

# A Combinatorial Approach to Jumping Particles: The Parallel TASEP

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**ABSTRACT:** In this article, we continue the combinatorial study of models of particles jumping on a row of cells which we initiated with the standard totally asymmetric simple exclusion process or TASEP (Duchi and Schaeffer, *Journal of Combinatorial Theory, Series A*, 110(2005), 1–29). We consider here the parallel TASEP, in which particles can jump simultaneously. On the one hand, the interest in this process comes from highway traffic modeling: it is the only solvable special case of the Nagel-Schreckenberg automaton, the most popular model in that context. On the other hand, the parallel TASEP is of some theoretical interest because the derivation of its stationary distribution, as appearing in the physics literature, is harder than that of the standard TASEP.

We offer here an elementary derivation that extends the combinatorial approach we developed for the standard TASEP. In particular, we show that this stationary distribution can be expressed in terms of refinements of Catalan numbers. © 2008 Wiley Periodicals, Inc. *Random Struct. Alg.*, 33, 434–451, 2008

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## 1. JUMPING PARTICLES AND THE TASEP FAMILY

The aim of this article is to continue the combinatorial study of a family of models of particles jumping on a row of cells that are known in the physics and probability literature as one-dimensional totally asymmetric simple exclusion processes (TASEPs for short). To define TASEPs, we first introduce a set of configurations and some rules.



Fig. 1. A TASEP configuration with  $n = 10$  cells, 5 black particles, and 5 white particles.

A TASEP configuration is a row of  $n$  cells, separated by  $n + 1$  walls (the leftmost and rightmost ones are borders). Each cell is occupied by one particle, and each particle has a type, black or white (see Fig. 1). Equivalently, a TASEP configuration can be viewed as an element of  $\{0, 1\}^n$  (with 0 for white and 1 for black particles). We denote  $S_n = \{0, 1\}^n$ , the set of TASEP configurations with  $n$  cells.

The transitions of the TASEP are based on a mapping  $\vartheta : S_n \times \{0, \dots, n\} \rightarrow S_n$  that, given a configuration  $\tau$  and a wall  $i$ , modifies  $\tau$  near  $i$  to produce a configuration  $\vartheta(\tau, i)$  according to the following rules:

- a. Rule  $\bullet|\circ \rightarrow \circ|\bullet$ : If the wall  $i$  separates a black particle (on its left) and a white particle (on its right), then two particles swap to give  $\vartheta(\tau, i)$ .
- b. Rule  $|\circ \rightarrow |\bullet$ : If the wall is the left border ( $i = 0$ ) and the leftmost cell contains a white particle, this white particle leaves the row and it is replaced by a black particle.
- c. Rule  $\bullet| \rightarrow \circ|$ : If the wall is the right border ( $i = n$ ) and the rightmost cell contains a black particle, this black particle leaves the row and it is replaced by a white particle.
- d. In the other cases, nothing happens,  $\vartheta(\tau, i) = \tau$ .

In the literature, white particles are sometimes viewed as empty cells: rules b and c can then be rephrased in terms of entrance (or creation) and exit (or destruction) of a particle. It is convenient for our purpose to maintain explicit the symmetry between black and white particles.

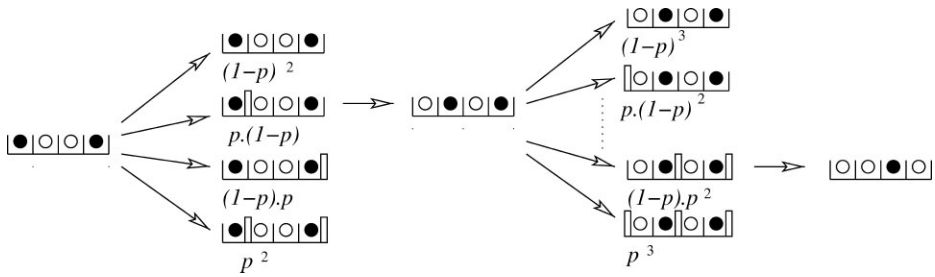
Given a configuration  $\tau$ , let  $M(\tau)$  be the set of walls on which the previous mapping  $\vartheta$  can effectively do something: inner walls with a neighborhood of the form  $\bullet|\circ$ , or borders with a neighborhood of the form  $|\circ$  or  $\bullet|$ . The definition of  $\vartheta$  can be extended to any subset  $\{i_1, \dots, i_k\}$  of  $M(\tau)$  by setting  $\vartheta(\tau, i_1, \dots, i_k) = \vartheta(\vartheta(\tau, i_1, \dots, i_{k-1}), i_k)$ . Observe that this extended mapping can be interpreted as performing moves at walls  $i_1, \dots, i_k$  in parallel since the basic mapping acts only locally and  $M(\tau)$  never contains two adjacent walls. A pair  $(\tau, A)$  with  $A \subset M(\tau)$  will be referred to as an active configuration, and from now on  $\vartheta$  is considered as a mapping from the set of active configurations into the set of TASEP configurations.

In the previous work [4], we dealt with several variants of sequential TASEP. In particular, the standard sequential TASEP with open boundaries is a Markov chain  $S^{\text{seq}}$  on the set  $S_n$  of TASEP configurations with  $n$  cells whose dynamic is defined as follows in terms of  $\vartheta$ :

- Let  $\tau = S^{\text{seq}}(t)$  be the current configuration.
- Choose uniformly a random wall  $i$  in  $\{0, \dots, n\}$ .
- Set  $S^{\text{seq}}(t + 1) = \vartheta(\tau, i)$ .

It should be observed that our description, although phrased in terms of particles, jumps and selection of a random wall, is a rigorous definition of a Markov chain on the set  $S_n$ . In particular, one can immediately use it to write down the transition matrix: the transition probability  $M^{\text{seq}}[\tau, \tau']$  from  $\tau$  to  $\tau'$  is given above as

$$M^{\text{seq}}[\tau, \tau'] = \frac{|\{i \mid \vartheta(\tau, i) = \tau'\}|}{n + 1}, \text{ for all } \tau \text{ and } \tau' \text{ in } S_n.$$



**Fig. 2.** A possible evolution, with  $n = 4$ : at each step, some active walls are selected with the given probabilities, and they trigger a transition.

However, we prefer to stick with descriptions in terms of jumping particles, which we believe convey a better intuition of the combinatorics of the problem.

The aim of the present article is to extend our approach to a more general model in which particles are allowed to jump simultaneously: the parallel TASEP is a Markov chain  $S^\parallel$  on the set of TASEP configurations whose dynamic is defined as follows in terms of  $\vartheta$  and of a fixed parameter  $0 < p < 1$ :

- Let  $\tau = S^\parallel(t)$  be the current configuration,  $M = M(\tau)$ , and  $m = m(\tau) = |M|$ .
- Choose a random subset  $A$  of  $M$  by independently giving to each wall of  $M$  probability  $p$  to be taken. In other terms, the probability that  $A = \{i_1, \dots, i_k\}$  for some given distinct elements  $i_1, \dots, i_k$  of  $M$  is  $p^k(1 - p)^{m-k}$ . The walls in  $A$  are referred to as the *active walls* of the active configuration  $(\tau, A)$ .
- Then set  $S^\parallel(t + 1) = \vartheta(\tau, A)$ .

Figure 2 illustrates the application of these rules, with active walls appearing as  $\parallel$ . Observe that the transformation  $\vartheta$  makes black particles travel from left to right, and makes white particles do the opposite. The transition probabilities of the chain  $S^\parallel$  can be alternatively written as

$$M^\parallel[\tau, \tau'] = \sum_{A|\vartheta(\tau, A)=\tau'} p^{|A|}(1 - p)^{m(\tau)-|A|}, \quad \text{for all } \tau \text{ and } \tau' \text{ in } S_n.$$

The difference between the sequential and the parallel TASEP is thus that in the first process, only one wall can trigger a move at a time, while in the second simultaneous moves can occur. Observe that if  $p$  is very small, it is unlikely that more than one particle jump at a time (since  $p^2 \ll p$ ). This implies that in the limit  $p \rightarrow 0$ , the parallel TASEP reduces to a (very slow) sequential TASEP.

