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Fighting Fish: enumerative properties

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Abstract. Fighting fish were very recently introduced by the authors as combinatorial structures made of square tiles that form two dimensional branching surfaces. A main feature of these fighting fish is that the area of uniform random fish of size n scales like $n^{5/4}$ as opposed to the typical $n^{3/2}$ area behavior of the staircase or directed convex polyominoes that they generalize.

In this extended abstract we focus on enumerative properties of fighting fish: in particular we provide a new decomposition and we show that the number of fighting fish with i left lower free edges and j right lower free edges is equal to

$$\frac{(2i+j-2)!(2j+i-2)!}{i!j!(2i-1)!(2j-1)!}.$$

These numbers are known to count rooted planar non-separable maps with $i+1$ vertices and $j+1$ faces, or two-stack-sortable permutations with respect to ascending and descending runs, or left ternary trees with respect to vertices with even and odd abscissa. However we have been unable until now to provide any explicit bijection between our fish and such structures. Instead we provide new refined generating functions for left ternary trees to prove further equidistribution results.

Keywords: enumerative combinatorics, exact formulas, algebraic generating functions

1 Introduction

In a recent paper [7] we introduced a new family of combinatorial structures which we call *fighting fish* since they are inspired by the aquatic creatures known under the same name (see this page on Betta Splendens fish). The easiest description of fighting fish is that they are built by gluing together unit squares of cloth along their edges in a directed way that generalize the iterative construction of directed convex polyominoes [9].

More precisely, we consider 45 degree tilted unit squares which we call *cells*, and we call the four edges of these cells *left upper edge*, *left lower edge*, *right upper edge* and *right lower edge* respectively. We view a cell as a simple surface with a (square) boundary, and

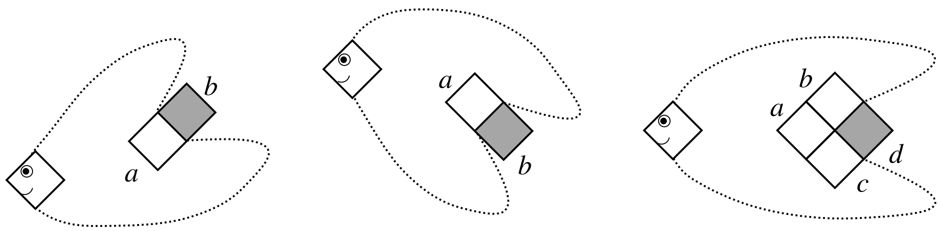


Figure 1: The three ways to grow a fish by adding a cell.

we intend to create larger and larger surfaces by gluying cells along edges. We call *free* an edge of a cell which it is not glued to an edge of another cell. All fighting fish are then obtained from an initial cell called the *head* by attaching cells one by one in one of the three following ways: (see Figure 1)

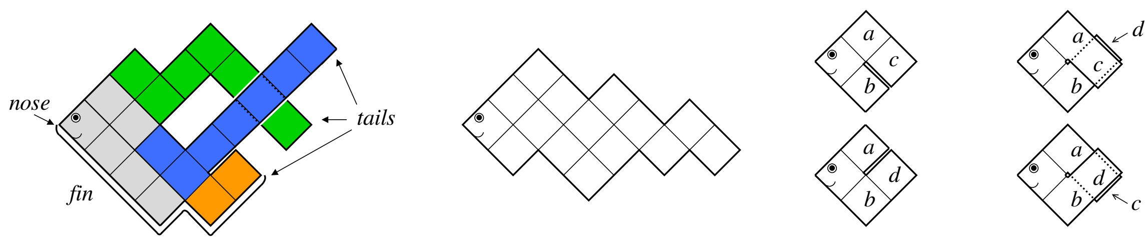
- Let a be a cell already in the fish whose right upper edge is free; then glue the left lower edge of a new cell b to the right upper edge of a.
- Let a be a cell already in the fish whose right lower edge is free; then glue the left upper edge of a new cell b to the right lower edge of a.
- Let a, b and c be three cells already in the fish and such that b (resp. c) has its left lower (resp. upper) edge glued to the right upper (resp. lower) side of a, and b (resp. c) has its right lower (resp. right upper) edge free; then simultaneously glue the left upper and lower edges of a new cell d respectively to the right lower edge of b and to the right upper edge of c.

