# Cost functions with several order of magnitudes and the use of Relative Internal Set Theory

Thomas Colcombet

***Abstract*—Cost monadic logic extends monadic second-order logic with the ability to measure the cardinal of sets. In particular, it allows to decide problems related to boundedness questions. In this paper, we provide new decidability results allowing the systematic investigation of questions involving "relative boundedness". The first contribution in this work is to introduce a suitable logic for such questions. The second is to show the decidability of this logic over finite words. The third contribution is the use of non-standard analysis: we advacate that developing the proofs in the axiomatic system of "relative internal set theory" entails a significant simplification of the proofs.**

## I. INTRODUCTION

The results of Büchi-Elgot-Trakhtenbrot stating the decidability of the monadic second-order theory over finite words is a central result in the verification of infinite structures [1], [2], [3]. The major extensions of these results were due to Büchi and Rabin, extending this decidability to infinite words and infinite trees [4]. The culmination of these form of results is probably the preservation of monadic second-order logic under the iteration construction of Muchnik-Walukiewicz [5].

Another branch of automata related techniques is concerned quantitative forms of verification. More specifically, we are interested by the contributions intiated by Hashiguchi [6] when he proved the decidability of the limitedness problem for functions computed by distance automata, automata that associate to the each input word a value in $\mathbb{N}\cup\{\infty\}$. These results have then be reproved and improved by Leung and Simon [7], [8], and then extended to more expressive forms of automata by Kirsten [9] and to finite trees in [10]. This line of research has been fruitful for solving difficult problems in language theory such as the finite power property and the emblematic star-height problem [11], [9], the star height problem for trees [10], or deciding the boundedness of fixpoints for monadic logic [12].

Inspired by ideas from [13], these two branches of research were unified in the theory of cost-functions [14], [15] where an extension of monadic second-order logic that can capture the quantitive aspects of distance autamata and their extensions was introduced and proved decidable. This theory was also successfully extended to finite trees [16] and infinite words. The case of infinite trees remains open (solving it would solve another important open problem in automata theory: the decidability of the Mostowski hierarchy [17]), and only the weak fragment of the logic (and a bit more) is understood so far [18], [19].

There are several reasons to study such extensions of monadic logic. The first one is indeed to solve difficult questions related to monadic second-order logic and automata theory. Another one is concerned with its consequences in verification: such techniques open the door for new results of model checking with quantitative objectives: question such as "is the system able to achieve a goal while keeping its resource consumption reasonable (to be understood as bounded here)?" At a general level, these techniques raise the results concerning monadic logic, which are Boolean in nature, to versions that have mild quantitative capacities (knowing that almost all versions of monadic logic with quantitative capabilities either turn out to be undecidable, or to be no-more expressive than the original logic). This approach may also appear useful in contexts of theoretical computer science other than monadic second-order logic.

We are concerned here with finite words, and extensions of the results and the techniques in [14], [15]. In these works, **cost monadic logic** was introduced. This logic extends monadic (second-order) logic with the ability to test whether a set has size at most $n$ (using the construct $|X|\leq n$ using always a single number variable $n$) providing this test occurs positively in the formula (below an even number of negation). Such a formula $\varphi$ is used to describe a function $[\![\varphi]\!]$ which to each word $u$ associates the least $n$ that makes the formula true over $u$. What is shown decidable is the boundedness problem, *i.e.*, $\exists n\forall u\,[\![\varphi]\!](u)\leq n$. This can also simply be stated as $\exists n\forall u\,\varphi$ if we consider $\varphi$ as implicitly referring to $u$. More generally what is shown decidable is the **domination** relation between two formulae of cost monadic logic, $\varphi$ and $\psi$: on every set of words, if $[\![\psi]\!]$ is bounded, then so is $[\![\varphi]\!]$. This is equivalent to $\forall m\exists n\forall u\,(([\![\psi]\!](u)\leq m)\rightarrow([\![\varphi]\!](u)\leq n))$ (see [15] for a thorough description). Equivalently, this can be rephrased as deciding $\forall m\exists n\forall u\,(\psi\rightarrow\varphi)$.

We continue to develop this path of research in several ways. The contributions of the paper can listed as follows.

- We introduce a new logic, *magnitude monadic logic*. In this logic, formulae can begin with an arbitrary block of number quantifier $\exists m_1\forall m_2\ldots\forall m_{2k}$ followed with a formula of monadic logic that can furthermore use predicates of the form $|X|\leq m_i$ providing these appear with a suitable constraint of positivity. In particular, the domination between formulae of cost monadic logic can be expressed in this logic.
- We generalise the proof techniques used for cost-functions to this extended framework, introducing magnitude monoids (that extend stabilization monoid), and showing that, in a suitable sense, these have the same expressive power as magnitude monadic logic. We derive decision procedure from it. In particular this extends all decidability results concerning cost functions over finite words in the literature.
- We perform our proofs in the framework of Relative Internal Set Theory (RIST): a conservative extension of ZFC introduced by Péreire [20] which implements the ideas of

non-standard analysis. This change in approach significantly simplifies the technical aspects of the proof. This part is motivated by ideas from Toruńczyk, who used profinite words for modelling and solving cost-functions and describe other objects [21], [22].

The rest of the paper is organised as follows. In Section II, we introduce the notion of magnitude formulae, and in particular magnitude monadic logic. In Section III, we present briefly relative internal set theory, and describe how its use drastically simplifies many concepts related to magnitude logics. In Section IV we introduce the algebraic object of magnitude monoids. In Section V we develop several notions necessary for giving semantics to magnitude monoids. In Section VI we finally present the notion of recognizability by monoid, show its equivalence with magnitude monadic logic, and solve it.

## II. Magnitude formulae and monadic logic

### A. Magnitude formulae

We start by introducing the principle of magnitude formulae in a general context of some first-order logic. We expect the reader familiar with logic terminology. First-order logic, is defined here as usual using first-order variables $x, y, \dots$, and it is allowed to quantify over them existentially ($\exists x\, \varphi$), universaly ($\forall x\, \varphi$), to use the boolean connectives ($\vee$, $\wedge$, $\neg$, $\to$) and terms constructed from the ($a(x)$, $R(x, y, f(z))$, $\dots$). We do not want to be precise concerning signatures, and details should be clear from the context.

Given a variable $n$, ranging over non-negative integers (from now, variables $m, n, m_1, \dots$ implicitly range over non-negative integers), we say that a formula $\varphi(n)$ **uses** $n$ **as an upper bound** if, for all valuations of other variables and all $n \leq m$, if $\varphi(n)$ holds $\varphi(m)$ also holds. A formula **syntactically uses** $n$ **as an upper bound** if, when negations are pushed to the leaves, all occurrences of the variable $n$ appear in constructions of the form $t \leq n$ where $t$ is a term. In particular $t > n$ or $t = n$ are disallowed constructions. Dually $\varphi$ **(syntactically) uses** $n$ **as a lower bound** if its negation (syntactically) uses $n$ as an upper bound. Remark that it is easy to transform a formula $\varphi(n)$ using $n$ as a lower bound into an equivalent one syntactically using it as a lower bound, namely $\psi(n) := \exists m\, (m \leq n \wedge \varphi(m))$.

A **magnitude formula** is a formula of the form:

$$Q_1 m_1\ Q_2 m_2\ \dots Q_k m_k\ \varphi\ ,$$

where $Q_i$ is either $\exists$ or $\forall$, $m_1 \dots m_k$ are variables ranging over non-negative integers, and $\varphi$ is a formula that uses $m_i$ as an upper bound if $Q_i$ is $\exists$ and as a lower bound otherwise, *i.e.*, if $Q_i$ is $\forall$. The formula $\varphi$ is called a **$\bar{Q}\bar{m}$-formula**. Usually, we abbreviate the sequence of quantifiers $Q_1 m_1\ Q_2 m_2\ \dots Q_k m_k$ as $\bar{Q}\bar{m}$. We call $\bar{Q}\bar{m}$ the **quantifier context**. It will most of the time be fixed. The **dual** ${}^d\bar{Q}\bar{m}$ of a quantifier context $\bar{Q}\bar{m}$ is obtained by exchanging $\exists$ for $\forall$ and vice versa.