We got interested in the TASEP because Derrida et al. [2, 3] proved that the stationary distribution of this Markov chain involves Catalan numbers. In [4], we gave a combinatorial explanation of this fact. In the present paper we extend our combinatorial approach to derive the stationary distribution of the parallel TASEP. Another extension, to the partially symmetric sequential TASEP, was recently developed in [7, 8]. Although we concentrate here on TASEPs with open boundary conditions, it is worth indicating that variants with periodic boundaries can be defined and studied similarly (identifying walls 0 and  $n$  and concentrating on rule  $a$ ). For these variants an alternative combinatorial interpretation was proposed in [1].

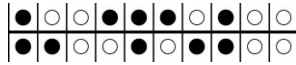


Fig. 3. A complete configuration with  $n = 14$ .

The stationary distribution of the parallel TASEP was first obtained by Schadschneider et al. (see [9] and references therein) in the easier case of periodic boundaries, and by Evans et al. [5] in the case with open boundaries. This last derivation is based on the same *matrix ansatz* approach developed by Derrida et al. for the sequential TASEP [3], but requires the introduction of a quartic (instead of quadratic) algebra. This extra complexity reflects in our combinatorial approach, in the sense that, with respect to [4], new ingredients are necessary to construct the covering Markov chain on which we rely. However, a nice feature of our approach is that we are able to remain within the realm of Catalan combinatorial structures.

To conclude this introduction, it is worth stressing the fact that the determination of the stationary distribution allows to compute some physical quantities related to the model (densities, flows, phase diagrams, . . .). Our approach, while providing a new simpler derivation and a combinatorial interpretation of the stationary distribution, leads then to the same computations as far as these next steps are concerned. We thus do not reproduce the corresponding discussions, which can be found in [5].

## 2. THE COMBINATORIAL APPROACH

Our method to study the TASEP consists in the construction of a new covering Markov chain  $X^\#$  on a set  $\Omega_n$  of complete configurations, which satisfies two main requirements: on the one hand, the stationary distribution of the TASEP chain  $S^\#$  can be simply expressed in terms of that of the covering chain  $X^\#$ ; on the other hand, the stationary distribution of the covering chain  $X^\#$  can be expressed by means of a combinatorial weight defined on the set  $\Omega_n$ .

### 2.1. The Complete Markov Chain

Since the parallel TASEP  $S^\#$  yields back the sequential TASEP  $S^{\text{seq}}$  for  $p \rightarrow 0$ , we first try to adapt directly the construction of [4] to simultaneous jumps.

Following [4], define a complete configuration of  $\Omega_n$  to be a pair of rows of particles satisfying the following constraints: (i) there is an equal number of black and white particles (the balance condition); (ii) on the left hand side of any vertical wall there are at least as many black particles as white ones (the positivity condition). An example of complete configuration is given in Fig. 3. The number of elements of  $\Omega_n$  is the  $n$ th Catalan number  $\frac{1}{n+1} \binom{2n}{n}$  (see [4], although readers with a background in combinatorics may as well recognize directly bicolored Motzkin paths in disguise).

Given a configuration  $\omega$  of  $\Omega_n$ , let  $\text{top}(\omega)$  denote its first row, which is a TASEP configuration. Still in the steps of [4], we look for a covering chain  $X^\#$  on  $\Omega_n$  that mimics in the top row the TASEP  $S^\#$ . More precisely we would like to define  $X^\#$  exactly as  $S^\#$  with  $\vartheta$  replaced by a mapping  $T$  that has nice combinatorial properties and that extends  $\vartheta$  in the following sense: given a configuration  $\omega$  of  $\Omega_n$  and a set of active walls  $A$ ,  $T$  should produce a new configuration  $\omega'$  of  $\Omega_n$  in such a way that if  $\text{top}(\omega) = \tau$ , then  $\text{top}(\omega') = \vartheta(\tau, A)$ . However such a direct extension of the construction of [4] does not seem to be sufficient to

account for the more complex dynamic of the parallel TASEP. Instead we need to introduce a further randomization step that is conveniently described in terms of colors.

A well colored configuration  $(\omega, A, R, G)$  consists of a complete configuration  $\omega$ , a set  $A$  of active walls, a set  $R$  of red walls and a set  $G$  of green walls, such that:

- The set  $A$  of active wall is a subset of  $M(\omega) = M(\text{top}(\omega))$ , the set of walls around which the local configuration in the first row is  $\bullet|\circ, |\circ$  or  $\bullet|$ ,
- The set  $R$  of red walls and the set  $G$  of green walls form a partition  $(R, G)$  of the subset  $C(\omega, A)$  of  $A$  consisting of walls around which the local configuration is  $\bullet|\circ$ .

Observe that  $G$  is redundant in the notation  $(\omega, A, R, G)$  since  $G = C(\omega, A) \setminus R$ , however it will be more convenient to keep it explicit.

We now define a weight  $Q$  on complete configurations and its “randomized” version  $q$  on the set of well colored configurations. Given  $(\omega, A, R, G)$  a well-colored configuration, we set

$$Q(\omega) = (1 - p)^{h(\omega)-m(\omega)}, \quad \text{and} \quad q(\omega, A, R, G) = p^{|A|}(1 - p)^{h(\omega)-|A|} \cdot p^{|R|}(1 - p)^{|G|},$$

where  $h(\omega)$  denote the number of columns of the form  $|\bullet|$ ,  $|\circ|$ , and  $|\circ|$  in  $\omega$ , and  $m(\omega) = |M(\omega)|$ . Again, readers with a background in enumerative combinatorics will recognize the statistic  $h(\omega)$  as the number of non decreasing steps in the bicolored Motzkin path associated to the configuration  $\omega$ .

Observe that

$$q(\omega, A, R, G) = Q(\omega) \cdot p^{|A|}(1 - p)^{m(\omega)-|A|} \cdot p^{|R|}(1 - p)^{|G|}. \tag{1}$$

Given  $\omega$ , we can thus apply the binomial formula  $\sum_{U \subset V} x^{|U|}y^{|V|-|U|} = (x + y)^{|V|}$  to sum over all partitions  $(R, G)$  of  $C(\omega, A)$ , and then again to sum over all subsets  $A \subset M(\omega)$ :

$$\sum_{A, R, G} q(\omega, A, R, G) = Q(\omega), \tag{2}$$

where the summation is over all triples  $(A, R, G)$  such that  $(\omega, A, R, G)$  is a well-colored configuration.

The key of our combinatorial approach is that we can construct a mapping that behaves nicely with respect to the weight  $q$ . The construction is given in Section 3.

**Theorem 1.** *There is a bijection  $\bar{T}$  from the set of well colored configurations onto itself such that:*

- The mapping  $T$ , defined as the first component of  $\bar{T}$ , mimics in the top row the mapping  $\vartheta$ : more explicitly, if  $(\omega', A', R', G') = \bar{T}(\omega, A, R, G)$  is the image of a well colored configuration  $(\omega, A, R, G)$  by  $\bar{T}$ , then  $\omega' = T(\omega, A, R, G)$  satisfies

$$\text{top}(\omega') = \vartheta(\text{top}(\omega), A).$$

- The weight  $q$  is preserved by the bijection  $\bar{T}$ : for any well colored configuration  $(\omega, A, R, G)$ ,

$$q(\bar{T}(\omega, A, R, G)) = q(\omega, A, R, G).$$

- The bijection  $\bar{T}$  leaves each colorless configuration  $(\omega, \emptyset, \emptyset, \emptyset)$  invariant.

Observe that the first property of  $\bar{T}$  completely defines its action on the first row of configurations: in particular,  $T$  must move black particles from left to right in the first row. We shall see in Section 3, when we explicitly describe  $\bar{T}$  that in the second row it will move black particles in the opposite direction, from right to left.

With Theorem 1 at hand, we are in the position to define our Markov chain  $X^\parallel$  on the set  $\Omega_n$ :

- Let  $\omega = X^\parallel(t)$  be the current configuration,  $M = M(\omega) = M(\text{top}(\omega))$ , and  $m = m(\omega) = |M|$ .
- Choose a random subset  $A$  of  $M$  by independently giving to each wall of  $M$  probability  $p$  to be taken. In other terms, the probability that  $A = \{i_1, \dots, i_k\}$  for some given distinct elements  $i_1, \dots, i_k$  of  $M$  is  $p^k(1-p)^{m-k}$ . The walls in the set  $A$  are referred to as active walls, and the pair  $(\omega, A)$  as an active configuration.
- Next, let  $C = C(\omega, A)$  be the subset of  $A$  consisting of walls around which the local configuration is  $\begin{smallmatrix} \bullet & | & \circ \\ \circ & | & \circ \end{smallmatrix}$ . Then each wall of  $C$  is colored red with probability  $p$  or green with probability  $1-p$ . In other terms we randomly partition  $C$  into  $R$  (red walls) and  $G$  (green walls), and associate to the active configuration  $(\omega, A)$  a well-colored configuration  $(\omega, A, R, G)$ .
- Then set  $X^\parallel(t+1) = T(\omega, A, R, G)$ .