Observe that in the first two cases the new cell is attached to the existing fish by one edge only (see also Figure 2(c)), while in the second case the cell is attached by two edges.

While this description is iterative we are interested in the objects that are produced, independently of the order in which cells are added: a *fighting fish* is a surface formed of collection of cells glued together that *can* be obtained by the iterative process above. The *head* of the fighting fish is the only cell with two free left edges, its *nose* is the leftmost point of the head; a *final cell* is a cell with two free right edges, and the corresponding *tail* is its rightmost point; the *fin* is the path that starts from the nose of the fish, follows its border counterclockwise, and ends at the first tail it meets (see Figure 2(a)).

The *size* of a fighting fish is the number of lower free edges (which is also equal to the number of upper free edges). Moreover, the *left size* (resp. *right size*) of a fighting fish is its number of left lower free edges (resp. right lower free edges). Clearly, the left and right size of a fish sum to its size. The *area* of a fighting fish is the number of its cells.

Examples of fighting fish are parallelogram polyominoes (aka staircase polyominoes), directed convex polyominoes, and more generally simply connected directed polyominoes in the sense of [9]. However, one should stress the fact that fighting fish are not



(a) (b) (c) (d)

Figure 2: (a) A fighting fish which is not a polyomino; (b) A parallelogram polyomino; (c) The two fighting fish with area 4 and size 5 that are not polyominoes; (d) Two different representations of the unique fighting fish with area 5 and size 6 not fitting in the plane.

necessarily polyominoes because cells can be adjacent without being glued together and more generally cells are not constrained to fit in the plane and can cover each other, as illustrated by Figures 2(a) and 2(d). The two smallest fighting fish which are not polyominoes have size 5 and area 4: as illustrated by Figure 2(c), they are obtained by gluing a square a to the upper right edge of the head, a square b to the right lower edge of the head and either a square c to the right lower edge of a, or a square d to the upper right edge of b. The smallest fighting fish not fitting in the plane has size 6 and area 5, it is obtained by gluing both c to a and d to b: in the natural projection of this fighting fish onto the plane, squares c and d have the same image. Observe that we do not specify whether c is above or below d; rather we consider that the surface has a branch point at vertex $c \cap d$ (see Figure 2(d)).

In [7] we obtained the generating functions of fighting fish using essentially Temperley's approach, that is, a decomposition in vertical slices. This allowed us to prove:

Theorem 1 ([7])**.** *The number of fighting fish with $n+1$ lower free edges is*

$$\frac{2}{(n+1)(2n+1)}\binom{3n}{n}. \tag{1.1}$$

We showed moreover that the average area of fighting fish of size n is of order $n^{5/4}$. This behavior suggests that, although fighting fish are natural generalizations of directed convex polyominoes, they belong to a different universality class: indeed the order of magnitude of the area of most classes of convex polyominoes is $n^{3/2}$ instead [12].

In the present extended abstract we explore further the remarkable enumerative properties of fighting fish. We propose in Section 2 a new decomposition that extends to fighting fish the classical wasp-waist decomposition of polyominoes [2]. Using the resulting equation we compute in Section 3.1 the generating functions of fighting fish with respect to the numbers of left and right lower free edges, fin length and number of tails, and use the resulting explicit parametrization to prove the following bivariate extension of Theorem 1:

Theorem 2. *The number of fighting fish with i left lower free edges and j right lower free edges is*

$$\frac{(2i+j-2)!(2j+i-2)!}{i!j!(2i-1)!(2j-1)!} = \frac{1}{ij}\binom{2i+j-2}{j-1}\binom{2j+i-2}{i-1}. \tag{1.2}$$

We also discuss in Section 3.2 several remarkable relations between fighting fish with marked points of various types. In particular we prove:

Theorem 3. *The number of fighting fish with i left lower free and j right lower free edges with a marked tail is*

$$\frac{(2i+2j-3)}{(2i-1)(2j-1)}\binom{2i+j-3}{j-1}\binom{2j+i-3}{i-1}. \tag{1.3}$$

