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Fighting fish*

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Abstract

We introduce new combinatorial structures, called fighting fish, that generalize directed convex polyominoes by allowing them to branch out of the plane into independent substructures.

On the one hand the combinatorial structure of fighting fish appears to be particularly rich: we show that their generating function with respect to the perimeter and number of tails is algebraic, and we conjecture a mysterious multivariate equidistribution property with the left ternary trees introduced by Del Lungo *et al*

On the other hand, fighting fish provide a simple and natural model of random branching surfaces which displays original features: in particular, we show that the average area of a uniform random fighting fish with perimeter $2n$ is of order $n^{5/4}$: to the best of our knowledge this behaviour is non-standard and suggests that we have identified a new universality class of random structures.

Keywords: enumeration, generating functions, exactly solved models

(Some figures may appear in colour only in the online journal)

1. Introduction

The aim of this article is to launch the study of a new class of branching surfaces that appears to display remarkable probabilistic properties. We do this by introducing a simple model, inspired on the one hand by the rich literature on polyominoes brilliantly discussed in the recent book 'Polygons, Polyominoes and Polycubes' edited by Guttmann [6], and on the other hand by the fantastic aquatic creatures commonly called *fighting fish* (see figure 1).

* Dedicated to Tony Guttmann on the occasion of his 70th birthday.

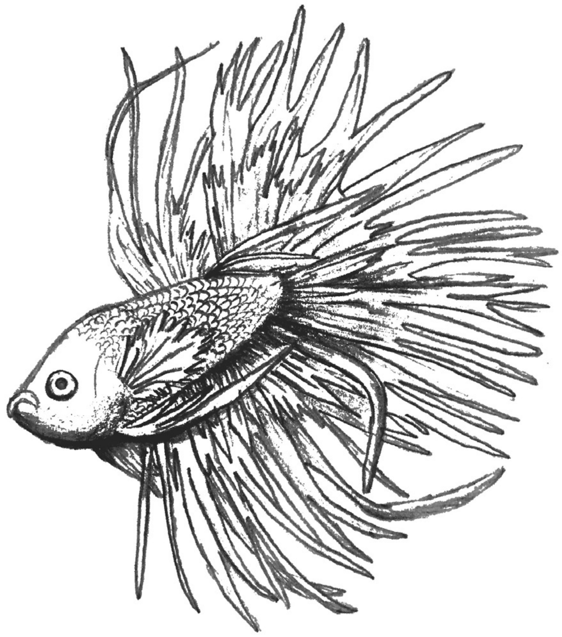

Figure 1. Siamese fighting fish. As reported in the `Encyclopaedia Britannica` 'the *Siamese fighting fish* (*Betta splendens*) is a freshwater tropical fish of the family Osphronemidae (order Perciformes), noted for the pugnacity of the males toward one another. The Siamese fighting fish, a native of Thailand, was domesticated there for use in contests. Combat consists mainly of fin nipping and is accompanied by a display of extended gill covers, spread fins, and intensified colouring'.

A first description of our combinatorial fighting fish is that they are built by gluing together unit squares of paper along their edges in a directed way. More precisely, as illustrated by figure 2, we consider 45 degree tilted unit squares which we call *cells* and view them as made of two triangular halves which we call the *left scale* and the *right scale*. In other words, the four edges of our squares are distinguished and called the *upper left edge*, *lower left edge*, *upper right edge* and *lower right edge*.

An edge of a cell in a fighting fish is *free* if it is not glued to the edge of another cell. All fighting fish can then be obtained from an initial cell called the *head* by attaching cells one by one in one of the three following ways: (see figure 2(b))

- Let a be a cell already in the fish whose upper right edge is free; then glue the lower left edge of a new cell b to the upper right side of a.
- Let a be a cell already in the fish whose lower right edge is free; then glue the upper left edge of a new cell b to the lower right edge of a.
- Let a, b and c be three cells already in the fish and such that b (resp. c) has its lower (resp. upper) left edge glued to the upper (resp. lower) right side of a, and b (resp. c) has its lower right (resp. upper right) edge free; then simultaneously glue the upper and lower left edge of a new cell d respectively to the lower right edge of b and to the upper right edge of c.

While this description is iterative we are interested in the objects that are produced, independently of the order in which cells are added: a *fighting fish* is a collection of cells glued

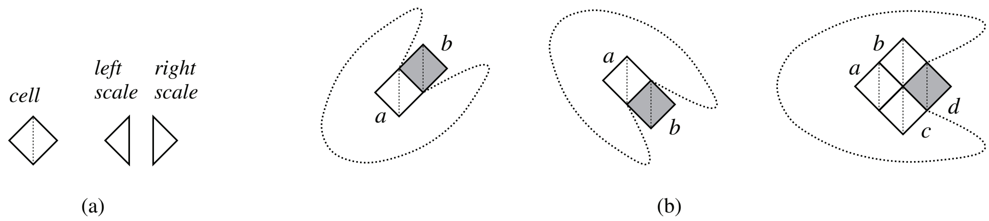

Figure 2. (a) The left and right scales of a cell; (b) the three ways to add a cell.

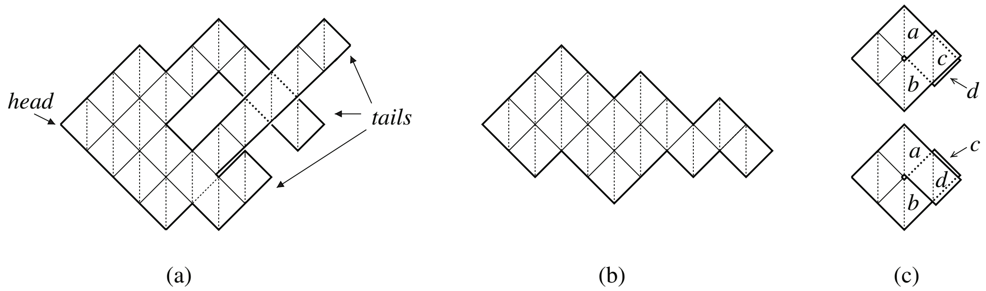

Figure 3. (a) A fighting fish which is not a polyomino; (b) A parallelogram polyomino; (c) Two different representations of the unique fighting fish with area 5 not fitting in the plane.

together edge by edge that *can* be obtained by the iterative process above. The *head* of the fighting fish is the only cell with two free left edges; a *tail* is a cell with two free right edges, and the *size* of a fighting fish is the number of free upper edges (which is easily seen to be equal to the number of free lower edges). The *area* of a fighting fish is the number of its cells.