See Figs. 4 and 5 for an illustration. The transition matrix  $M$  of the chain  $X^\parallel$  can be equivalently written, for all  $\omega$  and  $\omega'$  in  $\Omega_n$ ,

$$M[\omega, \omega'] = \sum_{(A,R,G) | T(\omega,A,R,G)=\omega'} p^{|A|} (1-p)^{m(\omega)-|A|} \cdot p^{|R|} (1-p)^{|G|} = \sum_{(A,R,G) | T(\omega,A,R,G)=\omega'} \frac{q(\omega, A, R, G)}{Q(\omega)},$$

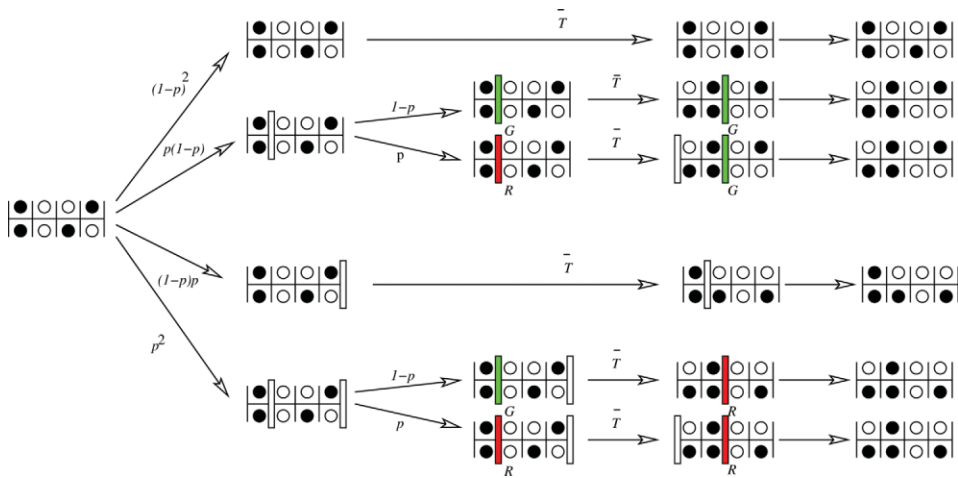
where the second expression follows from Formula (1).

Let us compare the dynamic of  $X^\parallel$  and  $S^\parallel$ . In the chain  $X^\parallel$ , a supplementary random coloring step is performed that does not exist in the chain  $S^\parallel$ . In particular, we allow the action of  $T$  to depend on this distinction between colors, and this will be used in the actual construction of the mapping  $T$  in Section 3. However, the colors only affect the bottom row: the action of  $T$  on the top row depends only on  $A$  and, as already observed, is identical to the action of  $\vartheta$ . Hence, when considering only the top row, the coloring step in the definition of  $X^\parallel$  can be ignored, and we obtain the following relation between  $X^\parallel$  and  $S^\parallel$ .

**Proposition 1.** *The process  $\text{top}(X^\parallel)$  is a Markov chain. It has the same transition probabilities as  $S^\parallel$ , and thus the same stationary distributions.*

*Proof.* Let us prove that  $\text{top}(X^\parallel)$  is a Markov chain identical to  $S^\parallel$  by comparing their transition probabilities. Let  $\tau$  and  $\tau'$  be two configurations of the TASEP and  $\omega$  be a complete configuration with  $\text{top}(\omega) = \tau$ . Let us compute the probability to go from the complete configuration  $\omega$  to a configuration with top row  $\tau'$ :

$$\Pr(\text{top}(X^\parallel(t+1)) = \tau' \mid X^\parallel(t) = \omega) = \sum_{\omega' | \text{top}(\omega') = \tau'} \Pr(X^\parallel(t+1) = \omega' \mid X^\parallel(t) = \omega)$$



**Fig. 4.** The possible transitions for a configuration  $\omega$  with  $M(\omega) = \{1, 4\}$  and  $n = 4$ . [Color figure can be viewed in the online issue, which is available at [www.interscience.wiley.com](http://www.interscience.wiley.com).]

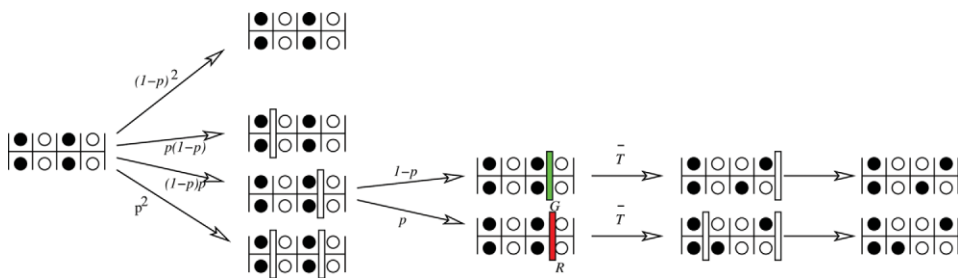
Making explicit the transition probabilities of  $X^\parallel$ , this reads

$$\Pr(\text{top}(X^\parallel(t + 1)) = \tau' \mid X^\parallel(t) = \omega) = \sum_{(A,R,G) \mid \text{top}(T(\omega,A,R,G)) = \tau'} p^{|A|} (1 - p)^{|M| - |A|} p^{|R|} (1 - p)^{|G|}$$

Since the choice of  $R$  and  $G$  has no effect on the top row, this can be rewritten as

$$\begin{aligned} \Pr(\text{top}(X^\parallel(t + 1)) = \tau' \mid X^\parallel(t) = \omega) &= \sum_{A \mid \theta(\text{top}(\omega), A) = \tau'} p^{|A|} (1 - p)^{|M| - |A|} \sum_{R,G} p^{|R|} (1 - p)^{|G|} \\ &= \sum_{A \mid \theta(\tau, A) = \tau'} p^{|A|} (1 - p)^{|M| - |A|} \\ &= \Pr(S^\parallel(t + 1) = \tau' \mid S^\parallel(t) = \tau). \end{aligned}$$

In particular this quantity depends only on  $\tau$  and  $\tau'$  (and not on the second row of  $\omega$ ) so that  $\text{top}(X^\parallel)$  can be seen as a Markov chain, with the same transition probabilities as  $S^\parallel$ . ■



**Fig. 5.** Two possible transitions for a configuration  $\omega$  with  $M(\omega) = \{1, 3\}$  and  $n = 4$ , illustrating the fact that the choice of the color can lead to two different configurations. [Color figure can be viewed in the online issue, which is available at [www.interscience.wiley.com](http://www.interscience.wiley.com).]

### 2.2. The Stationary Distribution

As we shall see in Section 3, the transitions of the complete chain  $X^\#$  that originate from exactly one wall and such that this wall is not red are exactly the transitions of the chain  $X$  studied in [4] in relation with the sequential TASEP. The chain  $X$  was proved irreducible there and this implies that the chain  $X^\#$ , which has more transitions, is irreducible as well for  $0 < p < 1$ . Moreover, since  $T(\omega, \emptyset, \emptyset, \emptyset) = \omega$  for all  $\omega$ , there is a positive probability to stay in any configuration, and the chain  $X^\#$  is aperiodic. A classical result of the theory of finite Markov chains is that an irreducible aperiodic chain has a unique stationary distribution to which it converges [6]. Our main result is then the following theorem.

**Theorem 2.** *The stationary distribution of the Markov chain  $X^\#$  is proportional to the weight*

$$Q(\omega) = (1 - p)^{h(\omega) - m(\omega)},$$

where  $h(\omega)$  is the number of columns of the form  $|\bullet|$ ,  $|\circ|$  or  $|\circ|$  and  $m(\omega)$  the number of walls at which a transition could occur in the first row (i.e walls around which the local configuration in the top row is  $\bullet|\circ$ ,  $\bullet|$  or  $|\circ$ ). In other terms,

$$\text{Prob}(X^\#(t) = \omega) \xrightarrow{t \rightarrow \infty} \frac{1}{Z_n(1 - p)} (1 - p)^{h(\omega) - m(\omega)} \quad \text{where} \quad Z_n(x) = \sum_{\omega \in \Omega_n} x^{h(\omega) - m(\omega)}.$$

In particular,  $Z_n(x)$  is a combinatorial refinement of the Catalan numbers  $Z_n(1) = \frac{1}{n+1} \binom{2n}{n}$ .

