
- (in Case 5 above) a triple $(\rho, (u_i, v_i), (u'', v''))$, where $\rho = \sigma(1)\ldots\sigma(i-1) \in \mathcal{S}^{\triangleleft}_{i-1}$ with $2 \leq i \leq n$, $(u_i, v_i) \in \{(D, L), (D, R)\}$, and (u'', v'') is a pair of words of length $n - i - 1$, where $u'' = u_2\ldots u_{n-i}X$ and $v'' = v_{i+1}\ldots v_{n-1}Y$.

Then we have the following theorem:

Theorem 3. The horizontal/vertical encoding is a bijection

$$\mathcal{S} \equiv \mathcal{M} - \mathcal{S}^{\lhd} \cdot \{(U, R), (D, L)\} \cdot \mathcal{W} \cdot \{(X, Y)\} - \mathcal{S}^{\triangleleft} \cdot \{(D, L), (D, R)\} \cdot \mathcal{W} \cdot \{(X, Y)\}.$$

In particular this theorem provides us with our interpretation of the form of the formula for the generating function $S(x)$ as a difference between a rational term, the generating function of the rational set of words $\mathcal{M}$, and an algebraic one, the series of non coding words, which decomposes in terms of Catalan structures. Recall indeed that $\mathcal{M}$ and $\mathcal{W}$ are rational languages while by symmetry $\mathcal{S}^{\triangleleft} \equiv \mathcal{S}^{\lhd} \equiv \mathcal{S}^{\triangleright}$, where $\mathcal{S}^{\triangleright}$ admits, as we have seen in Section 3.2.3, the following $\mathbb{N}$-algebraic decomposition:

$$\begin{aligned}
\mathcal{S}^{\triangleright} &\equiv \{\bullet\} + (1+\mathcal{S}^{/\!\!/}) \times \{\bullet\} \times \mathcal{S}^{\triangleright} + \{\bullet\} \times \mathcal{S}^{\triangleright\bullet} \\
\mathcal{S}^{\triangleright\bullet} &\equiv \mathcal{S}^{\triangleright} + (1+\mathcal{S}^{/\!\!/}) \times \{\bullet\} \times \mathcal{S}^{\triangleright\bullet} \\
\mathcal{S}^{/\!\!/} &\equiv (\{\bullet\} + \{\bullet\} \times \mathcal{S}^{/\!\!/}) \times (1+\mathcal{S}^{/\!\!/}).
\end{aligned}$$

Observe that the link between the central binomial structures $\mathcal{S}^{\triangleright}$ and the Catalan structures $\mathcal{S}^{/\!\!/}$ can also be recovered directly from the special case of Theorem 3 for square permutations starting with a fixed point:

$$\{\bullet\} \oplus \mathcal{S}^{\triangleright} \;\equiv\; \{(X,Y)\} \cdot \mathcal{W} \cdot \{(X,Y)\} \;-\; \{\bullet\} \oplus \mathcal{S}^{/\!\!/} \cdot \{(D,L),(D,R)\} \cdot \mathcal{W} \cdot \{(X,Y)\}.$$

In terms of generating function we obtain

$$\begin{aligned}
S^{/\!\!/}(t) &= t(1+S^{/\!\!/}(t))^2 \;=\; \frac{1-\sqrt{1-4t}}{2t} \\
S^{\triangleright}(t) &= W(t)\cdot t - S^{/\!\!/}(t)\cdot 2t \cdot W(t) \cdot t \;=\; \frac{t}{1-4t} - \frac{t(1-\sqrt{1-4t})}{(1-4t)} \;=\; \frac{t}{\sqrt{1-4t}} \\
S^{\triangledown}(t) &= S^{\triangleleft}(t) \;=\; S^{\triangleright}(t) \\
\\
S(t) &= M(t) \;-\; S^{\triangleleft}(t)\cdot 2t \cdot W(t) \cdot t \;-\; S^{\lhd}(t) \cdot 2t \cdot W(t) \cdot t
\end{aligned}$$

from which we recover Formula (3.4) for square permutations according to their length:

$$S(t) \;=\; \frac{2t}{1-4t}\left(2+\frac{2t}{1-4t}\right) \;-\; \frac{4t^3}{(1-4t)^{3/2}}$$

The encoding for convex permutominoes We can adapt the previous encoding to convex permutominoes by using the bijection ϕ between convex permutominoes and colored co-indecomposable square permutations presented in 2.3.3. Indeed the previous encoding can easily be adapted to take into account co-indecomposability and colored points. Given a colored co-indecomposable permutation σ its horizontal profile $H(\sigma) = Xu_2 \dots u_{n-1}X$ and its vertical profile $V(\sigma) = Yv_2 \dots v_{n-1}Y$ are defined as follows:

$$u_i = \begin{cases} U, & \text{if } (i,\sigma(i)) \text{is an upper point that is not colored;} \\ D, & \text{otherwise,} \end{cases}$$

with $i \in \{2, \dots, n-1\}$, and

$$v_i = \begin{cases} L, & \text{if } (\sigma(i), i) \text{ is a left point which is not an upper right one or which is not colored} \\ R, & \text{otherwise,} \end{cases}$$

with $i \in \{2, \dots, n-1\}$.

Then the decoding algorithm for convex permutominoes, viewed as colored co-indecomposable square permutations, is similar to the decoding algorithm for square permutations, with slightly different stopping conditions (see Figure 3.43 for an example):

- in Cases 3 and 4 above, one should stop as soon as $\sigma(1)\dots\sigma(i-1)$ is contained in the grid $(i-1)\times(i-1)$ attached to the left-upper corner, regardless of the other conditions, to eliminate co-decomposable permutations;

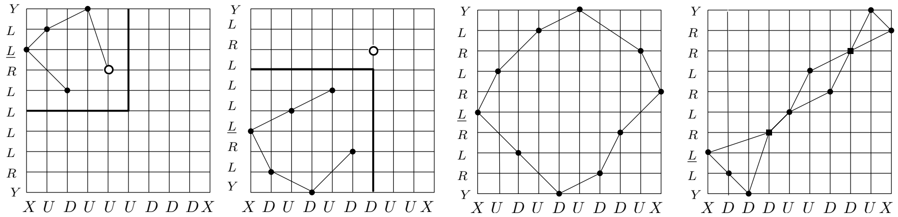


Figure 3.43: Four examples of triples (u, v, ℓ). The third one and the fourth one belong to the class of colored co-indecomposable square permutations.

- in Case 5 above, one should declare the insertion invalid only if $j \neq i$, to allow for colored fixed points.

As a result we obtain the following theorem:

Theorem 4. The horizontal/vertical encoding yields a bijection:

$$\mathcal{Q} \;\equiv\; \bar{\mathcal{Q}} \;\equiv\; \mathcal{M} \;-\; \bar{\mathcal{Q}}^{\triangleleft} \cdot \{(D,L)\} \cdot W \cdot \{(X,Y)\} \;-\; \bar{\mathcal{Q}}^{\triangleleft +} \cdot W \cdot \{(X,Y)\}$$

where $\bar{\mathcal{Q}}^{\triangleleft}$ denotes the subclass of colored permutations that can be obtained from colored square permutations ending with a maximal value by removing this maximal value, and $\mathcal{Q}^{\triangleleft +}$ denotes the subclass of co-indecomposable permutations that can be obtained from a colored square permutations ending with its minimal value by removing this last value and standardizing the permutation.

In order to deal with $\mathcal{Q}^{\triangleleft}$ and $\mathcal{Q}^{\triangleleft +}$ we observe that the transposed permutation classes $\mathcal{Q}^{\triangleright}$ and $\mathcal{Q}^{\triangleright +}$ can be obtained as special cases of the above construction of $\mathcal{Q}$:

- The set $\bullet\bar{\mathcal{Q}}^{\triangleright}$ of colored square permutations starting with the minimal value ($\sigma \in \mathcal{Q}$ s.t. $\sigma_1 = 1$), is obtained when the decoding is applied to pairs of the form $(w, 1)$:

  $$\bullet\bar{\mathcal{Q}}^{\triangleright} \;\equiv\; \{(w,1) \in \mathcal{M}\} \;-\; \bullet \cdot (1 + \bar{\mathcal{Q}}^{\sslash}) \cdot \{(D,L)\} \cdot W \cdot \{(X,Y)\}$$

  where $\bar{\mathcal{Q}}^{\sslash}$ denotes the class of permutations that are obtained from colored permutations of Q that start with the minimal value and end with the maximal value by removing these extremal values.

- The set $\bullet\bar{\mathcal{Q}}^{\triangleright +}$ of co-indecomposable colored square permutations starting with the maximal value ($\sigma_1 = n$ for a permutation of size n) is obtained when the decoding is applied to pairs of the form $(w, |w|)$ (upon waiving the co-indecomposability check at the first step to allow starting with the maximum):

  $$\bullet\bar{\mathcal{Q}}^{\triangleright +} \equiv \{(w,|w|) \in \mathcal{M}\} \;-\; \bullet \cdot \{(U,L),(D,R)\} \cdot W \cdot \{(X,Y)\} \;-\; \bullet \cdot \bar{\mathcal{Q}}^{\backslash\backslash +} \cdot W \cdot \{(X,Y)\}$$

  where $\bar{\mathcal{Q}}^{\backslash\backslash +}$ denotes the subclass of co-indecomposable permutations in the class of permutations obtained from permutations of $\bar{\mathcal{Q}}$ that start with the maximal value and end with the minimal value by removing these extremal values.

Finally the classes $\bar{\mathcal{Q}}^{\sslash}$ and $\bar{\mathcal{Q}}^{\backslash\backslash +}$ are simple variants of the class of parallel permutations. Then the previous equation translates into a system of equations which lead us to recover the formulas for convex permutominoes (see [41] for more details).

Refinements We would like to point out that we can re-use the same horizontal/vertical encoding to obtain refined generating functions, taking into account the number of upper points and the number of lower points in square permutations (resp. the number of upper sides and the number of left sides in convex permutominoes).

3.4.4 Some remarks about differences

As already announced, apart from explaining the difference between a rational and an algebraic part, the last two theorems also explain the common rational term between the formulas for square permutations and convex permutominoes. On the other hand the approach of Bousquet-Mélou and Guttmann to count convex polyominoes using loops, together with our adaptation to convex permutominoes using permutominides gives a common interpretation of the respective rational and algebraic nature of two terms. It would have been interesting to find a common ingredient to deal with all three families of structures.

Part II

Three examples of more advanced combinatorial constructions

The second part of this manuscript consists of three chapters detailing some more advanced combinatorial constructions: the first chapter presents an extension of the linear construction we introduced for square permutations to a more general class of permutations, having a fixed number of internal points; the second one presents a tricky linear construction for another class of permutations, avoiding a particular pattern, that can be viewed as a generalization of the class of parallel permutations; the third one introduces a newborn class of combinatorial objects, called the *fighting fish*, that is generalization of the class of directed polyominoes. In particular two constructions are given in this chapter leading to the enumeration of fighting fish: the first one is a construction by slices *à la* Temperley but non linear, while the second one is a generalization of the grammar we have seen in the first part for parallelogram polyominoes, but where we need introduce a catalytic variable to encode a marked operation, so that we finally do not have either a grammar-like decomposition (as indeed could be expected from the asymptotic of the class of fighting fish, which is not directly compatible with the existence of a $\mathbb{N}$-algebraic decomposition).
We would like to point out that there are other examples of more advanced constructions linked to the methods in the first part that we would have liked to show, but we decided to restrict to these three examples that are more recents and more linked with the objects that we have seen in the first part.

Chapter 4

Permutations with few internal points

Permutations without internal points have already been introduced in the first part under the name of square permutations (see Section 2.2 of Chapter 2). In [38] we extended the linear recursive construction for square permutations that we gave in [47] in order to enumerate the class ${\mathcal{S}_n}^{(i)}$ *of permutations with i internal points. In this chapter we will present this construction and we will see that for each fixed i the generating function of permutations with i internal entries with respect to the length is algebraic of degree 2. More precisely it is a rational function in the Catalan generating function. Our approach is constructive, yielding a polynomial uniform random sampling algorithm, and it can be refined to enumerate permutations with respect to each of the four types of records.*

4.1 The standard generating tree for permutations

Given a permutation σ of $\{1,\dots,n\}$ and two indices i,j of $\{1,\dots,n+1\}$, the *inflated permutation* $\sigma_{+i\,j}$ is the permutation of $\{1,\dots,n+1\}$ defined by

$$\sigma_{+i\,j}(k)=\begin{cases}\sigma(k') & \text{if } \sigma(k')<j\\ \sigma(k')+1 & \text{if } \sigma(k')\geq j\end{cases}\quad\text{where } k'=\begin{cases}k & \text{if } k<i\\ k-1 & \text{if } k>i\end{cases}\quad\text{and } \sigma_{+i\,j}(i)=j$$

As illustrated by Figure 4.1, $\sigma_{+i\,j}$ is obtained from σ by adding a column on the right of the vertical line i, a row on the top of the horizontal line j and by inserting a point in the cell (i,j).

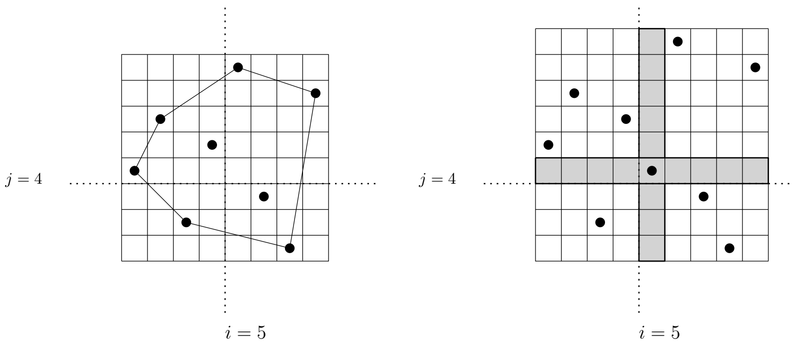


Figure 4.1: (a) A permutation σ with two internal points. (b) The permutation $\sigma_{+5\,4}$ with three internal points obtained by inflating the one on its left in the point $(5,4)$.

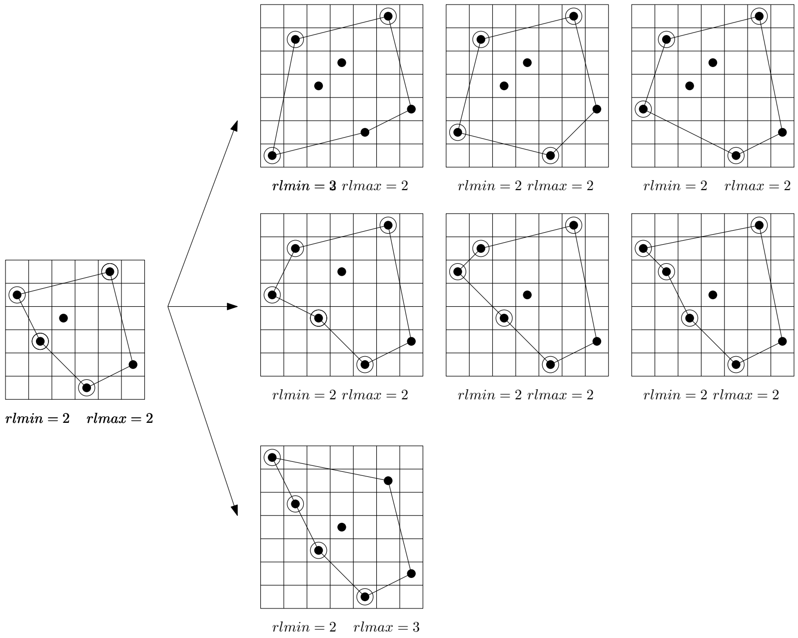


Figure 4.2: The standard construction applied to permutation 5 3 4 1 6 2. The sequences of lrmin and lrmax are circled, while the new values for rlmin et rlmax are written under each permutation.

Conversely, given a permutation σ of $\{1, \ldots, n+1\}$ and an entry (i, j) with $j = \sigma(i)$, the permutation $\sigma_{/ij}$ is the permutation

$$\sigma_{/i\,j}(k) = \begin{cases} \sigma(k') & \text{if } \sigma(k') < j \\ \sigma(k') - 1 & \text{if } \sigma(k') > j \end{cases} \quad \text{where } k' = \begin{cases} k & \text{if } k < i \\ k+1 & \text{if } k \geq i \end{cases}$$

obtained by removing column i and line j and standardizing the permutation on $\{1, \ldots, n\}$. By construction $(\sigma_{+ij})_{/ij} = \sigma$.

