OVERPARTITIONS, LATTICE PATHS, AND ROGERS-RAMANUJAN IDENTITIES

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Abstract. We extend partition-theoretic work of Andrews, Bressoud, and Burge to overpartitions, defining the notions of successive ranks, generalized Durfee squares, and generalized lattice paths, and then relating these to overpartitions defined by multiplicity conditions on the parts. This leads to many new partition and overpartition identities, and provides a unification of a number of well-known identities of the Rogers-Ramanujan type. Among these are Gordon's generalization of the Rogers-Ramanujan identities, Andrews' generalization of the Göllnitz-Gordon identities, and Lovejoy's "Gordon's theorems for overpartitions."

1. Introduction

In 1961 Gordon established his celebrated combinatorial generalization of the Rogers-Ramanujan identities:

Theorem 1.1. [26] *Let $B_{k,i}(n)$ denote the number of partitions of n of the form $(\lambda_1, \lambda_2, \ldots, \lambda_s)$, where $\lambda_\ell - \lambda_{\ell+k-1} \geq 2$ and at most $i-1$ of the parts are equal to 1. Let $A_{k,i}(n)$ denote the number of partitions of n into parts not congruent to $0, \pm i$ modulo $2k+1$. Then $A_{k,i}(n) = B_{k,i}(n)$.*

Over the years, a number of other combinatorial functions have been found to be equal to the $A_{k,i}(n)$ in Gordon's theorem. Most notable, perhaps, are two results of Andrews that employ Atkin's successive ranks [11] and Andrews' own new idea of Durfee dissection :

Theorem 1.2. [5] *Let $C_{k,i}(n)$ be the number of partitions of n whose successive ranks lie in the interval $[-i+2, 2k-i-1]$ and let $D_{k,i}(n)$ be the number of partitions of n with $i-1$ successive Durfee squares followed by $k-i$ successive Durfee rectangles. Then*

$$A_{k,i}(n) = B_{k,i}(n) = C_{k,i}(n) = D_{k,i}(n).$$

An overpartition is a partition where the final occurrence of a part can be overlined [18]. For example, there exist 8 overpartitions of 3

$$(3),\ (\overline{3}),\ (2,1),\ (\overline{2},1),\ (2,\overline{1}),\ (\overline{2},\overline{1}),\ (1,1,1),\ (1,1,\overline{1}).$$

In recent years overpartitions have been heavily studied, sometimes under different names and guises. They have been called joint partitions [12], or dotted partitions [13], and they are closely related to 2-modular diagrams [32], jagged partitions [23, 24] and superpartitions [21]. Overpartitions arise in the study of the combinatorics of basic hypergeometric series identities [19, 24, 28, 29, 38], congruences properties of modular forms [23, 33], supersymmetric functions [21], Lie algebras [27] and mathematical physics [21, 23, 24].

In 2003 Lovejoy [28] proved an overpartition identity wherein one of the functions closely resembles the $B_{k,i}(n)$ in Gordon's theorem:

Theorem 1.3 ("Gordon's theorem for overpartitions")**.** [28] *Let $\overline{B}_k(n)$ denote the number of overpartitions of n of the form $(\lambda_1, \lambda_2, \ldots, \lambda_s)$, where $\lambda_\ell - \lambda_{\ell+k-1} \geq 1$ if $\lambda_{\ell+k-1}$ is overlined and $\lambda_\ell - \lambda_{\ell+k-1} \geq 2$ otherwise. Let $\overline{A}_k(n)$ denote the number of overpartitions of n into parts not divisible by k. Then $\overline{A}_k(n) = \overline{B}_k(n)$.*

2000 *Mathematics Subject Classification.* Primary 11P81; Secondary 05A17.

Key words and phrases. Partitions, overpartitions, Rogers-Ramanujan identities, lattice paths.

The authors are partially supported by the ACI Jeunes Chercheurs "Partitions d'entiers à la frontière de la combinatoire, des q-séries et de la théorie des nombres."

Notice that Lovejoy's result can be seen as an overpartition analogue of Gordon's theorem, in the sense that the conditions on the $\overline{B}_k(n)$ reduce to the conditions on the $B_{k,k}(n)$ if the overpartition has no overlined parts and is indeed a partition.

Two questions naturally arise. First, given the similarities between Theorems 1.1 and 1.3, is there some general framework which contains these two theorems? Second, is there an analogue for overpartitions of Andrews' result, Theorem 1.2? In this paper, both of these questions shall be answered in the affirmative. Moreover, our results contain many other partition and overpartition identities, such as Andrews' generalization of the Göllnitz-Gordon identities [6].

It is well understood combinatorially that $B_{k,i}(n) = C_{k,i}(n) = D_{k,i}(n)$ and this result was established by some beautiful work of Burge [16, 17] using some recursive arguments. This work was reinterpreted by Andrews and Bressoud [8] who showed that Burge's argument could be rephrased in terms of lattice paths with two kinds of steps and that Gordon's theorem can be established thanks to these combinatorial arguments and the Jacobi Triple product identity [25]. Finally Bressoud [14] reinterpreted these in terms of different lattice paths with three kinds of steps and gave some direct bijections between the objects counted by $B_{k,i}(n)$, $C_{k,i}(n)$, $D_{k,i}(n)$ and the lattice paths.

With our main theorem we extend the main results of the above works [8, 14, 16, 17] to overpartitions. In particular, we generalize all of the combinatorial definitions to overpartitions and successfully adapt the methods of proof. This is the result that provides a unifying framework for Theorems 1.1, 1.2, and 1.3.

Theorem 1.4.

- *Let $\overline{B}_{k,i}(n,j)$ be the number of overpartitions of n of the form $(\lambda_1, \lambda_2, \dots, \lambda_s)$ with j overlined parts, where $\lambda_\ell - \lambda_{\ell+k-1} \geq 1$ if $\lambda_{\ell+k-1}$ is overlined and $\lambda_\ell - \lambda_{\ell+k-1} \geq 2$ otherwise and at most $i-1$ parts are equal to 1.*
- *Let $\overline{C}_{k,i}(n,j)$ be the number of overpartitions of n with j non-overlined parts in the bottom row of their Frobenius representation and whose successive ranks lie in $[-i+2, 2k-i-1]$.*
- *Let $\overline{D}_{k,i}(n,j)$ be the number of overpartitions of n with j overlined parts and $i-1$ successive Durfee squares followed by $k-i$ successive Durfee rectangles, the first one being a generalized Durfee square/rectangle.*
- *Let $\overline{E}_{k,i}(n,j)$ be the number of paths that use four kinds of unitary steps with special (k,i)-conditions, major index n, and j South steps.*

Then $\overline{B}_{k,i}(n,j) = \overline{C}_{k,i}(n,j) = \overline{D}_{k,i}(n,j) = \overline{E}_{k,i}(n,j)$.

All of the combinatorial notions in this theorem will be defined in detail in Section 2. The addition of the generalized lattice paths counted by $\overline{E}_{k,i}(n,j)$ is the key step which allows us to prove Theorem 1.4 combinatorially. In terms of generating functions, we have:

Theorem 1.5. *The generating function $\overline{\mathcal{E}}_{k,i}(a,q) = \sum_{n,j} \overline{E}_{k,i}(n,j) q^n a^j$ is:*

$$\overline{\mathcal{E}}_{k,i}(a,q) = \frac{(-aq)_\infty}{(q)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n a^n q^{kn^2+(k-i+1)n} \frac{(-1/a)_n}{(-aq)_n}. \tag{1.1}$$

Here we have used the classical q-series notations:

$$\begin{aligned} (a)_\infty &= (a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i), \\ (a)_n &= (a)_\infty/(aq^n)_\infty, \\ (a_1,\dots,a_k;q)_\infty &= (a_1;q)_\infty \dots (a_k;q)_\infty. \end{aligned}$$

In several cases, we can use the Jacobi Triple Product identity [25]:

$$(-1/z, -zq, q; q)_\infty = \sum_{n=-\infty}^{\infty} z^n q^{\binom{n+1}{2}} \tag{1.2}$$

to show that this generating function has a very nice form. For example,

Corollary 1.6.

$$\overline{\mathcal{E}}_{k,i}(0,q) = \frac{(q^i, q^{2k+1-i}, q^{2k+1}; q^{2k+1})_\infty}{(q)_\infty} \tag{1.3}$$

$$\overline{\mathcal{E}}_{k,i}(1/q,q^2) = \frac{(q^2;q^4)_\infty (q^{2i-1}, q^{4k+1-2i}, q^{4k}; q^{4k})_\infty}{(q)_\infty} \tag{1.4}$$

$$\overline{\mathcal{E}}_{k,i}(1,q) = \frac{(-q)_\infty}{(q)_\infty} \sum_{j=0}^{2(k-i)} (-1)^j (q^{i+j}, q^{2k-i-j}, q^{2k}; q^{2k})_\infty \tag{1.5}$$

$$\begin{aligned} \overline{\mathcal{E}}_{k,i}(1/q,q) &= \frac{(-q)_\infty}{(q)_\infty}((q^i, q^{2k-i}, q^{2k}; q^{2k})_\infty \\ &\quad +(q^{i-1}, q^{2k+1-i}, q^{2k}; q^{2k})_\infty) \end{aligned} \tag{1.6}$$

Hence our result gives a general view of different problems on partitions and overpartitions and shows how they are related.

- Equation (1.3) corresponds to the Andrews-Gordon identities [5].
- Equation (1.4) corresponds to Andrews's generalization of the Gordon-Göllnitz identities [6, 8].
- Equation (1.5) with $i = k$ and (1.6) with $i = 1$ correspond to the two Gordon's theorems for overpartitions of Lovejoy [28].

Therefore our extension of the work on the Andrews-Gordon identities [8, 14, 16, 17] to the case of overpartitions includes these identities, but it also includes Andrews's generalization of the Gordon-Göllnitz identities and Gordon's theorems for overpartitions. We prove this Corollary and deduce some new partition theorems in Section 7.

In Section 2 we present all the necessary notions. In Section 3 we present the paths counted by $\overline{E}_{k,i}(n,j)$ and compute the generating function. In Section 4 we present a direct bijection between the paths counted by $\overline{E}_{k,i}(n,j)$ and the overpartitions counted by $\overline{C}_{k,i}(n,j)$. In Section 5 we present a recursive bijection between the paths counted by $\overline{E}_{k,i}(n,j)$ and the overpartitions counted by $\overline{B}_{k,i}(n,j)$. We also give a generating function proof. In Section 6, we present a combinatorial argument that shows that the paths counted by $\overline{E}_{k,i}(n,j)$ and the overpartitions counted by $\overline{D}_{k,i}(n,j)$ are in bijection. All these bijections are refinements of Theorem 1.4. The number of peaks of the paths will correspond respectively to the number of columns of the Frobenius representations, the length of the multiplicity sequence and the size of the generalized Durfee square. In Section 7 we prove Corollary 1.6 and interpret it combinatorially. We conclude in Section 8 with further questions.

2. Definitions on overpartitions

We will define all the notions in terms of overpartitions. We refer to [3] for definitions for partitions. In all of the cases the definitions coincide when the overpartition has no overlined parts.

An overpartition of n is a non-increasing sequence of natural numbers whose sum is n in which the final occurrence (equivalently, the first occurrence) of a number may be overlined. Alternatively n can be called the weight of the overpartition. Since the overlined parts form a partition into distinct parts and the non-overlined parts form an ordinary partition, the generating function for overpartitions is $\frac{(-q)_\infty}{(q)_\infty}$. The Ferrers diagram of an overpartition is a classical Ferrers diagram where the corners can be marked (see Figure 1). A 2-modular diagram is a Ferrers diagram of an overpartition where the marked corners are filled with ones and the other cells are filled with twos (see Figure 2). The weight is the sum of the entries.

The *multiplicity* of the part j of an overpartition, denoted by f_j, is the number of occurrences of this part. We overline the multiplicity if the part appears overlined. For example, the multiplicity of the part 4 in the overpartition $(6,6,5,4,4,\overline{4},3,\overline{1})$ is $f_4 = \overline{3}$. The multiplicity sequence is the sequence $(f_1, f_2, \ldots)$. For example the previous overpartition has multiplicity sequence $(\overline{1}, 0, 1, \overline{3}, 1, 2)$.

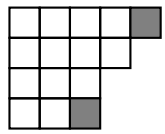

FIGURE 1. Ferrers diagram of the overpartition $\lambda = (\overline{5}, 4, 3, \overline{3})$ of weight 15.

2	2	2	2	1
2	2	2	2	
2	2	2		
2	2	1		

FIGURE 2. Example of a 2-modular diagram of weight 28.

The Frobenius representation of an overpartition [18, 30] of n is a two-rowed array

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_N \\ b_1 & b_2 & \cdots & b_N \end{pmatrix}$$

where $(a_1, \ldots, a_N)$ is a partition into distinct nonnegative parts and $(b_1, \ldots, b_N)$ is an overpartition into nonnegative parts where the *first* occurrence of a part can be overlined and $N + \sum(a_i + b_i) = n$. Following [18], we call $p_{Q,\mathcal{O}}(n)$ the number of such two-rowed arrays. We call this the Frobenius representation of an overpartition because it is in bijection with overpartitions. This was proved in [18] and we now recall the algorithm used for that proof.