*Proof.* Our aim is to show that the unnormalized distribution  $Q(\omega)$  is stationary. Let us thus assume that  $\text{Prob}(X^\#(t) = \omega) = c_0 Q(\omega)$  for some constant  $c_0$  and all  $\omega \in \Omega_n$ , and let us compute  $\text{Prob}(X^\#(t + 1) = \omega')$ .

By construction, the probability to be in configuration  $\omega'$  at time  $t + 1$  is the sum over the probability to be at time  $t$  in a configuration  $\omega$  multiplied by the transition probability between  $\omega$  and  $\omega'$ :

$$\begin{aligned} \text{Prob}(X^\#(t + 1) = \omega') &= \sum_{(w,A,R,G) \in T^{-1}(\omega')} \text{Prob}(X^\#(t) = \omega) p^{|A|} (1 - p)^{m(\omega) - |A|} p^{|R|} (1 - p)^{|G|}, \\ &= \sum_{(w,A,R,G) \in T^{-1}(\omega')} c_0 q(w, A, R, G), \end{aligned}$$

where the second line follows from Formula (1) and the hypothesis  $\text{Prob}(X^\#(t) = \omega) = c_0 Q(\omega)$ .

In view of Theorem 1,  $T$  is the first component of the bijection  $\bar{T}$ , so that  $T^{-1}(\omega')$  is the inverse image by  $\bar{T}$  of the set of well colored configurations of the form  $(\omega', A', R', G')$ . This implies

$$\text{Prob}(X^\#(t+1) = \omega') = \sum_{A',R',G'} c_0 q(\bar{T}^{-1}(w', A', R', G')) = \sum_{A',R',G'} c_0 q(w', A', R', G') = c_0 Q(\omega').$$

where the summations are over all triples  $(A', R', G')$  such that  $(\omega', A', R', G')$  is a well colored configuration, the second equality follows from the invariance of  $q$  under the action of  $\bar{T}$ , as stated in Theorem 1, and the last equality is Formula (2). ■



An immediate consequence of this theorem is the following interpretation of the stationary distribution of the Markov chain  $S^\parallel$  in terms of weighted complete configurations with fixed top row.

**Theorem 3.** *The stationary distribution of the Markov chain  $S^\parallel$  is proportional to*

$$\pi(\tau) = \sum_{\text{top}(\omega)=\tau} (1-p)^{h(\omega)-m(\tau)}.$$

*Proof.* The chains  $S^\parallel$  and  $\text{top}(X^\parallel)$  are identical, and any stationary distribution of the chain  $X^\parallel$  induces a stationary distribution of  $\text{top}(X^\parallel)$ , which, in view of the finiteness of the statespace, is obtained by summing over configurations with identical projections. ■

Observe that to describe a complete configuration  $\omega$  such that  $\text{top}(\omega) = \tau$ , it is sufficient to indicate the positions of the columns of the form  $\bullet \mid \circ$  and  $\circ \mid \circ$ . Indeed let  $\sigma = \{i_1, i_2, \dots, i_{2r}\}$ ,  $i_1 < \dots < i_{2r}$  be a subset of  $\{1, \dots, n\}$  such that the subconfiguration  $\tau_{i_1} \dots \tau_{i_{2r}}$  contains  $r$  black and  $r$  white particles, and the number of black is greater or equal the one of white particles in each of the subconfiguration  $\tau_{i_1} \dots \tau_{i_k}$ ,  $k = 1, \dots, 2r$ . In this case write  $\sigma \vdash \tau$  and  $|\sigma| = r$ . Then  $\sigma$  corresponds to a unique complete configuration  $\omega$  with  $\text{top}(\omega) = \tau$  and  $\pi(\tau)$  can be rewritten as

$$\pi(\tau) = (1-p)^{n-m(\tau)} \sum_{\sigma \vdash \tau} (1-p)^{-|\sigma|}.$$

For instance, for a configuration of the type  $\circ \dots \circ \bullet \dots \bullet$ , we obtain

$$\pi(\underbrace{\circ \dots \circ}_k \underbrace{\bullet \dots \bullet}_{n-k}) = (1-p)^{n-2}, \quad \text{for all } 0 < k < n,$$

which is independent of  $k$ , whereas for  $\bullet \dots \bullet \circ \dots \circ$ , the probability depends on  $k$  as follows:

$$\pi(\underbrace{\bullet \dots \bullet}_k \underbrace{\circ \dots \circ}_{n-k}) = \sum_{r=0}^{\min(k, n-k)} \binom{k}{r} \binom{n-k}{r} (1-p)^{n-1-r}, \quad \text{for all } 0 < k < n.$$

Observe that for different values of  $p$ , the parallel TASEPs indexed by these  $p$  are defined on the same finite statespace  $S_n$ , which is also the statespace of  $S^{\text{seq}}$ . In the limit  $p \rightarrow 0$ , the transition probabilities of the parallel TASEP indexed by  $p$  converge to those of the sequential TASEP, and, in view of the previous remark, the stationary distributions clearly converge as well. In particular, for  $p \rightarrow 0$ , we get  $Q(\omega) = 1$  for all  $\omega$ , and  $Z_n(1) = |\Omega_n| = \frac{1}{n+1} \binom{2n}{n}$ .

**Corollary 1.** *The stationary distribution of the sequential TASEP is:*

$$\Pr(S^{\text{seq}}(t) = \tau) \xrightarrow{t \rightarrow \infty} \frac{|\{\omega \mid \text{top}(\omega) = \tau\}|}{\frac{1}{n+1} \binom{2n}{n}}.$$

### 3. THE BIJECTION $\bar{T}$

In this section, we prove Theorem 1 by describing a bijection  $\bar{T}$  that transports the weight  $q$ . We first give the definitions of some local operations, and use them to describe an intermediate mapping  $\psi$ . Finally, we present  $\bar{T}$  and check that  $\bar{T}$  satisfies the requirements.

### 3.1. Local Operations

We shall use two types of local operations: deletions map configurations of  $\Omega_n$  to configurations of  $\Omega_{n-1}$ , while insertions map configurations of  $\Omega_{n-1}$  to configurations of  $\Omega_n$ . In the following definitions, the numbering of walls always refers to the configuration of  $\Omega_n$ :

- A *right deletion* at  $i \neq 0$  consists in
  - if  $i \neq n$ , removing a  $\begin{smallmatrix} | & \circ \\ \bullet & | \end{smallmatrix}$ -column on the right of  $i$ , that is,  $\omega_1 | \begin{smallmatrix} | & \circ \\ \bullet & | \end{smallmatrix} \parallel_x^y | \omega_2 \mapsto \omega_1 | \begin{smallmatrix} | & \circ \\ \bullet & | \end{smallmatrix} | \omega_2$ .
  - if  $i = n$ , removing a  $\begin{smallmatrix} | & \circ \\ \bullet & | \end{smallmatrix}$ -column at the right border, that is,  $\omega_1 | \begin{smallmatrix} | & \circ \\ \bullet & | \end{smallmatrix} \parallel_n \mapsto \omega_1 |$ .
- A *left deletion* at  $i \neq n$  consists in
  - if  $i \neq 0$ , removing a  $\begin{smallmatrix} \bullet & | \\ \circ & | \end{smallmatrix}$ -diagonal around  $i$ , that is,  $\omega_1 | \begin{smallmatrix} \bullet & | \\ \circ & | \end{smallmatrix} \parallel_x^y | \omega_2 \mapsto \omega_1 | \begin{smallmatrix} \bullet & | \\ \circ & | \end{smallmatrix} | \omega_2$ .
  - if  $i = 0$ , removing a  $\begin{smallmatrix} | & \circ \\ \bullet & | \end{smallmatrix}$ -column on the left border, that is,  $\parallel_0^{\circ} | \omega_2 \mapsto | \omega_2$ .
- A *right insertion* at  $j \neq n$  consists in
  - if  $j \neq 0$ , inserting a  $\begin{smallmatrix} | & \circ \\ \bullet & | \end{smallmatrix}$ -column on the right of wall  $j$ , that is,  $\omega_1 | \begin{smallmatrix} | & \circ \\ \bullet & | \end{smallmatrix} | \omega_2 \mapsto \omega_1 | \begin{smallmatrix} | & \circ \\ \bullet & | \end{smallmatrix} \parallel_x^y | \omega_2$ .
  - if  $j = 0$ , inserting a  $\begin{smallmatrix} | & \circ \\ \bullet & | \end{smallmatrix}$ -column on the left border, that is,  $| \omega_2 \mapsto \parallel_0^{\circ} | \begin{smallmatrix} | & \circ \\ \bullet & | \end{smallmatrix} | \omega_2$ .
- A *left insertion* at  $j \neq 0$  consists in
  - if  $j \neq n$ , inserting a  $\begin{smallmatrix} \bullet & | \\ \circ & | \end{smallmatrix}$ -diagonal around column  $j$ , that is,  $\omega_1 | \begin{smallmatrix} \bullet & | \\ \circ & | \end{smallmatrix} | \omega_2 \mapsto \omega_1 | \begin{smallmatrix} \bullet & | \\ \circ & | \end{smallmatrix} \parallel_x^y | \omega_2$ .
  - if  $j = n$ , inserting a  $\begin{smallmatrix} | & \circ \\ \bullet & | \end{smallmatrix}$ -column at the right border, that is,  $\omega_1 | \begin{smallmatrix} | & \circ \\ \bullet & | \end{smallmatrix} | \mapsto \omega_1 | \begin{smallmatrix} | & \circ \\ \bullet & | \end{smallmatrix} \parallel_n$ .