A standard way to grow permutations is to insert a new point in position $(1, j)$, with $j \in \{1, \ldots, n+1\}$, see the example in Figure 4.2 for permutation 5 3 4 1 6 2. The resulting generating tree is very regular: each σ of size n has $n+1$ sibblings, and this implies that the number of permutations at depth n in the tree is $n!$, as expected. Moreover the evolution of the parameters rlmin and rlmax can easily be tracked: any σ with k rlmin and ℓ rlmax generates one permutation with $k+1$ rlmin and ℓ rlmax, $n-1$ with k rlmin and ℓ rlmax and one with k rlmin and $\ell+1$ rlmax.

However the numbers of lrmax and lrmin in the sibblings of a permutation σ does not behave so simply: it depends on the exact positions of the lrmax and lrmin of σ as we can see in Figure 4.2. As a consequence, this simple generating tree does not allow to study the number of internal points in permutations.

4.2 A new generating tree for permutations

4.2.1 The generating tree for permutations without internal points

In order to circumvent the problem we grow permutations in a way that will ensure that the number of internal points increases at most by one at each insertion. The idea is to apply the standard construction only in a smaller area in order to control the evolution of lrmin and lrmax without keeping track of their ordinates.

A way to do this is to insert in the first column a point between the ordinate of the first point and the ordinate of the first lrmin that is not *double*, i.e. that is not both a lrmin and rlmax.

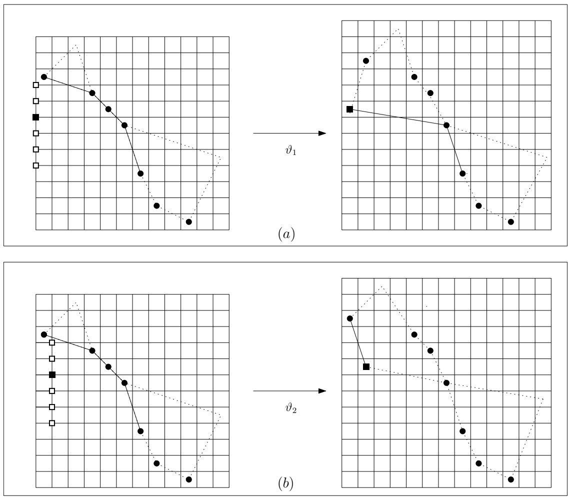


Figure 4.3: A permutation, its active zone and the expansion in a given point

We call this zone the *active zone*. Observe that this lrmin is not on the antidiagonal since a double point $(i, \sigma(i))$ must be on the antidiagonal by definition, i.e. $\sigma(i) = n - i$. Remark that, by inserting in this zone, the number of lrmax always increases by 1 while the number of lrmin decreases by $i \in \{0, \dots, \ell\}$, where ℓ is the length of the first sequence of double points starting from the left. Let us denote by ϑ_1 the operator associated with this construction. In Figure 4.3(a) there is an example, where the active zone is indicated on the left of the grid and a new point is inserted in the permutation σ. Observe that, by applying ϑ_1, we obtain an *ascending permutation*, that is a permutation σ' with $\sigma'(1) < \sigma'(2)$ (indeed we inserted a point with ordinate less or equal to $\sigma(1)$).

In order to obtain also *descending permutations* σ', with $\sigma'(1) > \sigma'(2)$, we have to extend our active zone by allowing to insert points $(2, j)$, where j still varies between the ordinate of the first point of σ and the ordinate of the first lrmin that is not double (see Figure 4.3,(b)) . The resulting operator ϑ_2 produces descending permutations since we insert a point $(2, j)$ on the right of $(1, \sigma(1))$ with $j < \sigma(1)$. By inserting in this zone the number of lrmax does not change while the number of lrmin decreases by $i - 1$, with $i \in \{0, \dots, \ell\}$ (increases by 1 if $i = 0$).

In order to describe the operators more formally, let us define the *non extremal double points* of σ as the double points $(i, \sigma(i))$ with $1 < i < n$, and the *special point* $(i_0, \sigma(i_0))$ as the leftmost lrmin which is not a non extremal double point (i_0 exists since $(\sigma^{-1}(1), 1)$ is always a lrmin which is not a non extremal double point).

Moreover let

$$j_0 = \begin{cases} \sigma(i_0) + 1 & \text{if } \sigma(i_0) > 1 \\ 1 & \text{otherwise} \end{cases}$$

and let the *external active zone* (see Figure 4.4) be the set of points of the grid with coordinates (i, j) with $i \in \{1, 2\}$ and $j_0 \leq j \leq \sigma(1)$:

$$\mathcal{E}(\sigma) = \{1, 2\} \times \{j_0, \dots, \sigma(1)\}.$$

Finally we define the *basepoint* u_σ of a permutation σ as the lowest point among its two first entries $(1, \sigma(1))$ and $(2, \sigma(2))$:

$$u_\sigma = (i_1, \sigma(i_1)) \quad \text{with} \quad i_1 = \begin{cases} 1 & \text{if} \quad \sigma(1) < \sigma(2), \\ 2 & \text{if} \quad \sigma(1) > \sigma(2). \end{cases}$$

Then by definition the operator $\vartheta = \vartheta_1 \cup \vartheta_2$ applied to a permutation σ generates the set of permutations $\vartheta(\sigma) = \{\sigma_{+ij} \mid (i, j) \in \mathcal{E}(\sigma)\}$: in other terms, for each point (i, j) of the external zone of σ, one permutation is created by inserting (i, j) in σ (see Figure 4.5).

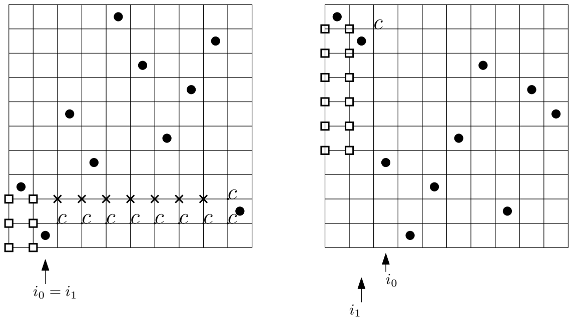


Figure 4.4: Example of external and internal active zones, the external active points are marked with an empty square, the internal active points are marked with a cross, while the blocking points that do not belong to the internal active zone are marked with a c. Observe that in the second picture the internal active zone is empty.

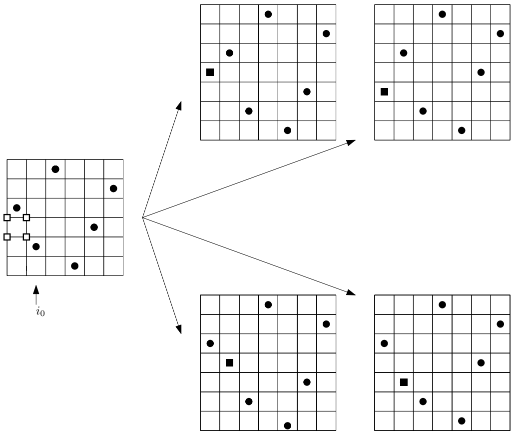


Figure 4.5: A permutation with its external active zone and the corresponding 4 children.

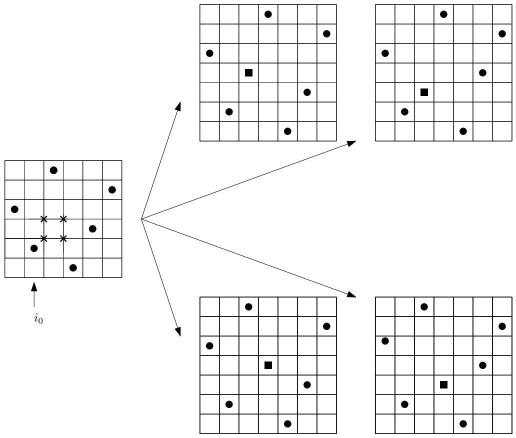


Figure 4.6: A permutation with its internal active zone and the corresponding 4 children.

Applied to a permutation without internal point, that is a square permutation (square permutations have been introduced in Section 2.2 of Chapter 2), the operator ϑ produces permutations without internal points: we in fact just gave a description of the recursive construction for square permutations given by Duchi and Poulalhon in [47]:

Theorem 5 ([47]). The operator ϑ induces a generating tree for square permutations.

Proof. The proof essentially follows from the observation that the point inserted by ϑ to produce a permutation σ' is precisely the basepoint of σ'. In other terms, if $\sigma' \in \vartheta(\sigma)$ then $\sigma'_{/u_{\sigma'}} = \sigma$. This allows to recover the father of each square permutation in the generating tree. □

We would like to point out more generally the fact that these operations can be applied to permutations with internal points. Indeed, given a permutation σ with internal points, we can define $\vartheta(\sigma)$ as above and we still have the property that if a permutation σ' can be generated by ϑ then its father is obtained by removing $u_{\sigma'}$, *i.e.*, if $\sigma' \in \vartheta(\sigma)$ then $\sigma'_{/u_{\sigma'}} = \sigma$.

The resulting operator ϑ changes the number of records but keeps the number of internal points in the produced permutations. We therefore need an extra operation that changes the number of internal points.

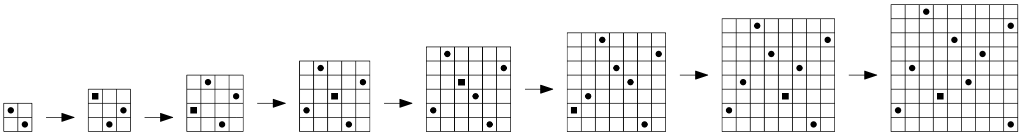


Figure 4.7: A part of a branch in $\mathcal{T}$

4.2.2 Producing internal points

We now want to insert internal points directly, in a way that produces the permutations that we do not know how to generate. In order to do this we say that a point (i,j) of σ' is a *blocking point* if $u_{\sigma'} = (i_1, \sigma'(i_1))$ is the only external point (i', j') of σ' such that $i' < i$ and $j' < j$.

Proposition 45. A permutation σ' can be generated by ϑ if and only if σ' has no internal blocking point.

Similarly, let us say that a point (i,j) of the grid is a blocking insertion in σ if $u_\sigma = (i_1, \sigma(i_1))$ is the only external point (i', j') of σ such that $i' < i$ and $j' < j$. Observe that a point (i,j) of the grid is a blocking insertion of σ if and only if (i,j) is a blocking point of σ_{+ij}.

This allows us to define the *internal active zone* of σ, $\mathcal{I}(\sigma)$ (see Figure 4.4), as the set of points (i,j) of the grid that simultaneously:

- satisfy $i > 2$,
- are a blocking insertion of σ,
- contain at least one point of σ both in their right-upper quadrant $\{(h,k) : h \geq i, k \geq j\}$ and in their right-lower quadrant $\{(h,k) : h \geq i, k < j\}$,
- and are on the left hand side of all already existing internal blocking points of σ,

Then the operator ϑ' is defined as follows: $\vartheta'(\sigma) = \{\sigma_{+ij} \mid (i,j) \in \mathcal{I}(\sigma)\}$. In other terms, for each point (i,j) of the internal active zone of σ, one permutation is created by inserting (i,j) in σ (see Figures 4.5 and 4.6).

Theorem 6. The operator $\vartheta^* = \vartheta \cup \vartheta'$ satifies Proposition 16 of Chapter 3 for the set of all permutations, and consequently it produces a generating tree $\mathcal{T}$ for permutations.

Proof. Let (i,j) be a point of the internal active zone $\mathcal{I}(\sigma)$ of a permutation σ. The first three properties above ensure that (i,j) will be an internal point of σ_{+ij}, and the last one allows to identify the last inserted point (i,j) as the leftmost internal blocking point in σ_{+ij}, thus making the insertion reversible: given a permutation σ', either it has no internal blocking point, so that σ' is generated by ϑ and its father is $\sigma'_{/u_{\sigma'}}$; or σ' has a leftmost internal blocking point (i,j) and the father of σ' is $\sigma'_{/(i,j)}$. □

Figure 4.7 represents a part of a branch in the generating tree.

4.3 The shape of the generating tree

We have defined a generating tree, now we need to show that it has enough regularities to admit a description in terms of finitely many parameters.

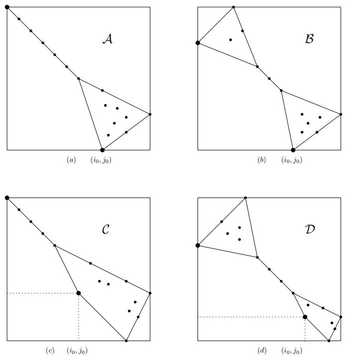


Figure 4.8: The classes $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$

4.3.1 A classification of permutations and the related parameters

In order to describe the regularities of the generating tree we will need to classify permutations according to the following properties:

- the ordinate $\sigma(1)$ of the first point is minimal, maximal or neither of the two;
- the ordinate $\sigma(i_0)$ of the special point is minimal or not;
- σ is descending ($\sigma(1) > \sigma(2)$), ascending ($\sigma(2) > \sigma(1) > 1$) or angular $\sigma(1) = 1$;
- the cover type is *covered* (if there is a point $(k, \sigma(k))$ such that $k > i_0$ and $\sigma(k) > \sigma(1)$) or *uncovered* (otherwise).

We start by giving a more intuitive classification taking into account only the first two properties: we denote by $\mathcal{A}$ (resp. $\mathcal{B}$) the class of permutations with $\sigma(1) = n$ (resp. $\sigma(1) \neq n$) and such that the point $(i_0, \sigma(i_0))$ is minimal that is, $\sigma(i_0) = 1$, and by $\mathcal{C}$ (resp. $\mathcal{D}$) the class of permutations with $\sigma(1) = n$ (resp. $\sigma(1) \neq n$) and such that the point $(i_0, \sigma(i_0))$ is not minimal (see Figure 4.8).

Then we refine our classifications taking into account the other properties, some of which are incompatible, so that the set of all permutations partitions into the following nine classes:

name	$\sigma(1)$	$\sigma(i_0)$	shape	type
A_{du}	n	1	desc	uncov
C_{du}	n	$\neq 1$	desc	uncov
$D_{d,c}$	$\neq n$	$\neq 1$	desc	cov
$D_{a,c}$	$\neq n$	$\neq 1$	asc	cov
$D_{a,u}$	$\neq n$	$\neq 1$	asc	uncov

name	$\sigma(1)$	$\sigma(i_0)$	shape	type
B_{ang}	1	1	ang	cov
$B_{d,c}$	$\neq 1, n$	1	desc	cov
$B_{a,u}$	$\neq 1, n$	1	asc	uncov
$B_{a,c}$	$\neq 1, n$	1	asc	cov

In order to describe the shape of the generating tree we will associate to the node corresponding to σ the following parameters (see Figure 4.9):

i) $k_\sigma = \sigma(l) - \sigma(i_0)$, where $(l, \sigma(l))$ is the first lrmin on the left of the special point $(i_0, \sigma(i_0))$.

ii) $i_\sigma = i_0 - 1$, the (reduced) abscissa of the special point $(i_0, \sigma(i_0))$.

iii) ℓ_σ, the number of non extremal double points preceeding the special point $(i_0, \sigma(i_0))$.

iv) p_σ, the number of points in the leftmost vertical line of the internal active zone (see Figure 4.9(c)).

v) q_σ, the number of points in the rightmost vertical line of the internal active zone.

vi) r_σ, the number of points in the bottom horizontal line of the internal active zone.