We use the notion of a *hook*. Given a positive integer a and a non-negative integer b, $h(a, b)$ is the hook that corresponds to the partition $(a, 1, \ldots, 1)$ where there are b ones. Combining a hook $h(a, b)$ and a partition α is possible if and only if $a > \alpha_1$ and $b \geq l(\alpha)$, where $l(\alpha)$ denotes the number of parts of α. The result of the union is $\beta = h(a, b) \cup \alpha$ with $\beta_1 = a$, $l(\beta) = b + 1$ and $\beta_i = \alpha_{i-1} + 1$ for $i > 1$.

Now take a two-rowed array ν counted by $p_{Q,\mathcal{O}}(n)$, increase the entries on the top row by 1 and initialize α and β to the empty object, ϵ. Beginning with the rightmost column of ν, we proceed to the left, building α into an ordinary partition and β into a partition into distinct parts. At the ith column, if b_i is overlined, then we combine the hook $h(a_i, b_i)$ and α. Otherwise, we add the part b_i to α' (the conjugate of α) and the part a_i to β. Joining the parts of α together with the parts of β gives the overpartition λ. An example is given below starting with $\nu = \begin{pmatrix} 7 & 5 & 4 & 2 & 0 \\ 6 & \overline{4} & 4 & 3 & \overline{1} \end{pmatrix}$.

ν	α	β
$\begin{pmatrix} 8 & 6 & 5 & 3 & 1 \\ 6 & \overline{4} & 4 & 3 & \overline{1} \end{pmatrix}$	ϵ	ϵ
$\begin{pmatrix} 8 & 6 & 5 & 3 \\ 6 & \overline{4} & 4 & 3 \end{pmatrix}$	$(1, 1)$	ϵ
$\begin{pmatrix} 8 & 6 & 5 \\ 6 & \overline{4} & 4 \end{pmatrix}$	$(2, 2, 1)$	(3)
$\begin{pmatrix} 8 & 6 \\ 6 & \overline{4} \end{pmatrix}$	$(3, 3, 2, 1)$	$(5, 3)$
$\begin{pmatrix} 8 \\ 6 \end{pmatrix}$	$(6, 4, 4, 3, 2)$	$(5, 3)$
ϵ	$(7, 5, 5, 4, 3, 1)$	$(8, 5, 3)$

We get $\lambda = (\overline{8}, 7, 5, 5, \overline{5}, 4, 3, \overline{3}, 1)$. The reverse bijection can be easily described. See [18].

We now define the successive ranks.

Definition 2.1. The *successive ranks* of an overpartition can be defined from its Frobenius representation. If an overpartition has Frobenius representation

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_N \\ b_1 & b_2 & \cdots & b_N \end{pmatrix}$$

then its ith successive rank r_i is $a_i - b_i$ minus the number of non-overlined parts in $\{b_{i+1}, \ldots, b_N\}$.

This definition is an extension of Lovejoy's definition of the rank [30]. For example, the successive ranks of $\begin{pmatrix} 7 & 4 & 2 & 0 \\ \overline{3} & 3 & 1 & \overline{0} \end{pmatrix}$ are $(2, 0, 1, 0)$.

We say that the *generalized Durfee square* of an overpartition λ has size N if N is the largest integer such that the number of overlined parts plus the number of non-overlined parts greater or equal to N is greater than or equal to N (see Figure 3).

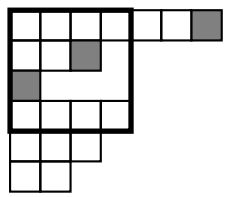

FIGURE 3. The generalized Durfee square of $\lambda = (\overline{7}, 4, 3, \overline{3}, 2, \overline{1})$ has size 4.

Proposition 2.2. *There exists a bijection between overpartitions whose Frobenius representation has N columns and whose bottom line has j overlined parts and overpartitions with generalized Durfee square of size N and $N - j$ overlined parts.*

Proof. An overpartition with generalized Durfee square of size N can be decomposed into a partition α into at most N parts (the conjugate of the partition under the generalized Durfee square) and an overpartition γ into N parts whose non-overlined parts are $\geq N$ (the rest). For example, the overpartition on Figure 3 gives $\alpha = (2, 2, 1)$ and $\gamma = (\overline{7}, 4, \overline{3}, \overline{1})$. An overpartition whose Frobenius representation has N columns can be decomposed into a partition β into N distinct parts (β is obtained by adding 1 to each part of the top line), a partition δ into distinct parts which lie between 0 and $N-1$ and a partition α into at most N parts. δ and α are obtained from the bottom line as follows: we first initialize α to the bottom line, then if the ith part of the bottom line is overlined, we take off its overline, we decrease the first $(i-1)$ parts of α by 1 and add a part $i-1$ to δ.

Now there exists a bijection between ordered pairs (β, δ) and overpartitions γ. This bijection is defined as follows: we overline all the parts of β, then for each part i in δ, we add i to β_{i+1} and we remove the overlining. We then reorder the parts, which gives us γ. This is easily invertible and is very similar to the Algorithm Z [9].

The decompositions of the first paragraph and the bijection of the second paragraph show that there is indeed a bijection between overpartitions whose Frobenius representation has N columns and overpartition with generalized Durfee square of size N. If there are j overlined parts in the bottom line of the Frobenius symbol, there are j parts in δ and by the bijection, there are $N - j$ overlined parts in γ. The proposition is thus established. □

Example 2.3. Let $\nu = \begin{pmatrix} 7 & 5 & 4 & 2 & 0 \\ 6 & \overline{4} & 4 & 3 & \overline{1} \end{pmatrix}$. We thus have $N = 5$ and $j = 2$ in this example. By adding 1 to each part of the top line, we get $\beta = (8, 6, 5, 3, 1)$. From the bottom line, we get $\delta = (4, 1)$ and $\alpha = (4, 3, 3, 2, 1)$. By applying the bijection described in the second paragraph of the above proof, we get $\gamma = (\overline{8}, 7, 5, \overline{5}, \overline{3})$. Since $\alpha = (4, 3, 3, 2, 1)$, the resulting overpartition is $\lambda = (\overline{8}, 7, 5, 5, \overline{5}, 4, 3, \overline{3}, 1)$. It has $N - j = 3$ overlined parts and its generalized Durfee square is of size 5.

This decomposition shows that the generating function for overpartitions with generalized Durfee square of size N where the exponent of q counts the weight and the exponent of a counts the number of overlined parts is

$$\frac{a^N q^{\binom{N+1}{2}}(-1/a)_N}{(q)_N(q)_N}.$$

Definition 2.4. The *successive Durfee squares* of an overpartition are its generalized Durfee square and the successive Durfee squares of the partition below the generalized Durfee square, if we represent the partition as in Figure 3, with the overlined parts above the non-overlined ones. We can also define similarly the *successive Durfee rectangles* by dissecting the overpartition with $d\times(d+1)$-rectangles instead of squares (see Figure 4). In this case, we also impose the condition that the partition on the right of a $d\times(d+1)$-rectangle cannot have more than d parts.

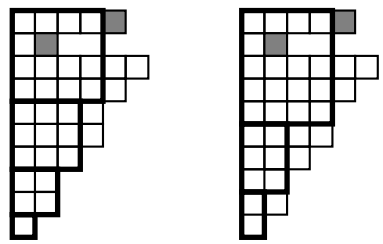

FIGURE 4. Successive Durfee squares and successive Durfee rectangles of $(6,5,\overline{5},4,4,3,2,2,\overline{2},1)$.

These definitions imply that the generating function for overpartitions with $i-1$ successive Durfee squares followed by $k-i$ successive Durfee rectangles (the first one being a generalized Durfee square/rectangle) is

$$\sum_{n_1\geq\ldots\geq n_{k-1}\geq 0} \frac{q^{\binom{n_1+1}{2}+n_i+\ldots+n_{k-1}}(-1/a)_{n_1}a^{n_1}}{(q)_{n_1}} \times \left(q^{n_2^2}\begin{bmatrix}n_1\\n_2\end{bmatrix}_q\right)\left(q^{n_3^2}\begin{bmatrix}n_2\\n_3\end{bmatrix}_q\right)\cdots\left(q^{n_{k-1}^2}\begin{bmatrix}n_{k-2}\\n_{k-1}\end{bmatrix}_q\right) \tag{2.1}$$

where

$$\begin{bmatrix}n\\k\end{bmatrix}_q=\frac{(q)_n}{(q)_k(q)_{n-k}}$$

is the generating function of partitions into at most k parts less or equal to $n-k$.

3. Paths and generating function

In this section we will define the paths, compute their generating function and therefore prove Theorem 1.5. This part is an extension of papers of Andrews and Bressoud [8, 14] based on ideas of Burge [16, 17]. We study paths in the first quadrant, that start on the y-axis, end on the x-axis, and use four kinds of unitary steps:

- North-East NE: $(x,y)\to(x+1,y+1)$,
- South-East SE: $(x,y)\to(x+1,y-1)$,
- South S: $(x,y)\to(x,y-1)$,
- East E: $(x,0)\to(x+1,0)$.

The *height* corresponds to the y-coordinate. A South step can only appear after a North-East step and an East step can only appear at height 0. The paths are either empty or end with a South-East step or a South step. A *peak* is a vertex preceded by a North-East step and followed by a South step (in which case it will be called a *NES peak*) or by a South-East step (in which case it will be called a *NESE peak*). The *major index* of a path is the sum of the x-coordinates of its peaks (see Figure 5 for an example). Let k and i be positive integers with $i\leq k$. We say that a path satisfies the *special* (k,i)*-conditions* if it starts at height $k-i$ and its height is less than k. When the path has no South steps, this is the definition of the paths in [14].

FIGURE 5. This path has four peaks: two NES peaks (located at $(2,2)$ and $(6,1)$) and two NESE peaks (located at $(4,1)$ and $(7,1)$). Its major index is $2+4+6+7=19$.

Let $\overline{E}_{k,i}(n,j,N)$ be the number of paths of major index n with N peaks and j South steps which satisfy the special (k,i)-conditions. Let $\overline{\mathcal{E}}_{k,i}(N)$ be the generating function for these paths, that is $\overline{\mathcal{E}}_{k,i}(N) = \overline{\mathcal{E}}_{k,i}(N,a,q) = \sum_{n,j} \overline{E}_{k,i}(n,j,N)a^j q^n$. Moreover, for $0 \le i < k$, let $\overline{\Gamma}_{k,i}(N)$ be the generating function for paths obtained by deleting the first NE step of a path which is counted in $\overline{\mathcal{E}}_{k,i+1}(N)$ and begins with a NE step.

Then

Proposition 3.1.

$$\overline{\mathcal{E}}_{k,i}(0) = 1; \tag{3.1}$$

$$\overline{\mathcal{E}}_{k,i}(N) = q^N\overline{\mathcal{E}}_{k,i+1}(N) + q^N\overline{\Gamma}_{k,i-1}(N); \quad i<k \tag{3.2}$$

$$\overline{\Gamma}_{k,i}(N) = q^N\overline{\Gamma}_{k,i-1}(N) + (a+q^{N-1})\overline{\mathcal{E}}_{k,i+1}(N-1); \quad i>0 \tag{3.3}$$

$$\overline{\mathcal{E}}_{k,k}(N) = \frac{q^N}{1-q^N}\overline{\Gamma}_{k,k-1}(N); \tag{3.4}$$

$$\overline{\Gamma}_{k,0}(N) = 0. \tag{3.5}$$

Proof. If $N=0$ the only path counted in $\overline{\mathcal{E}}_{k,i}(0)$ is the path with only South-East steps, which starts at $(0,k-i)$ and ends at $(k-i,0)$ (if $i=k$, it is just the empty path, starting and ending at $(0,0)$). This path has no peaks and its major index is thus 0. This proves (3.1). Now if the path has at least one peak, then we take off its first step. If $i<k$, then a path counted in $\overline{\mathcal{E}}_{k,i}(N)$ starts with a North-East (defined by $q^N\overline{\Gamma}_{k,i-1}(N)$) or a South-East step ($q^N\overline{\mathcal{E}}_{k,i+1}(N)$). This gives (3.2). If $i>0$, $\overline{\Gamma}_{k,i}(N)$ is the generating function for paths counted by $\overline{\mathcal{E}}_{k,i+1}(N)$ that start with a North-East step where the first step was deleted. These paths can start with a North-East step ($q^N\overline{\Gamma}_{k,i-1}(N)$), a South step ($a\overline{\mathcal{E}}_{k,i+1}(N-1)$) or a South-East step ($q^{N-1}\overline{\mathcal{E}}_{k,i+1}(N-1)$) and we get (3.3). If $i=k$ then a path counted by $\overline{\mathcal{E}}_{k,k}(N)$ starts with a North-East ($q^N\overline{\Gamma}_{k,k-1}(N)$) or an East step ($q^N\overline{\mathcal{E}}_{k,k}(N)$). This gives (3.4). The height of the paths is less than k, therefore no path which starts at height $k-1$ can start with a North-East step and $\overline{\Gamma}_{k,0}(N)=0$ as in (3.5). □

These recurrences uniquely define the series $\overline{\mathcal{E}}_{k,i}(N)$ and $\overline{\Gamma}_{k,i}(N)$. We get that:

Theorem 3.2.

$$\overline{\mathcal{E}}_{k,i}(N) = a^N q^{\binom{N+1}{2}}(-1/a)_N \sum_{n=-N}^{N} (-1)^n \frac{q^{kn^2+n(k-i)-\binom{n}{2}}}{(q)_{N-n}(q)_{N+n}}$$

$$\overline{\Gamma}_{k,i}(N) = a^N q^{\binom{N}{2}}(-1/a)_N \sum_{n=-N}^{N-1} (-1)^n \frac{q^{kn^2+n(k-i)-\binom{n+1}{2}}}{(q)_{N-n-1}(q)_{N+n}}$$

Proof. Let

$$\begin{cases} \overline{\mathcal{E}'}_{k,i}(N) = a^N q^{\binom{N+1}{2}}(-1/a)_N \displaystyle\sum_{n=-N}^{N} (-1)^n \frac{q^{kn^2+n(k-i)-\binom{n}{2}}}{(q)_{N-n}(q)_{N+n}} \\ \overline{\Gamma'}_{k,i}(N) = a^N q^{\binom{N}{2}}(-1/a)_N \displaystyle\sum_{n=-N}^{N-1} (-1)^n \frac{q^{kn^2+n(k-i)-\binom{n+1}{2}}}{(q)_{N-n-1}(q)_{N+n}} \end{cases}$$

Note that $\overline{\mathcal{E}'}_{k,i}(0)=1$.