Given a **quantifier context** $\bar{Q}\bar{m}$, it induces an order over $\mathbb{N}^k$ defined by $\bar{m} \leq_{\bar{Q}} \bar{n}$ if, $m_i \leq n_i$ for all $i$ such that $Q_i$ is $\exists$, and $n_i \leq m_i$ otherwise. This order has the property that whenever $\bar{m} \leq_{\bar{Q}} \bar{n}$, $\varphi(\bar{m}) \to \varphi(\bar{n})$ for all $\bar{Q}\bar{m}$-formula $\varphi$. Remark also that "dualising" the quantifier context reverses the order.

Examples of magnitude formula include boundedness formulas $\exists m\, \phi$, domination between cost function, the definition of nowhere dense graphs.

Let us describe informally why we name such formulae "magnitude formulae". One can see the evaluation of such a formula as a $k$-rounds game involving players $\exists$ and $\forall$. In this game, the player $Q_1$ plays firsts and choose $m_1 \in \mathbb{N}$, then player $Q_2$ chooses $m_2$, and so on. After $k$ rounds, the existential player wins if the resulting valuation of $\bar{m}$ makes the formula $\varphi$ true. Naturally $\exists$ wins such a game if and only if $\bar{Q}\bar{m}\ \varphi$ holds. Under this view, the positivity and negativity requirements in the use of $m_1, \dots, m_k$ in $\varphi$ have as direct consequence that, at each step of the game, the player to move has all interest in choosing the highest value possible. Indeed, if a player $Q$ wins the game, and that in some branch of his winning strategy he plays at round $i$ the value $m_i$, then the strategy will still be winning playing any $m'_i \geq m_i$ instead. For this reason it is sufficient to reason with strategies where $m_1$ is chosen "very large", $m_2$ is chosen "very large" in front of $m_1$, $m_3$ is chosen "very large" in front of $m_1$ and $m_2$, etc… In other words, $m_1, m_2, \dots$ can be considered as having *increasing orders of magnitude*. This intuition will be made precise thanks to the use of Relative Internal Set Theory (see section III).

### B. Magnitude monadic second-order logic

In this paper, we are more specifically interested in formulae of magnitude monadic logic. Let us recall that **monadic** (second-order) **logic** has the syntax of first-order logic, where variables are split into **first order variables** $x, y, \dots$ interpreted as elements of the structure and **monadic** (second-order) **variables** interpreted as sets of elements. A special relation $x \in Y$ interpreted as "$x$ belongs to $Y$" is allowed. All the predicates of the structure are used with first order variables. For instance on di-graphs (seen as a structure with vertices as elements and using a signature with sole symbol $\texttt{edge}(x, y)$ interpreted as the edge relation), the formula

$$\forall Z\, x \in Z \wedge (\forall z \forall z'\, z \in Z \wedge \texttt{edge}(z, z') \to z' \in Z) \to y \in Z$$

expresses that every set that contains $x$ and is closed under the edge relation also contains $y$. In other words, it expresses the existence of a path from $x$ to $y$.

It is common to use monadic logic–and we will be doing the same for magnitude monadic logic–over words. For doing this, words need be seen as relational structures. Formally, for a given alphabet $\mathbb{A}$, a word $u \in \mathbb{A}^*$ is seen as the relational structure of elements $1, \dots, |u|$ (the positions in the word) and is equipped with (a) the binary symbol $<$ interpreted as the natural order on integers, and (b) of the unary symbol $a$ interpreted as the set of positions of the words carrying letter $a$ for all $a \in \mathbb{A}$. A **language** is a set of words over a given alphabet. A language $L$ is **definable in monadic logic** if there is a formula $\varphi$ of monadic logic such that for all words $u$, $u \in L$ if and only if $\varphi$ is true on $u$ (written $u \models \varphi$ read "$u$ models $\varphi$"). The seminal Büchi-Elgot-Trakhtenbrot result states that the languages of words definable in monadic logic are exactly the regular languages, furthermore, these equivalences are effective.

Let us turn ourselves to the definition of magnitude monadic logic. Given a quantifier context $\bar{Q}\bar{m}$, a **$\bar{Q}\bar{m}$-monadic formula** $\varphi$ is a $\bar{Q}\bar{m}$-formula in the syntax of monadic logic extended with the ability to use the variables $m_1, \ldots, m_k$ in new predicates of the form "$|X| \leq m_i$" where $X$ is a monadic variable, and $|X|$ denotes the cardinality of $X$. Of course, each formula being a $\bar{Q}\bar{m}$-formula, the use of the new constructions is subject to the constraints of positivity inherited from magnitude logic: every predicate of the form $|X| \leq m_i$ has to appear positively in $\varphi$ if $Q_i$ is $\exists$, and negatively otherwise.

A formula of **magnitude logic** is of the form:

$$\bar{Q}\bar{m}\,\varphi$$

where $\bar{Q}\bar{m}$ is a quantifier context, and $\varphi$ is a $\bar{Q}\bar{m}$-monadic formula. Our objective is to solve questions such as $(\mathbb{N}, <) \models \bar{Q}\bar{m}\,\varphi$, or $(\mathbb{Q}, <) \models \bar{Q}\bar{m}\,\varphi$ (we are expecting to present these results in the future journal version of the paper).

However, in this paper, we do not want to be involved with infinite models. Indeed, the core of the technique deals with finite words, and treating infinite words would simply mean mixing the techniques with non immediately related notions such as Wilke algebras [23] or $\circ$-monoids [24]. This would result in many non-essential complications in the proofs. Then what about solving satisfiability over finite words? Asking this question would simply result in making magnitude monadic logic trivial. Indeed, consider a $\bar{Q}\bar{m}$-monadic formula $\varphi$, and consider $\varphi^*$ obtained from $\varphi$ by syntactically replacing every construct $|X| \leq m_i$ by $\mathtt{true}$. It is easy to check that on any finite model, $\bar{Q}\bar{m}\,\varphi$ holds if and only if the monadic formula $\varphi^*$ holds. Thus, this cannot be seen as the correct problem concerning magnitude monadic logic over finite words.

For this reason, we introduce a variant of magnitude monadic logic for finite words as follows. A **magnitude formula (for finite words)** is a formula of the form:

$$\bar{Q}\bar{m}\,\exists u\,\varphi\,,$$

where $\bar{Q}\bar{m}$ is a quantifier context, $\exists u$ is a quantification over finite words, and $\varphi$ is a $\bar{Q}\bar{m}$-monadic formula, which is interpreted over the word $u$. This kind of formulae exactly capture the essence of magnitude monadic logic over finite words. Remark that over infinite words, this quantification $\exists u$ is for free. Indeed, using standard encoding techniques, it can be replaced by a block of existential monadic quantifiers in $\varphi$. In this paper, we show the decidability of magnitude formula *for finite words*.

An example of such formulae concerns cost-functions (see introduction or [15]), if $\varphi(n), \psi(n)$ are formula of cost monadic logic, then

$$\exists m \forall n\,\exists u\,\neg\psi(n) \wedge \varphi(m)\ ,$$

expresses the non-domination of $[\![\psi]\!]$ by $[\![\varphi]\!]$.

## III. Relative Internal Set Theory

We advocate in this paper the use of Relative Internal Set Theory as a convenient context for developing the proofs. We first present the general framework of Relative Internal Set Theory in Section III-A, and then study some of its consequences on magnitude formulae in Section III-B.

### A. The framework

**Relative Internal Set Theory**[1] (**RIST**) is a conservative extension of ZFC. This means that we allow some new notions, *i.e.*, new keywords, to be used in the discourse, that have no interpretation in ZFC. These new keywords are subject to the application of specific axioms. However, any statement that does not involve the new symbols is provable in ZFC if and only if it is provable in RIST. Hence it is as valid to perform proofs in this conservative extension as in usual ZFC. This means that, despite the fact that we cannot prove more results than in ZFC, we can use steps of reasoning that are meaningless in ZFC, and that can make the proofs simpler. In particular, RIST will permit to compare the order of magnitudes of integers in a perfectly formal manner.

In this paper, we will not present RIST in its generality since it would lead us to introduce notions that are out of the scope of the present work. We will just present a conservative extension of ZFC that has sufficient features for our purpose (and is a serious restriction of RIST).