We observe that the vertical edges of the scales of a fighting fish form vertical paths that cut it into vertical strips: each strip consists of an alternating sequence of left and right scales, starting with a free (left or right) lower edge and ending with a free (left or right) upper edge. In particular a fighting fish has the same number of upper and lower free edges.

Examples of fighting fish are *parallelogram polyominoes* (aka *staircase polyominoes*), directed convex polyominoes, and more generally simply connected directed polyominoes in the sense of [6]. However, one should stress the fact that fighting fish are not necessary polyominoes because they are not constrained to fit in the plane, as illustrated by figure 3(c).

The smallest fighting fish not fitting in the plane is obtained by gluing a square a to the upper right edge of the head, a square b to the lower right edge of the head, a square c to the upper right edge of a, and a square d to the lower right edge of b: in the natural projection of this fighting fish onto the plane, squares c and d have the same image. Observe that we do not specify whether c is above or below d; rather we consider that the surface has a branch point at vertex $c \cap d$ (see figure 3(c)). A list of all fighting fish of area at most 4 is given in figure 4.

In this paper we are mainly concerned with the enumeration of fighting fish with respect to their size and area. The first result we prove is a remarkable formula that suggests rich combinatorial structure and surprising connections with various known objects:

Theorem 1. *The number of fighting fish with $n+1$ free upper edges is*

$$\frac{2}{(n+1)(2n+1)}\binom{3n}{n}.$$

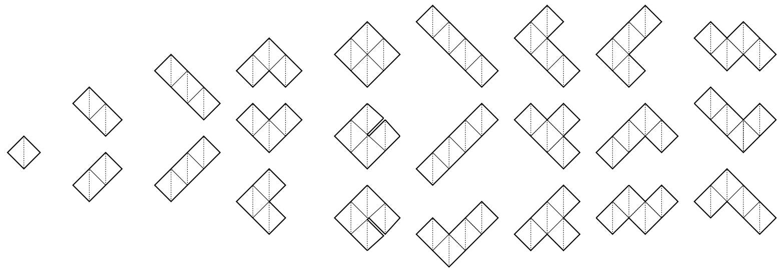

Figure 4. Fighting fish of area at most 4.

These numbers form sequence A000139 in [8] and they count, for instance, the number of 2-stack sortable permutations on n letters [13, 14], the number of rooted non-separable planar maps with n edges [3, 7, 12], and the number of left ternary trees having n nodes [4]. We have not found a direct bijection between fighting fish and any previously known structures but we offer some refined equienumeration conjectures at the end of the paper.

Another highlight of our paper is the following probabilistic consequence:

Theorem 2. *Let A_n denote the average area of uniform random fighting fish with $n+1$ free upper edges. Then, as n goes to infinity,*

$$A_n \sim \frac{3^{1/4}}{4\sqrt{2\pi}\,\Gamma(-\frac{1}{4})} \cdot n^{5/4}.$$

This results is worth comparing with table 11.1 in [9]: the area of uniform random polyominoes with perimeter n in all classical non-trivial solvable models of polyominoes behaves like $n^{3/2}$. Fighting fish clearly belong to a different universality class.

1.1. Organization of the paper

We first give in section 2 a recursive description of the slightly more general *fighting fish tails*, and a bijection with certain Motzkin labelled trees. Then we establish in section 3.1 a master equation for the generating function of fighting fish tails with respect to their number of free upper edges (variable t), number of tails (variable x) and area (variable q). We then solve in section 3.2 the equation in the case $q=1$ and give explicit formulas in section 3.3. Next we move in section 3.4 to the computation of the total area of fighting fish. Finally we briefly discuss ongoing further work and conjectures.

2. Fish tails and Motzkin trees

2.1. A recursive definition

For enumerative purpose it will be useful to give a recursive description of fighting fish. Let us define the set of *fish tails* and their heights inductively as follows:

Basis. The empty fish is the unique fish tail with height 0.
Inductive step. We define three operations:

- **operation** u: Given two fish tails f_1 of height $\ell \geqslant 0$ and f_2 of height $k \geqslant 0$, then a new fish tail f of height $\ell+1+k$ is obtained by adding a strip of $\ell+1$ right scales and ℓ left scales to f_1 and attaching this strip to the end point of the leftmost vertical path of f_2.

- **operations** h, h': Given two fish tails f_1 of height $\ell \geqslant 1$ and f_2 of height $k \geqslant 0$, then a new fish tail f of height $\ell + k$ is obtained by adding a strip of ℓ right scales and ℓ left scales to f_1 and attaching this strip to the end point of the leftmost vertical path of f_2. Observe that there are two ways to do this operation, depending whether the added strip starts with a right (operation h) or a left scale (operation h').
- **operation** d: Given two fish tails f_1 of height $\ell \geqslant 2$ and f_2 of height $k \geqslant 0$, then a new fish tail f of height $\ell - 1 + k$ is obtained by adding a strip of $\ell - 1$ right scales and ℓ left scales to f_1 and attaching this strip to the end point of the leftmost vertical path of f_2.

Observe that, like fighting fish, fish tails are not constrained to fit in the plane.

The following is straightforward.