We first prove that $\overline{\mathcal{E}'}_{k,i}(N)$ and $\overline{\Gamma'}_{k,i}(N)$ satisfy $\overline{\mathcal{E}'}_{k,i}(N) = q^N\overline{\mathcal{E}'}_{k,i+1}(N) + q^N\overline{\Gamma'}_{k,i-1}(N)$ for $1 \leq i \leq k$:

$$\begin{aligned}
&q^N\overline{\mathcal{E}'}_{k,i+1}(N) + q^N\overline{\Gamma'}_{k,i-1}(N)\\
=&a^N q^{\binom{N+1}{2}}(-1/a)_N \sum_{n=-N}^{N} (-1)^n \frac{q^{kn^2+n(k-i-1)-\binom{n}{2}}}{(q)_{N-n}(q)_{N+n}} q^N\\
&+ a^N q^{\binom{N}{2}}(-1/a)_N \sum_{n=-N}^{N-1} (-1)^n \frac{q^{kn^2+n(k-i+1)-\binom{n+1}{2}}}{(q)_{N-n-1}(q)_{N+n}} q^N\\
=&a^N q^{\binom{N+1}{2}}(-1/a)_N \sum_{n=-N}^{N-1} (-1)^n \frac{q^{kn^2+n(k-i)-\binom{n}{2}}}{(q)_{N-n}(q)_{N+n}} (q^{N-n} + (1-q^{N-n}))\\
&+ a^N q^{\binom{N+1}{2}}(-1/a)_N (-1)^N \frac{q^{kN^2+N(k-i-1)-\binom{N}{2}}}{(q)_0(q)_{2N}} q^N\\
=&a^N q^{\binom{N+1}{2}}(-1/a)_N \sum_{n=-N}^{N} (-1)^n \frac{q^{kn^2+n(k-i)-\binom{n}{2}}}{(q)_{N-n}(q)_{N+n}}\\
=&\overline{\mathcal{E}'}_{k,i}(N)
\end{aligned}$$

We then prove, quite similarly, that they satisfy (3.3):

$$\begin{aligned}
&(a+q^{N-1})\overline{\mathcal{E}'}_{k,i+1}(N-1) + q^N\overline{\Gamma'}_{k,i-1}(N)\\
=&a^{N-1} q^{\binom{N}{2}}(-1/a)_{N-1} \sum_{n=-N+1}^{N-1} (-1)^n \frac{q^{kn^2+n(k-i-1)-\binom{n}{2}}}{(q)_{N-1-n}(q)_{N-1+n}} (a+q^{N-1})\\
&+ a^N q^{\binom{N}{2}}(-1/a)_N \sum_{n=-N}^{N-1} (-1)^n \frac{q^{kn^2+n(k-i+1)-\binom{n+1}{2}}}{(q)_{N-n-1}(q)_{N+n}} q^N\\
=&a^N q^{\binom{N}{2}}(-1/a)_N \sum_{n=-N+1}^{N-1} (-1)^n \frac{q^{kn^2+n(k-i)-\binom{n+1}{2}}}{(q)_{N-n-1}(q)_{N+n}} ((1-q^{N+n}) + q^{N+n})\\
&+ a^N q^{\binom{N}{2}}(-1/a)_N (-1)^{-N} \frac{q^{kN^2-N(k-i+1)-\binom{-N+1}{2}}}{(q)_{2N-1}(q)_0} q^N\\
=&a^N q^{\binom{N}{2}}(-1/a)_N \sum_{n=-N}^{N-1} (-1)^n \frac{q^{kn^2+n(k-i)-\binom{n+1}{2}}}{(q)_{N-n-1}(q)_{N+n}}\\
=&\overline{\Gamma'}_{k,i}(N)
\end{aligned}$$

For (3.4), we prove that $\overline{\mathcal{E}'}_{k,k+1}(N) = \overline{\mathcal{E}'}_{k,k}(N)$:

$$\begin{aligned}
\overline{\mathcal{E}'}_{k,k+1}(N) =&a^N q^{\binom{N+1}{2}}(-1/a)_N \sum_{n=-N}^{N} (-1)^n \frac{q^{kn^2-n-\binom{n}{2}}}{(q)_{N-n}(q)_{N+n}}\\
=&a^N q^{\binom{N+1}{2}}(-1/a)_N \sum_{p=-N}^{N} (-1)^{-p} \frac{q^{k(-p)^2+p-\binom{-p}{2}}}{(q)_{N+p}(q)_{N-p}}
\end{aligned}$$

where $p=-n$

$$
\begin{aligned}
&=a^N q^{\binom{N+1}{2}}(-1/a)_N \sum_{p=-N}^{N}(-1)^p \frac{q^{kp^2+p-\frac{(-p)(-p-1)}{2}}}{(q)_{N+p}(q)_{N-p}}\\
&=a^N q^{\binom{N+1}{2}}(-1/a)_N \sum_{p=-N}^{N}(-1)^p \frac{q^{kp^2+p-\frac{(p)(p+1)}{2}}}{(q)_{N+p}(q)_{N-p}}\\
&=a^N q^{\binom{N+1}{2}}(-1/a)_N \sum_{p=-N}^{N}(-1)^p \frac{q^{kp^2+p-\binom{p+1}{2}}}{(q)_{N+p}(q)_{N-p}}\\
&=a^N q^{\binom{N+1}{2}}(-1/a)_N \sum_{p=-N}^{N}(-1)^p \frac{q^{kp^2-\binom{p}{2}}}{(q)_{N+p}(q)_{N-p}}\\
&=\overline{\mathcal{E}'}_{k,k}(N)
\end{aligned}
$$

Hence we have, using the fact that $\overline{\mathcal{E}'}_{k,i}(N)$ satisfies (3.2) for $i=k$:

$$\overline{\mathcal{E}'}_{k,k}(N)=q^N\overline{\mathcal{E}'}_{k,k}(N)+q^N\overline{\Gamma'}_{k,k-1}(N).$$

Finally, we have:

$$
\begin{aligned}
&\overline{\Gamma'}_{k,0}(N)=a^N q^{\binom{N}{2}}(-1/a)_N \sum_{n=-N}^{N-1}(-1)^n \frac{q^{kn^2+nk-\binom{n+1}{2}}}{(q)_{N-n-1}(q)_{N+n}}\\
&=a^N q^{\binom{N}{2}}(-1/a)_N \left(\sum_{n=0}^{N-1}(-1)^n \frac{q^{kn^2+nk-\binom{n+1}{2}}}{(q)_{N-n-1}(q)_{N+n}}+\sum_{n=-N}^{-1}(-1)^n \frac{q^{kn^2+nk-\binom{n+1}{2}}}{(q)_{N-n-1}(q)_{N+n}}\right)\\
&=a^N q^{\binom{N}{2}}(-1/a)_N \left(\sum_{n=0}^{N-1}(-1)^n \frac{q^{kn^2+nk-\binom{n+1}{2}}}{(q)_{N-n-1}(q)_{N+n}}\right.\\
&\quad\left.+\sum_{p=0}^{N-1}(-1)^{-1-p}\frac{q^{k(-1-p)^2+(-1-p)k-\binom{-p}{2}}}{(q)_{N+p}(q)_{N-1-p}}\right)
\end{aligned}
$$

where $p=-1-n$

$$
\begin{aligned}
&=a^N q^{\binom{N}{2}}(-1/a)_N \left(\sum_{n=0}^{N-1}(-1)^n \frac{q^{kn^2+nk-\binom{n+1}{2}}}{(q)_{N-n-1}(q)_{N+n}}-\sum_{p=0}^{N-1}(-1)^p \frac{q^{kp^2+kp-\binom{p+1}{2}}}{(q)_{N+p}(q)_{N-1-p}}\right)\\
&=0
\end{aligned}
$$

Since $\overline{\mathcal{E}'}_{k,i}(N)$ and $\overline{\Gamma'}_{k,i}(N)$ satisfy the recurrences of Proposition 3.1, we thus have $\overline{\mathcal{E}}_{k,i}(N)=\overline{\mathcal{E}'}_{k,i}(N)$ and $\overline{\Gamma}_{k,i}(N)=\overline{\Gamma'}_{k,i}(N)$. □

We just need the following proposition to prove Theorem 1.5.

Proposition 3.3. *For any $n\in\mathbb{Z}$*

$$\sum_{N\geq|n|}\frac{(-aq)_n(-q^n/a)_{N-n}q^{\binom{N+1}{2}-\binom{n+1}{2}}a^{N-n}}{(q)_{N+n}(q)_{N-n}}=\frac{(-aq)_\infty}{(q)_\infty}.$$

Proof. We only prove the case $n \geq 0$. The case $n < 0$ is identical as $(a)_{-n} = (-1/a)^n q^{\binom{n+1}{2}}/(aq)_n$ and therefore

$$\frac{(-aq)_n(-q^n/a)_{N-n}q^{\binom{N+1}{2}-\binom{n+1}{2}}a^{N-n}}{(q)_{N+n}(q)_{N-n}} = \frac{(-aq)_{-n}(-q^{-n}/a)_{N+n}q^{\binom{N+1}{2}-\binom{-n+1}{2}}a^{N+n}}{(q)_{N+n}(q)_{N-n}}.$$

We present an analytical proof and a combinatorial one.

Analytical proof:

$$\begin{aligned}
&\sum_{N=n}^{\infty} \frac{(-aq)_n(-q^n/a)_{N-n}q^{\binom{N+1}{2}-\binom{n+1}{2}}a^{N-n}}{(q)_{N+n}(q)_{N-n}} \\
=&\sum_{N=0}^{\infty} \frac{(-aq)_n(-q^n/a)_{N}q^{\binom{N+n+1}{2}-\binom{n+1}{2}}a^{N}}{(q)_{N}(q)_{N+2n}} \\
=&\frac{(-aq)_n}{(q)_{2n}} \sum_{N=0}^{\infty} \frac{q^{Nn+\binom{N+1}{2}}(-q^n/a)_N a^N}{(q)_N(q^{2n+1})_N}
\end{aligned}$$

We now apply the q-Gauss summation (Corollary 2.4 of [3] with $n \to N$, $a \to -q^n/a$, $b \to -\infty$ and $c \to q^{2n+1}$)

$$\begin{aligned}
=&\frac{(-aq)_n}{(q)_{2n}} \frac{(-aq^{n+1})_\infty}{(q^{2n+1})_\infty} \\
=&\frac{(-aq)_\infty}{(q)_\infty}
\end{aligned}$$

Combinatorial proof: Let n be a fixed non-negative integer and let N be the greatest integer such that the sum of the number of overlined parts greater than n and of the number of non-overlined parts greater than or equal to $N+n$ is greater than or equal to $N-n$. By definition, N, if it exists, is unique. We check that $N = n$ satisfies the condition: the sum of the number of overlined parts $> n$ and of the number of non-overlined parts $\geq 2n$ is indeed ≥ 0. Therefore, N exists and it is unique. We call N the size of the generalized n-Durfee square. Note that if $n = 0$, N is the size of the generalized Durfee square and this bijection is the same as the one presented in the proof of Proposition 2.2. For example, if $n = 2$ then the overpartition $(8, 8, \bar{6}, 5, 5, 3, 3, \bar{3}, \bar{1})$ gives $N = 6$.

We now show that the generating function for the overpartitions with generalized n-Durfee square N is:

$$\frac{(-aq)_n(-q^n/a)_{N-n}q^{\binom{N+1}{2}-\binom{n+1}{2}}a^{N-n}}{(q)_{N+n}(q)_{N-n}}.$$

The factor $(-aq)_n$ corresponds to the overlined parts $\leq n$ and the factor $\frac{1}{(q)_{N+n}}$ corresponds to the non-overlined parts $\leq N+n$.