In the case of the sequential TASEP [4], we used these operations to construct a bijection from the set of complete configurations with exactly one active wall, onto itself. The bijection essentially consisted in applying one deletion at the active wall and one insertion at another nearby wall. Since the main difference between the sequential and the parallel TASEP is the fact that there may be several active walls, one could just try to apply the bijection of [4] to all these walls in parallel. However, this naive approach fails when there are two active walls that are too close, because the transformations applied to nearby walls may interfere.

In order to circumvent this ambiguity a new operation is needed:

- A *block deletion* at a wall  $i$  of type  $\begin{smallmatrix} | & \circ \\ \bullet & | \end{smallmatrix} \parallel_x^y |$  consists in removing the block around  $i$ :  $\omega_1 | \begin{smallmatrix} | & \circ \\ \bullet & | \end{smallmatrix} \parallel_x^y | \omega_2 \mapsto \omega_1 | \omega_2$ .

Using the corresponding insertion, it would be possible to describe the bijection  $\bar{T}$  directly as the simultaneous application of some deletions and insertions near active walls. It will however prove more convenient to give a sequential description in terms of a partial mapping  $\psi$ .

### 3.2. The Mapping $\psi$

Given a configuration  $(\omega, A, R, G)$ , a pointer is an element of  $A \cup \{-, \rightarrow\}$ . The value  $\rightarrow$  and  $-$  are, respectively, interpreted as positions of the pointer to the left and to the right of the configuration. A pair  $(\omega, A, R, G; i)$  is a right admissible configuration if one of the following two conditions is satisfied:

- $i \in A \cup \{-, \rightarrow\}$  and the configuration  $(\omega, A, R, G)$  is a well colored configuration, or
- $i \in A, i + 1 \in A$ , the configuration  $(\omega, A \setminus \{i + 1\}, R, G)$  is a well colored configuration and the local configuration between walls  $i$  and  $i + 2$  is  $\begin{smallmatrix} | & \circ \\ \bullet & | \end{smallmatrix} \parallel_x^y |$ .

Left admissible configurations are defined in exactly the same way with  $\neg$  replaced by  $\vdash$ .

**Remark 1.** In the second case of the definition, the right admissible configuration  $(\omega, A, R, G; i)$  is obtained from a well colored configuration  $(\omega, A_0, R, G)$  by selecting an active wall  $i \in A_0$  and adding the wall  $i + 1$  to the set of active walls,  $A = A_0 \cup \{i + 1\}$ . In particular  $(\omega, A, R, G)$  cannot be itself a well colored configuration, since it has two adjacent active walls  $i$  and  $i + 1$ .

In this case the active wall  $i + 1$  is called an *abnormal* active wall. In particular there is at most one abnormal active wall in an admissible configuration and then it is immediately on the right of the pointer. Admissible configuration with an abnormal wall will serve as intermediate steps in the iterative definition of the mapping.

**Remark 2.** A left admissible configuration  $(\omega, A, R, G; i)$  is also right admissible unless  $i = \perp$ . Left admissible configurations of the form  $(\omega, A, R, G; \vdash)$ , as well as right admissible configurations of the form  $(\omega, A, R, G; \neg)$ , are in bijection with well colored configurations (in particular, they have no abnormal active walls).

We are now ready to describe a mapping  $\psi$ , that maps right admissible configurations onto left admissible ones. The image  $(\omega', A', R', G'; r')$  of a right admissible configuration  $(\omega, A, R, G; r)$  by  $\psi$  is obtained by applying some local operations near the pointer.

The value  $\neg$  serves as a initialization case: the image of a pair  $(\omega, A, R, G; \neg)$  by  $\psi$  is  $(\omega, A, R, G; \max(A))$  if  $A \neq \emptyset$ , and  $(\omega, A, R, G; \vdash)$  otherwise. When the pointer  $i$  is in  $A$ , the image  $(\omega', A', R', G'; i')$  of  $(\omega, A, R, G; i)$  depends on the local configuration around  $i$ , and, if it exists, on the rightmost active wall  $m < i$  on the left of  $i$  (the three cases are illustrated by Figs. 6–8):

**A. Cases  $\bullet \parallel_i^\circ$ , or  $\bullet \parallel_i^\circ$  with  $i$  green, or  $\bullet \parallel_0^\circ$  with  $i = 0$ :** (see Fig. 6)

Two operations are performed on  $\omega$  to produce its image  $\omega'$ . The first one is a left deletion at  $i$ . For the second let  $j_2 > i$  be the rightmost wall such that there are only black particles in the top row between walls  $i + 1$  and  $j_2$ . The second operation is a left insertion at  $j_2$ . There are two possibilities:

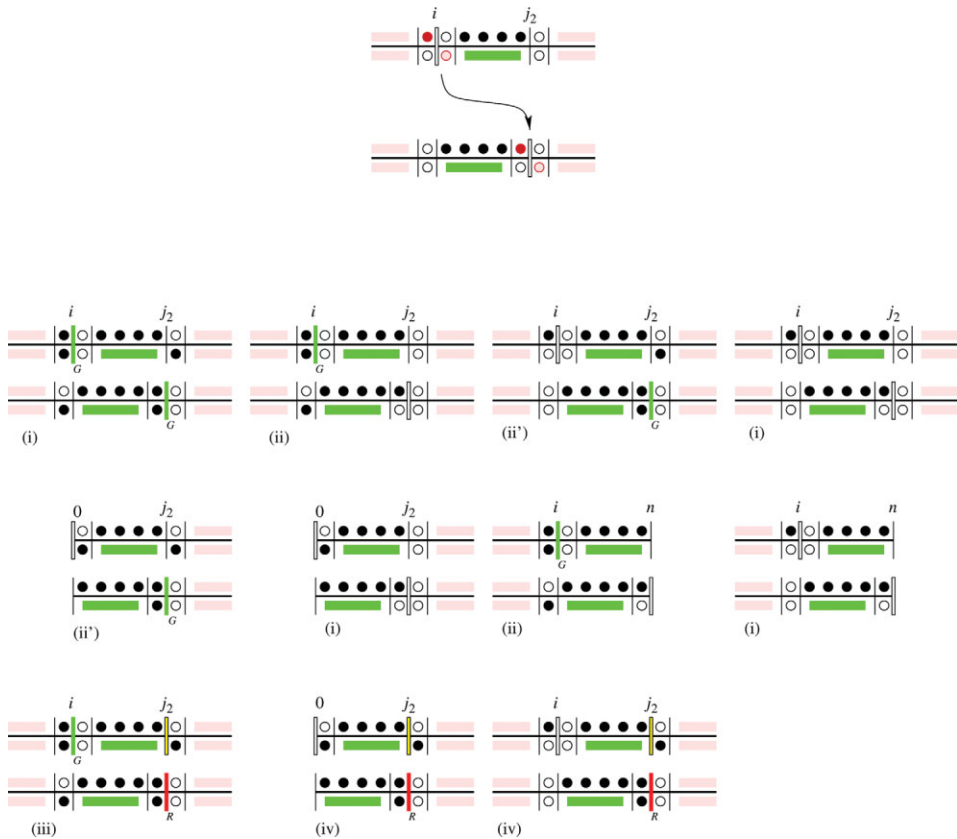
- The wall  $j_2$  was not active ( $j_2 \notin A$ ). The wall  $j_2$  replaces the wall  $i$  in the set of active walls:  $A' = A \cup \{j_2\} \setminus \{i\}$ . Moreover, if there is a black particle on its bottom right in  $\omega$ , the wall  $j_2$  is colored green:  $G' = G \cup \{j_2\} \setminus \{i\}$ .
- The wall  $j_2$  was active ( $j_2 \in A$ ). Then  $i$  is removed from the set of active walls and the wall  $j_2$  is colored red:  $A' = A \setminus \{i\}$ ,  $R' = R \cup \{j_2\}$ ,  $G' = G \setminus \{i\}$ .