Proposition 1. *Every fish tail can be obtained in a unique way using operations u, h, h', and d.*

Proof. The proof is by induction: There is a unique empty fish tail. By definition a non empty fish tail can be produced from smaller fish tails using one of the four operation, and only one operation can have been used since the four operations result in different forms of upper leftmost strip. The result follows by induction on f_1 and f_2. □

Proposition 2. *Fighting fish with $n + 1$ free upper edges are in one-to-one correspondence with fish tails with height 1 and n free upper edges.*

Proof. For each $k \geqslant 0$, let us call $\mathcal{F}_k$ the set of objects that can be built from an initial sequence of k right scales attached along a vertical line, by single cell additions of the three types of figure 2(b). Elements of $\mathcal{F}_1$ are clearly in bijection with fighting fish: given a fighting fish, the corresponding element of $\mathcal{F}_1$ is obtained by removing the left scale of its head. To prove the proposition we show more generally by induction that the set of fish tails is exactly $\mathcal{F} = \cup_k \mathcal{F}_k$.

First observe that the empty fish tail can be identified with the empty sequence of right scales, that is the unique element of $\mathcal{F}_0$. Then given an element f of $\mathcal{F}_k$ of size n, we prove that f is a fish tail. Assume that the first cut point p along the initial vertical line from the top of f occurs after $h \geqslant 1$ cells: by definition of p, in the initial right scales above p, two successive right scales have been glued to a common square using the third cell addition rule (otherwise there would be a cut point higher than p). These right and left scales form a strip S that can be used to decompose f into f_1 and f_2 using one of the four operations u, h, h' or d above: since f_1 and f_2 are disconnected from one another, the sequence of cell additions that produces f can be split into a sequence s_1 of cell additions that produces S and f_1, and a sequence s_2 of cell addition that produces f_2. Moreover in s_1 the cell additions that produce the squares incident to S can be performed first: $s_1 = s'_1 s'_2$ with s'_2 producing f_2 from an initial sequence of right scales. This shows that f_1 and f_2 both belong to $\mathcal{F}$ so that the inductive hypothesis can be applied: f_1 and f_2 are fish tails, and so is f.

Conversely, given a fish tail f of height k obtained from two fish tails f_1 and f_2 using operation u, h, h' or d. Then by the inductive hypothesis there are sequences of cell additions s_1 and s_2 producing f_1 and f_2, and these sequences can be combined with an initial sequence of cell additions applied to the k initial right scales to produce f: this implies that f belongs to $\mathcal{F}$ and concludes the proof. □

2.2. An alternative approach via a generalization of bicoloured Motzkin words

Recall that a *parallelogram polyomino* of semi-perimeter $n + 1$ is a pair (P, Q) of lattice paths with up steps $(1, 1)$ and down steps $(1, -1)$, starting from the origin and never meeting before their common end point (n, k), for some integer k. Parallelogram polyominoes are well known combinatorial objects [6, 11], and in particular it is known that the number of parallelogram polyominoes with semi-perimeter $n+1$ is equal to the nth Catalan number $\frac{1}{n+1}\binom{2n}{n}$.

Proposition 3. *Fighting fish with one tail are in one-to-one correspondance with parallelogram polyominoes.*

Proof. According to proposition 2, in order to construct a fighting fish with only one tail in any operation of the inductive definition of fish tails, f_2 must be the empty fish. The resulting definition is precisely the Temperley decomposition of parallelogram polyominoes provided for instance in [2, section 3.3.3.1]. □

We now propose an alternative characterization of fighting fish building on the standard correspondance between parallelogram polyominoes and bicoloured Motzkin words. Consider the four letter alphabet $\Sigma = \{u, d, h, h'\}$ and let δ be the morphism $(\Sigma^*, \cdot) \to (\mathbb{Z}, +)$ defined by $\delta(u) = 1, \delta(d) = -1$ and $\delta(h) = \delta(h') = 0$. A word w on Σ is a *bicoloured Motzkin word* if and only if $\delta(w) = 1$ and $\delta(v) \geqslant 1$, for all factorizations $w = zv$ with $|v| \geqslant 1$.

Proposition 4. *There is a bijection between parallelogram polyominoes with semi-perimeter $n+1$ and bicoloured Motzkin words of length n.*

Proof. Let (P, Q) be a parallelogram polyomino of semi-perimeter $n+1$, and P_i (resp. Q_i) be the ith step of P (resp. Q). The word w corresponding to (P, Q) is built so that $w_i \in \Sigma$ describes the pair (P_{i+1}, Q_{i+1}). Precisely, the ith term w_i is defined as follows:

- $w_i = h$ (resp. $w_i = h'$) if both P_{i+1} and Q_{i+1} are up steps (resp. down steps). In this case the width of the polyomino remains the same: $\delta(w_i) = 0$.
- $w_i = u$ if P_{i+1} is an up step and Q_{i+1} is a down step. In this case the width of the polyomino decreases by one from right to left: $\delta(u) = -1$.
- $w_i = d$ if P_{i+1} is a down step and Q_{i+1} is an up step. In this case the width of the polyomino increases by one from right to left: $\delta(d) = +1$. □

In order to extend this correspondance to fish tails, we introduce certain trees that can be regarded as an extension of bicoloured Motzkin words.

A *fish bone tree* (see figure 6) is a rooted plane tree T where each edge is labelled by a letter of Σ such that:

- the sum of $\delta(e)$ for all edges e of T is positive, where $\delta(e)$ stands for δ applied to the label of e;
- let e be an edge of T, with vertices i, j such that i is the father of j; then the sum of $\delta(f)$ running over all edges f in the subtree rooted at j, plus $\delta(e)$, is positive.