The remaining factors correspond to an overpartition into $N-n$ parts whose overlined parts are $> n$ and whose non-overlined parts are $\geq N+n$. To prove this, let us show that there exists a bijection between such overpartitions θ and triples $(\varepsilon, \zeta, \eta)$ where ε is the overpartition $(\overline{N}, \overline{N-1}, \ldots, \overline{n+1})$ which has $N-n$ parts (corresponding to the factor $a^{N-n}q^{\binom{N+1}{2}-\binom{n+1}{2}}$), ζ is a partition into distinct parts which lie between n and $N-1$ (corresponding to the factor $(-q^n/a)_{N-n}$) and η is a partition into $N-n$ non-negative parts (corresponding to the factor $\frac{1}{(q)_{N-n}}$). This bijection (similar to Algorithm Z [9]) is defined as follows: first for all i we set $\theta_i = \varepsilon_i + \eta_i$. Then for each part $n+i$ in ζ, we add $n+i$ to θ_{i+1} and we remove the overlining of that part. This implies that the non-overlined parts are $\geq N+n$. Finally, we reorder the parts. It is easy to see that this is a bijection as there only a unique ordering of the parts of θ which allows, if θ_{i+1} is not overlined, to take off $n+i$ from it and overline it and get a partition into distinct overlined parts.

For example for $N = 7$ and $n = 3$, if $\varepsilon = (\bar{7}, \bar{6}, \bar{5}, \bar{4})$, $\eta = (3, 3, 1, 0)$ and $\zeta = (6, 3)$, then $\theta = (13, \bar{9}, \bar{6}, 10) = (13, 10, \bar{9}, \bar{6})$.

□

Summing on N we get

$$\begin{aligned}
\sum_{N\geq 0}\overline{\mathcal{E}}_{k,i}(N) &= \sum_{N\geq 0} a^N q^{\binom{N+1}{2}}(-1/a)_N \sum_{n=-N}^{N}(-1)^n \frac{q^{kn^2+n(k-i)-\binom{n}{2}}}{(q)_{N-n}(q)_{N+n}} \\
&= \sum_{N\geq 0}\sum_{n=-N}^{N} a^N q^{\binom{N+1}{2}}(-1/a)_N(-1)^n \frac{q^{kn^2+n(k-i)-\binom{n}{2}}}{(q)_{N-n}(q)_{N+n}} \\
&= \sum_{n=-\infty}^{\infty}\sum_{N\geq |n|} a^N q^{\binom{N+1}{2}}(-1/a)_N(-1)^n \frac{q^{kn^2+n(k-i)-\binom{n}{2}}}{(q)_{N-n}(q)_{N+n}} \\
&= \sum_{n=-\infty}^{\infty}(-1)^n a^n q^{kn^2+(k-i+1)n}\frac{(-1/a)_n}{(-aq)_n}\cdot \\
&\qquad \sum_{N\geq |n|}\frac{(-aq)_n(\frac{-q^n}{a})_{N-n}q^{\binom{N+1}{2}-\binom{n+1}{2}}a^{N-n}}{(q)_{N-n}(q)_{N+n}} \\
&= \sum_{n=-\infty}^{\infty}(-1)^n a^n q^{kn^2+(k-i+1)n}\frac{(-1/a)_n}{(-aq)_n}\frac{(-aq)_\infty}{(q)_\infty} \\
&\quad \text{by Proposition 3.3} \\
&= \frac{(-aq)_\infty}{(q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n a^n q^{kn^2+(k-i+1)n}\frac{(-1/a)_n}{(-aq)_n}.
\end{aligned}$$

This is equation (1.1) of Theorem 1.5.

4. Paths and successive ranks

In this section we prove the case $\overline{C}_{k,i}(n,j)=\overline{E}_{k,i}(n,j)$ of Theorem 1.4. In fact, we prove a refinement of this case:

Proposition 4.1. *There exists a one-to-one correspondence between the paths of major index n with j south steps counted by $\overline{E}_{k,i}(n,j)$ and the overpartitions of n with j non-overlined parts in the bottom line of their Frobenius representation and whose successive ranks lie in $[-i+2,2k-i-1]$ counted by $\overline{C}_{k,i}(n,j)$. This correspondence is such that the paths have N peaks if and only if the Frobenius representation of the overpartition has N columns.*

This proposition can be proved with a recursive argument. We can show that $\overline{\mathcal{C}}_{k,i}(N)$, the generating function for overpartitions whose Frobenius representation has N columns and whose successive ranks lie in $[-i+2,2k-i-1]$, follows the same recurrences as $\overline{\mathcal{E}}_{k,i}(N)$, as done by Burge [16, 17] for the case of partitions. Details are given in [34]. We propose here a direct mapping that is a generalization of a bijection of Bressoud [14].

Given a lattice path which starts at $(0,a)$ and a peak (x,y) with u South steps to its left, we map this peak to the pair (s,t) where

$$\begin{aligned} s &= (x+a-y+u)/2 \\ t &= (x-a+y-2-u)/2 \end{aligned}$$

if there are an even number of East steps to the left of the peak (we then say that the peak is of *type 0*), and

$$\begin{aligned} s &= (x+a+y-1+u)/2 \\ t &= (x-a-y-1-u)/2 \end{aligned}$$

if there are an odd number of East steps to the left of the peak (we then say that the peak is of *type 1*). Moreover, we overline t if the peak is a NESE peak. In both cases, s and t are integers and we have $s+t+1=x$. In the case of partitions treated in [14], u is always 0.

Let N be the number of peaks in the path and j the number of South steps of the paths. Let (x_i, y_i) be the coordinates of the ith peak from the right and (s_i, t_i) be the corresponding pair.

Proposition 4.2. *The sequence $(s_1, s_2, \ldots, s_N)$ is a partition into distinct nonnegative parts and the sequence $(t_1, t_2, \ldots, t_N)$ is an overpartition into nonnegative parts with j non-overlined parts.*

Proof. We need to prove the following results:

- $s_N \geq 0$. If the last peak from the right (x_N, y_N) is of type 0, then $s_N = (x_N + a - y_N + u_N)/2$ and it is sufficient to prove that $x_N - y_N \geq -a$ since $u_N = 0$. It is obvious that any vertex has a greater (or equal) value of $x-y$ than the previous vertex in the path. Since the path begins at $(0, a)$, we have $x - y = -a$ at the beginning of the path and thus we have $x - y \geq -a$ for all vertices. Now if the peak is of type 1, then $x_N \geq 2$ and $s_N = (x_N + a + y_N - 1 + u_N)/2 \geq 0$.
- $t_N \geq 0$. If the last peak from the right (x_N, y_N) is of type 0, then $t_N = (x_N - a + y_N - 2 - u_N)/2$. We have $u_N = 0$ and $x_N > 0$. If $y_N \geq a$, then $t_N \geq 0$. Otherwise it is easy to see that $x_N \geq 2 + a - y_N$ (see Figure 6) and thus $t_N \geq 0$. Now if this peak is of type 1, then $t_N = (x_N - a - y_N - 1 - u_N)/2$. We have $u_N = 0$ and there is at least one East step to the left of the peak and therefore at least a South-East steps before the East step and at least y_N North-East steps after the East step. Hence $x_N - y_N - a > 0$ and $t_N \geq 0$.

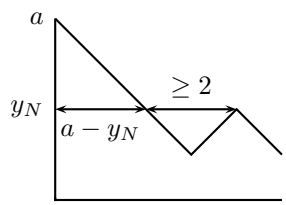

FIGURE 6. We show that $x_N \geq 2 + a - y_N$ in the case $y_N \leq a$.

- *The sequence s is a partition into distinct parts.* We need to prove that for all i, $s_i - s_{i+1} > 0$. If the ith peak from the right and the $(i+1)^{st}$ peak are both of type 0, it is clearly true since $x_i - y_i > x_{i+1} - y_{i+1}$ (two peaks cannot have the same value of $x-y$), and $u_i \geq u_{i+1}$ (remember that u_i is the number of South steps to the left of the ith peak). If the ith peak from the right and the $(i+1)^{st}$ peak are both of type 1, then $s_i - s_{i+1} = (x_i + y_i + u_i - x_{i+1} - y_{i+1} - u_{i+1})/2$. It is easy to see that that $x_i + y_i > x_{i+1} + y_{i+1}$ and $u_i \geq u_{i+1}$. If the ith peak is of type 0 and the $(i+1)^{st}$ is of type 1, we have $s_i - s_{i+1} = \frac{1}{2}(x_i - x_{i+1} - y_i - y_{i+1} + u_i - u_{i+1} + 1)$. Since $u_i \geq u_{i+1}$, it is sufficient to prove that $x_i - x_{i+1} - y_i - y_{i+1} \geq 0$. This comes from the fact that there is an East step between the two peaks (see Figure 7). If the ith peak is of type 1 and the $(i+1)^{st}$ is of type 0, the proof is similar.

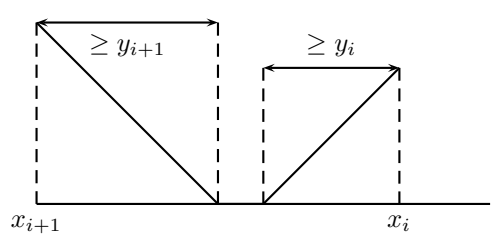

FIGURE 7. If the ith peak of is type 0 and the $(i+1)^{st}$ peak is of type 1, we have $x_i - x_{i+1} \geq y_i + y_{i+1}$.

- *The sequence t is an overpartition (where the first occurrence of a part can be overlined).* We need to prove that for all i, $t_i - t_{i+1} > 0$ if t_{i+1} is overlined and $t_i - t_{i+1} \geq 0$ otherwise. The fact that we always have $t_i - t_{i+1} \geq 0$ is proved in the same way as with the s_i. If t_{i+1} is overlined, then it corresponds to a NESE peak, so we have $x_i - x_{i+1} \geq 2$ and $u_{i+1} = u_i$. By considering the expression of $t_i - t_{i+1}$ in the four cases (the peaks i and $i+1$ are both of type 0, both of type 1, of type 0 and of type 1, or of type 1 and of type 0), the result is easily shown.

□

Therefore $\begin{pmatrix} s_1 & s_2 & \cdots & s_N \\ t_1 & t_2 & \cdots & t_N \end{pmatrix}$ is the Frobenius representation of an overpartition whose weight is

$$\sum_{i=1}^{N}(s_i + t_i + 1) = \sum_{i=1}^{N} x_i$$

i.e. the major index of the corresponding path.

As an example, the path in Figure 8 corresponds to the overpartition

$$\begin{pmatrix} 14 & 11 & 6 & 4 & 2 \\ 7 & \overline{6} & \overline{5} & 4 & \overline{3} \end{pmatrix}.$$

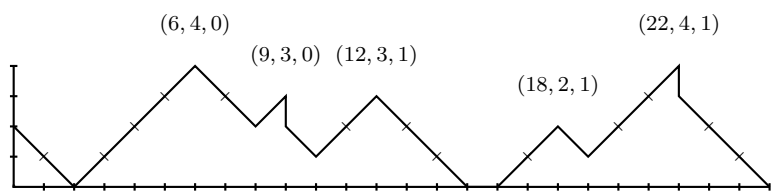

FIGURE 8. Illustration of the correspondence between paths and successive ranks. The values of x, y and u are given for each peak.

The peaks all have height at least one, thus for a peak (x, y) which is preceded by an even number of East steps, we have:

$$\begin{aligned} & 1 \leq y = a + 1 + t - s + u \leq k - 1 \\ \Leftrightarrow\ & a - k + 2 \leq s - t - u \leq a \\ \Leftrightarrow\ & \text{the corresponding successive rank is} \geq a - k + 2 \text{ and} \leq a \end{aligned}$$

and if the peak is preceded by an odd number of East steps, we have:

$$\begin{aligned} & 1 \leq y = s - t - u - a \leq k - 1 \\ \Leftrightarrow\ & a + 1 \leq s - t - u \leq k + a - 1 \\ \Leftrightarrow\ & \text{the corresponding successive rank is} \geq a + 1 \text{ and} \leq k + a - 1. \end{aligned}$$

Thus, given a Frobenius representation of an overpartition and a nonnegative integer a, there is a unique corresponding path which starts at $(0, a)$.

In our paths, $a = k - i$, therefore in the first case the successive rank $r \in [-i + 2, k - i]$ and in the second case $r \in [k - i + 1, 2k - i - 1]$.

The map is easily reversible. This proves Proposition 4.1.

5. Paths and multiplicities

In this section we prove the case $\overline{B}_{k,i}(n, j) = \overline{E}_{k,i}(n, j)$ of Theorem 1.4. We even prove a refinement:

Proposition 5.1. *There exists a one-to-one correspondence between the paths counted by $\overline{E}_{k,i}(n, j)$ and the overpartitions counted by $\overline{B}_{k,i}(n, j)$. This correspondence is such that the paths have N peaks if and only if the length of the multiplicity sequence of the overpartition is N (see Section 5.2 for the definition).*

We will first give a generating function proof of that proposition (without the refinement). Then we will give the sketch of a combinatorial proof which is a generalization of Burge's correspondence for partitions presented in [16].