Other colorings are left unchanged, and in both cases the pointer is set to  $m$  if it exists, to  $\perp$  otherwise.

**B. Cases  $\bullet \parallel_n^\circ$ , or  $\bullet \parallel_n^\circ$  with  $i = n$ :** (see Fig. 7)

Two operations are performed on  $\omega$  to produce its image  $\omega'$ . The first one is a right deletion at  $i$ . To describe the second operation, let  $j_1 < i$  be the leftmost wall on the left of  $i$  in  $\omega$  such that there are only white particles in the top row between walls  $j_1$  and  $i - 1$ . There are again several possibilities:

- The wall  $j_1$  was active (so that  $m = j_1$ ). Then the second operation is a right insertion at wall  $m + 1$ , which becomes active:  $A' = A \cup \{m + 1\} \setminus \{i\}$ . (This is the only case in which a wall with local configuration  $\circ \mid \circ$  can become active. As required by the definition of admissible configurations, the pointer will be set on the left of this abnormal active wall.)



**Fig. 6.** The mapping  $\psi$  in cases of type A. The rule is first presented informally. It is then instantiated for all possible local configurations to allow for easy verification of proofs. The index under each local configuration refers to the proof of Lemma 2. [Color figure can be viewed in the online issue, which is available at [www.interscience.wiley.com](http://www.interscience.wiley.com).]

- Otherwise, the second operation is a right insertion at the wall  $j_1$ . The wall  $j_1$  replaces the wall  $i$  in the set of active walls:  $A' = A \cup \{j_1\} \setminus \{i\}$ .

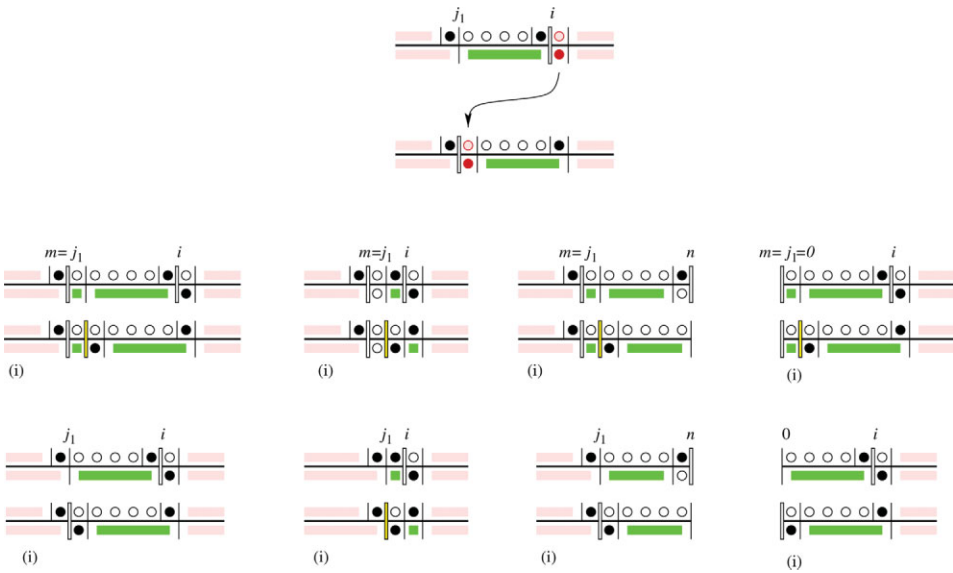
Other colorings are left unchanged, and in all cases the pointer is set to  $m$  if it exists and to  $\vdash$  otherwise.

**C. Case  $\bullet \parallel_i^\circ$  with  $i$  red:**(see Fig. 8)

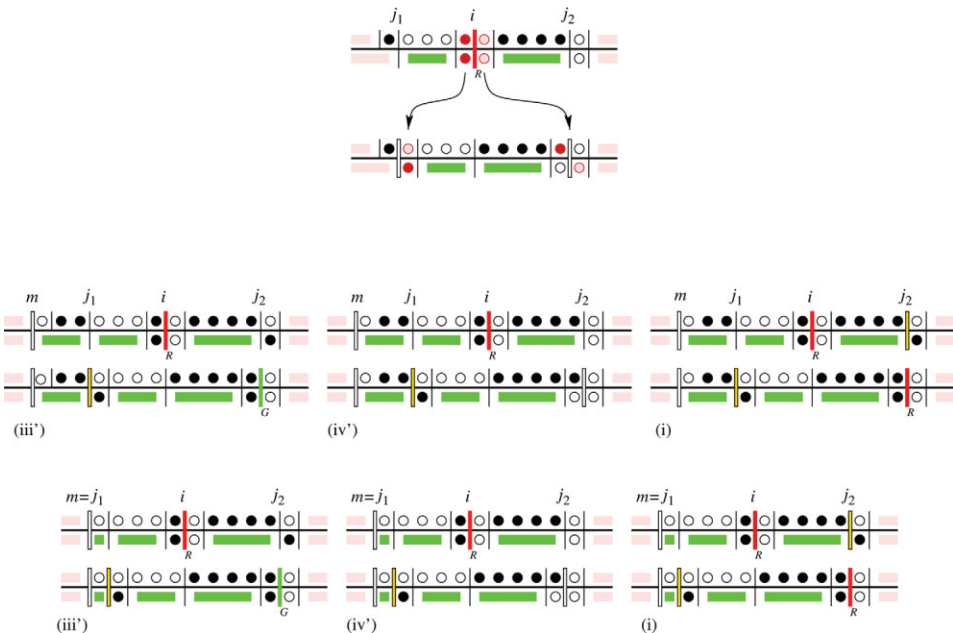
Three operations are performed on  $\omega$  to produce its image  $\omega'$ . The first one consists in removing the block  $\bullet \parallel_i^\circ$  around  $i$ . The second operation is a left insertion at  $j_2$  as for the cases of type A above, with the same two possibilities for the changes of colors. The third operation is a right insertion at  $m + 1$  or  $j_2$  as for the cases of type B above. Again the pointer is set to  $m$  if  $m$  exists and to  $\vdash$  otherwise.

**Lemma 1.** *The mapping  $\psi$  is a bijection from the set of right admissible configurations onto the set of left admissible configurations.*

*Proof.* All possible local configurations near the walls  $i$  and  $j_1$  or  $j_2$  are represented in Figs. 6–8 (up to variations of the length of the intermediate regions). The proof that no



**Fig. 7.** The mapping  $\psi$  in cases of type B. The rule is first presented informally. It is then instantiated for all possible local configurations to allow for easy verification of proofs. The index under each local configuration refers to the proof of Lemma 2. [Color figure can be viewed in the online issue, which is available at [www.interscience.wiley.com](http://www.interscience.wiley.com).]



**Fig. 8.** The mapping  $\psi$  in cases of type C. The rule is first presented informally. It is then instantiated for a selection of possible local configurations. (All configurations can be retrieved by combining type A to the right and type B to the left.) The index under each local configuration refers to the proof of Lemma 2. [Color figure can be viewed in the online issue, which is available at [www.interscience.wiley.com](http://www.interscience.wiley.com).]

cases were left aside is done by considering the content of the four columns around the two indices involved in each case, and checking that each possibility has been represented. The following two properties then prove the lemma.