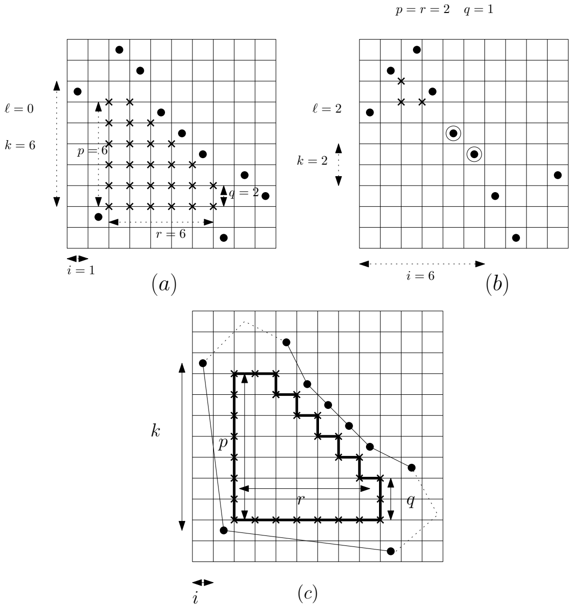


Figure 4.9: Parameters on two permutations. (a) A permutation belonging to the class $D_{d,c}$. (b) A permutation belonging to the class $D_{a,c}$, with $p = r = 2$ and $q = 1$. (c) The general shape of an internal active zone.

Observe that if $q = p$ the shape of the active zone is a rectangle, otherwise it consists in $r - (p - q)$ columns of p points followed by columns of respectively $p - 1$, down to q points.

The main feature of the generating tree induced by ϑ^* is that its shape can be described locally using "only" the above finitely many parameters:

Proposition 46. Given a class $E \in \{A, B_{ang}, B_{d,c}, B_{a,u}, B_{a,c}, C, D_{d,c}, D_{a,c}, D_{a,u}\}$ the number of sibblings of σ in the generating tree induced by ϑ^* depends only on the parameters $(E; k_\sigma, i_\sigma, \ell_\sigma, p_\sigma, q_\sigma, r_\sigma, n(\sigma))$, and more precisely the distribution of the parameters of these sibblings depends only on the above parameters of σ.

Proof. In order to prove the proposition one has to consider the various classes of permutations and describe how ϑ^* acts on it. For the sake of brievity we only give one here. ☐

4.3.2 One of several cases: the action of ϑ on Class $D_{d,c}$

Let us consider the class $D_{d,c}$: By definition a permutation σ of $D_{d,c}$ is descending and does not start with n: in particular $\sigma(2) < \sigma(1) < n$ so that $(2, \sigma(2))$ is not a double point and it is the special one, so that $i_0 = 2$ and $i_\sigma = 1$. Moreover there are no double points before the special point, so that $\ell_\sigma = 0$.

The sibblings of a permutation $\sigma \in D_{d,c}$ with parameters $(k, i = 1, \ell = 0, p, q, r, n)$ are those obtained by insertion of an external point and those obtained by insertion of an internal one, since the internal active zone can not be empty for this class. Let us describe more precisely both types of sibblings:

Insertion in the external active zone

- For each $j = 1, \ldots, k$, ϑ^* produces a permutation σ' by adding a point at position $(1, \sigma(i_0 = 2) + j)$: clearly $\sigma' \in D_{a,c}$ and the special point of σ' is $(i_0 + 1 = 3, \sigma(i_0))$, so that its first parameter is $k_{\sigma'} = j$, its second parameter is $i_{\sigma'} = i_\sigma + 1 = 2$, and its third parameter is $\ell_{\sigma'} = 0$ (that is, σ' does not have double points). Moreover the basepoint $u_{\sigma'}$ of σ' is

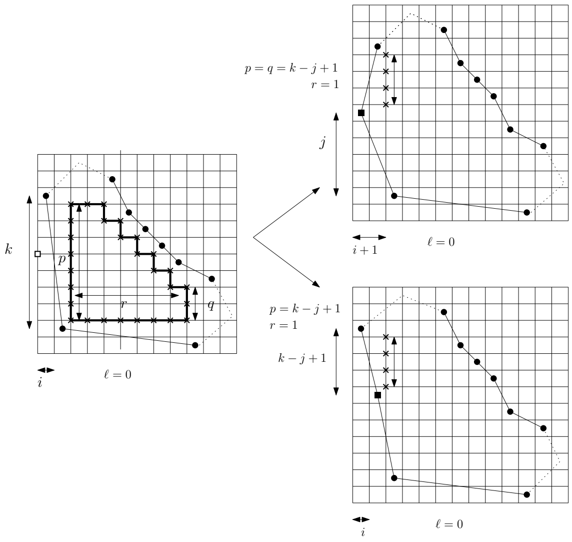


Figure 4.10: How paramaters change by inserting in the external active zone of a permutation $\sigma \in D_{d,c}$

$(1, \sigma'(1))$ and $p_{\sigma'} = k - j + 1$. On the other hand $r = 1$ since the special point of σ' has abscissa 3. Finally $q_{\sigma'} = p_{\sigma'}$ since there is only one column. (see Figure 4.10).

- For each $j = 1, \dots, k$, ϑ^* produces a permutation σ' by adding a point at position $(2, \sigma(i_0) + j)$: clearly $\sigma' \in D_{d,c}$, its first parameter is $k_{\sigma'} = k - j + 1$, its second parameter is $i_{\sigma'} = 1$, and its third parameter is 0. Moreover $p_{\sigma'} = q_{\sigma'} = k - j + 1$ and $r_{\sigma'} = 1$ as above (see Figure 4.10).

Class	k	i	ℓ	p	q	r	range
$D_{a,c}$	j	2	0	$k-j+1$	$k-j+1$	1	$j=1..k$
$D_{d,c}$	$k-j+1$	1	0	$k-j+1$	$k-j+1$	1	$j=1..k$

Insertion in the internal active zone

- By adding a point with abscissa $i_0 + i'$, with $i' \in \{i_0 \dots r - (p-q)\}$ (see Figure 4.11(a)), we get $k_{\sigma'} = k + 1$ while $i_{\sigma'} = i = 1$ and $\ell_{\sigma'} = \ell = 0$. Concerning internal parameters we have that $p_{\sigma'} = p + 1$, while $q_{\sigma'} = p + 1$ and $r_{\sigma'} = i'$ since the new internal active zone is on the left of the new point. Observe that, for a fixed i', we can obtain p different permutations with the same parameters.
- By adding a point with abscissa $i_0 + i'$, with $i' \in \{r - (p-q) + 1 \dots r\}$ (see Figure 4.11(b)), we have $k_{\sigma'} = k + 1$ while $i_{\sigma'} = i = 1$ and $\ell_{\sigma'} = \ell = 0$. Concerning internal parameters we have that $p_{\sigma'} = p + 1$, while the parameter $q_{\sigma'} = p + 1 - i'$, since i' is also the number of vertical steps in the staircase-shaped zone, while $r_{\sigma'} = r - p + q + i'$. Observe that, for a fixed i', we can obtain $p - i'$ different permutations with the same parameters.

We can resume these new classes with their parameters and their multiplicity in the following table:

Class	k	i	ℓ	p	q	r	range	multiplicity
$D_{d,c}$	$k+1$	1	0	$p+1$	$p+1$	i'	$i'=1..r-p+q$	p
$D_{d,c}$	$k+1$	1	0	$p+1$	$p+1-i'$	i'	$i'=r-p+q+1\dots r$	$p-i'$

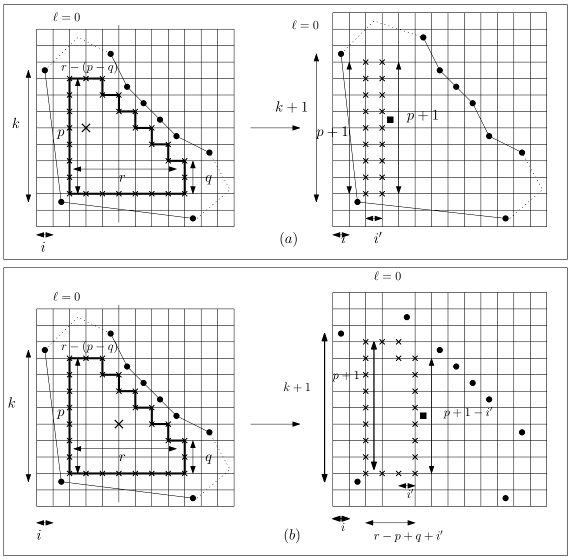


Figure 4.11: How paramaters change by inserting in the internal active zone of a permutation $\sigma \in D_{d,c}$

Then the succession rule (see Chapter 3, Section 3.1 for more details on succession rules) describing the productions of the class $D_{d,c}$ is the following:

$$
\left\{
\begin{array}{ll}
(k,j,0,p,q,r)_{D_{d,c}} \to & (1,j+1,0,k,k,j)_{D_{a,c}}(2,j+1,0,k-1,k-1,j)_{D_{a,c}}\dots(k,j+1,0,1,1,j)_{D_{a,c}} \\
 & \\
 & (k,1,0,k,k,j)_{D_{d,c}}(k-1,1,0,k-1,k-1,j)_{D_{d,c}}\dots(1,1,0,1,1,j)_{D_{d,c}} \\
 & \\
 & (k+1,j,0,p+1,p+1,1)^{p}_{D_{d,c}}(k+1,j,0,p+1,p+1,2)^{p}_{D_{d,c}}\dots \\
 & \dots(k+1,j,0,p+1,p+1,r-p+q)^{p}_{D_{d,c}} \\
 & \\
 & (k+1,j,0,p+1,p+1-(r-p+q+1),r-p+q+1)^{p-(r-p+q+1)}_{D_{d,c}} \\
 & (k+1,j,0,p+1,p+1-(r-p+q+2),r-p+q+2)^{p-(r-p+q+2)}_{D_{d,c}}\dots \\
 & \dots(k+1,j,0,p+1,p+1-r,r)^{p-r}_{D_{d,c}},
\end{array}
\right.
$$

where $(k,j,\ell,p,q,r)_E$, with $E \in \{A, B_{ang}, B_{d,c}, B_{a,u}, B_{a,c}, C, D_{d,c}, D_{a,c}, D_{a,u}\}$ represents a node of the generating tree labeled by the class it belongs to.

4.4 Taking advantage of regularities

4.4.1 Functional equations

Once we obtain the whole succession rule we can translate it into a system of functional equations. Let us associate a generating function to each class $E \in \{A, B_{ang}, B_{d,c}, B_{a,u}, B_{a,c}, C, D_{d,c}, D_{a,c}, D_{a,u}\}$ and each fixed $i \geq 0$, as follows:

$$E^{(i)}(t;u,v,w;x,y,z) = \sum_{\sigma \in E^{(i)}} t^{n(\sigma)} u^{k_\sigma} v^{j_\sigma} w^{l_\sigma} x^{p_\sigma} y^{q_\sigma} z^{r_\sigma},$$

where $E^{(i)}$ represents the class of pemutations E with i internal points. As usual the equation for a class $E^{(i)}$ is obtained from the whole succession rule upon summing over parent permutations the contribution of children that are labeled by E. The resulting system of equation is quite big so we only give here a synthetical statement describing its structure.

It will be usefull to introduce the following short hand notations: $\overline{z} \equiv 1/z$, and $E(u, v, w) \equiv E(t; u, v, w; 1, 1, 1)$, and $E[x, y, z] \equiv E(t; u, v, w; x, y, z)$.

Proposition 47. The family of generating functions $E^{(i)}(t; u, v, w; x, y, z), E \in \mathcal{C}$, satisfy a system of linear equations of the form

$$E^{(0)}[x, y, z] = tf_E\Big((F^{(0)}(u, v, w))_{F\in\mathcal{C}}; x, y, z\Big) + tuv\delta_{E=A_{du}} + t\delta_{E=B_{ang}}$$

and, for $i \geq 1$,

$$E^{(i)}[x, y, z] = tf_E\Big((F^{(i)}(u, v, w))_{F\in\mathcal{C}}; x, y, z\Big) + tu^{\delta_{E=E_{d^*}}} v^{\delta_{E=Ea*}} \Delta\big(E^{(i-1)}[x, y, z]\big)$$

where the f_E are linear operators acting by linear combination with rational coefficients in the variables u, v, w, x, y, z of the series $F^{(i)}(u, v, w)$ and their specializations using combinations of the substitutions $\{u \leftarrow uxy, v \leftarrow vz, w \leftarrow uxz, w \leftarrow w\overline{z}, u \leftarrow 1, v \leftarrow 1, w \leftarrow 1, x \leftarrow 1, y \leftarrow 1, z \leftarrow 1\}$, and where the linear differential operator Δ acts by $\Delta(G[x, y, z]) =$

$$xy^2z\left[\frac{x\frac{\mathrm{d}G}{\mathrm{d}x}[xy, 1, 1] - x\overline{z}\frac{\mathrm{d}G}{\mathrm{d}x}[xy\overline{z}, z, 1]}{1-z} + \frac{\frac{\mathrm{d}G}{\mathrm{d}y}[x, y, z] - x\overline{z}\frac{\mathrm{d}G}{\mathrm{d}x}[xy\overline{z}, z, z]}{z-y} + \frac{G[x, y, z] - G[xy\overline{z}, z, z]}{(z-y)^2}\right]$$

The operators $(f_E)_{E\in C}$ and the details of the proof are given in [36].

These equations encode multivariate linear recurrences for the coefficients of the generating functions that allow to compute first terms by iteration. Since the generating tree is controled by a finite number of parameters (k, j, ℓ, p, q, r, n) on a finite number of classes, the recursive method yields in principle a polynomial time random generating algorithm: due to the multivariate nature of the system, the algorithm is practical only for relatively small values of n (up to a few hundreds at most).

4.4.2 Enumerative results

Solving the system The equations for $E^{(0)}$ in Proposition 47 are refinements of the equations of [47]. They can be solved by repeated applications of the kernel method [5] and well chosen substitutions, to yield expression for the refined generating function of square permutations $P^{(0)}[x, y, z] = \sum_{E\in\mathcal{C}} E^{(0)}[x, y, z]$ which specializes in $P^{(0)}(1, 1, 1) = F^{(0)}(t) = R^{(0)}(C(t))$ with $R^{(0)}(u) = \frac{2(u-1)(u^3-u^2-3u+4)}{u(2-u)^4}$, in agreement with [47]. Observe that the equations for $E^{(i)}$ involves a first term which is the same as in the equation for $E^{(0)}$ and a second term which depends only on the generating functions $F^{(i-1)}$. Our strategy consistes in solving the equation:

$$E^{(i)}[x, y, z] = tf_E((F^{(i)}(u, v, w))_{F\in\mathcal{C}}; x, y, z) + h(x, y, z).$$

for a generic parameter function $h(x, y, z)$. Although this operation involves heavy symbolic manipulations, it still requires only subtle substitutions and applications of the kernel method. We were able to perform it using Maple and this yields a new set of equations expressing directly the $F^{(i)}(x, y, z)$ in terms of the $F^{(i-1)}(x, y, z)$. These new equations involve derivatives and specializations but clearly preserve the algebraicity of the multivariate generating functions and the rationality of the expression of the specialisation $E^{(i)}[1, 1, 1]$ in terms of the generating function $C(t)$. As a result we obtain:

Theorem 7. For all $i \geq 0$, the generating function $F^{(i)}(t) = \sum_{\sigma\in\mathcal{S}_n{}^{(i)}} t^n$ is algebraic of degree 2 and there exists a rational function $R^{(i)}(u)$ such that

$$F^{(i)}(t) = R^{(i)}(C(t)), \qquad \text{where C(t) is the Catalan generating function.}$$

In particular for $i = 1, 2, 3$,

$$R^{(i)}(u) = \frac{8(u-1)^{3+i}}{u^{1+i}(2-u)^{4+4i}}P^{(i)}(u),$$

where $P^{(1)}(u) = 3u^4 - 14u^3 + 17u^2 - 4u - 4$, $P^{(2)}(u) = 5u^8 - 21u^7 - 22u^6 + 246u^5 - 433u^4 + 291u^3 - 26u^2 - 60u + 24$. and $P^{(3)}(u) = 7u^{12} + 16u^{11} - 498u^{10} + 2108u^9 - 3255u^8 - 284u^7 + 6590u^6 - 7756u^5 + 3188u^4 + 960u^3 - 1856u^2 + 960u - 192$.