Recall that $\overline{B}_{k,i}(n,j)$ is the number of overpartitions λ of n with j overlined parts such that for all ℓ,

$$\begin{aligned} & f_1 < i \\ & \lambda_\ell - \lambda_{\ell+k-1} \geq \begin{cases} 1 & \text{if } \lambda_{\ell+k-1} \text{ is overlined} \\ 2 & \text{otherwise} \end{cases} \end{aligned}$$

or equivalently,

$$\begin{aligned} & f_1 < i \\ & \forall \ell,\ f_\ell + f_{\ell+1} < \begin{cases} k+1 & \text{if a part } \ell \text{ is overlined} \\ k & \text{otherwise} \end{cases} \end{aligned}$$

We will abbreviate this last condition with the notation $\forall \ell,\ f_\ell + f_{\ell+1} < k\overline{+1}$.

5.1. **A generating function proof.** Let $\overline{\mathcal{B}}_{k,i}(a,q) = \sum_{n,j\geq 0} \overline{B}_{k,i}(n,j)a^j q^n$. We prove that

$$\overline{\mathcal{B}}_{k,i}(a,q) = \overline{\mathcal{E}}_{k,i}(a,q) \tag{5.1}$$

We will generalize Lovejoy's proof of Theorem 1.1 of [28]. Let

$$J_{k,i}(a,x,q) = H_{k,i}(a,xq,q) - axqH_{k,i-1}(a,xq,q) \tag{5.2}$$

$$H_{k,i}(a,x,q) = \sum_{n=0}^{\infty} \frac{x^{kn}q^{kn^2+n-in}a^n(1-x^iq^{2ni})(axq^{n+1})_\infty(1/a)_n}{(q)_n(xq^n)_\infty}. \tag{5.3}$$

Andrews showed in [3, p. 106-107] that

$$H_{k,i}(a,x,q) - H_{k,i-1}(a,x,q) = x^{i-1}J_{k,k-i+1}(a,x,q) \tag{5.4}$$

$$J_{k,i}(a,x,q) - J_{k,i-1}(a,x,q) = (xq)^{i-1}(J_{k,k-i+1}(a,xq,q) - aJ_{k,k-i+2}(a,xq,q)). \tag{5.5}$$

We plug $i=0$ in equation (5.3) and obtain $H_{k,0}=0$. We then plug $i=1$ in equation (5.4) and obtain $H_{k,1}(a,x,q) = J_{k,k}(a,x,q)$. Then we plug $i=1$ in equation (5.2) and obtain $J_{k,1}(a,x,q) = J_{k,k}(a,xq,q)$. Finally we set $x=0$ in equations (5.2) and (5.3) and get $J_{k,i}(a,0,q)=1$. This implies for $1\leq i\leq k$, the following equations determine $J_{k,i}(a,x,q)$:

$$\begin{aligned} J_{k,i}(a,0,q) &= 1 \\ J_{k,i}(a,x,q) - J_{k,i-1}(a,x,q) &= (xq)^{i-1}(J_{k,k-i+1}(a,xq,q) - aJ_{k,k-i+2}(a,xq,q)); \\ J_{k,1}(a,x,q) &= J_{k,k}(a,xq,q). \end{aligned}$$

Let $\overline{\mathcal{B}}_{k,i}(a,x,q) = \sum_{n,j,p\geq 0} \overline{B}_{k,i}(n,j,p)a^jx^pq^n$ where $\overline{B}_{k,i}(n,j,p)$ is the number of overpartitions counted by $\overline{B}_{k,i}(n,j)$ with p parts. We show that

Lemma 5.2. *For $1\leq i\leq k$,*

$$\overline{\mathcal{B}}_{k,i}(a,x,q) = J_{k,i}(-a,x,q).$$

Proof. The only overpartition with zero parts is the empty one. Therefore $\overline{\mathcal{B}}_{k,i}(a,0,q)=1$. It is obvious that for $2\leq i\leq k$, $\overline{\mathcal{B}}_{k,i}(a,x,q) - \overline{\mathcal{B}}_{k,i-1}(a,x,q)$ is the generating function for overpartitions such that $\forall \ell,\ f_\ell + f_{\ell+1} < k\overline{+1}$ and $f_1 = i-1$. Moreover $\overline{\mathcal{B}}_{k,i}(a,xq,q)$ is the generating function for overpartitions such that $\forall \ell,\ f_\ell + f_{\ell+1} < k\overline{+1}$, $f_2 < i$ and $f_1 = 0$. Therefore $(xq)^{i-1}\overline{\mathcal{B}}_{k,k-i+1}(a,xq,q)$ is the generating function for overpartitions such that $\forall \ell,\ f_\ell + f_{\ell+1} < k\overline{+1}$, $f_1 = i-1$ and 1 is not overlined and $a(xq)^{i-1}\overline{\mathcal{B}}_{k,k-i+2}(a,xq,q)$ is the generating function for overpartitions such that $\forall \ell,\ f_\ell + f_{\ell+1} < k\overline{+1}$, $f_1 = i-1$ and 1 is overlined. We get, for $2\leq i\leq k$, $\overline{\mathcal{B}}_{k,i}(a,x,q) - \overline{\mathcal{B}}_{k,i-1}(a,x,q) = (xq)^{i-1}(\overline{\mathcal{B}}_{k,k-i+1}(a,xq,q) + a\overline{\mathcal{B}}_{k,k-i+2}(a,xq,q))$. Finally $\overline{\mathcal{B}}_{k,k}(a,xq,q)$ is the generating function for overpartitions such that $\forall \ell,\ f_\ell + f_{\ell+1} < k\overline{+1}$, $f_2 < k$ and $f_1 = 0$ and therefore is equal to $\overline{\mathcal{B}}_{k,1}(a,x,q)$. □

Proof. We can now prove equation (5.1). The lemma implies that

$$\overline{\mathcal{B}}_{k,i}(a,q) = J_{k,i}(-a,1,q).$$

Hence

$$\begin{aligned}
\overline{\mathcal{B}}_{k,i}(a,q) &= \frac{(-aq)_\infty}{(q)_\infty}\sum_{n=0}^{\infty}(-1)^n a^n \frac{q^{kn^2+n(k-i+1)}(-1/a)_n(1-q^{(2n+1)i})}{(-aq)_{n+1}} \\
&\quad + aq\frac{(-aq)_\infty}{(q)_\infty}\sum_{n=0}^{\infty}(-1)^n a^n \frac{q^{kn^2+n(k-i+2)}(-1/a)_n(1-q^{(2n+1)(i-1)})}{(-aq)_{n+1}} \\
&= \frac{(-aq)_\infty}{(q)_\infty}\left(\sum_{n=0}^{\infty}(-1)^n a^n \frac{q^{kn^2+n(k+1)}(-1/a)_n(q^{-in}+aq^{1-(i-1)n})}{(-aq)_{n+1}}\right. \\
&\quad \left. - \sum_{n=0}^{\infty}(-1)^n a^n \frac{q^{kn^2+n(k+1)}(-1/a)_n(q^{(n+1)i}+aq^{(n+1)(i-1)+1})}{(-aq)_{n+1}}\right) \\
&= \frac{(-aq)_\infty}{(q)_\infty}\left(\sum_{n=0}^{\infty}(-1)^n a^n \frac{q^{kn^2+n(k+1-i)}(-1/a)_n}{(-aq)_n}\right. \\
&\quad \left. - \sum_{n=0}^{\infty}(-1)^n a^{n+1} \frac{q^{kn^2+n(k+i)+i}(-1/a)_{n+1}}{(-aq)_{n+1}}\right) \\
&= \frac{(-aq)_\infty}{(q)_\infty}\left(\sum_{n=0}^{\infty}(-1)^n a^n \frac{q^{kn^2+n(k+1-i)}(-1/a)_n}{(-aq)_n}\right. \\
&\quad \left. + \sum_{n=-\infty}^{-1}(-1)^n a^{-n} \frac{q^{kn^2+n(k-i)}(-1/a)_{-n}}{(-aq)_{-n}}\right) \\
&= \frac{(-aq)_\infty}{(q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n a^n \frac{q^{kn^2+n(k+1-i)}(-1/a)_n}{(-aq)_n} \\
&= \overline{\mathcal{E}}_{k,i}(a,q)
\end{aligned}$$

□

5.2. **A combinatorial proof.** This part is a generalization of [16, Section 3]. We only give a sketch of the proof and details can be found in [35]. In this section we represent overpartitions by their multiplicity sequence $(f_0, f_1, f_2, \ldots)$. We include the multiplicity f_0 to simplify the definitions, although it is always equal to 0.

We say that a sequence $(f_m, \ldots, f_{m+\ell})$ is a *multuple* (tuple of multiplicity) if

- $f_{m+\ell} > 0$,
- f_m is not overlined, and
- f_{m+p} is overlined for $1 \le p \le \ell - 1$.

The *length* of a multuple $(f_m, \ldots, f_{m+\ell})$ is ℓ and its *weight* is $\sum_{i=0}^{\ell}(m+i)f_{m+i}$. We divide a multiplicity sequence of an overpartition into multuples going from the right to the left. When we find a positive multiplicity, we close a parenthesis to its right. We look for the next non-overlined multiplicity to its left and open a parenthesis to the left of the multiplicity. The *length* of a multiplicity sequence is the sum of the length of its multuples. For partitions, the length is called the number of pairs of multiplicities [16]. For example if the overpartition has for multiplicity sequence $(0, \bar{2}, 0, 2, \bar{1}, 1)$, then its multuples are $((0, \bar{2}), 0, (2, \bar{1}, 1))$. The first multuple has length 1 and the second 2. Therefore the length is 3.

We define a map F from multuples of weight n and length ℓ to multiplicity sequences of weight $n-\ell$ and length ℓ. Given a multuple $(f_m, \ldots, f_{m+\ell})$ then $F(f_m, \ldots, f_{m+\ell})$ is computed with the following algorithm :

- if $f_{m+\ell} = \bar{1}$, remove the overlining of $f_{m+\ell}$ and overline f_m,

- else if $\ell > 1$, remove the overlining of $f_{m+\ell-1}$ and overline f_m,
- $f_{m+\ell} \leftarrow f_{m+\ell} - 1$, and
- $f_m \leftarrow f_m + 1$.

For example if $m = 1$ and the multuple is $(1, \bar{1}, \bar{3})$, we have $n = 12$, $\ell = 2$ and $F(1, \bar{1}, \bar{3}) = (\bar{2}, 1, \bar{2})$ whose weight is 10.

Let $\overline{B}_{k,i}(n, j, N)$ be the number of overpartitions counted by $\overline{B}_{k,i}(n, j)$ of length N. Let $\overline{\mathcal{B}}_{k,i}(N) = \sum_{n,j} \overline{B}_{k,i}(n, j, N) q^n a^j$. Now we divide the multiplicity sequence of an overpartition $\lambda \in \overline{\mathcal{B}}_{k,i}(N)$ into multuples going from the right to the left and we apply F to each multuple. We call the result $F(\lambda)$. If $F(\lambda)$ has a zero part, this part is discarded. Note that if λ has weight n and length N then $F(\lambda)$ has weight $n - N$ and length N or $N - 1$. For example, $\lambda = (0, (0, \overline{1}), (1, \overline{1}, \overline{3}))$ has weight 24 and length 3 and $F(0, (0, \overline{1}), (1, \overline{1}, \overline{3})) = (0, \overline{1}, 0, \overline{2}, 1, \overline{2})$ whose weight is 21 and length 3.

Let $\overline{\mathcal{G}}_{k,i}(N)$ be the generating function for the overpartitions $\mu = F(\lambda)$ where λ is an overpartition in $\overline{\mathcal{B}}_{k,i+1}(N)$ and has a multuple $(f_0, \ldots, f_\ell)$ with $\ell = 1$ and $f_\ell \neq \bar{1}$.