All these results confirm the apparently close relation of fighting fish to the well studied combinatorial structures known as *non separable planar maps* [4], *two stack sortable permutations* [14, 13, 1], and *left ternary trees* [5, 10]. The closest link appears to be between fighting fish and left ternary trees, that is, ternary trees whose vertices all have non negative abscissa in the natural embedding [11]. We prove in Section 4 the following theorem, which was conjectured in [7]:

Theorem 4. *The number of fighting fish with size n and fin length k is equal to the number of left ternary trees with n nodes, k of which are accessible from the root using only left and middle branches.*

We prove this theorem by an independent computation of the generating functions of left ternary trees with respect to n and k (see Theorem 10), building on Di Francesco's method [6] for counting positively labeled trees. As discussed in Section 4 we conjecture that Theorem 4 extends to take into account the left and right size and the number of tails but the computation of the generating functions of left ternary trees according to these parameters appears to be harder and we have only been able to prove this bijectively in the case of fighting fish with at most two tails, and in the case of fighting fish with h tails but at most $h+2$ lower edges that are not in the fin.

2 A wasp-waist decomposition

Theorem 5. *Let P be a fighting fish. Then exactly one of the following cases (A), (B1), (B2), (C1), (C2) or (C3) occurs:*

(A) P consists of a single cell;

(B) P is obtained from a smaller size fighting fish P_1:

(B1) by gluing the right lower edge of a new cell to the upper left edge of the head of P_1 (Figure 3 (B1));

(B2) by gluing every left edge of the fin of P_1 to the upper right edge of a new cell, and gluing the left upper edge of every new cell to the right lower edge of the cell immediately on the left of its top vertex (Figure 3 (B2));

(C) P is obtained from two smaller size fighting fish, P_1 and P_2:

(C1) by performing on P_1 the operation described in (B2) and then gluing the upper left edge of the head of P_2 to the last edge of the fin of P_1 (Figure 3 (C1));

(C2) by choosing a right edge r on the fin of P_1 (last edge of the fin excluded) and gluing every left edge preceding r on the fin to the upper right edge of a new cell and, as above, gluing the left upper edge of every new cell to the right lower edge of the cell immediately on the left of its top vertex; Then, gluing the upper left edge of the head of P_2 to r (Figure 3 (C2));

(C3) by choosing a left edge ℓ on the fin of P_1 and gluing every left edge of the fish fin preceding ℓ (included) to the upper right edge of a new cell and, as above, gluing the left upper edge of every new cell to the right lower edge of the cell immediately on the left of its top vertex; Then, gluing the upper left edge of the head of P_2 to the right lower edge of the cell glued to ℓ (Figure 3 (C3)).

Moreover each of the previous operations, when applied to arbitrary fighting fish P_1 and if necessary P_2, produces a fighting fish.

Observe that Cases (A), (B1) and (B2) could have been alternatively considered as degenerate cases of Case (C1) where P_1 or P_2 would be allowed to be empty. Similarly Case (C1) and (C2) could have been merged into a unique case but we distinguish the two because in Case (C1) the first tail of P_1 is lost in the operation. Staircase polyominoes are exactly the fighting fish obtained using only Cases (A), (B1), (B2) and (C1).

Proof. Omitted (see [8]). □

Let $P(t, y, a, b, u) = \sum_P t^{\text{size}(P)-1} y^{\text{tails}(P)-1} a^{\text{rsize}(P)-1} b^{\text{lsize}(P)-1} u^{\text{fin}(P)-1}$ denote the generating functions of fighting fish with variables t, y, a, b and u marking the size, the number of tails, the right size, the left size, the fin length, all decreased by one. The variable t is redundant in view of variables a and b but it will be convenient to keep it.