We define the height $\delta(T)$ of a fish bone tree T as the sum of $\delta(e)$ for all edges e of T:

$$\delta(T) = \sum_{e \in T} \delta(e).$$

Proposition 5. *Fish tails of height h with n upper edges are in one-to-one correspondance with fish bone trees with height h and n edges, and in particular fighting fish with $n+1$ upper edges are in one-to-one correspondance with fish bone trees with n edges and height 1.*

Proof. Given a fighting fish we construct an abstract rooted planar tree whose vertices correspond to the vertical paths of the fish. Let j be a vertex and $v(j)$ be the associated vertical path. The father i of j is the vertex corresponding to the vertical path $v(i)$ which lies on the

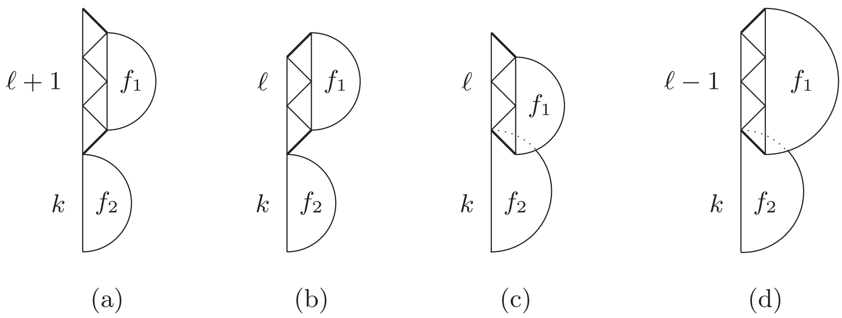

Figure 5. The recursive construction of fish tails: (a) operation u, (b), (c) operations h, h', (d) operation d.

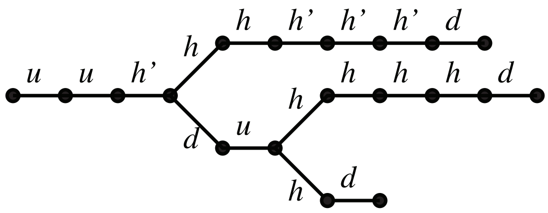

Figure 6. The fish bone tree corresponding to the fighting fish in figure 3(a).

other side of the vertical strip on the left $v(j)$. The label of the edge with vertices i, j indicates whether the strip starts and ends with left or a right scale, following the representation given in figure 5. Conversely given a fish bone tree we construct a fighting fish by gluing vertical strips starting from the leaves of the fish bone tree (which correspond to the tails of the fighting fish to be constructed) and proceeding bottom-up in the tree. □

In the special case of fighting fish with one tail, we recover the one-to-one correspondance of proposition 4 between parallelogram polyominoes and bicoloured Motzkin words, here presented as a chain (linear fish bone tree) of height 1.

3. Generating functions

3.1. The master equation

Let $\mathbf{F}(v,q) \equiv \mathbf{F}(v,q,x,t)$ denote the generating function of fish tails with variables t, v, x and q respectively marking the number of free upper edges (size), height, number of tails and area.

The following proposition immediately follows from the inductive definition of fish tails.

Proposition 6. *The series* $\mathbf{F}(v,q)$ *is the unique power series in* t *satisfying*

$$\mathbf{F}(v,q) = 1 + tvq\,\mathbf{F}(v,q)(\mathbf{F}(vq^2,q) - 1 + x) + 2t\,\mathbf{F}(v,q)(\mathbf{F}(vq^2,q) - 1) + \frac{t}{vq}\mathbf{F}(v,q)(\mathbf{F}(vq^2,q) - 1 - vq^2\,\mathbf{f}(q)), \tag{1}$$

where we have denoted $\mathbf{f}(q) = [v]\mathbf{F}(v,q)$.

Proof. The functional equation (1) for $\mathbf{F}(v,q)$ follows from the recursive definition of fish tails given in section 2.1. The unique fish tail with height 0 gives the term 1. Moreover, every non empty fish tail is obtained by applying one of the operation u, h, h' and d to two fish tails, and for each of these operations we have a different contribution to the expression of $\mathbf{F}(v,q)$:

- **operation** u: gives the term $tvq\,\mathbf{F}(v,q)(\mathbf{F}(vq^2,q)-1+x)$. In fact, the size (resp. height) of the obtained fish tail is given by the sum of the sizes (resp. heights) of f_1 and f_2 plus 1. Similarly, the number of tails of the obtained fish tail is given by the sum of the tails of f_1 and f_2, except for the case where f_1 is the empty fish. In this case, the number of tails is given by the number of the tails of f_2 plus 1. The area of the obtained fish tail is given by the sum of the areas of f_1 and f_2 plus $2\ell+1$, which explains the substitution $v := vq^2$ and the multiplication by q.
- **operations** h, h': give the term $2t\,\mathbf{F}(v,q)(\mathbf{F}(vq^2,q)-1)$. We use the same arguments as above, noticing that f_1 cannot to be the empty fish.
- **operation** d: gives the term $\frac{t}{vq}\mathbf{F}(v,q)(\mathbf{F}(vq^2,q)-1-vq^2\,\mathbf{f}(q))$. We use the same arguments as above, noticing that f_1 must be of height greater than 1. Then we have that the term $\mathbf{F}(vq^2,q)-1-v\,[v]\mathbf{F}(vq^2,q)$, where $v\,[v]\mathbf{F}(vq^2,q)$ corresponds to the case where f_1 has height equal to 1, can be rewritten as $vq^2[v]\mathbf{F}(v,q)$. The height of the obtained fish tail is given by the sum of the heights of f_1 and f_2 minus 1, whereas the area is given by the sum of the areas of f_1 and f_2 plus $2\ell-1$. □

Equation (1) can be rewritten in polynomial form as

$$\mathbf{P}(\mathbf{F}(v,q),\mathbf{F}(vq^2,q),\mathbf{f}(q),v,q)=0, \tag{2}$$

where $\mathbf{P}(w_1,w_2,w_3,v,q)\equiv\mathbf{P}(w_1,w_2,w_3,v,q,x,t)$ reads explicitely

$$-vqw_1+vq\;+\;tv^2q^2\,w_1(w_2-1+x)+\;2t\;vqw_1(w_2-1)+t\,w_1(w_2-1-vq^2\,w_3).$$

To the best of our knowledge, this type of polynomial catalytic q-equation has only been considered in the linear case. We point out that our ultimate aim is to study $\mathbf{f}(q)$, i.e. the generating function of fighting fish according to size, area and number of tails.