In the syntax of RIST, every expression usable in ZFC can be used. In particular elements are related thanks to the binary membership $\in$ predicate. In RIST, we are furthermore allowed to use the new unary symbols $\mathrm{St}_0, \mathrm{St}_1, \ldots$ The symbol $\mathrm{St}_i(x)$ means that "$x$ is standard at level $i$"[2]. We will say that $x$ is **$i$-standard**. This symbols can be applied to all objects: to natural numbers, to sets, to functions, or to any mathematical entity one can think of. An object which is $i$-standard but is not $i-1$-standard will be said **strictly $i$-standard**. The intuition behind this is that $0$-standard elements form, in themselves a model of ZFC, the part of "easily accessible or constructible objects". There are also elements that are 1-standard and not 0-standard. These elements are "unreachable from 0-standard elements" in the sense that it is impossible using 0-standard parameters in any formula to witness their existence or non-existence. Again, 1-standard elements form a model of ZFC. The strictly 1-standard integers, for instance, can be thought as "very large" natural numbers (in front of 0-standard numbers). Strictly 1-standard reals can be for instance "very large" or "infinitesimal". The 2-standard elements play the same role in front of 1-standard elements, and so on. We will see below that concerning integers, the notion of being $i$-standard can be thought of as "being of magnitude $i$".

Elements that are 0-standard are simply called **standard**. Standard elements are to be thought "simple", or in our case

[1] In fact there exist several variants of it, RIST, FRIST, GRIST… The choice of a specific instance does not make any difference for the subject in question here.

[2] This is a simplification of the real RIST in which a binary predicate is used, that means "being at least as standard as". This weaker presentation suits better our purpose.

representable in a computer. So formulae will implicitly be standard, finite monoids will also be standard, and more generally any object which is intended to be used inside a decision procedure will be standard. Other levels of "standardness" will be used in the proofs only.

A formula that does not involve any of these new symbols $\mathrm{St}_0, \mathrm{St}_1, \ldots$ is **internal**. The internal formula are exactly the formula of ZFC. By opposition, a non-internal formula is **external**. An **$i$-external formula** is a formula which does not use the predicates $\mathrm{St}_0, \ldots, \mathrm{St}_{i-1}$. We will note from now $\exists^{\mathrm{St}_i}\varphi$ for the formula $\exists x\ \mathrm{St}_i(x) \wedge \varphi$, and similarly $\forall^{\mathrm{St}_i}\ \varphi$ for the formula $\forall x\ \mathrm{St}_i(x) \rightarrow \varphi$.

Let us now describe what are the axioms of RIST.

**Extension of ZFC** every axiom of ZFC can be used. Thus, every statement provable in ZFC is also provable. This means in particular that any object that can be defined in ZFC can be defined in RIST, and has exactly the same internal properties. In particular it is possible to define the set of natural numbers $\mathbb{N}$, and as it is usual, it is possible to perform inductive proofs on them (providing the induction hypothesis is an internal formula!).

A particularly important example that is allowed is the comprehension axiom schema: given a set $E$, an internal formula $\varphi(x, \bar{y})$ and parameters $\bar{a}$, one can define the set $\{x \in E\ :\ \phi(x, \bar{a})\}$. For instance, one can construct the set of prime natural numbers. However *it is illegal to apply comprehension to external formulae*. For instance, $\{x \in \mathbb{N}\ :\ \mathrm{St}_0(x)\}$ is an *illegal notation* since $\mathrm{St}_0(x)$ is an external formula. This is as illegal as it is illegal to form in ZFC the "set of all sets". Concerning this example, we will even see Lemma 3 stating that if a set of natural numbers contain all standard numbers, it also contains a non-standard number. Working in RIST (as in any variant of non-standard analysis), requires to be extremely sensitive to such considerations.

**Order** Every $i$-standard element is $j$-standard for all $j \geq i$.

**Strictness** For all $i = 0, 1, \ldots$, there exists $n \in \mathbb{N}$ which is strictly $i$-standard. This axiom (which is again a weakening of the *idealisation axiom* of RIST ) guarantees that we have enhanced our universe of discourse. Without it, it would be impossible to prove that there exists a non-standard element.

**Transfer** For $\varphi(x, \bar{y})$ an internal formula and $i$-standard parameters $\bar{a}$, then

$$\forall x\ \varphi(x, \bar{a}) \quad \leftrightarrow \quad \forall^{\mathrm{St}_i} x\ \varphi(x, \bar{a})\ .$$
$$(\text{dually}\quad \exists x\ \varphi(x, \bar{a}) \quad \leftrightarrow \quad \exists^{\mathrm{St}_i} x\ \varphi(x, \bar{a}))$$

One can use this axiom iteratively on an internal formula, and get that $\varphi(\bar{a})$ holds if and only if $\varphi^{\mathrm{St}_i}(\bar{a})$ holds for all internal formula $\varphi$, where $\varphi^{\mathrm{St}_i}$ is the formula $\varphi$ relativised to the $i$-standard elements.

An important consequence of this axiom is that every object which is definable using an internal formula with $i$-standard parameters, is $i$-standard. For this reason, all usual mathematical objects, such as $\emptyset$, $\mathbb{N}$, $\mathbb{R}$, 0, 1 or $\pi$ are standard. It follows also from this argument that if $x$ is an $i$-standard element, then every element definable from it using an internal formula is also $i$-standard. For instance, given an $i$-standard integer, its successor, its predecessor (if relevant), its square, … are also $i$-standard.

**Standardization** Consider an $i$-external formula $\varphi(x, \bar{y})$ (*i.e.*, it can use the predicates $\mathrm{St}_i, \mathrm{St}_{i+1}, \ldots$ but not $\mathrm{St}_0, \mathrm{St}_1, \ldots, \mathrm{St}_{i-1}$), any parameters $\bar{a}$, and an $i$-standard set $X$, then, there exists an $i$-standard set $Y$ such that

$$\forall^{\mathrm{St}_i}\quad x \in Y\ \leftrightarrow\ (x \in X\ \wedge\ \varphi(x, \bar{y}))\ .$$

We denote this set $\{x \in X\ :\ \varphi(x, \bar{y})\}^{\mathrm{St}_i}$.

In fact, RIST has a richer set of axioms, that we do not disclose in this paper. Enhanced versions of this theory, with similar properties also exists, named FRIST and GRIST. In our case, these distinctions are irrelevant since we are only interested in relatively simple forms of proofs.

In any case, the important point is the correctness.

**Theorem 1.** *[Péraire 92] RIST is a conservative extension of ZFC.*

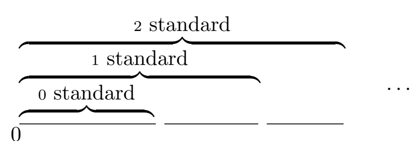
This signifies that any internal formula $\varphi$ provable in RIST is also provable in ZFC. For this reason, all proofs are (safely) performed in RIST from now, even if the original question is formulated in ZFC. The use of RIST simply gives rise to new powerful proof techniques.

Let us give now some immediate consequences of RIST. The first one states that the "standardness level" behaves monotonically with respect to natural numbers.

**Lemma 1.** *For $x \leq y$ natural numbers, if $y$ is $i$-standard, then $x$ is $i$-standard.*

*Proof:* By contradiction, assume that there exist natural numbers $x \leq y$ such that $y$ is $i$-standard but $x$ not. By *standardization*, consider the $i$-standard set $Z = \{z \in \mathbb{N}\ :\ z \leq x\}^{\mathrm{St}_i}$. Remark that every $i$-standard element in $Z$ is bounded by $y$. Thus by *transfer*ing the property "every set of natural numbers which has an upper bound has a maximal element", there is a maximal $i$-standard natural number in $Z$. Let $m$ be this number. By definition of $Z$, $m \leq x$. Since $m$ is $i$-standard while $x$ is not, this means $m < x$. Furthermore, "every natural number has a successor". So by transfer, $m$ has an $i$-standard successor. This successor is smaller or equal to $x$. This contradicts the maximality in the choice of $m$. ∎

According to this lemma, natural numbers can be seen as follows:




The 0-standard numbers have a minimal element, but no maximal element (because the successor of a 0-standard integer is 0-standard by transfer). Then for $i > 0$, for the same reason, strictly $i$-standard non-negative integers have no minimal nor maximal $i$-standard element. This structure is the reason why it is correct to see the successive levels of standardness for non-negative integers as order of magnitudes.