Corollary 6. *The generating functions $P(u) \equiv P(t, y, a, b, u)$ of fighting fish satisfies the equation*

$$P(u) = tu(1 + aP(u))(1 + bP(u)) + ytabuP(u)\frac{P(1) - P(u)}{1 - u}. \tag{2.1}$$

Proof. This is a direct consequence of the previous theorem (see [8]). □

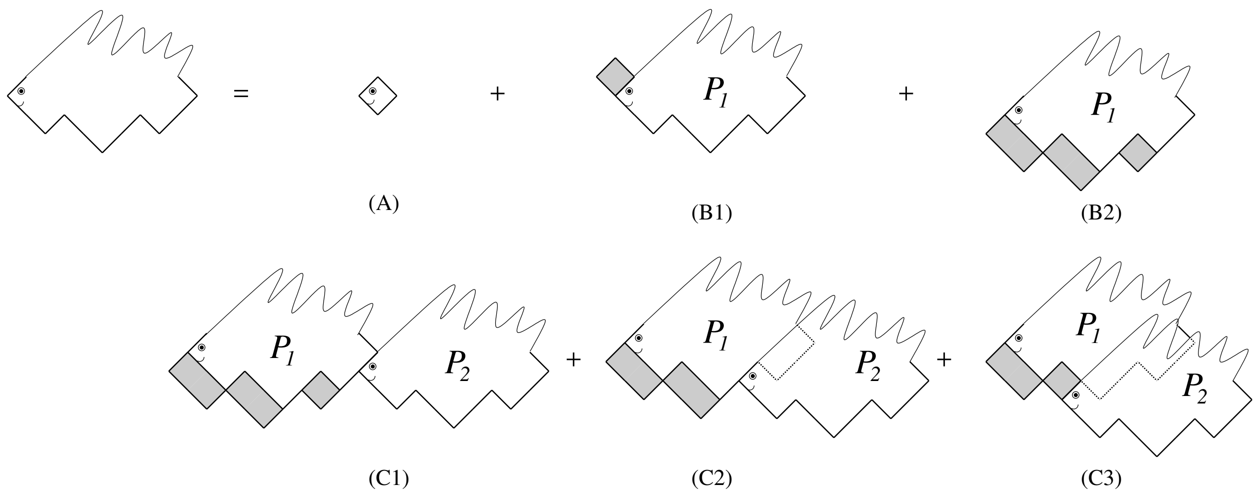


Figure 3: The wasp-waist decomposition.

3 Enumerative results for fish

3.1 The algebraic solution of the functional equation

The equation satisfied by fighting fish is a combinatorially founded polynomial equation with one catalytic variable: this class of equations was thoroughly studied by Bousquet-Mélou and Jehanne [3] who proved that they have algebraic solutions.

Theorem 7. *Let $\beta \equiv \beta(t, y, a, b)$ denote the unique power series solution of the equation*

$$\beta = t\left(1 + y\frac{ab\beta^2}{1 - ab\beta^2}\right)^2 (1 + a\beta)(1 + b\beta). \tag{3.1}$$

Then the generating function $P(1) \equiv P(t, y, a, b, 1)$ of fighting fish can be expressed as

$$P(1) = \beta - \frac{yab\beta^3(1 + a\beta)(1 + b\beta)}{(1 - ab\beta^2)^2}. \tag{3.2}$$

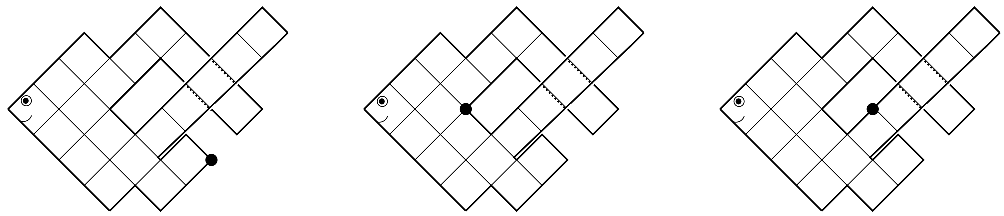
This theorem easily implies Theorem 2 using Lagrange inversion (see [8]).