3.2. Enumeration with respect to the perimeter and number of tails

Letting $q=1$ the master equation (1) reduces to

$$F(v)=1\;+\;tv\,F(v)(F(v)-1+x)\;+2t\,F(v)(F(v)-1)+\frac{t}{v}F(v)(F(v)-1-v\,f), \tag{3}$$

where $F(v)\equiv\mathbf{F}(v,1,x,t)$ and $f\equiv\mathbf{f}(1,x,t)$. Equivalently,

$$P(F(v),F(v),f,v,x,t)=0 \tag{4}$$

where $P(w_1,w_2,w_3,v)\equiv\mathbf{P}(w_1,w_2,w_3,v,1,x,t)$, or in explicit form:

$$P(w_1,w_2,w_3,v)=-vw_1+v\;+\;tv^2\,w_1(w_2-1+x)+\;2t\;vw_1(w_2-1)\;+t\,w_1(w_2-1-v\,w_3).$$

This equation is now a *polynomial equation with one catalytic variable*, in the sense of Bousquet-Mélou et Jehanne [1], and it admits an explicitely computable algebraic solution:

Theorem 3. *Let $V \equiv V(x,t)$ be the unique power series solution of the equation*

$$V = t \cdot \left(1 + V + x \cdot \frac{V^2}{1-V}\right)^2. \tag{5}$$

Then,

$$f = x \cdot V - x^2 \cdot \frac{V^3}{(1-V)^2}. \tag{6}$$

Proof. Upon differentiating (4) with respect to v we obtain:

$$\frac{\partial F}{\partial v}(v) \cdot \left[\frac{\partial P}{\partial w_1}(F(v), F(v), f, v) + \frac{\partial P}{\partial w_2}(F(v), F(v), f, v)\right] + \frac{\partial P}{\partial v}(F(v), F(v), f, v) = 0. \tag{7}$$

The equation

$$\left[\frac{\partial P}{\partial w_1}(F(v), F(v), f, v) + \frac{\partial P}{\partial w_2}(F(v), F(v), f, 1, v)\right] = 0$$

reads

$$v = t \cdot Q(F(v), F(v), f, v, x), \quad \text{with} \quad Q(w_1, w_2, w_3, v, x) = (1+v)^2(w_1 + w_2 - 1) + xv^2 - vw_3,$$

so that it admits a unique power series solution $V \equiv V(x,t)$. Then the series $V \equiv V(x,t)$, $f \equiv f(x,t)$ and $F_V \equiv F(V(x,t), x, t)$ are solutions of the system of polynomial equations:

$$\begin{cases} P(F_V, F_V, f, V) = 0, \\ V = t \cdot Q(F_V, F_V, f, V), \\ \dfrac{\partial P}{\partial v}(F_V, F_V, f, V) = 0. \end{cases} \tag{8}$$

Now solving for t, f and F_V by elimination we find that the algebraic curve (t, V, f, F_V) admits the parametrization

$$\left\{ t = \frac{V(1-V)^2}{(1-(1-x)V^2)^2}, \quad f = xV - x^2\frac{V^3}{(1-V)^2}, \quad F_V = 1 + x\frac{V^2}{1-V^2}, \right\} \tag{9}$$

from which the result follows. □

3.3. Explicit formulas

We are now in position to prove theorem 1, that is, that the number f_n of fighting fish with $n+1$ free upper edges is

$$f_n = \frac{2}{(n+1)(2n+1)}\binom{3n}{n}. \tag{10}$$

Proof. For $x = 1$ theorem 3 reads

$$f(1,t) = V(1,t) - \frac{V(1,t)^3}{(1-V(1,t))^2} \quad \text{with} \quad V(1,t) = t \cdot \frac{1}{(1-V(1,t))^2},$$

so that we can apply the Lagrange inversion formula to determine $[t^n]f(1,t)$, which gives the number of fighting fish of size $n+1$: for $f = \psi(V)$ where $V = t\phi(V)$, the Lagrange inversion formula states that $[t^n]f = \frac{1}{n}[v^{n-1}]\psi'(v)\phi(v)^n$. This yields:

$$\begin{aligned}[t^n]f(1,t) &= \frac{1}{n}[v^{n-1}]\,\psi'(v)\phi(v)^n = \frac{1}{n}[v^{n-1}]\left(v - \frac{v^3}{(1-v)^2}\right)' \frac{1}{(1-v)^{2n}} \\ &= \frac{1}{n}[v^{n-1}]\frac{1}{(1-v)^{2n+3}} - \frac{1}{n}[v^{n-1}]\frac{3v}{(1-v)^{2n+3}} \\ &= \frac{1}{n}\binom{3n+1}{2n+2} - \frac{3}{n}\binom{3n}{2n+2} = \frac{4\,(3n)!}{n!\,(2n+2)!}.\end{aligned}$$

□

The same approach can be applied to derive the number of fighting fish of size $n+1$ with one tail:

$$[x\,t^n]f = \frac{1}{n}[x\,v^{n-1}]\left(xv - x^2\frac{v^3}{(1-v)^2}\right)'\left(1+v+x\frac{v^2}{(1-v)^2}\right)^{2n} \tag{11}$$

$$= \frac{1}{n}[v^{n-1}](1+v)^{2n} = \frac{(2n)!}{n!\,(n+1)!}. \tag{12}$$

As expected the coefficient of $x\,t^n$ in f is equal to $c_n = \frac{1}{n+1}\binom{2n}{n}$, the nth Catalan number. Indeed, as stated in proposition 3 fighting fish of size $n+1$ with only one tail are in one-to-one correspondance with parallelogram polyominoes of semi-perimeter $n+1$.

More generally in view of equation (5), the coefficient V_ℓ of x^ℓ in V is rational in the Catalan generating function V_0, where V_0 is the unique power series solution of $V_0 = t(1+V_0)^2$, and in view of equation (6) the same holds for the generating function of fighting fish with ℓ tails. However explicit expressions are not particularly simple and we do not report them here.