An application of standardization is the ability to use variants of the notion of induction that work only at a standard level.

**Lemma 2.** *Assume $P(n)$ is an $i$-external property, such that $P(0)$ holds, and $P(n)$ implies $P(n+1)$, then $P(n)$ holds for all $i$-standard non-negative integers $n$.*

*Proof:* By standardization, there exists an $i$-standard set $Z$ that contains an $i$-standard $n$ if and only if $P(n)$ does not hold. For the sake of contradiction, assume there is some $i$-standard $n \in Z$, then there exists a minimal $n$ in $Z$ (by transfer), which furthermore is $i$-standard by Lemma 1. If $n = 0$ this contradicts the hypothesis that $P(0)$ holds. If $n > 0$, then by minimality in its construction $P(n-1)$ holds, but not $P(n)$. This is again a contradiction to the hypotheses. Hence all $i$-standard natural numbers $n$ belongs to $Z$, and thus satisfy $P(n)$. ∎

In particular, any induction of standard length is valid.

The following lemma is one of the most important ones in (any version of) non-standard analysis. It characterizes the fact that it is impossible to separate the "levels of standardness" by means of sets.

**Lemma 3** (Overspill)**.** *If a set contains all $i$-standard non-negative integers then it contains a non-negative integer that is not $i$-standard. Dually, if a set contains all non $i$-standard non-negative integers, it also contains some $i$-standard non-negative integer.*

*Proof:* Consider a set $N$ which contains all $i$-standard natural numbers. Let $x$ be a natural number that is not $i$-standard (it exists) and define $M = \{n \leq x \ : \ n \in N\}$. This set $M$ is bounded and hence (by transfer) has a maximal element $m$. For the sake of contradiction, assume that $m$ is $i$-standard. Then (a) $m+1 \in N$ since by transfer $m+1$ is $i$-stanard and $N$ contains all $i$-standard elements, and (b) $m < x$ since $m \leq x$ by definition and $x$ is not $i$-standard while $m$ is. Hence by (a) and (b) $m+1 \in N$. This contradicts the maximality of $m$. ∎

We will use the following consequence of it.

**Lemma 4.** *If $\psi(n)$ is $i{+}1$-external (possibly with parameters), then if $\psi(n)$ holds for all non-$i$-standard natural numbers $n$, it also holds for some $i$-standard natural number $n$.*

*Proof:* By standardization, consider the $i+1$-standard set $Z = \{n \in \mathbb{N} \ : \ \psi(n)\}^{\mathrm{St}_{i+1}}$. Assume $\psi(n)$ holds for all non-$i$-standard natural numbers $n$, this implies that all strictly $i+1$-standard natural numbers belong to $Z$. By transfer, since $Z$ is $i+1$ standard, this means that all non-$i$-standard natural numbers belong to $Z$. Hence, by overspill (Lemma 3), $Z$ contains an $i$-standard natural number. Thus (definition of $Z$), $\psi(n)$ holds for some $i$-standard natural number. ∎

### *B. Magnitude formulae in RIST*

In this section we show why the use of RIST helps manipulating magnitude formulae. These arguments are valid for any magnitude formula, be it of monadic logic or not. In particular, we will use them for general logical formulae.

**Caution:** we have introduced the notion of magnitude formulae and $\bar{Q}\bar{m}$-formulae before RIST, *i.e.*, as a notion of ZFC. This means that these formula are *always internal*. It is not allowed to uses standardness predicates in them.

Let us fix from now a quantifier context $\bar{Q}\bar{m} := Q_1m_1 \ldots Q_km_k$. We denote by $\bar{Q}^{\mathrm{St}}\bar{m}$ the sequence of quantifiers

$$Q_1^{\mathrm{St}_1}m_1\, Q_2^{\mathrm{St}_2}m_2 \ldots Q_k^{\mathrm{St}_k}m_k \ .$$

**Lemma 5.** *Given a magnitude formula $\bar{Q}\bar{m}\ \varphi$ without parameters,*

$$\bar{Q}\bar{m}\ \varphi \text{ holds} \quad \text{if and only if} \quad \bar{Q}^{\mathrm{St}}\bar{m}\ \varphi \text{ holds.}$$

*Proof:* We prove, by induction (of standard length) on $\ell = 0, 1, \ldots, k$, that $\bar{Q}\bar{m}\,\varphi$ is equivalent to $Q_1^{\mathrm{St}_1}m_1 \ldots Q_\ell^{\mathrm{St}_\ell}m_\ell Q_{\ell+1}m_{\ell+1} \ldots Q_km_k\,\varphi$. Each step is by transfer. ∎

For the moment, it is not clear why exchanging $\bar{Q}^{\mathrm{St}}\bar{m}$ for $\bar{Q}\bar{m}$ is of any help. This will be shown by the following lemma. To state it, let us define, for $\varphi(\bar{m})$ a syntactic $\bar{Q}\bar{m}$-formula, the formula $\varphi(\bar{\iota})$ which is obtained from $\varphi(\bar{m})$ by replacing syntactically every construction of the form $f(\bar{x}) \leq m_i$ by $\mathrm{St}_i(f(\bar{x}))$ (in the case of $\bar{Q}\bar{m}$-monadic formulae, every construction $|X| \leq m_i$ is replaced by $\mathrm{St}_i(|X|)$). This formula is of course not internal anymore. The idea behind the notation is that $\iota_j$ would be something like an "integer" element that would be above every $j$-standard natural number, and below every non $j$-standard natural number. Of course, such an element does not exist, but it is very convenient to use it as a notation. Since every $\bar{Q}\bar{m}$-formula can be transformed into a syntactically $\bar{Q}\bar{m}$-formula, we also allow ourselves to use the notation $\varphi(\bar{\iota})$ in this case (but using directly the above definition would be incorrect here).

Remark that $\bar{n} \leq_{\bar{Q}} \bar{m}$ is a $\bar{Q}\bar{m}$-formula. This means that we can use the notation $\bar{n} \leq_{\bar{Q}} \bar{\iota}$. Dually, we can write $\bar{n} \geq_{\bar{Q}} \bar{\iota}$. which is a ${}^d\bar{Q}\bar{m}$-formula.

**Proposition 6.** *For every $\bar{Q}\bar{m}$-formula $\varphi(\bar{m})$,*

$$\begin{aligned}\bar{Q}^{\mathrm{St}}\bar{m}\,\varphi(\bar{m}) \quad &\text{if and only if} \quad \varphi(\bar{\iota}) \ ,\\ &\text{if and only if} \quad \exists \bar{m} \leq_{\bar{Q}} \bar{\iota}\ \varphi(\bar{m}) \ ,\\ &\text{if and only if} \quad \forall \bar{m} \geq_{\bar{Q}} \bar{\iota}\ \varphi(\bar{m}) \ .\end{aligned}$$

*Proof:* Assume $\bar{Q}\bar{m}$ is $Q_1m_1 \ldots Q_km_k$. We prove by downward induction on $\ell = 0 \ldots k$ that for all $m_1, \ldots, m_\ell$ that are respectively 1-standard, …, $\ell$-standard,

$$\psi_\ell := Q_{l+1}^{\mathrm{St}}m_{\ell+1} \ldots Q_k^{\mathrm{St}}m_k\,\varphi(\bar{m}) \quad \text{holds}$$
$$\text{if and only if} \quad \varphi_\ell := \varphi(\bar{\iota}_{[l+1,\ldots,k]}) \text{ holds} \ ,$$

where $\varphi(\bar{\iota}_{[\ell+1,\ldots,k]})$ is obtained from $\varphi(\bar{m})$ by replacing syntactically every construction of the form $f(\bar{x}) \leq m_i$ for some $i = \ell{+}1 \ldots k$ by $\mathrm{St}_i(f(\bar{x}))$. Of course, the induction hypothesis holds for $\ell = k$ since $\psi_k = \varphi_k$. Assume now that the induction hypothesis holds for $\ell > 0$ and that $m_1, \ldots, m_{\ell-1}$ are fixed as in the induction hypothesis. Assume also, without loss of generality, that $Q_\ell$ is $\exists$ (the case of $\forall$ is dual).