Extensions and conjecture The generating tree above also allows to keep track of the numbers of records of each type $(lrmin, lrmax, rlmin, rlmax)$ in the generated permutations. Introducing four further parameters in the generating functions one can come up with a refined system of equations which can in principle be solved along the same lines to prove that the generating functions remain algebraic at fixed i. However performing explicitly the computations with all 10 parameters involves formidable intermediary steps and we have not been able until now to obtain the minimal polynomial even for the case $i = 0$ of the generating function of square permutations with respect to the size and four types of records. We have not been able either to find a general explicit formula for all $i \geq 0$, or for the bivariate generating function $F(t, s) = \sum_{i\geq 0} s^i F^{(i)}(t)$, which is obviously non algebraic ($F(t, 1)$ is the ordinary generating function of permutations). We were instead able to formulate the following conjecture for all $i \geq 1$:

Conjecture 1. For any fixed i, the number of permutations of size n with i internal points satisfies

$$|S_n^{(i)}| \underset{n\to\infty}{\sim} \frac{i!}{2^{i+3}(2i)!} \cdot n^{2i+1} \cdot 4^n.$$

Chapter 5

A Generating Tree for Permutations Avoiding the Pattern $122^{+}3$

In this chapter we are going to study the family of permutations avoiding the pattern $122^{+}3$ *(trivially equivalent to those avoiding* 1–23–4*), which extend the popular* 123*-avoiding permutations (that are parallel permutations we introduced in Chapter 2, up to vertical simmetry). In particular we provide [43] an algorithmic description of a generating tree for these permutations. The algorithm leads to a recursive bijection between* 1–23–4*-avoiding permutations and Dyck paths with marked valleys. It extends a known bijection between* 123*-avoiding permutations and Dyck paths, and makes explicit the connection between these objects that was earlier obtained by Callan through a series of non-trivial bijective steps. In particular the construction is simple enough to allow for efficient exhaustive generation.*

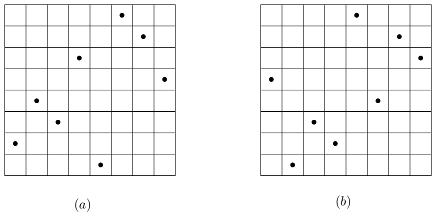
5.1 Pattern avoidance

Patterns in permutations have been occasionally studied for over a century, but in the last two decades this area has grown, with quite a number of published papers. The study of permutations avoiding some pattern has been motivated both by its combinatorial difficulty and by its appearance in some data structuring problems in computer science.

It is well known that the number of permutations of length n avoiding any one classical pattern of length 3 is the nth Catalan number, which counts a large amount of different combinatorial objects [81]. There are many other results in this direction, relating pattern avoiding permutations to various other combinatorial structures, either via bijections, or by analytic approaches.

Recently, the notion of generalized pattern in permutations has been introduced and considered. Whereas an occurrence of a classical pattern p in a permutation π is simply a subsequence of π whose letters are in the same relative order (of size) as those in p, in an occurrence of a generalized pattern, some letters of that subsequence may be required to be adjacent in the permutation. For example, the classical pattern 1234 simply corresponds to an increasing subsequence of length four, whereas an occurrence of the generalized pattern 1–23–4 would require the middle two letters of that sequence to be adjacent in π. Thus, the permutation 23145 contains 1234 but not 1–23–4.

Generalized patterns provide a significant addition to classical patterns and their connections to other combinatorial structures. In fact the non-classical generalized patterns are likely to provide richer connections to other combinatorial structures than the classical ones do. Other than combinatorics, generalized patterns find applications in other scientific areas such as, for instance, in the genome rearrangement problem, which is one of the major trends in bioinformatics and biomathematics [4, 86], and discrete tomography [50].



(a)

(b)

Figure 5.1: (a) A 122^+3-avoiding permutation; (b) A 1–23–4-avoiding permutation.

In this chapter we will study permutations avoiding the generalized pattern 122^+3, which is equivalent to the pattern 1–23–4. These patterns are of great importance since they constitute a neat but non trivial generalization of the popular pattern 123, and they have been considered in some previous papers [27, 54] leaving several open problems.

In [43] we provided an algorithmic description of a generating tree (see 3.1 for definitions on generating trees) for these permutations. Our algorithm leads to a recursive bijection between $1-23-4$-avoiding permutations and valley-marked Dyck paths. It extends a known bijection between 123-avoiding permutations and Dyck paths, and makes explicit the connection between these objects that was earlier obtained by Callan [27] through a series of non-trivial bijective steps. In particular it is simple enough to allow for efficient exhaustive generation.

5.2 Getting started

We start recalling some basic facts on pattern avoiding permutations. For more details we refer the reader to [15]. A permutation $\sigma = \sigma_1\sigma_2\dots\sigma_n$ contains $\tau = \tau_1\tau_2\dots\tau_k$ if there exist $i_1 < i_2 < \dots < i_k$ such that $\sigma_{i_a} < \sigma_{i_b}$ if and only if $\tau_a < \tau_b$. Otherwise, σ avoids τ, and we denote $AV_n(\tau)$ the set of permutations σ of $\{1,\dots,n\}$ that avoids a fixed permutation τ.

Among the several existing notions which extend the classical pattern avoidance, here we are interested in the so-called *bivincular patterns* [83, 30], and more precisely we consider the patterns 122^+3 and 1–23–4:

i) a permutation π contains the pattern 122^+3 if there are four indices $i < j < k < t$ such that $\pi_i < \pi_j < \pi_k = \pi_j + 1 < \pi_t$. We would like to point out that in the standard but heavier notation of bivincular patterns [8] these correspond to $(\pi, \emptyset, \{2,3\})$;

ii) a permutation π contains the pattern 1–23–4 if there are three indices $i < j < k$ such that $\pi_i < \pi_j < \pi_{j+1} < \pi_k$. In the standard notation of bivincular patterns [8] these correspond to $(\pi, \{2,3\}, \emptyset)$.

For instance, the permutation 4 1 7 3 5 2 6 contains the pattern 1–23–4 because of the entries $\pi_2 = 1$, $\pi_4 = 3$, $\pi_5 = 5$ $\pi_7 = 6$, whereas π does not contain the pattern 122^+3. We immediately point out that 122^+3-avoiding permutations are in one to one correspondance with 1–23–4-avoiding ones via group inversion, i.e. $\pi \in AV(122^+3)$ if and only if $\pi^{-1} \in AV(1–23–4)$, so we can study one or the other set equivalently (see Figure 5.1 for an example, the two permutations are inverse each other.)

Permutations avoiding 123 and permutations avoiding 122^+3. Our interest in permutations avoiding 122^+3 is motivated by the fact that they form a natural extension of the class of 123-avoiding permutations, which on their turn are perhaps the simplest and widely studied class of permutations counted by Catalan numbers.

We recall that 123-avoiding permutations,denoted by $AV_n(123)$, and 122^+3 permutations, denoted by $AV_n(122^+3)$, have a simple characterization in terms of left-to-right minima and right-to-left maxima:

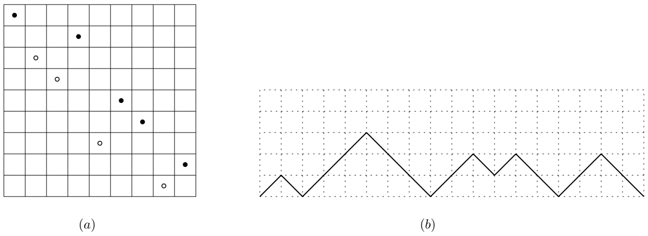

(a) (b)

Figure 5.2: (a) The 123-avoiding permutation 9 7 6 8 3 5 4 1 2; (b) The corresponding Dyck path.

Lemma 4. A permutation π of length n belongs to $AV_n(123)$ if and only if for every pair (i, j) such that $1 \leq i < j \leq n$ and $\pi_i < \pi_j$,

$$\pi_i \text{ is a left-to-right minimum } \textbf{and } \pi_j \text{ is a right-to-left maximum.}$$

Observe that this class of permutations is equivalent to the class of parallel permutations up to vertical simmetry. The previous characterization can be extended to $AV(122^+3)$ as follows.

Proposition 48. A permutation π of length n belongs to $AV_n(122^+3)$ if and only if for every pair (i, j) such that $1 \leq i < j \leq n$ and $\pi_j = \pi_i + 1$,

$$\pi_i \text{ is a left-to-right minimum } \textbf{or } \pi_j = \pi_i + 1 \text{ is a right-to-left maximum.}$$

Krattenthaler's bijection. Let us consider a Dyck path and denote up steps by $U = (1, 1)$ and down steps by $D = (1, -1)$.

In [71] Krattenthaler shows the following bijection between 123-avoiding permutations and Dyck paths. Let π be a 123-avoiding permutation, decomposed as $\pi = w_r m_r w_{r-1} m_{r-1} \dots w_1 m_1$, where $m_1, \dots, m_r$ are the right-to-left maxima of π from right to left. From the previous lemma we have that for all i the entries in w_i are left-to-right minima, so they are in decreasing order and smaller than all the entries of w_{i+1}. Krattenthaler's bijection consists in reading π from right to left and building the corresponding Dyck path form right to left as follows: each right-to-left maximum m_i is translated into $m_i - m_{i-1}$ down steps (with $m_0 = 0$), whereas each factor w_i is translated into $|w_i| + 1$ up steps. Figure 5.2 (a) shows 123-avoiding permutation 9 7 6 8 3 5 4 1 2 with its right-to-left maxima depicted in black, which is mapped in the Dyck path in Figure 5.2 (b). Observe that we already have seen this bijection in Section 3.3.1, while describing the bijection between parallel permutations and parallelogram permutominoes: indeed a parallel permutation is a 321-avoiding permutation up to vertical simmetry and a parallelogram permutomino is uniquely described by its upper Dyck path. The bijection described in Section 3.3.1 between parallel permutations and Dyck paths is the Krattenthaler's bijection.

The sequence A113227 and Dyck paths with marked valleys. As already mentioned, the sequence counting 1–23–4 avoiding permutations is not new in combinatorics. In [54] Elizalde obtained asymptotic bounds for these numbers. Moreover in [27] Callan proved that the family of AV(1–23–4)-avoiding permutations is counted by the sequence A113227 in [64], whose first terms are:

$$1, 2, 6, 23, 105, 549, 3207, 20577, 143239, 1071704, 8555388, 72442465, \dots$$

More precisely, according to [27] the nth term p_n of the sequence above can be expressed as $p_n = \sum_{j=0}^{n} p_{n,j}$, where the terms $p_{n,j}$ satisfy the recurrence relation

$$\begin{cases} p_{0,0} = 1 \\ p_{n,0} = 0 \text{ if } n \geq 1, \\ p_{n,j} = p_{n-1,j-1} + j \sum_{i=j}^{n-1} p_{n-1,i} \;\text{ for } n \geq j \geq 1. \end{cases} \tag{5.1}$$

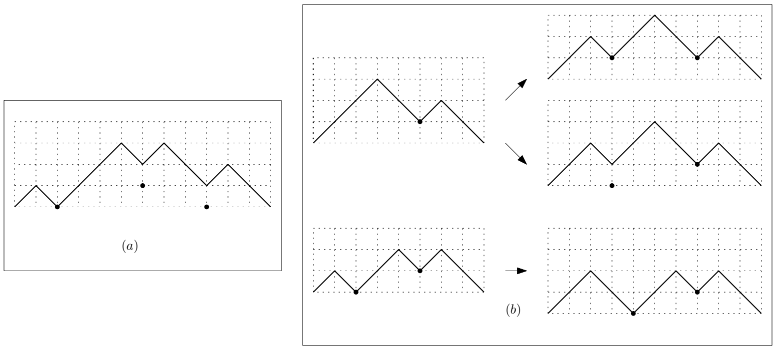


Figure 5.3: (a) A VM-Dyck path; (b) A VM-Dyck path whose first ascent has height greater or equal to 2 producing two VM-Dyck paths whose first ascent has height 2 and a VM-Dyck path whose first ascent has height 1 producing a VM-Dyck path whose first ascent has height 2.

On the one hand Callan points out that there are some "simple" families of objects counted by sequence A113227, for which it is easy to prove Recurrence (5.1): in particular Callan considers increasing ordered trees with increasing leaves and Dyck paths with marked valleys [27]. Let us briefly illustrate Callan's simple proof on the latter structures: *Dyck paths with marked valleys* are Dyck paths where every valley (i.e. factor DU) at level h is marked at a given level between 0 to h (see Figure 5.3 (a)). One can easily prove that the number of Dyck paths with marked valleys (briefly, VM-Dyck paths) of semi-length n is p_n, and more precisely, the number $p_{n,j}$ of VM-Dyck paths of length n and with first ascent of length j satisfies recurrence (5.1). Indeed a VM-Dyck path of semi-length $n > 1$ and with first ascent of length $j \geq 1$ can be uniquely obtained:

i) from a VM-Dyck path of semi-length $n-1$ and with first ascent of length $j-1$, by adding a peak UD on the top of the first ascent, thus giving the first term of (5.1) (see Figure 5.3 (b));

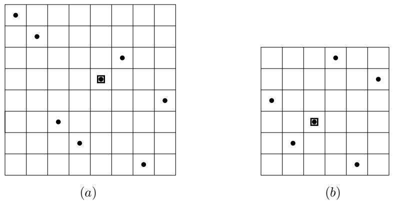
ii) from a VM-Dyck path of semi-length $n-1$ and with first ascent of length $i \geq j$ by adding a peak UD at level j of the first ascent. In this case, we produce a new valley at level $j-1$, and it can be marked in j different ways, thus giving the second term of (5.1) (see Figure 5.3 (b)).

On the other hand Recurrence (5.1) appears to be difficult to understand directly on AV_n(1–23–4). Therefore, in order to enumerate these permutations Callan presents in [27] a chain of several non trivial bijections going from AV_n(1–23–4) to increasing ordered trees with increasing leaves. In view of the various intermediary combinatorial structures involved, a natural question is to find a direct bijection between 1–23–4-avoiding permutations or 122^+3-avoiding permutations, and one of the "simple" families satisfying recurrence A113227.

5.3 Generation of $AV(122^+3)$

In this section we describe the algorithm `INSERTPOINT`, that we introduced in [43]. It takes as input a permutation π of $AV_n(122^+3)$ and two positive integers i, j, inserts a point in π in a position that depends on i and j, and return as output a permutation of $AV_{n+1}(122^+3)$.
Before describing the algorithm `INSERTPOINT` we need to define the following parameters on π:

- the length $h(\pi)$ of the (possible empty) maximal initial subsequence $n\,(n-1)\dots(n-h+1)$ of consecutive points of π, called *initial double points*. More precisely $h(\pi) = h$ if and only if $\pi = n\,n-1\dots n-h+1\,\pi_{h+1}\,\dots\,\pi_n$ and $\pi_{h+1} \neq n-h$.
- the *threshold index* of π, denoted by $t(\pi) = t$, where t is the smallest index among $\{1,\dots,n+1\}$ that satisfies the two conditions:



(a) (b)

Figure 5.4: (a) The permutation $\pi = 8\,7\,3\,2\,5\,6\,1\,4$, where $h(\pi) = 2$ and $t(\pi) = 5$; (b) The permutation $\sigma = 4\,2\,3\,6\,1\,5$, where $h(\sigma) = 0$ and $t(\sigma) = 3$. The threshold of both permutations is highlighted.

$8\,7\,3\,2\,\underline{5}\,6\,1\,4$ (2,2) $\longrightarrow$

$\underline{9}\,8\,7\,3\,2\,5\,6\,1\,4$ (3,2) $i{=}j{=}1$ — $8\,\underline{9}\,7\,3\,2\,5\,6\,1\,4$ (0,1) $i{=}j{=}2$ — $8\;7\,\underline{9}\,3\,2\,5\,6\,1\,4$ (0,2) $i{=}j{=}3$

$8\,7\,3\,\underline{9}\,2\,5\,6\,1\,4$ (0,3) $i{=}4, j{=}4$ — $8\,3\,6\,\underline{9}\,2\,5\,7\,1\,4$ (0,3) $i{=}4, j{=}3$

$8\,7\,3\,2\,\underline{9}\,5\,6\,1\,4$ (0,4) $i{=}5, j{=}5$ — $8\;3\,2\,6\,\underline{9}\,5\,7\,1\,4$ (0,4) $i{=}5, j{=}4$ — $8\,3\,6\,2\,\underline{9}\,5\,7\,1\,4$ (0,4) $i{=}5, j{=}3$

$4\,2\,\underline{3}\,6\,1\,5$ (0,2) $\longrightarrow$

$\underline{7}\,4\,2\,3\,6\,1\,5$ (1,2) $i{=}j{=}1$

$4\,\underline{7}\,2\,3\,6\,1\,5$ (0,1) $i{=}2, j{=}2$ — $4\,\underline{6}\,2\,3\,7\,1\,5$ (0,1) $i{=}2, j{=}1$

$4\,2\,\underline{7}\,3\,6\,1\,5$ (0,2) $i{=}3, j{=}3$ — $4\,2\,\underline{6}\,3\,7\,1\,5$ (0,2) $i{=}3, j{=}2$ — $5\;2\,\underline{4}\,3\,7\,1\,6$ (0,2) $i{=}3, j{=}1$

Figure 5.5: An example of the outputs of algorithm `INSERTPOINT` on a given permutation π and all the admissible values i, j. For each permutation the threshold is underlined.