Alternatively one can consider the total number of tails:

Corollary 1. *The number of fighting fish of size $n+1$ with a marked tail is:*

$$s_n = \frac{1}{n}\binom{3n-2}{n-1}. \tag{13}$$

Proof. In order to obtain the number of fighting fish with a marked tail we calculate the function $\frac{\partial f}{\partial x}$ and then, we use again the Lagrange inversion formula to compute

$$s_n = [t^n]\frac{\partial f}{\partial x}(1,t).$$

Differentiating the first and second equations in system (9) with respect to x and then setting $x = 1$ we obtain a system of two equations in the two unknown $\frac{\partial f}{\partial x}(1,t)$ and $\frac{\partial V}{\partial x}(1,t)$:

$$\begin{cases} \dfrac{\partial f}{\partial x}(1,t) = \dfrac{\frac{\partial V}{\partial x}(1,t)(1-3V(1,t)) + V(1,t)(1-3V(1,t)) + V(1,t)^2(1+V(1,t))}{(1-V(1,t))^3} \\ \dfrac{\partial V}{\partial x}(1,t) = \dfrac{2\,t(\frac{\partial V}{\partial x}(1,t) + V(1,t)^2 - V(1,t)^3)}{(1-V(1,t))^3}. \end{cases} \tag{14}$$

Using $t = V(1,\ t)(1 - V(1,\ t))^2$ to simplify the above system we get $\frac{\partial f}{\partial x} = V(1,t)$. The Lagrange inversion theorem then reads

$$\begin{aligned} [t^n]\frac{\partial f}{\partial x}(1,t) &= [t^n]V(1,t) = \frac{1}{n}\,[v^{n-1}]\phi(v)^n \\ &= \frac{1}{n}\,[v^{n-1}]\,\frac{1}{(1-v)^{2n}} = \frac{1}{n}\binom{3n-2}{2n-1}. \end{aligned}$$

□

Corollary 2. *The average number of tails of a uniform random fighting fish of size* $n+1$ *is* $\frac{(n+1)(2n+1)}{3\,(3n-1)}$.

3.4. The total area generating function

The generating function of the total area of fighting fish is the series $A \equiv A(x,t)$ given by

$$A = \left.\frac{\partial(q\mathbf{f}(q))}{\partial q}\right|_{q=1} = f + \frac{\partial \mathbf{f}}{\partial q}(1),$$

that counts fighting fish weighted by their area (recall that $\mathbf{f}(q)$ counts fish tails of height 1 by their area, hence the correction factor q to account for the head of fighting fish). The series $\frac{\partial \mathbf{f}}{\partial q}(1)$ appears in the derivative of the master equation $\mathbf{P}(\mathbf{F}(v,q), \mathbf{F}(vq^2,q), \mathbf{f}(q), v, q) = 0$ with respect to q:

$$\begin{aligned} &\frac{\partial}{\partial q}\mathbf{P}(\mathbf{F}(v,q), \mathbf{F}(vq^2,q), \mathbf{f}(q), v, q) \\ &\quad = \frac{\partial}{\mathbf{F}}\partial q(v,q)\frac{\partial \mathbf{P}}{\partial w_1}(\mathbf{F}(v,q), \mathbf{F}(vq^2,q), \mathbf{f}(q), v, q) \\ &\qquad + \left(\frac{\partial}{\mathbf{F}}\partial q(vq^2,q) + 2vq\frac{\partial}{\mathbf{F}}\partial v(vq^2,q)\right)\frac{\partial \mathbf{P}}{\partial w_2}(\mathbf{F}(v,q), \mathbf{F}(vq^2,q), \mathbf{f}(q), v, q) \\ &\qquad + \frac{\partial \mathbf{f}}{\partial q}(q)\frac{\partial \mathbf{P}}{\partial w_3}(\mathbf{F}(v,q), \mathbf{F}(vq^2,q), \mathbf{f}(q), v, q) + \frac{\partial \mathbf{P}}{\partial q}(\mathbf{F}(v,q), \mathbf{F}(vq^2,q), \mathbf{f}(q), v, q) = 0. \end{aligned} \tag{15}$$

Indeed, for $q = 1$ this equation can be rewritten as

$$\begin{aligned}
&\frac{\partial}{\partial q}\mathbf{P}(\mathbf{F}(v,q),\mathbf{F}(vq^2,q),\mathbf{f}(q),v,q)|_{q=1}\\
&\quad=\frac{\partial\mathbf{F}}{\partial q}(v,1)\frac{\partial P}{\partial w_1}(F(v),F(v),f,v)\\
&\qquad+\left(\frac{\partial}{\mathbf{F}}\partial q(v,1)+2v\frac{\partial F}{\partial v}(v)\right)\frac{\partial P}{\partial w_2}(F(v),F(v),f,v)\\
&\qquad+\frac{\partial\mathbf{f}}{\partial q}(1)\frac{\partial P}{\partial w_3}(F(v),F(v),f,v)+\frac{\partial\mathbf{P}}{\partial q}(F(v),F(v),f,v,1)=0.
\end{aligned}$$

Upon setting $v=V$, a simplification occurs as the coefficient of $\frac{\partial}{\mathbf{F}}\partial q(v,1)$ is precisely the defining equation for V:

$$\begin{aligned}
&2v\frac{\partial F}{\partial v}(V)\frac{\partial P}{\partial w_2}(F_V,F_V,f,V)\\
&\quad+\frac{\partial\mathbf{f}}{\partial q}(1)\frac{\partial P}{\partial w_3}(F_V,F_V,f,V)+\frac{\partial\mathbf{P}}{\partial q}(F_V,F_V,f,V,1)=0.
\end{aligned}\tag{16}$$

In order to obtain an equation for $\frac{\partial\mathbf{f}}{\partial q}(1)$ we thus need to determine $\frac{\partial F}{\partial v}(V)$.