Proof of Theorem 7. Our proof follows closely the approach of [3], so we omit the details (see [8]) and only present the strategy: Rewrite Equation (2.1) as

$$(u - 1)P(u) = tu(u - 1)(1 + aP(u))(1 + bP(u)) + ytuabP(u)(P(u) - P(1)). \tag{3.3}$$

and take the derivative with respect to u:

$$\begin{aligned} &P(u) - t(2u - 1)(1 + aP(u))(1 + bP(u)) - ytabP(u)(P(u) - P(1)) \\ &\quad = -\frac{\partial}{\partial u}P(u) \cdot \big(u - 1 - tu(u - 1)(a + b + 2abP(u)) - ytuab(2P(u) - P(1))\big) \end{aligned}$$



(a) (b) (c)

Figure 4: Fish with marked points: (a) a tail, (b) a branch point, (c) an upper flat point.

Now there clearly exists a power series U that cancels the second factor in the right hand side of the previous equation: U is the unique power series root of the equation

$$U - 1 = tU(U-1)(a+b+2abP(U)) + ytabU(2P(U) - P(1)). \tag{3.4}$$

The series U must also cancel the left hand side,

$$P(U) = t(2U-1)(1+aP(U))(1+bP(U)) + ytabP(U)(P(U) - P(1)), \tag{3.5}$$

and, for $u = U$, Equation (3.3) reads

$$(U-1)P(U) = tU(U-1)(1+aP(U))(1+bP(U)) + ytUabP(U)(P(U) - P(1)). \tag{3.6}$$

Solving the resulting system of 3 equations for the three unknowns U, $P(U)$ and $P(1)$ yields the theorem, with $P(U) = \beta$. □

The full series $P(u)$ is clearly algebraic of degree at most 2 over $\mathbb{Q}(y, a, b, u, \beta)$, but it admits in fact a parametrization directly extending the one of the theorem.

Corollary 8. *Let $B(u) \equiv B(t, y, a, b, u)$ be the unique power series solution of the equation:*

$$B(u) = tu\left(1 + aB(u) + yaB(u)\frac{b\beta(1+a\beta)}{1-ab\beta^2}\right)\left(1 + bB(u) + ybB(u)\frac{a\beta(1+b\beta)}{1-ab\beta^2}\right), \tag{3.7}$$

so that $B(1) = \beta$ as defined in Theorem 7, then

$$P(u) = B(u) - yabB(u)^2\beta\frac{(1+a\beta)(1+b\beta)(1-ab\beta^2+yab\beta^2)}{(1-ab\beta^2)^2(1-abB(u)\beta+yabB(u)\beta)}.$$

3.2 Fighting fish with marked points

Among the vertices on the boundary of a fish we already distinguished the nose and the tails: these are the only vertices of the boundary having no cells neither right above nor right below them. As illustrated by Figure 4, let us further call *branch point* a vertex of the boundary having cells of the fish both right above and right below it, and *upper (resp. lower) flat point* a vertex of the boundary having a cell right below but not right above (resp. right above but not right below). Let $P^{<}$ denote the generating functions of fish with a marked branch point, $P^{>}$ the generating functions of fish with a marked tail, $2P^{-}$ the generating functions of fish with a marked *flat* point, that is, a marked point which is neither the head nor a tail nor a branch point (observe that each fish has the same number of upper and lower flat points, hence the factor 2). The generating functions of fighting fish with a marked point is then:

$$P(1) + 2P^{-} + P^{>} + P^{<} = 2\frac{t\partial}{\partial t}(tP(1)). \tag{3.8}$$

From the fact that there is always one more tail than branch point we have

$$P(1) + P^{<} = P^{>} \tag{3.9}$$

so that we also have

$$P^{-} + P^{>} = P(1) + P^{<} + P^{-} = \frac{t\partial}{\partial t}(tP(1)). \tag{3.10}$$

Fighting fish with a marked branch point or tail can also be counted thanks to the variable y:

$$P^{<} = \frac{y\partial}{\partial y}P(1), \quad \text{or} \quad P^{>} = \frac{y\partial}{\partial y}(yP(1)). \tag{3.11}$$

Observe that differentiating Equation (3.3) with respect to y instead of u yields the same coefficient for the derivative of $P(u)$, which cancels for $u = U$. This simplification leads to the remarkable relations:

$$P^{<} = y\frac{\partial}{\partial y}P(1) = P(U) - P(1), \quad \text{and} \quad P^{>} = P(U). \tag{3.12}$$

The latter relation allows us to use bivariate Lagrange inversion on the parametrization $P(U) = \beta$ in Theorem 7 to prove Theorem 3, we omit the details (see [8]).