First, the image of a right admissible configuration by  $\psi$  is a left admissible configuration. Indeed, the fact that the image configuration  $\omega'$  satisfies the positivity condition follows immediately from the fact that the local operations of Section 3.1 preserve this condition. It should be observed also that if there is an abnormal active wall on the right of the pointer in  $\omega$  (i.e., an active wall with local configuration  $\circ|\circ$  in the top row), then the application of  $\psi$  brings a black particle to its left, hence turning it back into a normal active wall. This ensures that there is at most one abnormal active wall in the image of a right admissible configuration.

Second, a case analysis allows to check that any left admissible configuration has a preimage by  $\psi$ . Indeed, let us describe the image by  $\psi$  in each of the above cases A, B, C:

- A. The image of the configurations of this case by  $\psi$  consists of the all configurations  $(\omega', A', R', G'; m)$  such that the local configuration near the first active wall  $j_2$  on the right of  $m$  is  $\begin{smallmatrix} \circ & \circ \\ \circ & \circ \end{smallmatrix} \parallel_{j_2} \begin{smallmatrix} \circ \\ \circ \end{smallmatrix}$ , or  $\begin{smallmatrix} \circ & \circ \\ \bullet & \circ \end{smallmatrix} \parallel_{j_2} \begin{smallmatrix} \circ \\ \circ \end{smallmatrix}$ , or  $\begin{smallmatrix} \circ & \circ \\ \bullet & \circ \end{smallmatrix} \parallel_n$  with  $j_2 = n$ . Indeed in each of these cases the rules can be applied backward to get a valid right admissible configuration.
- B. The image of the configurations of this case by  $\psi$  consists of all the configurations  $(\omega', A', R', G'; m)$  such that the local configuration near the first active wall  $j_2$  on the right of  $m$  is  $\begin{smallmatrix} \circ & \circ \\ \bullet & \circ \end{smallmatrix} \parallel_{j_2} \begin{smallmatrix} \circ \\ \circ \end{smallmatrix}$ , or  $\parallel_0 \begin{smallmatrix} \circ \\ \circ \end{smallmatrix}$  with  $j_2 = 0$ , or  $\begin{smallmatrix} \circ & \circ \\ \circ & \bullet \end{smallmatrix} \parallel_{j_2} \begin{smallmatrix} \circ \\ \circ \end{smallmatrix}$  (abnormal active wall), and the next wall  $j_3$  to the right of  $j_2$  in  $M(\omega')$  is not active or of the form  $\begin{smallmatrix} \bullet & \circ \\ \circ & \bullet \end{smallmatrix} \parallel_{j_3} \begin{smallmatrix} \circ \\ \circ \end{smallmatrix}$ . Again in each of these cases the rules can be applied backward to get a valid right admissible configuration.
- C. The image of the configurations of this case by  $\psi$  consists of all the configurations  $(\omega', A', R', G'; m)$  such that the local configuration near the first active wall  $j_2$  on the right of  $m$  is  $\begin{smallmatrix} \bullet & \circ \\ \circ & \circ \end{smallmatrix} \parallel_{j_2} \begin{smallmatrix} \circ \\ \circ \end{smallmatrix}$ , or  $\parallel_0 \begin{smallmatrix} \circ \\ \circ \end{smallmatrix}$  with  $j_2 = 0$ , or  $\begin{smallmatrix} \circ & \circ \\ \circ & \bullet \end{smallmatrix} \parallel_{j_2} \begin{smallmatrix} \circ \\ \circ \end{smallmatrix}$  (abnormal active wall), and the next wall  $j_3$  to the right of  $j_2$  in  $M(\omega')$  is active and of the form  $\begin{smallmatrix} \bullet & \circ \\ \circ & \bullet \end{smallmatrix} \parallel_{j_3} \begin{smallmatrix} \circ \\ \circ \end{smallmatrix}$ , or  $\begin{smallmatrix} \bullet & \circ \\ \bullet & \circ \end{smallmatrix} \parallel_{j_3} \begin{smallmatrix} \circ \\ \circ \end{smallmatrix}$ , or  $\begin{smallmatrix} \bullet & \circ \\ \bullet & \circ \end{smallmatrix} \parallel_n$  with  $j_3 = n$ . Again in each of these cases the rules can be applied backward to get a valid right admissible configuration.

The union of these cases clearly covers all the possible left admissible configurations, so that  $\psi$  is surjective. Finally, since there is the same number of right and left admissible configurations  $\psi$  is bijective. ■

### 3.3. The Bijection $\bar{T}$

Observe that the mapping  $\psi$  always moves the pointer to the left and that a left admissible configuration obtained by  $\psi$  is also right admissible as long as the pointer has not reached  $\vdash$ . In particular we will define the mapping  $\bar{T}$  by iterating  $\psi$  so that the pointer goes from  $\dashv$  to  $\vdash$ :

- Let  $\ell := 0$  and  $(\omega_0, A_0, R_0, G_0) = (\omega, A, R, G)$ , and set the initial pointer to the right of the configuration:  $i_0 = \dashv$ .
- Repeat
  - let  $(\omega_{\ell+1}, A_{\ell+1}, R_{\ell+1}, G_{\ell+1}; i_{\ell+1}) = \psi(\omega_\ell, A_\ell, R_\ell, G_\ell; i_\ell)$  and  $\ell := \ell + 1$ .

until  $i_\ell$  reaches the value  $\vdash$ .

- The image  $\bar{T}(\omega, A, R, G)$  is  $(\omega_\ell, A_\ell, R_\ell, G_\ell)$ .

The following proposition follows easily from the properties of the mapping  $\psi$ .

**Proposition 2.** *The mapping  $\bar{T}$  is a bijection from the set of well colored configurations onto itself. Moreover if  $(\omega', A', R', G') = \bar{T}(\omega, A, R, G)$ , then  $\text{top}(\omega') = \vartheta(\text{top}(\omega), A)$ .*

*Proof.* Given a well colored configuration  $(\omega', A', R', G')$  we consider the naturally associated left admissible configuration  $(\omega', A', R', G'; \vdash)$ . From this configuration the mapping  $\psi$ , which is bijective, can be iterated backward in a unique way until a configuration  $(\omega, A, R, G; \dashv)$  is reached. This gives the preimage  $(\omega, A, R, G)$  of  $(\omega', A', R', G')$  by  $\bar{T}$ , and the fact that  $\psi$  is bijective implies that  $\bar{T}$  is also bijective.

Let us check that if  $(\omega', A', R', G') = \bar{T}(\omega, A, R, G)$ , then  $\text{top}(\omega') = \vartheta(\text{top}(\omega), A)$ . Let  $A = \{i_k, i_{k-1}, \dots, i_1\}$  with  $i_k < i_{k-1} < \dots < i_1$ . The mapping  $\psi$  moves the pointer from an active wall to the next one on the left, therefore during its iteration, the successive values of the pointer are  $\dashv, i_1, \dots, i_k, \vdash$ . Moreover, by construction of  $\psi$  the image  $(\omega_{\ell+1}, A_{\ell+1}, R_{\ell+1}, G_{\ell+1}; i_{\ell+1})$  of  $(\omega_\ell, A_\ell, R_\ell, G_\ell; i_\ell)$  is such that  $\text{top}(\omega_{\ell+1}) = \vartheta(\text{top}(\omega_\ell), \{i_\ell\})$ . The result follows since  $\vartheta(\tau, i_k, \dots, i_\ell) = \vartheta(\vartheta(\tau, i_\ell), i_k, \dots, i_{\ell+1})$ . ■

Although it is more convenient to describe  $\bar{T}$  in a sequential way, as we did in terms of  $\psi$ , it is worth observing again that  $\bar{T}$  essentially acts in parallel on all walls, with one pair of particles leaving or arriving at every non-red active wall, and two pairs of particles leaving or arriving at every red active wall.

Another remarkable feature of the bijection  $\bar{T}$  is that it can be interpreted as moving black particles to the right in the top row and in the opposite direction in the bottom row. Indeed, as can be checked on Figs. 6–8, each time a green region is moved to the right, a black particle is put on its left, so that the global move can be reinterpreted as the displacement of some black particles to the left. The reader is referred to [4], where a similar interpretation in terms of circulating particles is developed for the sequential case.