Recall that the derivative of the main equation with respect to v is

$$\frac{\partial F}{\partial v}(v)\cdot\left[\frac{\partial P}{\partial w_1}(F(v),F(v),f,v)+\frac{\partial P}{\partial w_2}(F(v),F(v),f,v)\right]+\frac{\partial P}{\partial v}(F(v),F(v),f,v)=0.\tag{17}$$

One cannot simply set $v=V$ in this equation to obtain $\frac{\partial F}{\partial v}(V)$ because the coefficient of $\frac{\partial F}{\partial v}(v)$ cancels at this point by definition of V. Instead we expand the equation at $v=V$ to the second order:

$$\begin{aligned}
&\underbrace{\frac{\partial F}{\partial v}(V)\cdot\left[\frac{\partial P}{\partial w_1}(.)+\frac{\partial P}{\partial w_2}(.)\right]+\frac{\partial P}{\partial v}(.)}_{=0}\\
&\quad+(v-V)\cdot\left(\frac{\partial^2 F}{\partial v^2}(V)\cdot\underbrace{\left[\frac{\partial P}{\partial w_1}(.)+\frac{\partial P}{\partial w_2}(.)\right]}_{=0}\right.\\
&\quad+\frac{\partial F}{\partial v}(V)\cdot\left[\frac{\partial F}{\partial v}(V)\cdot\left(\frac{\partial}{\partial w_1}+\frac{\partial}{\partial w_2}\right)+\frac{\partial}{\partial v}\right]\left(\frac{\partial P}{\partial w_1}+\frac{\partial P}{\partial w_2}\right)(.)\\
&\quad+\left.\left(\left[\frac{\partial F}{\partial v}(V)\cdot\left(\frac{\partial}{\partial w_1}+\frac{\partial}{\partial w_2}\right)+\frac{\partial}{\partial v}\right]\frac{\partial P}{\partial v}\right)(.)\right)\\
&=O((v-V)^2),
\end{aligned}$$

where (.) stands for the evaluation at (F_V,F_V,f,V). Since the coefficients are zero at all orders in this expansion at $v=V$, the coefficient of $(v-V)$ yields a quadratic equation, which turns out to uniquely define $\frac{\partial F}{\partial v}(V)$ in terms of V and x: there is a polynomial

$$R(w,v,x)=-2v(1-v^2)^2(1+v)^2w^2+2(1-v^2)^2(1-v^2+xv^2)w-2xv(1-v^2+xv^2),$$

quadratic in w, such that $\frac{\partial F}{\partial v}(V)$ is the unique power series solution of

$$R(\frac{\partial F}{\partial v}(V), x, V) = 0. \tag{18}$$

Together with equation (16) this equation allows us to obtain by elimination a quadratic equation satisfied by $\frac{\partial \mathbf{f}}{\partial q}(1)$ over $\mathbb{Q}(V, x)$. Using the expression of f in terms of V a similar result is obtained for the series A.

Proposition 7. *The generating function $A \equiv A(x, t)$ for the total area of fighting fish is algebraic of degree* 2 *over $\mathbb{Q}(V, x)$ and satisfies*

$$-V(1-V)^2A^2 + 2(1-V)^2(1-V^2+xV^2)A - 4xV(1-V^2+xV^2) = 0. \tag{19}$$

Extracting the coefficient of x in this equation yields

$$2(1-V_0)^2(1-V_0^2)A_1 - 4xV_0(1-V_0^2) = 0,$$

where $V_0 = [x^0]V$ is a Catalan generating function satisfying $V_0 = t(1+V_0)^2$. From this equation we recover the generating function A_1 for the total area of parallelogram polyominoes, viewed as fighting fish with one tail:

$$A_1 = \frac{2V_0}{(1-V_0)^2} = \frac{2t}{1-4t}.$$

The simplification to a rational function of t is a well known feature of parallelogram polyominoes [10]. Observe that it implies that the average area of parallelogram polyominoes of size n is $4^n/C_n$, that is of order $n^{3/2}$.

In general upon extracting the coefficient of x^ℓ in equation (19) and again using the rationality of coefficients of x^i in V we obtain that the generating function of the total area of fighting fish with ℓ tails as a rational function of the Catalan generating function V_0, the unique power series solution of $V_0 = t(1+V_0)^2$.

3.5. The average area of fighting fish

We now prove theorem 2, stating that the average area A_n of fighting fish with $n+1$ free upper edges grows like $n^{5/4}$.

Proof. The series $V(1,t)$ is by definition the unique power series solution of the equation $V = t\phi(V)$ with $\phi(x) = 1/(1-x)^2$. Standard results in analytic combinatorics [5, theorem VI.4, p 404] ensure that V has a finite radius of convergence t_c and a singular expansion of the generic square root type:

$$V = V_c - \gamma\sqrt{1-t/t_c} + O(1-t/t_c), \tag{20}$$

where the constant $V_c = 1/3$ is the smallest positive root of $\phi(x) - x\phi'(x) = (1-x)(1-3x)$, $t_c = V_c/\phi(V_c) = 4/27$ and $\gamma = \sqrt{\frac{2\phi(V_c)}{\phi''(V_c)}} = \sqrt{4/27} = \frac{2}{3\sqrt{3}}$.

Equation (19) for $x = 1$ reads:

$$V(1-V)^2A^2 - 2(1-V)^2A + 4V = 0,$$

or equivalently

$$A = \frac{1}{V} - \frac{\sqrt{(1+V)(1-3V)}}{V(1-V)}.$$

Since A has positive coefficients, by Pringsheim's theorem [5, theorem IV.4, p 240] it has a positive dominant singularity, which is obtained for $V = 1/3$, that is, at $t = t_c$. From the above expression in terms of V and expansion (20) we have the singular expansion

$$A = 3 - \frac{\sqrt{\frac{4}{3} \cdot 3 \frac{2}{3\sqrt{3}} \sqrt{(1 - t/t_c)}}}{2/9} + O\sqrt{(1 - t/t_c)}$$
$$= 3 - 3^{5/4}\sqrt{2}\,(1 - t/t_c)^{1/4} + O(1 - t/t_c)^{1/2}.$$