(1) π_t is not a left-to-right minimum;

(2) $\pi_t = n$ OR $\pi_t + 1$ is on the left of π_t OR $\pi_t + 1 = n - h(\pi)$.

If there is no point π_t in π satisfying these two properties, then $t(\pi)$ is set to be equal to $n + 1$ (i.e. the threshold index is out of the permutation).

Figure 5.4 shows the threshold of two permutations. Observe that the condition $\pi_t + 1 = n - h(\pi)$ simply means that, if we remove the initial double points from π, then π_t is placed just below the maximal element (as in Figure 5.4 (a)). Note also that the threshold index $t(\pi)$ is larger than the parameter $h(\pi)$, because any double point is a left-to-right minimum. If $1 \leq t \leq n$, then point π_t is called the *threshold* of π.

- the parameter $s(\pi)$ such that $s(\pi) = t(\pi) - h(\pi) - 1$.

Moreover, we define the set of *admissible values* for π, as the set of pairs (i, j) such that $1 \leq i \leq t(\pi)$ and ($j = i$ or $h(\pi) < j < i$). This is the set of values of i and j for which `INSERTPOINT`(π, i, j) returns an output. Let `LENGTH` (resp. `THRESHOLD`, `DOUBLE`) be a function that computes the size n (resp. $t(\pi)$, $h(\pi)$) of $\pi \in AV_n(122^+3)$. The algorithm `INSERTPOINT` is given as Algorithm 1 below. Figure 5.5 illustrates the applications of the algorithm to the permutation $8\,7\,3\,2\,\underline{5}\,6\,1\,4$, whose threshold is underlined, for all the admissible values of i and j.

From now on most of the proofs are omitted, the reader will find the details in [43].

Algorithm 1: Insertion of a point in a 122^+3-avoiding permutation.

```
Input  : A permutation $\pi$ and two positive integers $i$ and $j$.
Output: A permutation $\tau$.
INSERTPOINT $(\pi, i, j)$;
$n$=LENGTH($\pi$);
$t$=THRESHOLD($\pi$);
$h$=DOUBLE($\pi$);
if $1 \leq i \leq t$ and $1 \leq j \leq i$ then
    if $h = 0$ then
        Order the array $[n+1, n, \pi_1, \ldots, \pi_{i-1}]$ decreasingly;
        Remove from it the last entry and call $S$ the resulting array of length $i$;
        Set $x$ be equal to the $(i+1-j)$th element of $S$;
        Build $\tau = \pi_{+ix} = \pi'_1 \ldots \pi'_{i-1}\, x\, \pi'_i \ldots \pi'_n$, where $\pi'_k = \pi_k$ if $\pi_k < x$, and $\pi'_k = \pi_k + 1$
          otherwise, for $k \in \{1, \ldots, n\}$;
        return $\tau$;
    else
        if $j = i$ then
            Build $\tau = \pi_{+i(n+1)} = \pi_1 \ldots \pi_{i-1}\,(n+1)\,\pi_i \ldots \pi_n$;
            return $\tau$;
        end
        if $h < j < i$ then
            Build $\tau = ((\pi_{-1n})_{+j(n-h)})_{+i(n+1)} =$
              $\pi'_2 \ldots \pi'_h\, \pi'_{h+1} \ldots \pi'_j\, (n-h)\, \pi'_{j+1} \ldots\, \pi'_{i-1}\, (n+1)\, \pi'_i \ldots \pi'_n$, where $\pi'_k = \pi_k$ if
              $\pi_k < n-h$, and $\pi'_k = \pi_k + 1$ otherwise, for $k \in \{1, \ldots, n\}$;
            return $\tau$;
        else
            return error;
        end
    end
else
    return error;
end
```

Proof. We start by proving that any permutation τ obtained as output of `INSERTPOINT` avoids 122^+3. We distinguish two cases:

- $h(\pi) = 0$; Let us recall that the element x is inserted in position i of the permutation π, with $i \leq t(\pi)$. If $x = n+1$ no occurrences of 122^+3 can be created. If $x = n$ then it cannot create the pattern 122^+3 by playing the role of the 2. It could create it by playing the role of 2^+ but this is impossible since $n-1$ is on the right of n, if this was not the case $n-1$ would satisfy the conditions of threshold of π and, at the same time, it would be on the left of $t(\pi)$ (since $i \leq t(\pi)$), thus leading to a contradiction. Otherwise if $x = \pi_s$, with $s \leq i$, then by construction $x+1$ lies on the left of x, that is x cannot create the pattern 122^+3 by playing the role of the 2. It could create it by playing the role of 2^+ but this is impossible. Indeed there are two possibilities: $x-1$ is on the right of x, then x can not play the role of the 2; $x-1$ is on the left of x, then it must be on the left of $x+1$ too, otherwise $x-1$ would satisfy the conditions of threshold in π and, at the same time, it would be on the left of $t(\pi)$. Then, if x creates the pattern 122^+3 in τ by playing the role of 2^+ it means that $x+1$, corresponding to x in the permutation σ, would play the role of the 2^+ too, so that we have a contradiction.
- If $h(\pi) \neq 0$ and $j = i$, with $i \leq t(\pi)$, then $n+1$ is inserted in position i and no occurrences

of 122^+3 can be produced. If $h < j < i$, then $n+1$ is inserted in position i, and the leftmost double point n is removed and the point $n-h$ is inserted in position $j < i$. Observe that since $n-h+1$ is a right-to-left maximum, then $n-h$ cannot create an occurrence of 122^+3 by playing the role of 2. Neither it could create an occurrence of 122^+3 by playing the role of the 2^+ since if $n-h-1$ is on the left of $n-h$ then it must be a left-to-right minimum, if this was not the case, it would satisfy the conditions of threshold of π and, at the same time, it would be on the left of $t(\pi)$.

Finally we introduced a second algorithm `REMOVEPOINT` (omitted here) which deletes a point from a permutation $\tau \in AV_n(122^+3)$ giving as output a triple (π, i, j), where $\pi \in AV_{n-1}(122^+3)$ and i, j depend on the deletion. Then we showed that these two algorithms are inverse one of the other, thus proving that each permutation of $AV_n(122^+3)$ is produced exactly once by the algorithm `INSERTPOINT`. Details are omitted and can be found in [43]. $\square$

Let $\vartheta : AV_n(122^+3) \to 2^{AV_n(122^+3)}$ be the operator that generates for a permutation π in $AV_n(122^+3)$ the set of permutations $\texttt{INSERTPOINT}(\pi, i, j)$ for i, j running over all admissible values for π.

Theorem 8. The operator ϑ produces a generating tree for $AV_n(122^+3)$ that can be described by means of the following *succession rule*:

$$\Omega_T = \begin{cases} (1,0) \\ \\ (h,s) \to (h+1,s)(0,1)\dots(0,h)(0,h+1)^2\dots(0,h+s)^{s+1}\,. \end{cases}$$

Proof. To obtain the succession rule we associate to each permutation π a *label* $(h(\pi), s(\pi))$, and we prove that the number and labels of the children of the node corresponding to π in the generation tree depend only on its label. More precisely, we check that the permutation $\pi = 1$ has label $(1,0)$, and for a permutation π with label (h, s), we check that the admissible values of i, j are those listed in table in Figure 5.6 together with the label of the corresponding permutations $\texttt{INSERTPOINT}(\pi, i, j)$ (Recall that the notation $(e, f)^g$ means that g permutations are produced with label (e, f)):

Admissible values	Output label
$i = j = 1$	$(h+1, s)$
$i = j = 2$	$(0, 1)$
$\vdots$	$\vdots$
$i = j = h+1$	$(0, h)$
$i = h+2,$ and $h+1 \leq j \leq h+2$	$(0, h+1)^2$
$\vdots$	$\vdots$
$i = h+s+1,$ and $h+1 \leq j \leq h+s+1$	$(0, h+s)^{s+1}$

Figure 5.6: All the admissible values of i, j for a permutation π and the label corresponding to the output of $\texttt{INSERTPOINT}(\pi, i, j)$.

Again, details of the proof can be found in [43]. $\square$

5.4 Generation of Dyck paths with marked valleys

In this section we deal with the class of VM-Dyck paths. We describe an algorithm `INSERTPEAK` such that assigning to each VM-Dyck path $\mathcal{P}$ a label (h, s) and considering all the admissible

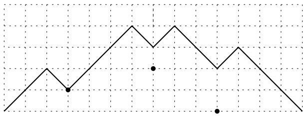


Figure 5.7: A VM-Dyck path whose representation is $UUD\,\mathbf{1}\,UUUD\,\mathbf{2}\,UDD\,\mathbf{0}\,UDDD$.

```
Algorithm 2: Insertion of a peak in a VM-Dyck path.
Input  : A VM-Dyck path $\mathcal{P}$ and two positive integers $i$ and $j$.
Output: A VM-Dyck path $\mathcal{T}$.
INSERTPEAK $(\mathcal{P}, i, j)$;
Write $\mathcal{P} = (U\,D\,\mathbf{0})^h\,U^k\,\mathcal{Q}$ according to (5.2);
if $1 \leq i \leq h+k$ and $j \leq i$ then
    if $h = 0$ then
        Build the VM-Dyck path $\mathcal{T} = U^i\,D\,(\mathbf{j-1})\,U^{k+1-i}\,\mathcal{Q}$;
        return $\mathcal{T}$;
    else
        if $j = i$ then
            Let $q = \min\{i, h+1\}$;
            Build the VM-Dyck path $\mathcal{T} = U^i\,D^q\,(\mathbf{j-q})\,(U\,D\,\mathbf{0})^{h+1-q}U^{k+q-i}\,\mathcal{Q}$;
            return $\mathcal{T}$;
        end
        if $h < j < i$ then
            Build the VM-Dyck path $\mathcal{T} = U^i\,D^{h+1}\,(\mathbf{j-h-1})\,U^{k+h+1-i}\,\mathcal{Q}$;
            return $\mathcal{T}$;
        else
            return error;
        end
    end
else
    return error;
end
```

outputs of `INSERTPEAK`$(\mathcal{P}, i, j)$, for any i, j, we obtain the same succession rule Ω_T. A VM-Dyck path $\mathcal{P}$ of semi-length n is a pair (P, v) where P is Dyck path of semi-length n (the *underlying Dyck path* of $\mathcal{P}$), and $v = (v_1, \ldots, v_m)$ is an array whose ith entry indicates the level at which the ith valley of P is marked (with m the number of valleys of P). We use a representation of $\mathcal{P}$, obtained by replacing the occurrence of the ith factor DU in P (i.e., the ith valley of P) by the factor $D\,\mathbf{v}_i\,U$ to indicate the mark level. See Figure 5.7 for an example. Let $\mathcal{P}$ be a VM-Dyck path. The path $\mathcal{P}$ can be decomposed in a unique way as:

$$\mathcal{P} = (UD\,\mathbf{0})^h U^k \mathcal{Q} \tag{5.2}$$

where $\mathcal{Q}$ is a right factor of Dyck path with marked valleys and $h \geq 0$ and $k > 0$ are maximal for this decomposition. Observe that this representation is not possible if the underlying Dyck path is a sequence of peaks, and in this case we write $\mathcal{P} = (UD\,\mathbf{0})^h\,UD$, and set $k = 0$.

Moreover, we introduce the parameter $s = k - 1$ if $k \neq 0$, or $s = 0$ otherwise.

In Figure 5.8 is shown an example of the application of `INSERTPEAK` to a given VM-Dyck path, for any admissible values i, j. By construction we have that:

Theorem 9. The algorithm `INSERTPEAK` generates VM-Dyck paths and it induces a generation tree that can be described by means of the rule Ω_T.

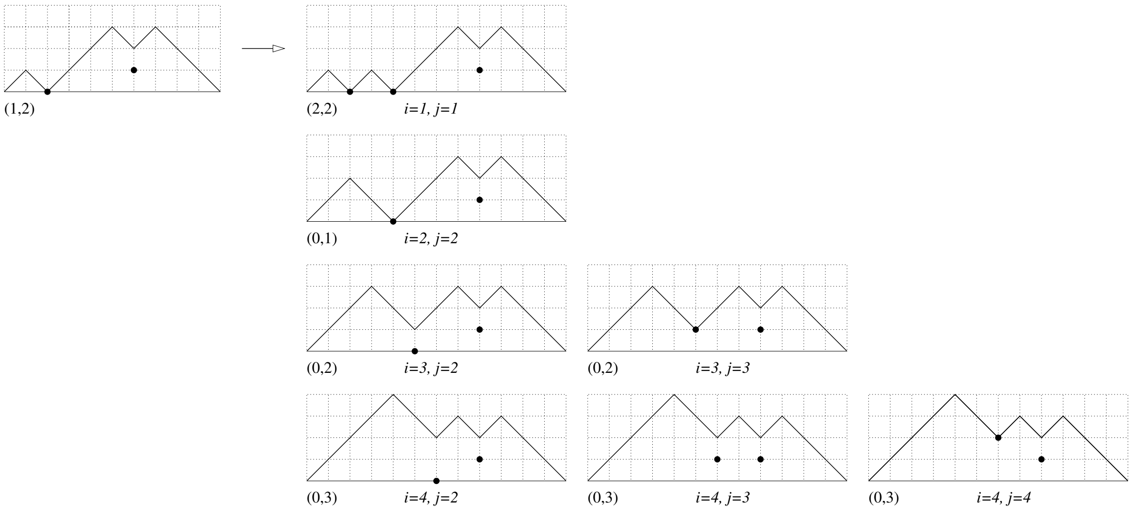


Figure 5.8: All the VM-Dyck paths produced by the VM-Dyck path $\mathcal{P}$ on the left through the application of $\texttt{INSERTPEAK}(\mathcal{P}, i, j)$. For each VM-Dyck path the corresponding label in Ω_T is reported.

Proof. Observe that the output $\mathcal{T}$ of `INSERTPEAK` is a VM-Dyck path. Indeed, roughly speaking, the operations of `INSERTPEAK` consist in adding a pick of height 1 at the beginning of a VM-Dyck path, or trasforming the first sequence of h picks of height 1 in a big pick of height $h+1$ and inserting it in the first ascent of a VM-Dyck path, or transforming i of the first h little picks in a big pick of height $i+1$, and finally by labeling the new valleys. Then the output is still a Dyck path with labeled valleys. Following the same lines as for permutations in $AV_n(122^+3)$, we introduced another algorithm `REMOVEPEAK` and prove that `INSERTPEAK` and `REMOVEPEAK` are inverse one of the other, thus proving that each VM-Dyck path is produced exactly once by the algorithm `INSERTPEAK`. The algorithm `REMOVEPEAK` and details about the succession rule are omitted and can be found in [43].

□

5.5 A bijection between $AV(122^+3)$ and Dyck paths with marked valleys

Now we can recursively define a function φ which maps a permutation in $AV_n(122^+3)$ onto a VM-Dyck path of size n:

$$\begin{cases} \varphi(1) = UD & \text{if } n = 1 \\ \\ \varphi(\texttt{INSERTPOINT}(\pi,\texttt{i},\texttt{j})) = \texttt{INSERTPEAK}(\varphi(\pi),\texttt{i},\texttt{j}) & \text{if } n > 1. \end{cases}$$

The fact that φ is a bijection follows from the fact that in the generating tree of Ω_T each 122^+3-avoiding permutation of size n (resp. VM-Dyck path of semi-length n) is uniquely identified by a path from the root to a node at level n. Thus φ maps a permutation in $AV_n(122^+3)$ onto the VM-Dyck path which corresponds to the same path in Ω_T (see Figure 5.9).

When restricting to 123-avoiding permutations, we observe that we recover a well known construction:

Corollary 1. The restriction of φ to 123-avoiding permutations determines a bijection between 123-avoiding permutations of length n and Dyck paths of semi-length n which is precisely Krattenthaler's bijection.