Assume first that $\psi_{\ell-1}$ holds. This means that there exists some $\ell$-standard $m_\ell$ such that $\psi_\ell$ holds. By induction hypothesis, this means that $\varphi_\ell$ holds. Then, whenever $f(\bar{x}) \leq m_\ell$ holds (for some term $f(\bar{x})$), $\mathrm{St}_\ell(f(\bar{x}))$ also holds. Since every such test occur positively in $\varphi_\ell$, we can replace everywhere in it $f(\bar{x}) \leq m_\ell$ constructs by $\mathrm{St}_\ell(f(\bar{x}))$, and obtain that $\varphi_{\ell-1}$ holds.

Conversely, assume $\varphi_{\ell-1}$ holds. Consider a non-$\ell$-standard $m_\ell$. Then by Lemma 1, $\mathrm{St}_\ell(f(\bar{x}))$ implies $f(\bar{x}) \leq m_\ell$. Hence $\varphi_\ell$ holds for all non $\ell-1$-standard $m_\ell$. Thus, since $\varphi_\ell$ is $\ell$-external, using Lemma 4, $\varphi_\ell$ holds for some $\ell$-standard $m_\ell$. We can apply the induction hypothesis and get that $\psi_\ell$ holds for some $\ell$-standard $m_\ell$, and hence $\psi_{\ell-1}$ holds.

The two other equivalences use the same arguments. ■

**Example 1.** We provide here an example of magnitude monadic logic for finite words. The formula is

$$\forall m_1 \exists m_2\, \exists u\, \varphi(\bar{m}) \quad \text{with} \quad \varphi(\bar{m}) := m_1 < |u|_a \wedge |u|_a \leq m_2$$

(here $|u|_a$ represents the number of occurrences of the letter $a$: it is simple to write $\varphi$ as a $\forall m_1 \exists m_2$-monadic formula). This formula is obviously true. A proof of it requires to give, for all $m_1$, values for $m_2$ and $u$ that make the formula true. For instance, one can choose $m_2 = m_1 + 1$ and $u = a^{m_2}$.

What have we shown by the above explanations? Simply that it is equivalent to solve the formula:

$$\exists u\, \varphi(\bar{\iota})\ , \quad \text{namely } \exists u\, \neg\mathrm{St}_1(|u|_a) \wedge \mathrm{St}_2(|u|_a)\ .$$

Hence, a single (non-standard) word such that the number of occurrences of the letter $a$ is strictly 2-standard is a witness of the truth of the formula. We see here all the interest of using RIST. By Skolemisation, finding a witness of the truth of some magnitude monadic logic requires to manipulate higher order objects (in the above example, a function which to each $m_1$ associates $m_2$ and $u$). Now that we have shifted the presentation to RIST, a witness is a single word (but non-standard in general: this is the price to pay).

## IV. Magnitude stabilisation monoids

The core of our approach for solving as well as understanding the expressive power of magnitude monadic logic over finite words is to provide an equivalent algebraic notion, magnitude monoids, that allows ourselves to define properties of words that are recognizable by magnitude monoids. This object is an extension of the notion of stabilisation monoids for cost functions.

### A. Semigroups and monoids

A **semigroup** $\mathcal{S} = (S, \cdot)$ is a set $S$ equipped with an associative operation "$\cdot$". A **monoid** is a semigroup such that the product has a **neutral element** 1, *i.e.*, such that $1 \cdot x = x \cdot 1 = x$ for all $x \in S$. We extend the product to products of arbitrary length by defining $\pi$ from $S^+$ to $S$ by $\pi(a) = a$ and $\pi(ua) = \pi(u) \cdot a$. If the semigroup is a monoid of neutral element 1, we further set $\pi(\varepsilon) = 1$. All semigroups are monoids, and conversely it is sometimes convenient to transform a semigroup $\mathcal{S}$ into a monoid $\mathcal{S}^1$ simply by the adjunction of a new neutral element 1.

An idempotent in $\mathcal{S}$ is an element $e \in S$ such that $e \cdot e = e$. We denote by $E(\mathcal{S})$ the set of idempotents in $\mathcal{S}$. An **ordered semigroup** $(S, \cdot, \leq)$ is a semigroup $(S, \cdot)$ together with an order $\leq$ over $S$ such that the product $\cdot$ is **compatible** with $\leq$; *i.e.*, $a \leq a'$ and $b \leq b'$ implies $a \cdot b \leq a' \cdot b'$. An **ordered monoid** is an ordered semigroup, the underlying semigroup of which is a monoid.

### B. Magnitude stabilisation monoids

Let us fix ourselves a quantifier context $\bar{Q}\bar{m} := Q_1 m_1 \ldots Q_k m_k$.

**Definition 1.** An **$\bar{Q}$-magnitude semigroup** is a structure $\mathcal{S} = (S, \cdot, \leq, \sharp_1, \ldots, \sharp_k)$ such that:

- $(S, \cdot, \leq)$ is an ordered monoid of finite standard size[3],
- For all $i = 1 \ldots l$, $\sharp_i$ is a mapping from $E(S)$ to $E(S)$ denoted exponentially. Implicitely we define $\sharp_0$ as the identity over $E(S)$. Each $\sharp_i$ is called the **stabilisation of level** $i$. The idempotents $e$ such that $e = e^{\sharp_i}$ are said **$i$-stable**. Idempotents that are stable for all $i = 1 \ldots k$ are said **stable**.
- For all $i, j$, and all idempotents $e$, $(e^{\sharp_i})^{\sharp_j} = e^{\sharp_{\max(i,j)}}$.
- If $e \leq f$ are two idempotents, then $e^{\sharp_i} \leq f^{\sharp_i}$.
- For all $i = 1 \ldots k$, if $Q_i$ is $\exists$, then $e^{\sharp_i} \leq e^{\sharp_{i-1}}$, and $e^{\sharp_{i-1}} \leq e^{\sharp_i}$ otherwise,
- For all $a, b$ such that $a \cdot b$ and $b \cdot a$ are idempotents, $(a \cdot b)^{\sharp_i} = a \cdot (b \cdot a)^{\sharp_i} \cdot b$.

A **magnitude monoid** $\mathcal{M}$ is a magnitude semigroup that has a stable neutral element 1.

The **dual** of a magnitude monoid is obtained by reversing the order. If $\mathcal{M}$ is a $\bar{Q}\bar{m}$-magnitude monoid then its dual ${}^d\mathcal{M}$ is a ${}^d\bar{Q}$-magnitude monoid, *i.e.*, for the dual quantifier context.

A stabilisation monoid (see [15]) exactly corresponds to a $\exists$-magnitude monoid. Though one cannot use directly the results concerning stabilisation monoids to establish results for magnitude monoids, most proofs are essentially similar.

We did not describe so far what was the semantics of magnitude monoid. This require several extra definitions (see below). Informally, what should be kept in mind is that $e^{\sharp}_i$ represents the effect of iterating a strictly $i$-standard number of times the idempotent $e$. Under this view, rules such as $(a \cdot b)^{\sharp_i} = a \cdot (b \cdot a)^{\sharp_i} \cdot b$ have a natural meaning.

What is a bit more subtle is the use of the order, and in particular the rule stating that if $Q_i$ is $\exists$, then $e^{\sharp_i} \leq e^{\sharp_{i-1}}$, and $e^{\sharp_{i-1}} \leq e^{\sharp_i}$ otherwise. This rule serves two purposes. On the one hand, it reflects the positivity assumptions in the use of cardinal predicates in magnitude monadic logic. At the same time this constraint is necessary for developing the tools describing the semantics of magnitude monoids: under-computations, over-computations and computations. This will be the subject of the next section.

**Example 2.** We expand on Example 1. Thus, $\bar{Q}\bar{m}$ is again $\forall m_1 \exists m_2$, and the formula expresses that there exists a word with more than $m_1$ occurrences of $a$'s, and at most $m_2$.

The idea is that we will separate the four following kind of words:

$\lambda$ is the neutral element, and it corresponds to all words consisting solely of letters different from $a$.

[3]This standardness assumption is to be used with RIST. Otherwise finiteness suffices.

1 corresponds to words that contain at least one $a$, but at most $m_1$ of them. Interpreted in RIST, this means words that have a 1-standard number of occurrences of $a$.

2 corresponds to words that contain more than $m_1$ occurrences of $a$, but at most $m_2$ of them. Interpreted in RIST, this means words that have a strictly 2-standard number of occurrences of $a$.