Similarly differentiating Equation (3.3) with respect to t and taking $u = U$ yields:

$$U = \frac{1}{1 - V} \quad \text{where} \quad V = ytab\frac{t\partial}{\partial t}P(1) = ytab(P^{-} + P^{<}). \tag{3.13}$$

Equations (3.12) and (3.13) admit direct combinatorial interpretations (see [8]).

4 Fighting fish and left ternary trees: the fin/core relation

A *ternary tree* is a finite tree which is either empty or contains a root and three disjoint ternary trees called the left, middle and right subtrees of the root. Given an initial root label j, a ternary tree can be naturally embedded in the plane in a deterministic way: the root has abscissa j and the left (resp. middle, right) child of a node with abscissa $i \in \mathbb{Z}$ has abscissa $i+1$ (resp, i, $i-1$). A *j-positive tree* is a ternary tree whose nodes all have non negative abscissa in the latter embedding; 0-positive trees were first introduced in the literature with the name *left ternary tree* [5, 10] (in order to be consistent with these papers one should orient the abscissa axis toward the left).

It is known that the number of left ternary trees with i nodes at even position and j nodes at odd position is given by Formula (1.2) [5, 10]. In order to refine this result we introduce the following new parameters on left ternary trees:

- Let the *core* of a ternary tree T be the largest subtree including the root of T and consisting only of left and middle edges.
- Let a *right branch* of a ternary tree be a maximal sequence of right edges.

In order to prove Theorem 4 we compute the generating functions of j-positive trees according to the number of nodes and nodes in the core.

4.1 A refined enumeration of j-positive trees

Let $\delta \equiv \beta(t, 1, 1, 1)$ be the specialization of the unique power series solution of Equation (3.1) for $y = a = b = 1$, so that

$$\delta = \frac{t}{(1-\delta)^2}.$$

Let furthermore $\tau \equiv \tau(t) = 1/(1-\delta)$ and $X \equiv X(t)$ be the unique formal power series solution of the equation

$$X = (1 + X + X^2)\delta.$$

The series τ satisfies $\tau = 1 + t\tau^3$, it is thus a generating function of ternary trees.

Building on Di Francesco's educated guess and check approach [6], Kuba obtained a formula for the generating functions of j-positive trees:

Theorem 9 ([6, 11])**.** *The generating functions $\tau_j \equiv \tau_j(t)$ of j-positive trees with respect to the number of nodes is given for all $j \geq 0$ by the explicit expression:*

$$\tau_j = \tau \frac{(1 - X^{j+5})(1 - X^{j+2})}{(1 - X^{j+4})(1 - X^{j+3})}.$$

Now the generating function of ternary trees is easily refined to take into account the number of nodes in the core: let

$$T(u) = 1 + tuT(u)^2\tau \quad \text{and} \quad D(u) = tuT(u)^2,$$

where $T(1) = \tau$ and a comparison with the specialization of Equation (3.7) for $y = a = b = 1$ shows that $D(u) \equiv B(t, 1, 1, 1, u)$. In particular the generating function of fighting fish according to the size and the fin length, given by Corollary 8, can be written as

$$P(u) = D(u) - D(u)^2 \frac{\delta}{(1-\delta)^2} = T(u)(1+\delta) - T(u)^2\delta - 1. \tag{4.1}$$

Theorem 10. *The generating functions $T_j(u) \equiv T_j(t, u)$ of j-positive trees with respect to the number of nodes and size of the core is given for $j \geq -1$ by*

$$T_j(u) = T(u) \frac{H_j(u)}{H_{j-1}(u)} \frac{1 - X^{j+2}}{1 - X^{j+3}} \tag{4.2}$$

where for all $j \geq -2$,

$$H_j(u) = (1 - X^{j+1})XT(u) - (1 + X)(1 - X^{j+2}).$$

Theorem 10, together with Corollary 8 as rewritten above, directly implies Theorem 4.