Proposition 5.3. *Let λ be an overpartition in $\overline{\mathcal{B}}_{k,i}(N)$ with $N > 0$. Then*

- *λ is an overpartition of n and has a multuple $(f_0, \ldots, f_\ell)$ with $\ell > 1$ or $f_\ell = \bar{1}$ if and only if $F(\lambda)$ is an overpartition of $n - N$, has one less overlined part than λ and is in $\overline{\mathcal{B}}_{k,i}(N - 1)$.*
- *λ is an overpartition of n and has a multuple $(f_0, \ldots, f_\ell)$ with $\ell = 1$ and $f_\ell \neq \bar{1}$ if and only if $F(\lambda)$ is an overpartition of $n - N$, has the same number of overlined parts as λ and is in $\overline{\mathcal{G}}_{k,i-1}(N)$.*
- *λ is an overpartition of n and has no multuple $(f_0, \ldots, f_\ell)$ if and only if $F(\lambda)$ is an overpartition of $n - N$, has the same number of overlined parts as λ and is in $\overline{\mathcal{B}}_{k,i+1}(N)$.*

Let λ be an overpartition in $\overline{\mathcal{G}}_{k,i}(N)$. Then

- *λ is an overpartition of n and has a multuple $(f_0, \ldots, f_\ell)$ with $\ell > 1$ or $f_\ell = \bar{1}$ if and only if $F(\lambda)$ is an overpartition of $n - N$, has one less overlined part than λ and is in $\overline{\mathcal{B}}_{k,i}(N - 1)$.*
- *λ is an overpartition of n and has a multuple $(f_0, \ldots, f_\ell)$ with $\ell = 1$ and $f_\ell \neq \bar{1}$ and $F(\lambda)$ is an overpartition of $n - N$, has the same number of overlined parts as λ and is in $\overline{\mathcal{G}}_{k,i-1}(N)$.*
- *λ is an overpartition of n and has no multuple $(f_0, \ldots, f_\ell)$ if and only if $F(\lambda)$ is an overpartition of $n - N + 1$, has the same number of overlined parts as λ and is in $\overline{\mathcal{B}}_{k,i+1}(N - 1)$.*

Proof. The full proof requires lots of details and is given in [35]. We give here the first ingredient of the proof, that is, if λ is an overpartition such that the condition $\forall \ell$, $f_\ell + f_{\ell+1} < k\overline{+1}$ holds, then this condition still holds for $F(\lambda)$. Indeed the only successive multiplicities f_m and f_{m+1}, where $f_m + f_{m+1}$ can increase are such that f_m or f_{m+1} is the leftmost entry of a multuple. If it is f_m then after the operation f_m is now overlined and $f_m + f_{m+1} < k\overline{+1}$ still holds. If it is f_{m+1} then $f_m + f_{m+1}$ increases by one, only if $f_m = 0$ before the operation. We know that $f_{m+1} < k - 1$ before the operation (as $f_{m+1} + f_{m+2} < k$ and $f_{m+2} > 0$) then $f_m + f_{m+1} < k$ still holds after the operation. □

Note that $\overline{\mathcal{B}}_{k,i}(0) = 1$ as the only overpartition of length 0 is the empty overpartition and that if an overpartition has a multuple $(f_0, \ldots, f_\ell)$ then $i > 1$. The previous remark and proposition imply that:

$$\begin{aligned}
\overline{\mathcal{B}}_{k,1}(N) &= q^N \overline{\mathcal{B}}_{k,2}(N); \\
\overline{\mathcal{B}}_{k,i}(N) &= q^N (\overline{\mathcal{B}}_{k,i+1}(N) + \overline{\mathcal{G}}_{k,i-1}(N) + a\overline{\mathcal{B}}_{k,i}(N - 1)), \text{ if } 1 < i \leq k; \\
\overline{\mathcal{G}}_{k,1}(N) &= q^{N-1} \overline{\mathcal{B}}_{k,2}(N - 1); \\
\overline{\mathcal{G}}_{k,i}(N) &= q^{N-1} \overline{\mathcal{B}}_{k,i+1}(N - 1) + q^N \overline{\mathcal{G}}_{k,i-1}(N) + aq^N \overline{\mathcal{B}}_{k,i}(N - 1), \text{ if } 1 < i < k; \\
\overline{\mathcal{B}}_{k,i}(0) &= 1.
\end{aligned}$$

Now we have all the ingredients to prove Proposition 5.1. We remark that $\overline{\mathcal{B}}_{k,k+1}(N) = \overline{\mathcal{B}}_{k,k}(N)$ and we set $\overline{\Gamma}_{k,i}(N) = \overline{\mathcal{G}}_{k,i}(N) + a\overline{\mathcal{B}}_{k,i+1}(N-1)$ if $i > 0$ and 0 otherwise. Then

$$\begin{aligned}
\overline{\mathcal{B}}_{k,i}(N) &= q^N(\overline{\mathcal{B}}_{k,i+1}(N) + \overline{\Gamma}_{k,i-1}(N)), \text{ if } i < k;\\
\overline{\mathcal{B}}_{k,k}(N) &= \overline{\Gamma}_{k,k-1}(N)/(1-q^N);\\
\overline{\Gamma}_{k,i}(N) &= (a+q^{N-1})\overline{\mathcal{B}}_{k,i+1}(N-1) + q^N\overline{\Gamma}_{k,i-1}(N), \text{ if } i > 0;\\
\overline{\mathcal{B}}_{k,i}(0) &= 1;\\
\overline{\Gamma}_{k,0}(N) &= 0.
\end{aligned}$$

These are the recurrences proven in Proposition 3.1. Therefore $\overline{\mathcal{B}}_{k,i}(N) = \overline{\mathcal{E}}_{k,i}(N)$ and Proposition 5.1 is proved.

6. Paths and successive Durfee squares

We prove in this section that

$$\overline{\mathcal{E}}_{k,i}(a,q) = \sum_{n_1 \geq \ldots \geq n_{k-1} \geq 0} \frac{q^{\binom{n_1+1}{2}+n_2^2+\cdots+n_{k-1}^2+n_i+\cdots+n_{k-1}}(-1/a)_{n_1}a^{n_1}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-2}-n_{k-1}}(q)_{n_{k-1}}}. \tag{6.1}$$

This gives the case $\overline{E}_{k,i}(n,j) = \overline{D}_{k,i}(n,j)$ of Theorem 1.4 as the right hand side of Equation (6.1) is the generating function for the overpartitions counted by $\overline{D}_{k,i}(n,j)$ (see Equation (2.1)). We give an analytical and a combinatorial proof.

6.1. An analytical proof.

We use the Bailey lattice structure from [1] to transform (6.1) into (1.1). Recall that a pair of sequences (α_n, β_n) form a Bailey pair with respect to a if for all $n \geq 0$ we have

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r}(aq)_{n+r}}.$$

We need the following lemma which is a special case of (3.8) in [1].:

Lemma 6.1. *If (α_n, β_n) is a Bailey pair with respect to q, then for all $0 \leq i \leq k$ we have*

$$\begin{aligned}
\frac{1}{(q,-aq)_\infty} &\times \sum_{n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{\binom{n_1+1}{2}+n_2^2+\cdots n_k^2+n_{i+1}+\cdots n_k}(-1/a)_{n_1}a^{n_1}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}}\beta_{n_k}\\
&= \frac{\alpha_0}{(q)_\infty^2} + \frac{1}{(q)_\infty^2}\sum_{n\geq 1}\frac{(-1/a)_n a^n q^{(n^2-n)(i-1/2)+in}(1-q)}{(-aq)_n}\\
&\times \left(\frac{q^{(n^2+n)(k-i)}}{(1-q^{2n+1})}\alpha_n - \frac{q^{((n-1)^2+(n-1))(k-i)+2n-1}}{(1-q^{2n-1})}\alpha_{n-1}\right)
\end{aligned} \tag{6.2}$$

Proof. We set $a = q$, $\rho_1 = -1/a$, and then let n as well as all remaining ρ_i and σ_i tend to ∞ in (3.8) of [1] to obtain (6.2). □

Proof of (6.1). We use the Bailey pair with respect to q [37, p.468, (B3)],

$$\beta_n = \frac{1}{(q)_\infty} \qquad \text{and} \qquad \alpha_n = \frac{(-1)^n q^{n(3n+1)/2}(1-q^{2n+1})}{(1-q)}.$$

Substituting into Lemma 6.1 and simplifying, we obtain

$$\begin{aligned}\sum_{n_1\geq\cdots\geq n_k\geq 0}&\frac{q^{\binom{n_1+1}{2}+n_2^2+\cdots+n_k^2+n_{i+1}+\cdots+n_k}(-1/a)_{n_1}a^{n_1}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}(q)_{n_k}}\\ =&\frac{(-aq)_\infty}{(q)_\infty}\left(1+\sum_{n=1}^{\infty}\frac{q^{(k+1)n^2+(k-i+1)n+}(-a)^n(-1/a)_n}{(-aq)_n}\right.\\ &\left.+\sum_{n=1}^{\infty}\frac{q^{(k+1)n^2-(k-i)n}(-a)^n(-1/a)_n}{(-aq)_n}\right)\\ =&\frac{(-aq)_\infty}{(q)_\infty}\left(1+\sum_{n=1}^{\infty}\frac{q^{(k+1)n^2+(k-i+1)n}(-a)^n(-1/a)_n}{(-aq)_n}\right.\\ &\left.+\sum_{n=-\infty}^{-1}\frac{q^{(k+1)n^2+(k-i+1)n}(-a)^n(-1/a)_n}{(-aq)_n}\right).\end{aligned}$$

Replacing k by $k-1$ and i by $i-1$ gives (1.1). □

6.2. **A combinatorial proof.** We will use and generalize the notion of relative height of a peak. This notion was defined by Bressoud in [14] for paths with no South steps and a simpler version of the definition was given in [10]. We adapt this definition for the paths with South steps.

Definition 6.2. The relative height of a peak (x, y) is the largest integer h for which we can find two vertices on the path, $(x', y-h)$ and $(x'', y-h)$, such that $x' < x \leq x''$ and such that between these two vertices there are no peaks of height $> y$ and every peak of height y has abscissa $\geq x$.

The original definition was the same except $x \leq x''$ was $x < x''$. Indeed when there are no South steps the case $x = x''$ is impossible.

For example, the relative heights in the path from Figure 9 are, from left to right: 1, 2, 1, 4, 3, 2.

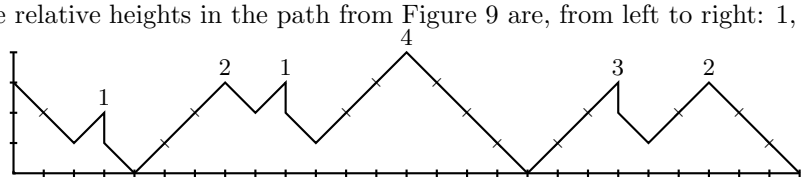

FIGURE 9. An example of a path and its relative heights.

We will prove here that

Proposition 6.3. *The coefficient of $q^n a^\ell$ in*

$$\frac{q^{\binom{n_1+1}{2}+n_2^2+\cdots+n_{k-1}^2+n_i+\cdots+n_{k-1}}(-1/a)_{n_1}a^{n_1}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-2}-n_{k-1}}(q)_{n_{k-1}}} \tag{6.3}$$

is the number of paths with major index n and ℓ South steps, starting at height $k-i$, whose height is less than k and having n_j peaks of relative height $\geq j$ for $1 \leq j \leq k-1$. Therefore,

$$\sum_{n_1\geq\cdots\geq n_{k-1}\geq 0}\frac{q^{\binom{n_1+1}{2}+n_2^2+\cdots+n_{k-1}^2+n_i+\cdots+n_{k-1}}(-1/a)_{n_1}a^{n_1}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-2}-n_{k-1}}(q)_{n_{k-1}}}=\overline{\mathcal{E}}_{k,i}(a,q).$$

To prove this proposition we will use a result of Bressoud [14]

Lemma 6.4. *The coefficient of q^n in*

$$\frac{q^{n_1^2+n_2^2+\cdots+n_{k-1}^2+n_i+\cdots+n_{k-1}}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-2}-n_{k-1}}(q)_{n_{k-1}}}$$

is the number of paths with major index n, no South steps, starting at height $k-i$, whose height is less than k and having n_j peaks of relative height $\geq j$ for $1 \leq j \leq k-1$.

An example of such a path, taken from [14], is shown on Figure 10. For that path, we have $k = 4$, $i = 1$, $n_1 = 3$, $n_2 = 1$ and $n_3 = 1$.

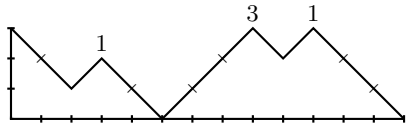

FIGURE 10. Another example of a path and its relative heights.

We can now move on to the proof of Proposition 6.3.

Proof. We generalize the argument of Bressoud in [14]. Consider a path with no South steps that starts at height $k-i$, whose height is less than $k-1$ and that has n_j peaks of relative height $\geq j-1$ for $2 \leq j \leq k-1$. By Lemma 6.4, such paths are counted by

$$\mathcal{P}_{k,i}(q) = \frac{q^{n_2^2+\cdots+n_{k-1}^2+n_i+\cdots+n_{k-1}}}{(q)_{n_2-n_3}\cdots(q)_{n_{k-2}-n_{k-1}}(q)_{n_{k-1}}} \tag{6.4}$$

where $2 \leq i \leq k$. Note that we have replaced n_j by n_{j+1} so that the terms in n_1 should be introduced by the algorithm described below, which will then give us the generating function in the same form as in (6.3).

For any given $k \geq 2$ and i such that $1 \leq i \leq k$, we describe an algorithm which generates a path counted by (6.3) from a path counted by $\mathcal{P}_{k,i}(q)$ if $i \geq 2$ or $\mathcal{P}_{k,2}(q)$ if $i = 1$, a partition λ into distinct parts which lie in $[0, n_1 - 1]$ and a partition b into $n_1 - n_2$ nonnegative parts.

We will need to prove that this contruction is uniquely reversible, that the algorithm generates all of our paths, that the distribution of relative heights is not modified (except at the first step where all the peaks will be raised by one) and that the algorithm affects the generating function in the appropriate way.

We first perform a "volcanic uplift" by inserting a NES peak at each peak (see Figure 11). This increases all the relative height by one.

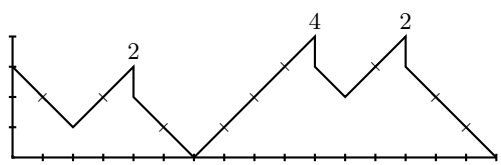

FIGURE 11. Effect of the volcanic uplift on the path from Figure 10.