3 corresponds to words that contain more than $m_2$ occurrences of $a$. Interpreted in RIST, this means words that have a non 2-standard number of occurrences of $a$.

It is quite clear how to construct the product. For instance $1 \cdot 2 = 2$ means that concatenating a word with a 1-standard number of occurrences of $a$ to a word with a strictly 2-standard number of occurrences of $a$ yields a word with a strictly 2-standard number of occurrences of $a$. Also, stabilisations are obvious. $\sharp_i$ should be thought to as iterating (an idempotent) a strictly $i$-standard number of times. Thus we immediately get, for instance, that $1^{\sharp_2} = 2^{\sharp_2} = 2^{\sharp_1} = 2$.

Overall, we get the following table:

| $x \backslash y$ | $\lambda$ | 1 | 2 | 3 | $\sharp_1$ | $\sharp_2$ | $\sharp_3$ |
|---|---|---|---|---|---|---|---|
| $\lambda$ | $\lambda$ | 1 | 2 | 3 | $\lambda$ | $\lambda$ | $\lambda$ |
| 1 | 1 | 1 | 2 | 3 | 1 | 2 | 3 |
| 2 | 2 | 2 | 2 | 3 | 2 | 2 | 3 |
| 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |

It remains to provide the order. What is mandatory according to the definition is $2 \geq 1$ since $Q_1$ is $\forall$ and $1 = 1^{\sharp_1} \leq 1^{\sharp_2} = 2$. Similarly $2 \geq 3$. This suffices.

Keeping the quantifiers $\forall m_1 \exists m_2$ in mind, this means that it is always better for the truth of a formula when the size of sets are in the interval $(m_1, m_2]$. Thus the order on the elements of the monoids reflect the positivity assumptions that are used in the logic.

## V. Semantics of magnitude monoids

It not yet clear how to work with magnitude monoids. Indeed, so far, we have no ways to "evaluate" a magnitude monoid on a particular input. The key objects in this context are the ones of computations, under-computations and over-computations. These are generalisations of the ideas from [15], themselves highly inspired from the work of Simon [29], all being adapted to the presentation in RIST.

### A. Computation trees

Consider $\bar{m}$ in $\mathbb{N}^{k-1}$, that is implicitely extended with $m_0 = 0$ and $m_k = \infty$ and increasing (*i.e.*, $m_0 < m_1 < \cdots < m_{k-1} < m_k$). An **$\bar{m}$-under computation $T$ for a word** $u \in S^+$ **of height h** is an unranked order tree such that each node is **labeled** with $T(x) \in S$ and:

- $T$ has $|u|$ leaves, and for all $i = 1 \ldots |u|$, $T(x_i) \leq a_i$ where $x_i$ is the $i$th leaf of $T$ when read from left to right, and $a_i$ is the $i$th letter of $u$,
- for every non-leaf node $x$ labeled $c := T(x)$ of children labeled $c_1, \ldots, c_n$ when read from left to right one of the following situation holds:

  **binary node**: $n = 2$ and $c \leq c_1 \cdot c_2$,

  **idempotent node**: $c_1 = c_2 = \cdots = c_n = e \in E(S)$, $n \geq 3$ and $c \leq e^{\sharp_i}$, where $i = 0 \ldots k$ is is the only index such that $m_i \leq n < m_{i+1}$,
- no branch has length more than $h$.

Unless specified, the height of an under-computation is always standard. Sometimes a better bound may be specified.

An **$\bar{n}$-over-computation** is an $\bar{n}$-under-computation for the dual magnitude monoid. This amounts the reverse the order of $\mathcal{S}$ in the definition. An **$\bar{n}$-computation** is at the same time an $\bar{n}$-under-computation and an $\bar{n}$-over-computation. This amounts to replace in the above definition the order of $\mathcal{S}$ for the equality.

**Proposition 7.** *The property "being an $\bar{m}$-under-computation of height at most $h$" is a $\bar{Q}\bar{m}$-formula.*

*Proof idea:* Assume $Q_i$ is $\exists$, and that $T$ is an $\bar{m}$-under-computation over some $u$. Now, replace $m_i$ by some $m'_i \geq m_i$, yielding the new tuple $\bar{m}'$. We claim that $T$ is also a $\bar{m}'$-under-computation.

Indeed, what could happen that would prevent that? The answer is that the index $i$ in the definition of an idempotent node may change. Inspecting more closely the definition, the only relevant situation is the one of an idempotent node of degree $n$ such that $m_i \leq n < m_{i+1}$, but after the change, $m_{i-1} \leq n < m'_i$. Let $c$ and $e$ be as in the definition of an idempotent node. The fact that $T$ is a $\bar{m}$-under-computation means that $c \leq e^{\sharp_i}$. But, since $Q_i$ is $\exists$, this means that $e^{\sharp_i} \leq e^{\sharp_{i-1}}$. Hence $c \leq e^{\sharp_{i-1}}$, which means that the definition of an idempotent node for $\bar{m}'$ is also satisfied.

The case when $Q_i$ is $\forall$ cannot be deduced from the $\exists$-case, but is similar.

Doing this for every suitable $m_i$, we prove that an $\bar{m}$-under-computation is a $\bar{m}'$-under-computation for all $\bar{m}' \geq_{\bar{Q}} \bar{m}$. ■

A consequence of this proposition, according to Proposition 6, is that it is valid to use **$\bar{\iota}$-under-computation**. This is the object that we will be using most of the time from now. In practice, an $\bar{\iota}$-under-computation is defined exactly as an $\bar{m}$-computation but for the following new rule which replaces the case of idempotent nodes:

**idempotent node'**: $c_1 = c_2 = \cdots = c_n = e \in E(S)$, $n \geq 3$ and $c \leq e^{\sharp_i}$, where $i$ is such that if $i > 1$, $n$ is $i$-standard, and if $i < k$, $n$ is not $i+1$-standard.

We will use this presentation from now. Still, we need to be careful: though tempting, it is not allowed to talk about $\bar{\iota}$-computations. A natural definition could be given, which would amount to replace in the above definition of $\bar{\iota}$-under-computation the order over the semigroup by an equality. However, we could not guarantee the existence of such an object for all words. This is why we have to work with under and over-computations.

**Example 3.** Consider the magnitude monoid $\mathcal{M}$ of Example 2. Following the intuition in this example, call a word over $M^*$ of "kind $\lambda$" if it contains only $\lambda$'s. Call it of "kind 1" if it contains at least one 1, a 1-standard number of them, and no 2 nor 3. Call it of "kind 2" if it does not contain a 3, it contains a 2-standard number of 1 or 2, and contains at least one occurrences of 2 or a strictly 2-standard number of occurrences of 1. Finally, call a word of "kind 3" in the remaining cases, which are if either it contains a 3, or it contains a non 2-standard number of occurrences of 1 or 2.

One can prove, by induction (of standard length, using Lemma 2) that an $\bar{\iota}$-under-computation (recall that these are implicitly of standard height) over a word $u$ of kind $x$ necessary has a value $c \leq x$. One can also prove that an $\bar{\iota}$-over-computation over $u$ has always a value $c \geq x$. Furthermore, one can prove that there exists always an $\bar{\iota}$-under-computation of value $x$ as well as an $\bar{\iota}$-over-computation of value $x$. The following results we show that this "kind" is always uniquely determined, and it is called the evaluation of the word in the magnitude monoid.

## B. Existence of computations

We have presented the notions of under-computations, over-computations and computations. This will be used as means for evaluating words over magnitude semigroups. The first result one provides states the existence of this object. This means that every word can be evaluated.

**Theorem 2** (existence of computations)**.** *For all words $u$ over a magnitude semigroup $\mathcal{S}$ and all $\bar{n}$, there exists an $\bar{n}$-computation for $u$ of height at most $3|S|$.*

This result generalizes Theorem 3.3 in [15] which holds only for statbilization monoids. Though it requires to be redone, the proof is exactly the same, *i.e.*, based on the analysis of Green's relations. In our case, since magnitude semigroups are of standard size, this means that the height of the computation is standard.

As we already mentioned, it would be invalid to talk about $\bar{\iota}$-computations. Despite that, we can derive a result in RIST from Theorem 2, as follows.