Proof of Theorem 4. We want to show that $T_0(u) = 1 + P(u)$: we have

$$T_0(u) = T(u) \frac{H_0(u)}{H_{-1}(u)} \frac{1 - X^2}{1 - X^3},$$

and by definition $H_{-1} = -(1 + X)(1 - X)$ and $H_{-2}(u) = (X - 1)T(u)$, so that

$$T_0(u) = T(u) \frac{(1 - X)XT(u) - (1 + X)(1 - X^2)}{-(1 - X^2)} \frac{1 - X^2}{1 - X^3} = -T(u)^2\beta + T(u)(1 + \beta),$$

which coincides with Equation (4.1) up to a constant term for the empty tree. □

Proof of Theorem 10. In order to prove the theorem it is sufficient to show that the series given by the right hand side of Equation (4.2) satisfies for all $j \geq -1$ the equation:

$$T_j(u) = 1 + tuT_{j+1}(u)T_j(u)\tau_{j-1} \tag{4.3}$$

where τ_j is given by Theorem 9, with the convention that $\tau_{-2} = 0$: indeed the system of Equations (4.3) clearly admits the generating functions of j-positive ternary trees as its unique power series solutions. The case $j = -1$ is immediate:

$$T_{-1}(u) = T(u) \frac{H_{-1}(u)}{H_{-2}(u)} \frac{1 - X}{1 - X^2} = 1.$$

Let now $j \geq 0$, then the right hand side of Equation (4.3) reads

$$\begin{aligned} 1+tuT(u)^2\tau \cdot \frac{H_{j+1}(u)}{H_{j-1}(u)} \cdot \frac{1-X^{j+4}}{1-X^{j+3}} \cdot \frac{1-X^{j+1}}{1-X^{j+4}} \\ = \frac{H_{j-1}(u)(1-X^{j+3}) + (T(u)-1)H_{j+1}(u)(1-X^{j+1})}{H_{j-1}(u)(1-X^{j+3})} \end{aligned} \tag{4.4}$$

and we want to show that this is equal to the right hand side of Equation (4.2). Now

$$\begin{aligned} H_{j-1}(u)(1-X^{j+3}) &= (1-X^{j+3})(1-X^j)XT(u) - (1+X)(1-X^{j+1})(1-X^{j+3}), \\ -H_{j+1}(u)(1-X^{j+1}) &= -(1-X^{j+1})(1-X^{j+2})XT(u) + (1+X)(1-X^{j+3})(1-X^{j+1}), \\ T(u)H_{j+1}(u)(1-X^{j+1}) &= (1-X^{j+1})(1-X^{j+2})XT(u)^2 - (1+X)(1-X^{j+3})(1-X^{j+1})T(u), \end{aligned}$$

while

$$T(u)H_j(u)(1-X^{j+2}) = (1-X^{j+2})(1-X^{j+1})XT(u)^2 - (1+X)(1-X^{j+2})^2 T(u).$$

The coefficients of $T(u)^2$ and $T(u)^0$ in the numerators of (4.4) and of (4.2) are clearly matching. Upon expanding all contributions to the coefficient of $T(u)$ in these numerators in powers of X, the various terms are seen to match as well. □

4.2 A refined conjecture

In view of Theorem 2 and Theorem 4, it is natural to look for a common generalization. Indeed one can even take the number of tails into account:

Conjecture 11. *The number of fighting fish with size n, fin length k, having h tails, with i left lower free edges and j right lower free edges is equal to the number of left ternary trees with n nodes, core size k, having h right branches, with $i+1$ non root nodes with even abscissa and j nodes with odd abscissa.*

This conjecture naturally calls for a bijective proof, however we have been unable until now to provide such a proof, except in two specific cases: The case of left ternary trees with at most one right branch, which are in bijection with fighting fish with at most two tails for all values of n, k, i and j, and the case of left ternary trees with h right branches and at most $h+2$ vertices not in the core, which are in bijection with fighting fish with h tails and $h+2$ lower edges that are not in the fin.

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