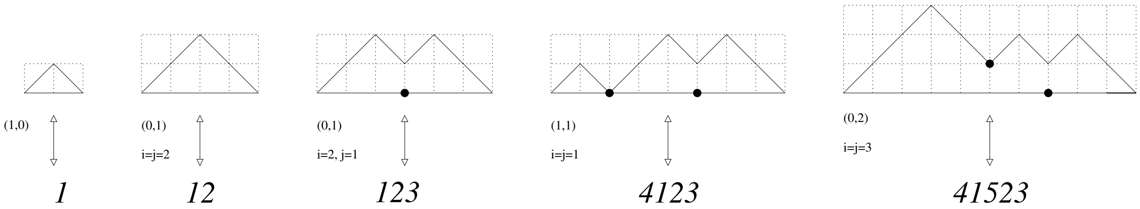


Figure 5.9: The growth of a VM-Dyck path starting from the path UD and the corresponding 122^+3-avoiding permutations via the bijection φ. At each step the values i, j and the label of the object in Ω_T is reported.

Proof. Omitted. □

The succession rule describing the growth of 123-avoiding permutations presented in Corollary 1, denoted by Ω_R, is a neat restriction of Ω_T:

$$\Omega_R = \begin{cases} (1,0) \\ \\ (h,s) \rightarrow (h+1,s)(0,1)\dots(0,h+s)\,. \end{cases}$$

Finally, adapting the general strategy of [3], the fact that we have a generating tree for $AV(122^+3)$ with finitely many labels (two labels in our case) and such that each node produces at least two valid children implies an amortized constant time generation algorithm (CAT) for codes of permutations of $AV_n(122^+3)$.

Chapter 6

Fighting fish

Fighting fish is a model that we very recently introduced in [44] as combinatorial structures made of square tiles that form two dimensional branching surfaces. A main feature of these fighting fish is that the area of uniform random fish of size n scales like $n^{5/4}$ *as opposed to the typical* $n^{3/2}$ *area behavior of the staircase or direct convex polyominoes that they generalize.*

In this chapter we concentrate on enumerative properties of fighting fish. First we will show how to obtain the generating functions of fighting fish using essentially Temperley's approach, that is a decomposition in vertical slices, but with a branching structure that leads to non linear equations. Then will present another decomposition, that extends to fighting fish the classical wasp-waist decomposition of polyominoes [21]. Using the resulting equation we will obtain a new refined generating function according to different parameters in the fighting fish, that are the numbers of left and right-lower free edges, fin length and number of tails. In particular we will show that the number of fighting fish with i *left-lower free edges and* j *right-lower free edges is equal to*

$$\frac{(2i+j-2)!(2j+i-2)!}{i!j!(2i-1)!(2j-1)!}.$$

These numbers are known to count rooted planar non-separable maps with $i+1$ *vertices and* $j+1$ *faces, or two-stack-sortable permutations with respect to ascending and descending runs, or left ternary trees with respect to vertices with even and odd abscissa. However there is not an explicit bijection between our fish and such structures.*

As reported in the `Encyclopaedia Britannica` "the *Siamese fighting fish* (*Betta splendens*) is a freshwater tropical fish of the family Osphronemidae (order Perciformes), noted for the pugnacity of the males toward one another. The Siamese fighting fish, a native of Thailand, was domesticated there for use in contests. Combat consists mainly of fin nipping and is accompanied by a display of extended gill covers, spread fins, and intensified colouring".

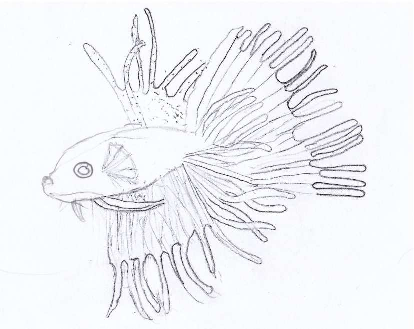


Figure 6.1: Siamese fighting fish by Camilla.

6.1 Introduction to fighting fish

In a recent paper [44] we introduced a new family of combinatorial structures which we called *fighting fish* since they are inspired by the aquatic creatures known under the same name (see

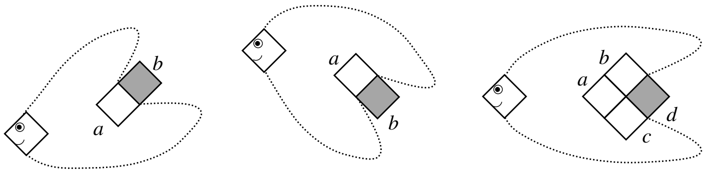


Figure 6.2: The three ways to grow a fish by adding a cell.

Figure 6.1).

The easiest description of fighting fish is that they are built by gluing together unit squares of cloth along their edges in a directed way that generalize the iterative construction of directed convex polyominoes and directed polyominoes without holes (see Chapter 2 for definitions on polyominoes).

More precisely, we consider 45 degree tilted unit squares which we call *cells*, and we call the four edges of these cells *left-upper edge*, *left-lower edge*, *right-upper edge* and *right-lower edge* respectively. We intend to glue cells along edges, and we call *free* an edge of a cell which it is not glued to an edge of another cell. All fighting fish are then obtained from an initial cell called the *head* by attaching cells one by one in one of the three following ways: (see Figure 6.2)

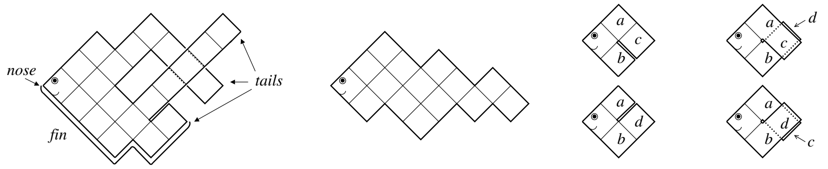
- Let a be a cell already in the fish whose right-upper edge is free; then glue the left-lower edge of a new cell b to the right-upper edge of a.
- Let a be a cell already in the fish whose right-lower edge is free; then glue the left-upper edge of a new cell b to the right-lower edge of a.
- Let a, b and c be three cells already in the fish and such that b (resp. c) has its left-lower (resp. upper) edge glued to the right-upper (resp. lower) side of a, and b (resp. c) has its right-lower (resp. right-upper) edge free; then simultaneously glue the left-upper and lower edges of a new cell d respectively to the right-lower edge of b and to the right-upper edge of c.

While this description is iterative we are interested in the objects that are produced, independently of the order in which cells are added: a *fighting fish* is a collection of cells glued together edge by edge that *can* be obtained by the iterative process above. The *head* of the fighting fish is the only cell with two free left edges, its *nose* is the leftmost point of the head; a *final cell* is a cell with two free right edges, and the corresponding *tail* is its rightmost point; the *fin* is the path that starts from the nose of the fish, follows its border counterclockwise, and ends at the first tail it meets (see Figure 6.3 (a)).

The *size* of a fighting fish is the number of lower free edges (which is easily seen to be equal to the number of upper free edges). Moreover, the *left size* (resp. *right size*) of a fighting fish is its number of left-lower free edges (resp. right lower free edges). Clearly, the left and right size of a fish sum to its size. The *area* of a fighting fish is the number of its cells. Each cell of a fighting fish can be divided by a vertical edge into two scales, called respectively the left and the right scale (see Figure 6.4).

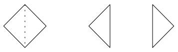
We observe that the vertical edges of the scales of a fighting fish form vertical paths that cut it into vertical strips: each strip consists of an alternating sequence of left and right scales, starting with a free (left-lower or right-lower) edge and ending with a free (left-upper or right-upper) edge. In particular a fighting fish has the same number of upper and lower free edges.

Examples of fighting fish are parallelogram polyominoes, directed convex polyominoes, and more generally directed polyominoes without holes. However, one should stress the fact that fighting fish are not necessarily polyominoes because cells can be adjacent without being glued together and more generally cells are not constrained to fit in the plane and can cover each other, as illustrated by Figure 6.3 (a),(c), and (d). The two smallest fighting fish which are not polyominoes



(a) (b) (c) (d)

Figure 6.3: (a) A fighting fish which is not a polyomino; (b) A parallelogram polyomino; (c) The two fighting fish with area 4 that are not polyominoes; (d) Two different representations of the unique fighting fish with area 5 not fitting in the plane.



(a) (b) (c)

Figure 6.4: (a) A cell. (b) A left scale. (c) A right scale.

have size 5 and area 4: as illustrated by Figure 6.3 (c), they are obtained by gluing a square a to the right-upper edge of the head, a square b to the right-lower edge of the head and either a square c to the right-lower edge of a, or a square d to the right-upper edge of b. The smallest fighting fish not fitting in the plane has size 6 and area 5, it is obtained by gluing both c to a and d to b: in the natural projection of this fighting fish onto the plane, squares c and d have the same image. Observe that we do not specify whether c is above or below d; rather we consider that the surface has a branch point at vertex $c \cap d$ (see Figure 6.3 (d)). A list of all fighting fish of area at most 4 is given in Figure 6.5.

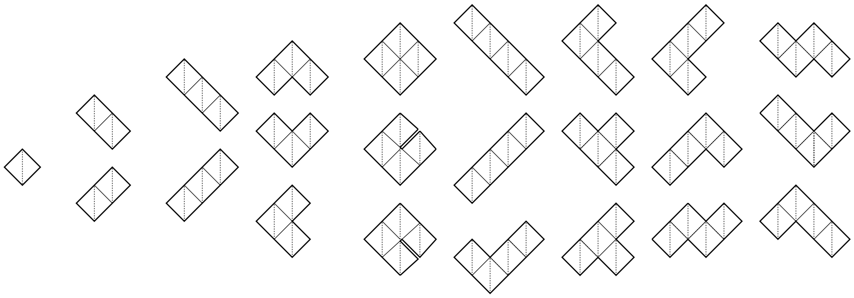


Figure 6.5: Fighting fish of area at most 4.

6.2 Enumeration of fighting fish according to their size and area

In this section we are mainly concerned with the enumeration of fighting fish with respect to their size and area. The first result we are going to prove is a remarkable formula that we obtained in[44], that suggests rich combinatorial structure and surprising connections with various known objects:

Theorem 10. The number of fighting fish with size $n+1$ is

$$\frac{2}{(n+1)(2n+1)}\binom{3n}{n}.$$

These numbers form sequence A000139 in [64] and they count, for instance, the number of 2-stack sortable permutations on n letters [87, 88, 14], the number of rooted non-separable planar maps with n edges [25, 66, 85], and the number of left ternary trees having n nodes [31].

Another highlight of our paper is the following probabilistic consequence:

Theorem 11. Let A_n denote the average area of uniform random fighting fish with size $n+1$. Then, as n goes to infinity,

$$A_n \sim \frac{3^{1/4}}{4\sqrt{2\pi}\Gamma(-\frac{1}{4})} \cdot n^{5/4}.$$

This results is worth comparing with Table 11.1 in [79]: the area of uniform random polyominoes with perimeter n in all classical non-trivial solvable models of polyominoes behaves like $n^{3/2}$. Fighting fish clearly belong to a different universality class.

6.2.1 A recursive definition

For enumerative purpose it will be useful to give a recursive description of fighting fish. We start by giving a recursive description of the slightly more general *fighting fish tails.* Let us define the set of fish tails and their heights inductively as follows:

Basis. The empty fish is the unique fish tail with height 0.

Inductive step. We define three operations:

- **operation u:** Given two fish tails f_1 of height $\ell \geq 0$ and f_2 of height $k \geq 0$, then a new fish tail f of height $\ell + 1 + k$ is obtained by adding a strip of $\ell + 1$ right scales and ℓ left scales to f_1 and attaching this strip to the end point of the leftmost vertical path of f_2.
- **operations h, h':** Given two fish tails f_1 of height $\ell \geq 1$ and f_2 of height $k \geq 0$, then a new fish tail f of height $\ell + k$ is obtained by adding a strip of ℓ right scales and ℓ left scales to f_1 and attaching this strip to the end point of the leftmost vertical path of f_2. Observe that there are two ways to do this operation, depending whether the added strip starts with a right (operation h) or a left scale (operation h').
- **operation d:** Given two fish tails f_1 of height $\ell \geq 2$ and f_2 of height $k \geq 0$, then a new fish tail f of height $\ell - 1 + k$ is obtained by adding a strip of $\ell - 1$ right scales and ℓ left scales to f_1 and attaching this strip to the end point of the leftmost vertical path of f_2.

Observe that, like fighting fish, fish tails are not constrained to fit in the plane.

The following is straightforward.

Proposition 49. Every fish tail can be obtained in a unique way using operations u, h, h', and d.

Proof. The proof is by induction: There is a unique empty fish tail. By definition a non empty fish tail can be produced from smaller fish tails using one of the four operations, and only one operation can have been used since the four operations result in different forms of upper leftmost strip. The result follows by induction on f_1 and f_2. □

Proposition 50. Fighting fish with $n+1$ free upper edges are in one-to-one correspondence with fish tails with height 1 and n free upper edges.

Proof. For each $k \geq 0$, let us call $\mathcal{F}_k$ the set of objects that can be built from an initial sequence of k right scales attached along a vertical line, by single cell additions of the three types of Figure 6.2 (b). Elements of $\mathcal{F}_1$ are clearly in bijection with fighting fish: given a fighting fish, the corresponding element of $\mathcal{F}_1$ is obtained by removing the left scale of its head. To prove the proposition we show more generally by induction that the set of fish tails is exactly $\mathcal{F} = \bigcup_k \mathcal{F}_k$.

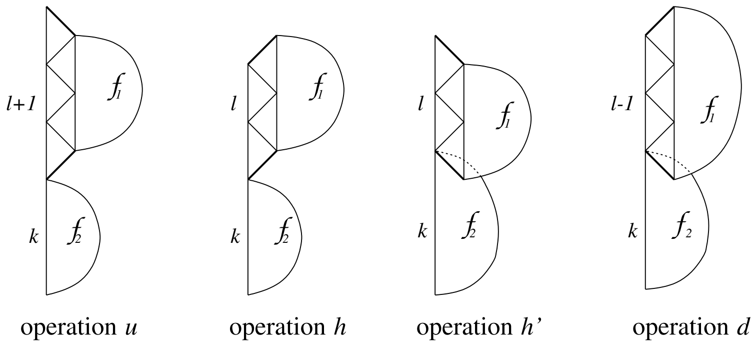


Figure 6.6: The recursive construction of fish tails: (a) operation u, (b),(c) operations h, h', (d) operation d.

First observe that the empty fish tail can be identified with the empty sequence of right scales, that is the unique element of $\mathcal{F}_0$. Then given an element f of $\mathcal{F}_k$ of size n, we prove that f is a fish tail. Assume that the first cut point p along the initial vertical line from the top of f occurs after $h \geq 1$ cells: by definition of p, in the initial right scales above p, two successive right scales have been glued to a common square using the third cell addition rule (otherwise there would be a cut point higher than p). These right and left scales form a strip S that can be used to decompose f into f_1 and f_2 using one of the four operations u, h ,h' or d above: since f_1 and f_2 are disconnected from one another, the sequence of cell additions that produces f can be split into a sequence s_1 of cell additions that produces S and f_1, and a sequence s_2 of cell addition that produces f_2. Moreover in s_1 the cell additions that produce the squares incident to S can be performed first: $s_1 = s'_1 s'_2$ with s'_2 producing f_2 from an initial sequence of right scales. This shows that f_1 and f_2 both belong to $\mathcal{F}$ so that the inductive hypothesis can be applied: f_1 and f_2 are fish tails, and so is f.

Conversely, given a fish tail f of height k obtained from two fish tails f_1 and f_2 using operation u, h, h' or d, then by the inductive hypothesis there are sequences of cell additions s_1 and s_2 producing f_1 and f_2, and these sequences can be combined with an initial sequence of cell additions applied to the k initial right scales to produce f: this implies that f belongs to $\mathcal{F}$ and concludes the proof. □

6.2.2 Generating functions

The master equation Let $\mathbf{F}(v,q) \equiv \mathbf{F}(v,q,x,t)$ denote the generating function of fish tails with variables t, v, x and q respectively marking the number of free upper edges (size), height, number of tails and area.

The following proposition immediately follows from the inductive definition of fish tails.