**Proposition 8.** *For all words $u \in S^+$, there exists an element $a$ such that*

- *there is an $\bar{m}$-computation for $u$ of height at most $3|S|$ and value $a$ for some $\bar{m} \leq_{\bar{Q}} \bar{\iota}$,*
- *there is an $\bar{m}$-computation for $u$ of height at most $3|S|$ and value $a$ for some $\bar{m} \geq_{\bar{Q}} \bar{\iota}$.*

*Proof:* All computations in this proof have height at most $3|S|$.

Fix a word $u \in S^+$. Call a **limit value** for $u$ some $x \in M$ such that "for all $\bar{m} \leq_{\bar{Q}} \bar{\iota}$ there exists an $\bar{n}$-computation of value $x$ for $u$ for some $\bar{m} \leq_{\bar{Q}} \bar{n} \leq_{\bar{Q}} \vec{\iota}$". Assume that such an $x$ would not exist, this would mean that there exists some $\bar{m} \leq_{\bar{Q}} \bar{\iota}$ such that there is no $\bar{m}$-computation for $u$ (we use here the standardness of the magnitude semigroup to exhaust all possible candidate to be limit values). This would contradict Theorem 2.

Hence there exists some limit value $x$ for $u$. Being a limit value means that there exists some $\bar{m}$-computation for $u$ of value $x$ for some $\bar{m} \leq_{\bar{Q}} \bar{\iota}$.

Consider now the following formula $\psi(\bar{m})$: "there exists an $\bar{n}$-computation for $u$ of value $x$ for some $\bar{n} \geq_{\bar{Q}} \bar{m}$". Of course, the higher is $\bar{m}$ for $\leq_{\bar{Q}}$, the less true is the property. This means that it is a ${}^{d}\bar{Q}\bar{m}$-formula. Hence we can apply Proposition 6 on the formula $\forall \bar{m} \leq_{\bar{Q}} \bar{\iota}\ \psi(\bar{m})$ and get that $\exists m \geq_{\bar{Q}} \bar{\iota}\ \psi(\bar{m})$. It follows that there exists an $\bar{m}$-computation for some $\bar{m} \geq_{\bar{Q}} \bar{\iota}$. ■

The problem is that so far, we do not know if this value is unique. This is the subject of the next section.

## C. Unicity of computation

Now that we have seen that a computation witnessing the value of a word can always be produced, we have to prove that this is meaningful, and in particular, this means that several different computations essentially yield the same result. This is formalised by the following statement.

**Theorem 3** (unicity)**.** *For all words $u \in S^+$, all $\bar{\iota}$-under-computations for $u$ of value $a$ and and all $\bar{\iota}$-over-computations for $u$ of value $b$,*

$$a \leq b\ .$$

This theorem, at the same time generalises the result for cost-functions, Theorem 3.4 in [15]. This is the most involved part of the proof, and in particular the place where it is needed to analyse the structure of under and over-computations. It follows essentially the same structure of proof as in [15], that would be slightly complexified by the use of several order of magnitudes, and clearly simplified by the use of RIST.

When combined with Proposition 8, we obtain the following fundamental definition.

**Definition 2.** For all words $u$, there exists a unique element $a \in S$ such that Proposition 8 holds. This element is noted $\rho_{\mathcal{S}}(u)$ and is called the **evaluation of $u$ in $\mathcal{S}$**. It is also the sole element which is at the same time the value of an $\bar{\iota}$-under-evaluation for $u$ and of an $\bar{\iota}$-over-evaluation for $u$.

Let us remark that the definition were given so far for magnitude semigroups. Extending it to magnitude monoids requires to allow occurrences of the neutral element everywhere in a computation. There is essentially no difficulty here. See for instance [15] where this is entirely done for cost functions.

# VI. Recognizable $\bar{Q}^{\mathrm{St}}\bar{m}$-languages

## A. Definition

A $\bar{Q}^{\mathrm{St}}\bar{m}$**-language** is an external set of words over a given alphabet definable by a formula of the form $\bar{Q}^{\mathrm{St}}\bar{m}\,\varphi$ where $\varphi$ is a $\bar{Q}\bar{m}$-formula. An **external set** means that it is not strictly speaking a set, but rather a definable property (indeed $\bar{Q}^{\mathrm{St}}\bar{m}\,\varphi$ is an external formula and hence it is disallowed to form the set of words that satisfy it). Despite its external nature, we use some set terminology for $\bar{Q}^{\mathrm{St}}\bar{m}$-languages, and in particular we use union, intersection or projection of such objects. This is valid since these operations are in fact first order constructions that can be handled at the level of the logic.

Given a $\bar{Q}\bar{m}$-magnitude monoid $\mathcal{M}$, a mapping $h$ from an alphabet $\mathbb{A}$ to $M$ and a filter $F \subseteq M$ (an upward closed subset for the order of $\mathcal{M}$), we say that a word $u \in \mathbb{A}^+$ is $\bar{\iota}$**-accepted by** $\mathcal{M}, h, F$ if

$$\rho_{\mathcal{M}}(\tilde{h}(u)) \in F\ ,$$

where $\tilde{h}$ is the extension of $h$ into a mapping from $\mathbb{A}^*$ to $M^*$. The $\bar{Q}^{\rm St}\bar{m}$**-language recognized** is the external set $[\![\mathcal{M}, h, F]\!]$ that contains $u$ if $u$ is $\bar{\iota}$-accepted by $\mathcal{M}, h, F$.

### B. Elementary closure properties

Boolean closures are easy to obtain.

**Lemma 9.** *If $K$ and $L$ are recognizable $\bar{Q}^{\rm St}\bar{m}$-languages, then the same holds for $K \cap L$ and $K \cup L$. Furthermore $\complement P$ is a recognizable ${}^d\bar{Q}^{\rm St}\bar{m}$-language.*

We do not develop these points. Complement is obtained by dualizing the magnitude monoid, and exchanging the filter for its complement. Union and intersection are obtained by product construction as usual.

It is also important for us to have some constant $\bar{Q}^{\rm St}\bar{m}$-languages. We start with regular languages.

**Lemma 10.** *All (standard) regular languages are recognizable $\bar{Q}^{\rm St}\bar{m}$-languages.*

*Proof:* Let $L$ be a regular language. This means that it is recognizable by a monoid. Hence there exists a finite monoid $\mathcal{M} = (M, \cdot)$, a subset $F \subseteq M$, and a mapping $h$ from $\mathbb{A}$ to $M$ such that $u \in L$ if and only if $\pi(\tilde{h}(u))$.

We turn $\mathcal{M}$ into the $\bar{Q}\bar{m}$-magnitude monoid $\mathcal{M}' := (M, \cdot, Id_{E(M)}, \ldots, Id_{E(M)})$. Hence, we extend the monoid with a trivial equality order, and the stabilizations are set to the identity. This is the canonical way to transform a monoid into a magnitude monoid. It is easy to prove by induction on its height that for all under-computations for some $u \in M^*$, the value is $\pi(u)$. It follows that for all words $u$, $\rho_{\mathcal{M}}(u) = \pi(u)$, and as a consequence $[\![\mathcal{M}', h, F]\!] = L$. ∎

The other properties that we need to recognize are described by the following lemma.

**Lemma 11.** *If $Q_i$ is $\exists$ then the words that have an $i$-standard number of occurrences of a letter 'a' form a recognizable $\bar{Q}^{\rm St}\bar{m}$-language.*

*Dually, if $Q_i$ is $\forall$ then the words that have a non $i$-standard number of occurrences of a letter 'a' form a recognizable $\bar{Q}^{\rm St}\bar{m}$-language.*

*Proof:* We use a three element monoid, $\{1, a, 0\}$. The element $1$ is the usual neutral element and $0$ is an absorbing element. The element $a$ corresponds to the words that have at least one occurrence of the letter $a$, but only an $i$-standard number of them. The element $0$ gathers all the other words, namely the ones that have a non-$i$-standard number of occurrences of $a$. We obtain the following table.

| $x \backslash y$ | $1$ | $a$ | $0$ | $\sharp_1$ | $\ldots$ | $\sharp_{i-1}$ | $\sharp_i$ | $\ldots$ | $\sharp_k$ |
|---|---|---|---|---|---|---|---|---|---|
| $1$ | $1$ | $a$ | $0$ | $1$ | | $1$ | $1$ | | $1$ |
| $a$ | $a$ | $a$ | $0$ | $a$ | $\ldots$ | $a$ | $0$ | $\ldots$ | $0$ |
| $0$ | $0$ | $0$ | $0$ | $0$ | | $0$ | $0$ | | $0$ |

And the order is simply the least one such that $0 \leq a$. As expected $h$ sends all letters to $1$ but $a$ which is sent to $0$, and $F = \{1, a\}$. Proving the correctness of this construction requires an inductive analysis of the under and over-computations. We do not perform it here. ∎

### C. Magnitude expressions and decidability

Magnitude expressions are the natural extensions of Hashiguchi's $\sharp$-expressions to the case of magnitude monoids. These are used in order to explore the structure of magnitude monoids and to provide witnesses of non-emptiness.