We then insert $n_1 - n_2$ NES peaks at the beginning of the path (see Figure 12). Note that all these peaks have relative height one and that they are the only peaks of relative height one since the volcanic uplift has increased all the relative heights by one.

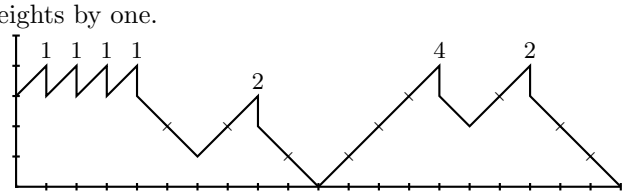

FIGURE 12. After adding the $n_1 - n_2 = 4$ NES peaks of relative height one to the path from Figure 11.

If $i = 1$, we introduce an extra SE step at the beginning of the path, from $(0, k-1)$ to $(1, k-2)$.

Now if λ contains a part $j-1$ ($1 \leq j \leq n_1$), we transform the jth NES peak from the right into a NESE peak (see Figure 13).

Finally, for $1 \leq j \leq n_1 - n_2$, we move the jth peak of relative height one from the right b_j times according to the rules illustrated in Figure 14.

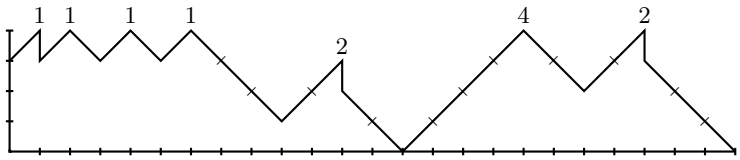

FIGURE 13. Effect of transforming some NES peaks into NESE peaks in the path from Figure 12. In this example, $\lambda = (5, 4, 3, 1)$.

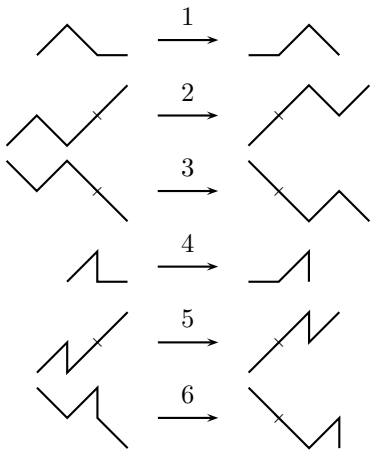

FIGURE 14. The rules for moving peaks of relative height one.

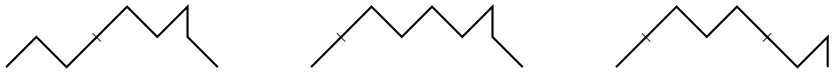

FIGURE 15. We want to move the leftmost peak to the right twice, but after the first move, we come up against a sequence of adjacent peaks. We then move the rightmost peak in this sequence.

When we move a peak, it can meet the next peak to the right. We say that a peak (x, y) meets a peak (x', y') if

$$x' - x = \begin{cases} 2 \text{ if } (x, y) \text{ is a NESE peak} \\ 1 \text{ if } (x, y) \text{ is a NES peak} \end{cases}.$$

If this happens, we abandon the peak we have been moving and move the next peak to the right. If we come up against a sequence of adjacent peaks, we move the rightmost peak in the sequence (see Figure 15).

To conclude the proof, we must show that the distribution of relative heights is not modified by the operations of Figure 14 (Proposition 6.5), that the construction procedure is uniquely reversible (Proposition 6.6), that we generate all of our paths (Proposition 6.7), and that the algorithm affects the generating function in the appropriate way (Proposition 6.8). □

Proposition 6.5. *The operations of Figure 14 preserve the number of peaks of relative height $\geq j$ for all j.*

Proof. Let us show it for each operation. We call p the peak which is moved. Remember that before the move, the relative height of p is 1.

For operations 1 and 4, the relative height of p clearly remains 1 after the move. The other peaks are not affected and their relative heights are therefore not modified.

For operations 2 and 5, it can be easily shown that the relative height of p remains 1 (see Figure 16 for an example) unless it meets a peak, in which case the two peaks will swap their relative heights (see Figure 17 for an example). In both cases, the relative heights of the other peaks are not modified.

For operations 3 and 6, the relative height of p will clearly remain 1 since the peak located to its left has a larger (or equal) height (see Figure 18).

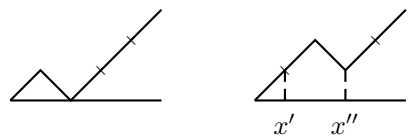

FIGURE 16. Case where the relative height of p is not modified.

FIGURE 17. When p meets a peak, their relative heights are inverted. This does not modify the number of peaks with a given relative height.

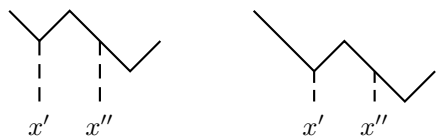

FIGURE 18. The relative height of p remains one when we apply the operation 3 or 6.

□

Proposition 6.6. *The construction used in the proof of Proposition 6.3 is uniquely reversible.*

Proof. We describe the algorithm to undo the algorithm used to prove Proposition 6.3. We start with a path counted by (6.3).

We first have to move the peaks of relative height 1 to the left. We begin with the leftmost of these peaks and we move it to the left end of the path. If, during this move, our peak becomes adjacent to another peak, we abandon the peak we have been moving and we move the peak to its left (we do the same if we come up against a sequence of contiguous peaks). The number of moves we had to perform to bring the peak to the beginning of the path gives us b_1. We proceed similarly for the other peaks of relative height 1, which gives us a partition b into $n_1 - n_2$ parts ≥ 0. Note that we can do this because of Proposition 6.5: since the number of peaks of relative height 1 is preserved when we move the peaks, the peaks of relative height 1 in the path counted by (6.3) correspond to these which were added by the volcanic uplift and the subsequent insertion of peaks.

At this step, among the n_1 peaks, some are NESE peaks. We transform these NESE peaks into NES peaks: if the jth peak from the right was transformed, it gives a part $j-1$ in λ.

If $i = 1$, we remove the first South-East step of the path. Finally, we remove the $n_1 - n_2$ NES peaks at the beginning of the path and we decrease by 1 the height of the remaining peaks by removing the NES peaks. The resulting path is counted by $\mathcal{P}_{k,2}(q)$ if $i = 1$ and by $\mathcal{P}_{k,i}(q)$ otherwise. □

Proposition 6.7. *Any path in $\overline{E}_{k,i}(n,j)$ can be generated by our algorithm.*

Proof. This is easy to see using the reverse algorithm described in the proof of Proposition 6.6. That algorithm can be applied to any path P counted by (6.3) and gives a Bressoud path P' (i.e. a path counted by (6.4)). If we apply the direct algorithm to P', we will obtain our initial path P counted by (6.3). Thus, for any path P in $\overline{E}_{k,i}(n,j)$, there exists a Bressoud path P' which gives that path. □

Proposition 6.8. *Our algorithm affects the generating function in the appropriate way.*

Proof. The volcanic uplift increases the major index of the path by

$$1 + 2 + \cdots + n_2 = \binom{n_2+1}{2}$$

and the relative height of each peak by one. Moreover, the n_2 NES peaks introduce a factor a^{n_2}.

The new peaks introduced after the uplift have total major index $\binom{n_1-n_2+1}{2}$ and they increase the abscissa of each of the old peaks by n_1-n_2. Since they are NES peaks, they also give a factor $a^{n_1-n_2}$. Altogether, the two operations introduce a factor

$$q^{\binom{n_2+1}{2}}a^{n_2}\times q^{\binom{n_1-n_2+1}{2}+n_2(n_1-n_2)}a^{n_1-n_2}=q^{\binom{n_1+1}{2}}a^{n_1}.$$

If $i=1$, we add an extra SE step at the beginning of the path, which introduces a factor q^{n_1}.

Transforming the jth peak from the right into a NESE peak increases the major index of the path by $j-1$ because the $j-1$ rightmost peaks are shifted by 1 to the right. We do that if there is a part $j-1$ in λ; altogether, the major index of the path increases precisely by the size of λ, which is a partition into distinct parts in $[0,n_1-1]$. Since we transform a NES peak into a NESE peak for each part of λ, this step introduces a factor $(-1/a)_{n_1}$.

Finally, when we move the jth peak of relative height one from the right b_j times, we increase the abscissa of the path by b_j. Altogether, the major index of the path increases precisely by the size of $b=(b_1,b_2,\dots,b_{n_1-n_2})$, which is a partition into n_1-n_2 nonnegative parts. Such partitions are counted by $\frac{1}{(q)_{n_1-n_2}}$. □

The multiple series in Proposition 6.3 can be re-expressed as (2.1), which is the generating function for overpartitions with $i-1$ successive Durfee squares followed by $k-i$ successive Durfee rectangles, the first one being a generalized Durfee square/rectangle.

7. New partition theorems

We first prove Corollary 1.6 and then extract its combinatorial information.

7.1. **Proof of Corollary 1.6.** We recall here the Jacobi Triple Product identity (JTP) given in equation (1.2):

$$(-1/z,-zq,q;q)_\infty=\sum_{n=-\infty}^{\infty}z^nq^{\binom{n+1}{2}}.$$

and the result of Theorem 1.5

$$\overline{\mathcal{E}}_{k,i}(a,q)=\frac{(-aq)_\infty}{(q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^na^nq^{2k\binom{n+1}{2}-n(i-1)}\frac{(-1/a)_n}{(-aq)_n}.$$

We first prove (1.3). Using Theorem 1.5, we get

$$\overline{\mathcal{E}}_{k,i}(0,q)=\frac{1}{(q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^nq^{(2k+1)\binom{n+1}{2}-ni}.$$

We substitute $q\to q^{2k+1}$, $z\to -q^{-i}$ in (1.2) and get

$$\overline{\mathcal{E}}_{k,i}(0,q)=\frac{(q^i,q^{2k+1-i},q^{2k+1};q^{2k+1})_\infty}{(q)_\infty}.$$

Second we prove (1.4). Using Theorem 1.5, we get

$$\overline{\mathcal{E}}_{k,i}(1/q,q^2)=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}(-1)^nq^{4k\binom{n+1}{2}-n(2i-1)}.$$

We substitute $q\to q^{4k}$, $z\to -q^{-2i+1}$ in (1.2) and get

$$\overline{\mathcal{E}}_{k,i}(1/q,q^2)=\frac{(q^2;q^4)_\infty(q^{2i-1},q^{4k+1-2i},q^{4k};q^{4k})_\infty}{(q)_\infty}.$$

Third we prove (1.5). Using Theorem 1.5, we get

$$\overline{\mathcal{E}}_{k,i}(1,q)\quad=\quad\frac{(-q)_\infty}{(q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^nq^{2k\binom{n+1}{2}-n(i-1)}\frac{2}{1+q^n}$$

Writing half of this series twice, once with n and once with $-n$, we have

$$\overline{\mathcal{E}}_{k,i}(1,q) = \frac{(-q)_\infty}{(q)_\infty}\left(\sum_{n=-\infty}^{\infty}(-1)^n q^{2k\binom{n+1}{2}-ni}\frac{q^n}{1+q^n} + \sum_{n=-\infty}^{\infty}(-1)^n q^{2k\binom{n+1}{2}+n(i-2k)}\frac{1}{1+q^n}\right)$$

Therefore

$$\overline{\mathcal{E}}_{k,k}(1,q) = \frac{(-q)_\infty}{(q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{2k\binom{n+1}{2}-nk};$$

and for $i<k$

$$\begin{aligned}\overline{\mathcal{E}}_{k,i}(1,q) &= \frac{(-q)_\infty}{(q)_\infty}\left(\sum_{n=-\infty}^{\infty}(-1)^n q^{2k\binom{n+1}{2}-ni} - \sum_{n=-\infty}^{\infty}(-1)^n q^{2k\binom{n+1}{2}-n(i+1)}\frac{q^n}{1+q^n}\right.\\ &\quad\left.+\sum_{n=-\infty}^{\infty}(-1)^n q^{2k\binom{n+1}{2}+n(i-2k)} - \sum_{n=-\infty}^{\infty}(-1)^n q^{2k\binom{n+1}{2}+n(i+1-2k)}\frac{1}{1+q^n}\right)\\ &= \frac{(-q)_\infty}{(q)_\infty}\left(\sum_{n=-\infty}^{\infty}(-1)^n q^{2k\binom{n+1}{2}}(q^{-ni}+q^{n(i-2k)})\right) - \overline{\mathcal{E}}_{k,i+1}(1,q)\\ &= \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{2(k-i)}(-1)^j\sum_{n=-\infty}^{\infty}(-1)^n q^{2k\binom{n+1}{2}-n(i+j)}.\end{aligned}$$

We substitute $q\to q^{2k}$, $z\to -q^{-i-j}$ in (1.2) and get

$$\overline{\mathcal{E}}_{k,i}(1,q) = \frac{(-q)_\infty}{(q)_\infty}\sum_{j=0}^{2(k-i)}(-1)^j(q^{i+j},q^{2k-i-j},q^{2k})_\infty.$$

Finally we prove (1.6).

$$\overline{\mathcal{E}}_{k,i}(1/q,q) = \frac{(-1)_\infty}{(q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{2k\binom{n+1}{2}-ni}\frac{1+q^n}{2}.$$

We substitute $q\to q^{2k}$, $z\to -q^{-i}$ and $q\to q^{2k}$, $z\to -q^{-i+1}$ in (1.2) and get

$$\overline{\mathcal{E}}_{k,i}(1/q,q) = \frac{(-q)_\infty}{(q)_\infty}(q^i,q^{2k-i},q^{2k};q^{2k})_\infty + (q^{i-1},q^{2k-i+1},q^{2k};q^{2k})_\infty.$$

Now we give some combinatorial interpretation of Equations (1.4), (1.5) and (1.6).