### 3.4. The Bijection and the Weight

To conclude the proof of Proposition 1, it remains to check that for any well colored configuration  $(\omega, A, R, G)$  we have

$$q(\bar{T}(\omega, A, R, G)) = q(\omega, A, R, G).$$

To do so, it is sufficient to prove that the weight  $q$  is left unchanged by the mapping  $\psi$ . Observe first that the definition of  $q$  immediately extends to admissible configurations with an abnormal active wall (the abnormal wall contributes to  $|A|$  like all active walls). Recall that

$$q(\omega, A, R, G) = p^{|A|} (1 - p)^{h(\omega) - |A|} \cdot p^{|R|} (1 - p)^{|G|}$$

where  $h(w)$  is  $n$  minus the number of  $|\circ|$  columns.

**Lemma 2.** *Let  $(\omega, A, R, G; r)$  be a right admissible configuration and  $(\omega', A', R', G'; r')$  be its image by  $\psi$ . Then*

$$q(\omega', A', R', G') = q(\omega, A, R, G).$$

*Proof.* All possible local configurations are listed in Figs. 6–8 and for each of them we have to check that the weight is invariant by  $\psi$ . This is done by identifying the way the parameters  $|A|$ ,  $h(\omega)$ ,  $|G|$  and  $|R|$  change under the application of  $\psi$ . All these parameters count some columns or walls defined by local conditions so that their variation can be computed from the local variations of the configuration, without knowledge of the unchanged regions. In particular since the exhaustive list of local variations of configurations is presented in Figs. 6–8, the action of  $\psi$  can be computed on the picture for each case. Doing this leads to the following seven cases (under each picture an index indicates in which case it falls):

- Case (i). The four parameters are invariant, so is the weight  $q$ .
- Case (ii).  $h(\omega') = h(\omega) + 1$  but  $|G'| = |G| - 1$ , so that  $q$  is invariant.
- Case (ii').  $h(\omega') = h(\omega) - 1$  but  $|G'| = |G| + 1$ , so that  $q$  is invariant.
- Case (iii).  $|G'| = |G| - 1$ ,  $|R'| = |R| + 1$ , but  $|A'| = |A| - 1$ , so that  $q$  is invariant.
- Case (iii').  $|G'| = |G| + 1$ ,  $|R'| = |R| - 1$ , but  $|A'| = |A| + 1$ , so that  $q$  is invariant.
- Case (iv).  $h(\omega') = h(\omega) - 1$  and  $|R'| = |R| + 1$ , but  $|A'| = |A| - 1$ , so that  $q$  is invariant.
- Case (iv').  $h(\omega') = h(\omega) + 1$  and  $|R'| = |R| - 1$ , but  $|A'| = |A| + 1$ , so that  $q$  is invariant.

The weight  $q$  is invariant in each case, as claimed by the lemma. ■

#### 4. CONCLUSION

The interest in the parallel TASEP originates in the Nagel-Schreckenberg automaton, which is a landmark of highway traffic flow modeling.

#### The Nagel-Schreckenberg Automaton

A configuration of this Markov chain consists of a row of  $n$  cells and  $n + 1$  walls containing some cars. Each cell can be occupied by a car and cars are numbered from left to right. The  $j$ th car is characterized at time  $t$  by its position  $x_j$  and its velocity  $v_j(t) \in \{0, 1, \dots, v_{\max}\}$ , where  $v_{\max}$  is the maximal velocity chosen for the system. Moreover, let us denote by  $d_j(t)$  the distance, i.e. the number of cells between the  $j$ th car and the  $(j + 1)$ th car.

- At time  $t = 0$ , the system is in a configuration  $NS(0)$  (possibly chosen at random).
- From time  $t$  to  $t + 1$ , the system evolves from the configuration  $NS(t)$  to the configuration  $NS(t + 1)$  by applying the following successive transformations to all cars in parallel (see Fig. 9):
  - A: Acceleration.** If the  $j$ th car is not at the maximal velocity then its velocity increases by one, i.e.  $v'_j(t) = \min(v_j(t) + 1, v_{\max})$ .
  - D: Safety deceleration.** If the distance  $d_j(t)$  to the next car is less than its velocity  $v'_j(t)$  then the latter decreases to  $d_j(t)$ , i.e.  $v''_j(t) = \min(d_j(t), v'_j(t))$ .
  - R: Random deceleration.** The  $j$ th car can decelerate by one with probability  $q$  if  $v''_j(t)$  is not zero:

$$v_j(t + 1) = \begin{cases} \max(v''_j(t) - 1, 0) & \text{with probability } q, \\ v''_j(t) & \text{otherwise.} \end{cases}$$



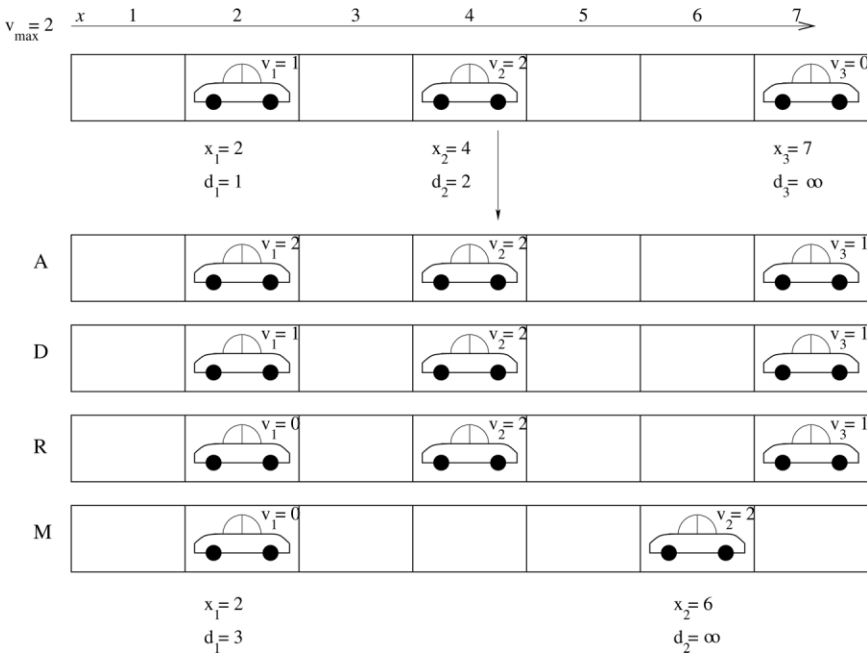


Fig. 9. An example of evolution for the Nagel-Schreckenberg model.

**M: Movement.** The  $j$ th car moves  $v_j(t + 1)$  cells to the right, i.e.  $x_j(t + 1) = x_j(t) + v_j(t + 1)$ .

The resulting  $(x_j(t + 1), v_j(t + 1))_j$  defines the new configuration  $NS(t + 1)$ , and in particular the new distances  $d_j(t + 1)$ .

These four rules are sufficient to define Nagel-Schreckenberg model with periodic boundary. Several entry rules are clearly possible to define the model with open boundaries, depending on the velocity and position at which cars are expected to enter the row.

Although the Nagel-Schreckenberg automaton is a Markov chain with very simple rules, it appears to be difficult to study from a mathematical point of view. In particular, its stationary distribution is only known for the particular case  $v_{\max} = 1$ : indeed if  $v_{\max} = 1$  then the Nagel-Schreckenberg automaton corresponds to the parallel TASEP, where cars are black particles, and where  $q = 1 - p$ . Indeed, when  $v_{\max} = 1$ , the possible velocities for each car are 0 or 1. Therefore, after step **A** all the velocities are equal to 1. At step **D**, only cars with  $d_j = 0$  (i.e. immediately followed by another car) decrease their velocity to 0. At step **R**, a car  $j$  such that  $v_j = 1$  keeps its non zero velocity with probability equal to  $1 - q = p$ . At step **M**, only cars with  $d_j \neq 0$  and  $v_j = 1$  move of one cell to the right. This is equivalent to saying that from time  $t$  to time  $t + 1$  cars that have a free cell on their right can move in there with probability  $p$ . This corresponds exactly to the dynamic of the parallel TASEP.

Regarding boundaries one should observe that our entry rule for the parallel TASEP correspond to a maximum entry flow situation, in which cars or particles are always available to enter the system. It is known in the literature [2, 5] that the TASEP also has an interesting

behavior when the entry or exit rate is different from the internal jumping probability. In [4] our combinatorial approach to the sequential TASEP was shown to allow for the computation of the stationary distribution of this more general model. We believe that a similar extension is possible for the parallel TASEP, but we retain as a more interesting challenge the open problem to compute the stationary distribution of the Nagel-Schreckenberg automaton for velocities larger than one.

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