Proposition 51. The series $\mathbf{F}(v,q)$ is the unique power series in t satisfying

$$\begin{aligned}\mathbf{F}(v,q) \;&=\; 1 + tvq\,\mathbf{F}(v,q)\left(\mathbf{F}(vq^2,q) - 1 + x\right) + 2t\,\mathbf{F}(v,q)\left(\mathbf{F}(vq^2,q) - 1\right) + \\ &\quad \tfrac{t}{vq}\,\mathbf{F}(v,q)\left(\mathbf{F}(vq^2,q) - 1 - vq^2\,\mathbf{f}(q)\right),\end{aligned} \tag{6.1}$$

where we have denoted $\mathbf{f}(q) = [v]\mathbf{F}(v,q)$.

Proof. Equation (6.1) for $\mathbf{F}(v,q)$ follows from the recursive definition of fish tails given in Section 6.2.1. The unique fish tail with height 0 gives the term 1. Moreover, every non empty fish tail is obtained by applying one of the operation u, h, h' and d to two fish tails, and for each of these operations we have a different contribution to the expression of $\mathbf{F}(v,q)$:

- **operation** u : gives the term $tvq\,\mathbf{F}(v,q)\,\big(\mathbf{F}(vq^2,q)-1+x\big)$. In fact, the size (resp. height) of the obtained fish tail is given by the sum of the sizes (resp. heights) of f_1 and f_2 plus 1. Similarly, the number of tails of the obtained fish tail is given by the sum of the tails of f_1 and f_2, except for the case where f_1 is the empty fish. In this case, the number of tails is given by the number of the tails of f_2 plus 1. The area of the obtained fish tail is given by the sum of the areas of f_1 and f_2 plus $2\ell+1$, which explains the substitution $v := vq^2$ and the multiplication by q.

- **operations** h, h' : give the term $2t\,\mathbf{F}(v,q)\,\big(\mathbf{F}(vq^2,q)-1\big)$. We use the same arguments as above, noticing that f_1 cannot to be the empty fish.

- **operation** d : gives the term $\frac{t}{vq}\,\mathbf{F}(v,q)\,\big(\mathbf{F}(vq^2,q)-1-vq^2\,\mathbf{f}(q)\big)$. We use the same arguments as above, noticing that f_1 must be of height greater than 1. Then we have that the term $\mathbf{F}(vq^2,q)-1-v\,[v]\mathbf{F}(vq^2,q)$, where $v\,[v]\mathbf{F}(vq^2,q)$ corresponds to the case where f_1 has height equal to 1, can be rewritten as $vq^2[v]\mathbf{F}(v,q)$. The height of the obtained fish tail is given by the sum of the heights of f_1 and f_2 minus 1, whereas the area is given by the sum of the areas of f_1 and f_2 plus $2\ell-1$.

□

Equation (6.1) can be rewritten in polynomial form as

$$\mathbf{P}(\mathbf{F}(v,q),\mathbf{F}(vq^2,q),\mathbf{f}(q),v,q)=0, \tag{6.2}$$

where $\mathbf{P}(w_1,w_2,w_3,v,q)\equiv\mathbf{P}(w_1,w_2,w_3,v,q,x,t)$ reads explicitely

$$-vqw_1+vq\,+\,tv^2q^2\,w_1\,(w_2-1+x)\,+\,2t\,vqw_1\,(w_2-1)+t\,w_1\,\big(w_2-1-vq^2\,w_3\big)\,.$$

To the best of our knowledge, this type of polynomial catalytic q-equation has only been considered in the linear case. We point out that our ultimate aim is to study $\mathbf{f}(q)$, *i.e.* the generating function of fighting fish according to size, area and number of tails.

Enumeration with respect to the perimeter and number of tails Letting $q=1$ the master equation (6.1) reduces to

$$F(v)=1\,+\,tv\,F(v)\,(F(v)-1+x)\,+\,2t\,F(v)\,(F(v)-1)+\frac{t}{v}\,F(v)\,(F(v)-1-v\,f)\,, \tag{6.3}$$

where $F(v)\equiv\mathbf{F}(v,1,x,t)$ and $f\equiv\mathbf{f}(1,x,t)$. Equivalently,

$$P(F(v),F(v),f,v,x,t)=0 \tag{6.4}$$

where $P(w_1,w_2,w_3,v)\equiv\mathbf{P}(w_1,w_2,w_3,v,1,x,t)$, or in explicit form:

$$P(w_1,w_2,w_3,v)=-vw_1+v\,+\,tv^2\,w_1\,(w_2-1+x)\,+\,2t\,vw_1\,(w_2-1)+t\,w_1\,(w_2-1-v\,w_3)\,.$$

This equation is now a *polynomial equation with one catalytic variable*, in the sense of Bousquet-Mélou et Jehanne [20], and it admits an explicitely computable algebraic solution:

Theorem 12. Let $V\equiv V(x,t)$ be the unique power series solution of the equation

$$V=t\cdot\left(1+V+x\cdot\frac{V^2}{1-V}\right)^2. \tag{6.5}$$

Then,

$$f=x\cdot V-x^2\cdot\frac{V^3}{(1-V)^2}. \tag{6.6}$$

Proof. Upon differentiating (6.4) with respect to v we obtain:

$$\tfrac{\partial F}{\partial v}(v) \cdot \left[\tfrac{\partial P}{\partial w_1}(F(v), F(v), f, v) \,+\, \tfrac{\partial P}{\partial w_2}(F(v), F(v), f, v)\right] \,+\, \tfrac{\partial P}{\partial v}(F(v), F(v), f, v) = 0. \qquad (6.7)$$

The equation

$$\left[\frac{\partial P}{\partial w_1}(F(v), F(v), f, v) \,+\, \frac{\partial P}{\partial w_2}(F(v), F(v), f, 1, v)\right] = 0$$

reads

$$v = t \cdot Q(F(v), F(v), f, v, x), \quad \text{with} \quad Q(w_1, w_2, w_3, v, x) = \,(1+v)^2(w_1 + w_2 - 1) + xv^2 - vw_3,$$

so that it admits a unique power series solution $V \equiv V(x,t)$. Then the series $V \equiv V(x,t)$, $f \equiv f(x,t)$ and $F_V \equiv F(V(x,t), x, t)$ are solutions of the system of polynomial equations:

$$\begin{cases} P(F_V, F_V, f, V) = 0, \\ V = t \cdot Q(F_V, F_V, f, V), \\ \tfrac{\partial P}{\partial v}(F_V, F_V, f, V) = 0. \end{cases} \qquad (6.8)$$

Now solving for t, f and F_V by elimination we find that the algebraic curve (t, V, f, F_V) admits the parametrization

$$\left\{ t = \frac{V(1-V)^2}{(1-(1-x)V^2)^2}, \quad f = xV - x^2\frac{V^3}{(1-V)^2}, \quad F_V = 1 + x\frac{V^2}{1-V^2}, \right\} \qquad (6.9)$$

from which the result follows. ☐

Explicit formulas We are now in position to prove Theorem 10, that is, that the number f_n of fighting fish with $n+1$ free upper edges is

$$f_n = \frac{2}{(n+1)(2n+1)}\binom{3n}{n}. \qquad (6.10)$$

Proof. For $x = 1$ Theorem 12 reads

$$f(1,t) = V(1,t) - \frac{V(1,t)^3}{(1-V(1,t))^2} \quad \text{with} \quad V(1,t) = t \cdot \frac{1}{(1-V(1,t))^2,}$$

so that we can apply the Lagrange inversion formula to determine $[t^n]f(1,t)$, which gives the number of fighting fish of size $n+1$: for $f = \psi(V)$ where $V = t\phi(V)$, the Lagrange inversion formula states that $[t^n]f = \frac{1}{n}[v^{n-1}]\psi'(v)\phi(v)^n$. This yields:

$$\begin{aligned} [t^n]f(1,t) &= \frac{1}{n}\,[v^{n-1}]\,\psi'(v)\phi(v)^n = \frac{1}{n}\,[v^{n-1}]\,\left(v - \frac{v^3}{(1-v)^2}\right)'\frac{1}{(1-v)^{2n}} \\ &= \frac{1}{n}\,[v^{n-1}]\,\frac{1}{(1-v)^{2n+3}} - \frac{1}{n}\,[v^{n-1}]\,\frac{3v}{(1-v)^{2n+3}} \\ &= \frac{1}{n}\binom{3n+1}{2n+2} - \frac{3}{n}\binom{3n}{2n+2} = \frac{4\,(3n)!}{n!\,(2n+2)!}\,. \end{aligned}$$

☐

The same approach can be applied to derive the number of fighting fish of size $n+1$ with one tail:

$$[x\,t^n]f = \frac{1}{n}[x\,v^{n-1}]\left(xv - x^2\frac{v^3}{(1-v)^2}\right)'\left(1+v+x\frac{v^2}{(1-v)^2}\right)^{2n} \tag{6.11}$$

$$= \frac{1}{n}[v^{n-1}]\,(1+v)^{2n} = \frac{(2n)!}{n!\,(n+1)!}. \tag{6.12}$$

As expected the coefficient of $x\,t^n$ in f is equal to $c_n = \frac{1}{n+1}\binom{2n}{n}$, the nth Catalan number. Indeed fighting fish of size $n+1$ with only one tail are in one-to-one correspondance with parallelogram polyominoes of semi-perimeter $n+1$.

More generally in view of Equation (6.5), the coefficient V_ℓ of x^ℓ in V is rational in the Catalan generating function V_0, where V_0 is the unique power series solution of $V_0 = t(1+V_0)^2$, and in view of Equation (6.6) the same holds for the generating function of fighting fish with ℓ tails. However explicit expressions are not particularly simple and we do not report them here.

Alternatively one can consider the total number of tails:

Corollary 2. The number of fighting fish of size $n+1$ with a marked tail is:

$$s_n = \frac{1}{n}\binom{3n-2}{n-1}. \tag{6.13}$$

Proof. In order to obtain the number of fighting fish with a marked tail we calculate the function $\frac{\partial f}{\partial x}$ and then, we use again the Lagrange inversion formula to compute

$$s_n = [t^n]\frac{\partial f}{\partial x}(1,t)\,.$$

Differentiating the first and second equations in System (6.9) with respect to x and then setting $x=1$ we obtain a system of two equations in the two unknown $\frac{\partial f}{\partial x}(1,t)$ and $\frac{\partial V}{\partial x}(1,t)$:

$$\begin{cases} \dfrac{\partial f}{\partial x}(1,t) = \dfrac{\frac{\partial V}{\partial x}(1,t)(1-3V(1,t)) + V(1,t)(1-3V(1,t)) + V(1,t)^2(1+V(1,t))}{(1-V(1,t))^3} \\[2ex] \dfrac{\partial V}{\partial x}(1,t) = \dfrac{2\,t(\frac{\partial V}{\partial x}(1,t) + V(1,t)^2 - V(1,t)^3)}{(1-V(1,t))^3}. \end{cases} \tag{6.14}$$

Using $t = V(1,t)(1-V(1,t))^2$ to simplify the above system we get $\frac{\partial f}{\partial x} = V(1,t)$. The Lagrange inversion theorem then reads

$$\begin{aligned} [t^n]\frac{\partial f}{\partial x}(1,t) &= [t^n]V(1,t) = \frac{1}{n}\,[v^{n-1}]\phi(v)^n \\ &= \frac{1}{n}\,[v^{n-1}]\,\frac{1}{(1-v)^{2n}} = \frac{1}{n}\binom{3n-2}{2n-1}. \end{aligned}$$

□

Corollary 3. The average number of tails of a uniform random fighting fish of size $n+1$ is $\frac{(n+1)(2n+1)}{3\,(3n-1)}$.

The total area generating function The generating function of the total area of fighting fish is the series $A \equiv A(x,t)$ given by

$$A = \frac{\partial(q\mathbf{f}(q))}{\partial q}\Big|_{q=1} = f + \frac{\partial \mathbf{f}}{\partial q}(1),$$

that counts fighting fish weighted by their area (recall that $\mathbf{f}(q)$ counts fish tails of height 1 by their area, hence the correction factor q to account for the head of fighting fish). The series $\frac{\partial \mathbf{f}}{\partial q}(1)$ appears in the derivative of the master equation $\mathbf{P}(\mathbf{F}(v,q), \mathbf{F}(vq^2,q), \mathbf{f}(q), v, q) = 0$ with respect to q:

$$\begin{aligned}
&\tfrac{\partial}{\partial q}\mathbf{P}(\mathbf{F}(v,q), \mathbf{F}(vq^2,q), \mathbf{f}(q), v, q) = \\
&\qquad \tfrac{\partial \mathbf{F}}{\partial q}(v,q)\tfrac{\partial \mathbf{P}}{\partial w_1}(\mathbf{F}(v,q), \mathbf{F}(vq^2,q), \mathbf{f}(q), v, q) \\
&\qquad + \left(\tfrac{\partial \mathbf{F}}{\partial q}(vq^2,q) + 2vq\tfrac{\partial \mathbf{F}}{\partial v}(vq^2,q)\right)\tfrac{\partial \mathbf{P}}{\partial w_2}(\mathbf{F}(v,q), \mathbf{F}(vq^2,q), \mathbf{f}(q), v, q) \\
&\qquad + \tfrac{\partial \mathbf{f}}{\partial q}(q)\tfrac{\partial \mathbf{P}}{\partial w_3}(\mathbf{F}(v,q), \mathbf{F}(vq^2,q), \mathbf{f}(q), v, q) + \tfrac{\partial \mathbf{P}}{\partial q}(\mathbf{F}(v,q), \mathbf{F}(vq^2,q), \mathbf{f}(q), v, q) = 0.
\end{aligned} \tag{6.15}$$

Indeed, for $q = 1$ this equation can be rewritten as

$$\begin{aligned}
&\tfrac{\partial}{\partial q}\mathbf{P}(\mathbf{F}(v,q), \mathbf{F}(vq^2,q), \mathbf{f}(q), v, q) \mid_{q=1} = \\
&\qquad \tfrac{\partial \mathbf{F}}{\partial q}(v,1)\tfrac{\partial P}{\partial w_1}(F(v), F(v), f, v) \\
&\qquad + \left(\tfrac{\partial \mathbf{F}}{\partial q}(v,1) + 2v\tfrac{\partial F}{\partial v}(v)\right)\tfrac{\partial P}{\partial w_2}(F(v), F(v), f, v) \\
&\qquad + \tfrac{\partial \mathbf{f}}{\partial q}(1)\tfrac{\partial P}{\partial w_3}(F(v), F(v), f, v) + \tfrac{\partial \mathbf{P}}{\partial q}(F(v), F(v), f, v, 1) = 0.
\end{aligned}$$

Upon setting $v = V$, a simplification occurs as the coefficient of $\frac{\partial \mathbf{F}}{\partial q}(v,1)$ is precisely the defining equation for V:

$$\begin{aligned}
&2v\tfrac{\partial F}{\partial v}(V)\tfrac{\partial P}{\partial w_2}(F_V, F_V, f, V) \\
&\qquad + \tfrac{\partial \mathbf{f}}{\partial q}(1)\tfrac{\partial P}{\partial w_3}(F_V, F_V, f, V) + \tfrac{\partial \mathbf{P}}{\partial q}(F_V, F_V, f, V, 1) = 0.
\end{aligned} \tag{6.16}$$

In order to obtain an equation for $\frac{\partial \mathbf{f}}{\partial q}(1)$ we thus need to determine $\frac{\partial F}{\partial v}(V)$.