7.2. **2-modular diagrams.** We state in details the result for the Andrews' generalization of the Gordon-Göllnitz identities which correspond to equation (1.4) of Corollary 1.6. The coefficient of q^n in this equation is the number of partitions of n with parts not congruent to $2 \mod 4$ and $0,\pm(2i-1) \mod 4k$. We make the change of variable $q\to q^2$ and $a\to 1/q$ in Theorem 1.4 and interpret it combinatorially in terms of 2-modular diagrams defined in Section 2. There exists an easy bijection ϕ between 2-modular diagrams of weight n with j ones and overpartitions of $(n+j)/2$ with j overlined parts. This bijection consists of erasing any 2 of the modular diagram and changing any 1 to a marked corner. With this bijection in hand, the successive ranks (resp. Durfee dissection) of a 2-modular diagram μ are the successive ranks (resp. Durfee dissection) of the corresponding overpartition $\phi(\mu)$.

Proposition 7.1. *All the following are equal:*

- *The number of partitions of n with parts not congruent to $2 \mod 4$ and $0,\pm(2i-1) \mod 4k$;*
- *The number of partitions of n of the form $(\lambda_1,\lambda_2,\dots,\lambda_s)$ with unrepeated odd parts, where $\lambda_\ell-\lambda_{\ell+k-1}\geq 3$ if $\lambda_{\ell+k-1}$ is even and 2 otherwise; and $f_1+f_2<i$;*
- *The number of 2-modular diagrams of n whose successive ranks lie in $[-i+2,2k-i-1]$;*
- *The number of 2-modular diagrams of n with $i-1$ successive Durfee squares followed by $k-i$ successive Durfee rectangles, the first one being a generalized Durfee square/rectangle.*

- *The number of paths that use four kinds of unitary steps with special (k,i)-conditions where n is twice the sum of the x-coordinates of the peaks minus the number of South steps.*

Remark. The first two parts of the theorem are the Andrews' generalization of the Gordon-Göllnitz identities. The interpretation in terms of successive ranks and Durfee dissection is new to our knowledge.

7.3. **Superpartitions.** We give new partition theorems related to Gordon's theorems for overpartitions [28] which are the combinatorial interpretation of the case $i=k$ of equation (1.5) and $i=1$ of equation (1.6). We now interpret combinatorially equations (1.5) and (1.6). Superpartitions [21] are overpartitions where the first occurrence of a part can be overlined and the part $\overline{0}$ can appear. Let $\overline{B}_{k,i}(n)$ be the number of overpartitions of n of the form $(\lambda_1,\lambda_2,\dots,\lambda_s)$, where $\lambda_\ell-\lambda_{\ell+k-1}\geq 1$ if $\lambda_{\ell+k-1}$ is overlined and $\lambda_\ell-\lambda_{\ell+k-1}\geq 2$ otherwise and at most $i-1$ parts are equal to 1.

Theorem 7.2. *For $1\leq i\leq k-1$, the number of overpartitions counted by $\overline{B}_{k,i}(n)$ plus the number of overpartitions counted by $\overline{B}_{k,i+1}(n)$ is equal to the number of superpartitions where the non-overlined parts are not congruent to $0,\pm i$ modulo $2k$.*

Proof. Theorem 1.4 tells us that $\overline{B}_{k,i}(n)+\overline{B}_{k,i+1}(n)$ is the coefficient of q^n of $\overline{\mathcal{E}}_{k,i}(1,q)+\overline{\mathcal{E}}_{k,i+1}(1,q)$.

Thanks to equation (1.5), we know that $\overline{\mathcal{E}}_{k,i}(1,q)+\overline{\mathcal{E}}_{k,i+1}(1,q)=$

$$\begin{aligned}
&= \frac{(-q)_\infty}{(q)_\infty}\left(\sum_{j=0}^{2(k-i)}(-1)^j(q^{i+j},q^{2k-i-j},q^{2k};q^{2k})_\infty+\sum_{j=0}^{2(k-i-1)}(-1)^j(q^{i+1+j},q^{2k-i-1-j},q^{2k};q^{2k})_\infty\right)\\
&= \frac{(-q)_\infty}{(q)_\infty}\left(\sum_{j=0}^{2(k-i)}(-1)^j(q^{i+j},q^{2k-i-j},q^{2k};q^{2k})_\infty-\sum_{j=1}^{2(k-i)-1}(-1)^j(q^{i+j},q^{2k-i-j},q^{2k};q^{2k})_\infty\right)\\
&= \frac{(-1)_\infty}{(q)_\infty}(q^i,q^{2k-i},q^{2k};q^{2k})_\infty.
\end{aligned}$$

The coefficient of q^n in that last equation is the number of superpartitions where the non-overlined parts are not congruent to $0,\pm i$ modulo $2k$. □

Theorem 7.3. *For $2\leq i\leq k-1$, the number of superpartitions of n of the form $(\lambda_1,\lambda_2,\dots,\lambda_s)$, where $\lambda_\ell-\lambda_{\ell+k-1}\geq 1$ if λ_j is overlined and $\lambda_\ell-\lambda_{\ell+k-1}\geq 2$ otherwise and where the number of non-overlined ones plus the number of $\overline{0}$ is at most $i-1$ is equal to the number of overpartitions of n where the non-overlined parts are not congruent to $0,\pm i$ modulo $2k$ plus the number of overpartitions of n where the non-overlined parts are not congruent to $0,\pm(i-1)$ modulo $2k$.*

Proof. We interpret combinatorially the coefficient of q^n in equation (1.6). This is the number of overpartitions of n where the non-overlined parts are not congruent to $0,\pm i$ modulo $2k$ plus the number of overpartitions of n where the non-overlined parts are not congruent to $0,\pm(i-1)$ modulo $2k$. Note that this is the interpretation of $\overline{\mathcal{E}}_{k,i}(1/q,q)$. This implies that all the overlined parts in Theorem 1.4 are decreased by one and the result follows. □

8. Conclusion

We showed in this work how the combinatorial interpretation of the Andrews-Gordon identities can be generalized to the case of overpartitions, when the combinatorial statistics (successive ranks, generalized Durfee square, length of the multiplicity sequence) are defined properly. There exist other generalizations of the Rogers-Ramanujan identities, see for example [15]. It was shown that the combinatorial interpretation in terms of lattice paths can also be done for these identities [2, 14, 16, 17]. Our work can also be extended in that direction and the results are presented in [20]. Recently Lovejoy and the second author have shown how to generalize the results presented in this paper and in the paper [20] to overpartition pairs. This work appears in [31]. Finally there exists an extension of the concept of successive ranks for partitions due to Andrews, Baxter, Bressoud, Burge, Forrester and Viennot [7] and our goal now is to extend that notion to overpartitions.

References

[1] A.K. Agarwal, G.E. Andrews, and D.M. Bressoud, The Bailey lattice, *J. Indian. Math. Soc.* **51** (1987), 57-73.

[2] A. K. Agarwal and D. M. Bressoud, Lattice paths and multiple basic hypergeometric series. *Pacific J. Math.* **135** (1989) 209–228.

[3] G. E. Andrews, The theory of partitions. Cambridge University Press, Cambridge, 1998.

[4] G. E. Andrews, An analytic generalization of the Rogers-Ramanujan identities for odd moduli. *Proc. Nat. Acad. Sci. USA* **71** (1974) 4082–4085.

[5] G. E. Andrews, Sieves in the theory of partitions. *Amer. J. Math.* **94** (1972) 1214–1230.

[6] G. E. Andrews, A generalization of the Göllnitz-Gordon partition theorems. *Proc. Amer. Math. Soc.* **8** (1967) 945–952.

[7] G. E. Andrews, R. J. Baxter, D. M. Bressoud, W. H. Burge, P. J. Forrester and G. Viennot, Partitions with prescribed hook differences, *European J. Combin.* (1987) **8** 341–350.

[8] G. E. Andrews and D. Bressoud, On The Burge correspondence between partitions and binary words. Number Theory (Winnipeg, Man., 1983). *Rocky Mountain J. Math.* **15** (1985), no. 2, 225–233.

[9] G. E. Andrews and D. M. Bressoud, Identities in Combinatorics III: further aspects of ordered set sorting. *Discrete Math* **49** (1984) 222–236.

[10] A. Berkovich and P. Paule, Lattice paths, q-multinomials and two variants of the Andrews-Gordon identities, *Ramanujan J.* **5** (2001), 409–425.

[11] A.O.L. Atkin, A note on ranks and conjugacy of partitions, *Quart. J. Math. Oxford Ser. (2)* **17** (1966) 335–338.

[12] C. Bessenrodt and I. Pak, Partition congruences by involutions, *European J. Combin.* **25** (2004) 1139–1149. *J. Combin. Theory Ser. A*, **113** (2006), no. 6, 1019–1030.

[13] F. Brenti, Determinants of Super-Schur Functions, Lattice Paths, and Dotted Plane Partitions *Adv. Math.*, **98** (1993), 27-64.

[14] D. Bressoud, Lattice paths and the Rogers-Ramanujan identities. Number Theory, Madras 1987, 140–172, *Lecture Notes in Math.* **1395**, Springer, Berlin, 1989.

[15] D. Bressoud, A generalization of the Rogers-Ramanujan identities for all moduli. *J. Comb. Theory* **27** (1979) 64–68.

[16] W. H. Burge, A correspondence between partitions related to generalizations of the Rogers-Ramanujan identities. *Discrete Math.* **34** (1981), no. 1, 9–15.

[17] W. H. Burge, A three-way correspondence between partitions. *European J. Combin.* **3** (1982), no. 3, 195–213.

[18] S. Corteel, J. Lovejoy, Overpartitions. *Trans. Amer. Math. Soc.* **356** (2004), no. 4, 1623–1635.

[19] S. Corteel, J. Lovejoy, Frobenius partitions and the combinatorics of Ramanujan's ${}_1\psi_1$ summation, *J. Comb. Theory Ser. A* **97** (2002), 177–183.

[20] S. Corteel, J. Lovejoy and O. Mallet, An extension to overpartitions of the Rogers-Ramanujan identities for even moduli, preprint.

[21] P. Desrosiers, L. Lapointe, and P. Mathieu, Jack superpolynomials, superpartition ordering and determinantal formulas. *Comm. Math. Phys.* **233** (2003), no. 3, 383–402

[22] J-F. Fortin, P. Jacob and P. Mathieu, Jagged partitions. *Ramanujan J.* **10** (2005), no. 2, 215–235.

[23] J.-F. Fortin, P. Jacob, and P. Mathieu, Generating function for K-restricted jagged partitions. *Electron. J. Comb.* **12**, No.1, Research paper R12, 17 p., (2005).

[24] J.-F. Fortin, P. Jacob, and P. Mathieu, Jagged partitions, *Ramanujan J.* **10** (2005), no. 2, 215–235.

[25] G. Gaspar and M. Rahman, Basic hypergeometric series, Cambridge University Press (1990).

[26] B. Gordon, A combinatorial generalization of the Rogers-Ramanujan identities, *Amer. J. Math.* **83** (1961), 393–399 .

[27] S-J. Kang, J-H. Kwon, Crystal bases of the Fock space representations and string functions. *J. Algebra* **280** (2004), no. 1, 313–349.

[28] J. Lovejoy, Gordon's theorem for overpartitions. *J. Combin. Theory Ser. A* **103** (2003), no. 2, 393–401.

[29] J. Lovejoy, Overpartition theorems of the Rogers-Ramanujan type *J. London Math. Soc.* **69** (2004), 562–574.

[30] J. Lovejoy, Rank and conjugation for the Frobenius representation of an overpartition. *Ann. Combin.* **9** (2005) 321–334.

[31] J. Lovejoy and O. Mallet, Overpartition pairs and two families of basic hypergeometric series, preprint.

[32] P. A. MacMahon, Combinatory Analysis, vol. 2, Cambridge Univ. Press, Cambridge, 1916.

[33] K. Mahlburg, The overpartition function modulo small powers of 2, *Discrete Math.* **286** (2004), no. 3, 263–267.

[34] O. Mallet, Rangs successifs, chemins et identités de type Rogers-Ramanujan, *Master's thesis, Paris 6* (2005).

[35] O. Mallet, *PhD thesis*, Paris 7, in preparation.

[36] L. J. Rogers, Second memoir on the expansion of certain infinite products, *Proc. London Math. Soc.* **25** (1894), 318–343.

[37] L.J. Slater, A new proof of Rogers's transformations of infinite series *Proc. London Math. Soc.* **53** (1951), 460-475.

[38] A. J. Yee, Combinatorial proofs of Ramanujan's ${}_1\psi_1$ summation and the q-Gauss summation, *J. Combin. Theory Ser. A*, **105** (2004), 63–77.

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