The standard function scale ([5, theorem VI.1, p 381]) and the transfert theorem ([5, theorem VI.3, p 390]) yield

$$[t^n]A \underset{n\to\infty}{\sim} \frac{3^{5/4}\sqrt{2}}{\Gamma(-\frac{1}{4})} n^{-5/4} t_c^{-n}.$$

Now returning to the expression of f in terms of V in system (9) and using again expansion (20) yields $f(t) = \frac{1}{4} - \frac{3}{4}(1 - t/t_c) + \frac{\sqrt{3}}{2}(1 - t/t_c)^{3/2} + O((1 - t/t_c)^2)$, so that

$$[t^n]f \underset{n\to\infty}{\sim} \frac{3}{8\sqrt{\pi}} n^{-5/2} t_c^{-n}.$$

(This results also follows immediately from theorem 1 using Stirling's formula). Finally the average area of fighting fish of size n is the ratio of the above two coefficients which is equivalent to $\frac{3^{1/4}}{4\sqrt{2\pi}\,\Gamma(-1/4)} \cdot n^{5/4}$ as n goes to infinity. □

4. Fighting fish versus left ternary trees

As already observed in the introduction the number f_n of fighting fish defines sequence A000139 in [8] and this sequence is known in the literature to count various combinatorial structures, including 2-stack sortable permutations on n letters [14], rooted non-separable planar maps with n edges [3, 12], and left ternary trees having n nodes [4].

One approach to provide a more combinatorial explanation of formula (10) would be to find a bijection between fighting fish and some of these known objects. Let us concentrate for instance on *left ternary trees*.

A *ternary tree* is finite set of nodes which is either empty or contains a root and three disjoint ternary trees, called the root's left, middle and right subtrees (see figure 7(a)). For each node y of a ternary tree T, there is a unique path $\gamma(y)$ from the root of T to y. The path $\gamma(y)$ consists of three steps, left, middle and right. The abscissa of y is the difference between $\gamma(y)$'s left and right steps. A *left tree* is a ternary tree whose nodes all have nonnegative abscissa (see figure 7(b)). In [4] it is bijectively proved that the number of left ternary trees with n nodes is given by $\frac{2}{(n+1)(2n+1)}\binom{3n}{n}$, which is equal to the number of fighting fish of size n. Moreover, from corollary 1 it follows that the number of fighting fish of size n with a marked tail is equal to the number of pairs of left ternary trees of size $n-1$ (sequence A006013 in [8]).

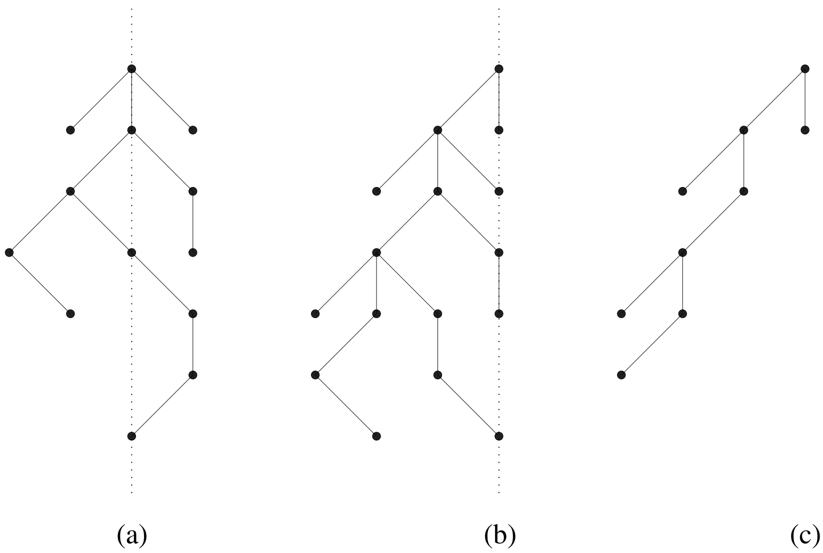

Figure 7. (a) A ternary tree; (b) a left ternary tree T; (c) the core $c(T)$ of the left ternary tree T in (b).

The equicardinality of the set of fighting fish of size n and the set of left ternary trees with n nodes suggests looking for an explicit bijection between theses objects but until now we have been unable to do so. Instead our effort in this direction has led us to formulate a refined equidistribution conjecture.

Let us call the *fin* of a fighting fish the path starting from the head of a fish with a lower edge, following its border, and ending the first time the path reaches a tail. On the other hand, let us denote by the *core* of a left ternary tree T, the binary tree $c(T)$ included in T and obtained from T by removing all right edges of T and their corresponding subtrees. Using analytic computations (not reported here), and by experimental evidence we are led to formulate the following conjecture:

Conjecture 1. The number of fighting fish with size n and fin length k is equal to the number of left ternary trees with n nodes and core size k.

5. Conclusion

We have identified a new universality class of combinatorial structures that model branching surfaces. We foresee several directions of research naturally arising from our work:

– *Algebraic decomposition.* Our first aim is to obtain a derivation of theorem 3 based on a direct algebraic decomposition of fighting fish with a marked tail or with a marked branchpoint (indeed the generating functions of these marked fish are indeed respectively xV and $x^2V^3/(1-V)^2$, which are clearly $\mathbb{N}$-algebraic series). The gf of unmarked fighting fish is then clearly the difference between these two gfs. By standard methods such a derivation would lead to a more efficient random sampling algorithm for fighting fish than the straighforward recursive approach applied to the master equation.

– *Area distribution.* Our approach to compute the mean area clearly extends to further moments and should lead to a characterization of the area distribution. Another interesting direction is to study directly the master equation as a polynomial catalytic q-equation and discuss the phase diagram of the model in q and t.

– *Universality.* Natural variants of our model can be designed, for instance by imposing local restrictions to fish bone trees (e.g. restrict to binary fish bone trees): we expect such models to behave quantitatively like fighting fish (number of objects growing like $\rho^n n^{-5/2}$, area of order $n^{5/4}$), and more generally to lead to branching surfaces with similar large-scale properties but we have chosen to concentrate on fighting fish due to their particularly nice enumerative properties.

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