Recall that the derivative of the main equation with respect to v is

$$\tfrac{\partial F}{\partial v}(v) \cdot \left[\tfrac{\partial P}{\partial w_1}(F(v), F(v), f, v) + \tfrac{\partial P}{\partial w_2}(F(v), F(v), f, v)\right] + \tfrac{\partial P}{\partial v}(F(v), F(v), f, v) = 0. \tag{6.17}$$

One cannot simply set $v = V$ in this equation to obtain $\frac{\partial F}{\partial v}(V)$ because the coefficient of $\frac{\partial F}{\partial v}(v)$ cancels at this point by definition of V. Instead we expand the equation at $v = V$ to the second order:

$$\begin{aligned}
&\underbrace{\tfrac{\partial F}{\partial v}(V) \cdot \left[\tfrac{\partial P}{\partial w_1}(.) + \tfrac{\partial P}{\partial w_2}(.)\right] + \tfrac{\partial P}{\partial v}(.)}_{=0} \\
&\qquad +(v-V) \cdot \Bigg(\tfrac{\partial^2 F}{\partial v^2}(V) \cdot \underbrace{\left[\tfrac{\partial P}{\partial w_1}(.) + \tfrac{\partial P}{\partial w_2}(.)\right]}_{=0} \\
&\qquad\qquad + \tfrac{\partial F}{\partial v}(V) \cdot \left[\tfrac{\partial F}{\partial v}(V) \cdot \left(\tfrac{\partial}{\partial w_1} + \tfrac{\partial}{\partial w_2}\right) + \tfrac{\partial}{\partial v}\right]\left(\tfrac{\partial P}{\partial w_1} + \tfrac{\partial P}{\partial w_2}\right)(.) \\
&\qquad\qquad + \left(\left[\tfrac{\partial F}{\partial v}(V) \cdot \left(\tfrac{\partial}{\partial w_1} + \tfrac{\partial}{\partial w_2}\right) + \tfrac{\partial}{\partial v}\right]\tfrac{\partial P}{\partial v}\right)(.)\Bigg) \\
&\qquad = O((v-V)^2),
\end{aligned}$$

where $(.)$ stands for the evaluation at (F_V, F_V, f, V). Since the coefficients are zero at all orders in this expansion at $v = V$, the coefficient of $(v - V)$ yields a quadratic equation, which turns out to uniquely define $\frac{\partial F}{\partial v}(V)$ in terms of V and x: there is a polynomial

$$R(w, v, x) = -2v(1-v^2)^2(1+v)^2w^2 + 2(1-v^2)^2(1-v^2+xv^2)w - 2xv(1-v^2+xv^2),$$

quadratic in w, such that $\frac{\partial F}{\partial v}(V)$ is the unique power series solution of

$$R(\tfrac{\partial F}{\partial v}(V), x, V) = 0\,. \tag{6.18}$$

Together with Equation (6.16) this equation allows us to obtain by elimination a quadratic equation satisfied by $\frac{\partial \mathbf{f}}{\partial q}(1)$ over $\mathbb{Q}(V, x)$. Using the expression of f in terms of V a similar result is obtained for the series A.

Proposition 52. The generating function $A \equiv A(x, t)$ for the total area of fighting fish is algebraic of degree 2 over $\mathbb{Q}(V, x)$ and satisfies

$$-V(1-V)^2A^2 + 2(1-V)^2(1-V^2+xV^2)A - 4xV(1-V^2+xV^2) = 0. \tag{6.19}$$

Extracting the coefficient of x in this equation yields

$$2(1-V_0)^2(1-V_0^2)A_1 - 4xV_0(1-V_0^2) = 0,$$

where $V_0 = [x^0]V$ is a Catalan generating function satisfying $V_0 = t(1+V_0)^2$. From this equation we recover the generating function A_1 for the total area of parallelogram polyominoes, viewed as fighting fish with one tail:

$$A_1 = \frac{2V_0}{(1-V_0)^2} = \frac{2t}{1-4t}.$$

The simplification to a rational function of t is a well known feature of parallelogram polyominoes. Observe that it implies that the average area of parallelogram polyominoes of size n is $4^n/C_n$, that is of order $n^{3/2}$.

In general upon extracting the coefficient of x^ℓ in Equation (6.19) and again using the rationality of coefficients of x^i in V we obtain that the generating function of the total area of fighting fish with ℓ tails as a rational function of the Catalan generating function V_0, the unique power series solution of $V_0 = t(1+V_0)^2$.

The average area of fighting fish We now prove Theorem 11, stating that the average area A_n of fighting fish with $n+1$ free upper edges grows like $n^{5/4}$.

Proof. The series $V(1, t)$ is by definition the unique power series solution of the equation $V = t\phi(V)$ with $\phi(x) = 1/(1-x)^2$. Standard results in analytic combinatorics [62, Theorem VI.4, p. 404] ensure that V has a finite radius of convergence t_c and a singular expansion of the generic square root type:

$$V = V_c - \gamma\sqrt{1-t/t_c} + O(1-t/t_c), \tag{6.20}$$

where the constant $V_c = 1/3$ is the smallest positive root of $\phi(x) - x\phi'(x) = (1-x)(1-3x)$, $t_c = V_c/\phi(V_c) = 4/27$ and $\gamma = \sqrt{\frac{2\phi(V_c)}{\phi''(V_c)}} = \sqrt{4/27} = \frac{2}{3\sqrt{3}}$.

Equation (6.19) for $x = 1$ reads:

$$V(1-V)^2A^2 - 2(1-V)^2A + 4V = 0,$$

or equivalently

$$A = \frac{1}{V} - \frac{\sqrt{(1+V)(1-3V))}}{V(1-V)}.$$

Since A has positive coefficients, by Pringsheim's theorem [62, Theorem IV.4, p. 240] it has a positive dominant singularity, which is obtained for $V = 1/3$, that is, at $t = t_c$. From the above expression in terms of V and Expansion (6.20) we have the singular expansion

$$A \;=\; 3 - \frac{\sqrt{\frac{4}{3} \cdot 3\frac{2}{3\sqrt{3}}\sqrt{1-t/t_c})}}{2/9} + O(\sqrt{1-t/t_c}) = 3 - 3^{5/4}\sqrt{2}(1-t/t_c)^{1/4} + O((1-t/t_c)^{1/2}.$$

The standard function scale ([62, Theorem VI.1, p. 381]) and the transfert theorem ([62, Theorem VI.3, p. 390]) yield

$$[t^n]A \underset{n\to\infty}{\sim} \frac{3^{5/4}\sqrt{2}}{\Gamma(-\frac{1}{4})} n^{-5/4} t_c^{-n}.$$

Now returning to the expression of f in terms of V in System (6.9) and using again Expansion (6.20) yields $f(t) = \frac{1}{4} - \frac{3}{4}(1 - t/t_c) + \frac{\sqrt{3}}{2}(1 - t/t_c)^{3/2} + O((1 - t/t_c)^2)$, so that

$$[t^n]f \underset{n\to\infty}{\sim} \frac{3}{8\sqrt{\pi}} n^{-5/2} t_c^{-n}.$$

(This results also follows immediately from Theorem 10 using Stirling's formula). Finally the average area of fighting fish of size n is the ratio of the above two coefficients which is equivalent to $\frac{3^{1/4}}{4\sqrt{2\pi}\Gamma(-1/4)} \cdot n^{5/4}$ as n goes to infinity. □

6.3 Refined generating function

In this section we explore further the remarkable enumerative properties of fighting fish. We propose a new decomposition, introduced in [45], that extends to fighting fish the classical wasp-waist decomposition of polyominoes [21]. Using the resulting equation we compute the generating function of fighting fish with respect to the numbers of left and right-lower free edges, fin length and number of tails, and use the resulting explicit parametrization to prove the following bivariate extension of Theorem 10:

Theorem 13. The number of fighting fish with i left-lower free edges and j right-lower free edges is

$$\frac{(2i+j-2)!(2j+i-2)!}{i!j!(2i-1)!(2j-1)!} = \frac{1}{ij}\binom{2i+j-2}{j-1}\binom{2j+i-2}{i-1}. \tag{6.21}$$

We will also discuss several remarkable relations between fighting fish with marked points of various types. In particular we prove:

Theorem 14. The number of fighting fish with i left-lower free and j right-lower free edges with a marked tail is

$$\frac{(2i+2j-3)}{(2i-1)(2j-1)}\binom{2i+j-3}{j-1}\binom{2j+i-3}{i-1} \tag{6.22}$$

All these results confirm the apparently close relation of fighting fish to the well studied combinatorial structures known as non separable planar maps [25], two stack sortable permutations and left ternary trees. The closest link appears to be between fighting fish and left ternary trees, that is, ternary trees whose vertices all have non negative abscissa in the natural embedding [72]. In [45] we proved the following theorem, which we conjectured in [44]:

Theorem 15. The number of fighting fish with size n and fin length k is equal to the number of left ternary trees with n nodes, k of which are accessible from the root using only left and middle branches.

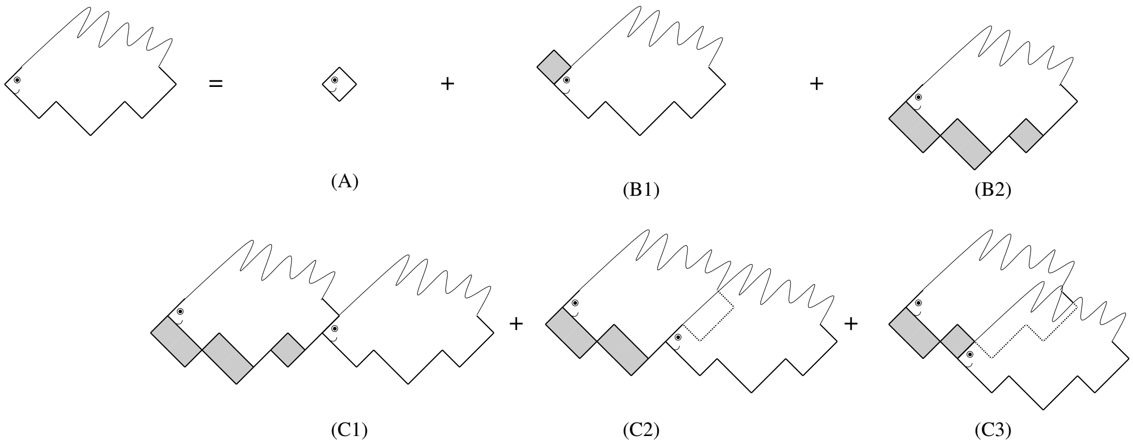
We proved the previous theorem by an independent computation of the generating function of left ternary trees with respect to n and k building on di Francesco's method [35] for counting positively labeled trees. In the rest of this section we are going to describe the new decomposition on fighting fish. We choose not to prove the previous Theorem 15 neither the following one. For readers who want to deepen this subject we refer to an extend version of our paper [46].

6.3.1 A wasp-waist decomposition

Theorem 16. Let P be a fighting fish. Then exactly one of the following cases (A), (B1), (B2), (C1), (C2) or (C3) occurs:

(A) P consists of a single cell;

(B) P is obtained from a smaller size fighting fish P_1:



(A) (B1) (B2)

(C1) (C2) (C3)

Figure 6.7: The wasp-waist decomposition.

- (B1) by gluing the right-lower edge of a new cell to the left-upper edge of the head of P_1 (Figure 6.7 (B1));
- (B2) by gluing every left edge of the fin of P_1 to the right-upper edge of a new cell, and gluing the right-lower edge and the left-upper edge of all pairs of adjacent new cells (Figure 6.7 (B2));

(C) P is obtained from two smaller size fighting fish, P_1 and P_2:

- (C1) by performing to P_1 the operation described in (B2) and then gluing the left-upper edge of the head of P_2 to the last edge of the fin of P_1 (Figure 6.7 (C1));
- (C2) by choosing a right edge r on the fin of P_1 (last edge of the fin excluded) and gluing every left edge preceding r on the fin to the right-upper edge of a new cell and, as above, gluing the right-lower edge and the left-upper edge of every pair of adjacent new cells; Then, gluing the left-upper edge of the head of P_2 to r (Figure 6.7 (C2));
- (C3) by choosing a left edge ℓ on the fin of P_1 and gluing every left edge of the fish fin preceding ℓ (included) to the right-upper edge of a new cell and, as above, gluing the right-lower edge and the left-upper edge of every pair of adjacent new cells; Then, gluing the left-upper edge of the head of P_2 to the right-lower edge of the cell glued to ℓ (Figure 6.7 (C3)).

Moreover each of the previous operations, when applied to arbitrary fighting fish P_1 and if necessarily P_2, produces valid a fighting fish.

Observe that Cases (A), (B1) and (B2) could have been alternatively considered as degenerate cases of Case (C1) where P_1 or P_2 would be allowed to be empty. Staircase polyominoes are exactly the fighting fish obtained using only Cases (A), (B1), (B2) and (C1).

Let $P(t, y, a, b; u) = \sum_P t^{\mathrm{size}(P)-1} y^{\mathrm{tails}(P)-1} a^{\mathrm{rsize}(P)-1} b^{\mathrm{lsize}(P)-1} u^{\mathrm{fin}(P)-1}$ denote the generating function of fighting fish with variables t, y, a, b and u respectively marking the size, the number of tails, the right size, the left size, the fin length, all decreased by one.

Corollary 4. The generating function $P(u) \equiv P(t, y, a, b; u)$ of fighting fish satisfies the equation

$$P(u) = tu(1 + aP(u))(1 + bP(u)) + ytabuP(u)\frac{P(1) - P(u)}{1 - u}. \tag{6.23}$$

Proof. This is a direct consequence of the previous theorem, details are omitted. □

6.3.2 The algebraic solution of the functional equation

The equation satisfied by fighting fish is a combinatorially funded polynomial equation with one catalytic variable: this class of equations was thoroughly studied by Bousquet-Mélou and Jehanne [20] who proved that they have algebraic solutions.

Theorem 17. Let $B \equiv B(t; y, a, b)$ denote the unique power series solution of the equation

$$B = t\left(1 + y\frac{abB^2}{1 - abB^2}\right)^2 (1 + aB)(1 + bB). \tag{6.24}$$

Then the generating function $P(1) \equiv P(t; y, a, b, 1)$ of fighting fish can be expressed as

$$P(1) = B - \frac{yabB^3(1 + aB)(1 + bB)}{(1 - abB^2)^2}. \tag{6.25}$$

This theorem easily implies Theorem 13 using Lagrange inversion.

Proof of Theorem 17. The proof follows closely the approach of [20], details are omitted, we only present the strategy: Rewrite Equation (6.23) as

$$(u - 1)P(u) = tu(u - 1)(1 + aP(u))(1 + bP(u)) + ytuabP(u)(P(u) - P(1)). \tag{6.26}$$

and take the derivative with respect to u:

$$\begin{aligned} &P(u) - t(2u - 1)(1 + aP(u))(1 + bP(u)) - ytabP(u)(P(u) - P(1)) \\ &= -\frac{\partial}{\partial u}P(u) \cdot (u - 1 - tu(u - 1)(a + b + 2abP(u)) - ytuab(2P(u) - P(1))) \end{aligned}$$

Now there clearly exists a unique power series U that cancels the second factor in the right hand side of the previous equation: U is the unique power series root of the equation

$$U - 1 = tU(U - 1)(a + b + 2abP(U)) + ytabU(2P(U) - P(1)). \tag{6.27}$$

Since U must also cancel the left hand side,

$$P(U) = t(2U - 1)(1 + aP(U))(1 + bP(U)) + ytabP(U)(P(U) - P(1)), \tag{6.28}$$

and, for $u = U$, Equation (6.26) reads

$$(U - 1)P(U) = tU(U - 1)(1 + aP(U))(1 + bP(U)) + ytUabP(U)(P(U) - P(1)). \tag{6.29}$$

Solving the resulting system of 3 equations for the three unknown U, $P(U)$ and $P(1)$ yields the theorem, with $P(U) = B$. ☐

The full series $P(u)$ is clearly algebraic of degree at most 2 over $\mathbb{Q}(u, B)$, but it admits in fact a parametrization directly extending the one of the theorem.

Corollary 5. Let $B(u)$ be the unique power series solution of the equation:

$$B(u) = tu\left(1 + aB(u) + yaB(u)\frac{bB(1 + aB)}{1 - abB^2}\right)\left(1 + bB(u) + ybB(u)\frac{aB(1 + bB)}{1 - abB^2}\right) \tag{6.30}$$

then

$$P(u) = B(u) - yabB(u)^2B\frac{(1 + aB)(1 + bB)(1 - abB^2 + yabB^2)}{(1 - abB^2)^2(1 - abB(u)B + yabB(u)B)}.$$

6.4 A refined conjecture

We conjectured that Theorem 15 extends to the following:

Conjecture 2. The number of fighting fish with size n, fin length k, having h tails, with i left lower free edges and j right lower free edges is equal to the number of left ternary trees with n nodes, core size k, having h right branches, with $i + 1$ non root nodes with even abscissa and j nodes with odd abscissa. This conjecture naturally calls for a bijective proof, however we have been unable to provide such a proof, except in two specific cases:

- The case of left ternary trees with at most one right branch, which are in bijections with fighting fish with at most two tails for all values of n, k, i and j
- The case of left ternary trees with h right branches and at most $h + 2$ vertices, which are in bijection with fighting fish with h tails and $h + 2$ lower edges that are not in the fin.

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