A **magnitude expression** over a (standard) set $X$ is a (standard) term built using the operations $\cdot$, $\sharp_1, \ldots, \sharp_k$, and using constants from $X$. Hence, this is a term in the signature of magnitude monoids. A magnitude expression over a magnitude monoid is **valid** if it can be evaluated in it. This means that $\sharp_i$ operations are applied only to idempotents. A valid expression $f$ has a **value** which is simply denoted $\mathtt{val}(f)$. Given some $X \subseteq M$, $\langle X \rangle^\sharp$ is the set of values of expressions over $X$.

An expression is said **of level** $i$ if it does not use the operations $\sharp_{i+1}, \ldots, \sharp_k$. Given some $X \subseteq M$, $\langle X \rangle^{\leq i}$ is the set of values of expressions of level $i$ over $X$. An expression is **strictly of level** $i$ if it is of level $i$ but not of level $i-1$. This means that it uses at least once the operation $\sharp_i$. Given some $X \subseteq M$, $\langle X \rangle^{=i}$ is the set of values of expressions strictly of level $i$ over $X$.

Given natural numbers $\bar{n} = (n_1, \ldots, n_k)$ and a magnitude expression $f$, the $\bar{n}$**-unfolding of** $f$ is the word obtained from $f$ by substituting $n_i$ syntactically for each $\sharp_i$, and evaluating the resulting expressions (here $u^{n_i}$ means repeating $n_i$ times the word $u$). It transforms an expression over $X$ into a word in $X^*$.

**Lemma 12.** *Given a valid magnitude expression $f$ over some magnitude monoid $\mathcal{M}$ and some $\bar{n}$, then there is an $\bar{n}$-evaluation for the $\bar{n}$-unfolding of $f$ of height at most $|f|$ (the size of $f$), and value $\mathtt{val}(f)$.*

*Proof:* By induction on the structure of $f$. ∎

These magnitude expressions are used for decision properties. This is done in the following lemma.

**Lemma 13.** *The problem, given a recognizable $\bar{Q}^{\rm St}\bar{m}$-language $L$ to decide whether there exists a word $u \in L$ is decidable. The problem to decide if there exists a word $u \in L$ of strictly $i$-standard length is also decidable.*

*Proof:* Assume $L$ recognized by $\mathcal{M}, h, I$. Consider now the (computable) set $Z := \langle h(\mathbb{A}) \rangle^\sharp$ where $h(\mathbb{A})$ is the set of images of letters of the alphabet $\mathbb{A}$ under $h$. We claim that

$$Z \cap F \neq \emptyset \text{ if and only if } P(u) \text{ holds for some } u,$$

from which the decidability follows.

Indeed, assume that $Z \cap F$ is non-empty. This means that there exists a $\bar{Q}^{\rm St}\bar{m}$-magnitude expression $f$ of value $a \in Z \cap F$. Consider some $0 < n_1 < \cdots < n_k$ such that $n_i \geq_{\bar{Q}} \bar{\iota}$ (this is possible). Let $v$ be the $\bar{n}$-unfolding of $f$. According to Lemma 12 there exists an $\bar{n}$-computation for $v$ of value $\mathtt{val}(f) \in F$. Since $\bar{n} \geq_{\bar{Q}} \bar{\iota}$, this is also an $\bar{\iota}$-under-computation for $v$. Hence $\rho_{\mathcal{M}}(v) \geq a \in F$, which means $\rho_{\mathcal{M}}(v) \in F$. It follows that any word $u \in \tilde{h}^{-1}(v)$ (and there are some) is a witness that $L$ is non-empty.

Conversely, assume that $u$ is accepted. This means $a := \rho_{\mathcal{M}}(\tilde{h}(u)) \in F$. From the fundamental definition, this means that there is an $\bar{m}$-computation for $\tilde{h}(u)$ of value $a$ for some $\bar{m}$. By induction (of standard length), ($\star$) the value of a computation over a word $v \in X^*$ for some $X \subseteq M$ belongs to $\langle X \rangle^\sharp$: indeed, each binary node in a computation corresponds to a product, and each idempotent node corresponds to a stabilisation. Hence $a \in Z$. Thus $a$ is a witness that $Z \cap F$ is non-empty.

For treating the case of strictly $i$-standard words, one uses $Z := \langle h(\mathbb{A}) \rangle^{=\sharp_i}$, *i.e.*, we restrict our attention to computations for words of strictly $i$-standard length. The remaining of the proof is the same, but for the fact that one should check instead of $(\star)$ that the value of a computation over a word $v \in X^*$ of strictly $i$-standard length for some $X \subseteq M$ belongs to $\langle X \rangle^{=\sharp_i}$. This is again a simple induction (of standard length). ■

### *D. Projection*

The last, and important, closure property is the projection.

**Lemma 14.** *If $L$ is a recognizable $\bar{Q}^{\mathrm{St}}\bar{m}$-language over alphabet $\mathbb{A}$, and $h$ a mapping from $\mathbb{A}$ to $\mathbb{B}^*$, then $\tilde{h}(L)$ is a recognizable $\bar{Q}^{\mathrm{St}}\bar{m}$-language.*

This is obtained by a form or powerset construction.

### *E. And the logic?*

We are finally able to put all pieces together. Recall the definition of magnitude monadic logic. Given a $\bar{Q}\bar{m}$-monadic formula $\varphi$ over the signature of words, it defines naturally the $\bar{Q}^{\mathrm{St}}\bar{m}$-language containing words such that $u \models \bar{Q}^{\mathrm{St}}\bar{m}\,\varphi$. We say that the $\bar{Q}^{\mathrm{St}}\bar{m}$-language is **definable in $\bar{Q}^{\mathrm{St}}\bar{m}$-monadic logic**.

**Theorem 4.** *A $\bar{Q}^{\mathrm{St}}\bar{m}$-language of finite words is definable in $\bar{Q}^{\mathrm{St}}\bar{m}$-monadic logic if and only if is recognizable by a $\bar{Q}^{\mathrm{St}}\bar{m}$-magnitude monoid.*

This is the standard technique used, e.g, by Büchi: For proving the decidability of monadic logic, one is required to provide effective closure of a class of languages under union, intersection, complement and projection as well as provide sufficient constant languages in it. This is exactly what we have done so far for recognizable $\bar{Q}^{\mathrm{St}}\bar{m}$-languages with Lemmas 9, 10, 11 and 14. The only novelty needed here, is, thanks to Proposition 6, to use the equivalence of formulae of the form $\bar{Q}^{\mathrm{St}}\bar{m}\,\exists X\,\psi$ with $\exists X\,\bar{Q}^{\mathrm{St}}\bar{m}\,\psi$ (and similarly for universal quantifiers).

For the converse direction this requires proving that the “existence of an $\bar{n}$-under-computation of height at most $3|M|$ and value in $F$” is definable in $\bar{Q}\bar{m}$-monadic logic. Indeed, such a tree being of bounded depth, it can be guessed using monadic variables.

If we combine it with Lemma 13, we obtain decidability.

**Theorem 5.** *The emptiness of a $\bar{Q}^{\mathrm{St}}\bar{m}$-language definable in $\bar{Q}^{\mathrm{St}}\bar{m}$-monadic logic is decidable. Magnitude monadic logic over finite words is decidable.*

For the second statement, consider a formula of magnitude monadic logic $\bar{Q}\bar{m}\,\exists u\,\varphi$. By Lemma 5, $\bar{Q}\bar{m}\,\exists u\,\varphi$ holds if and only if $\bar{Q}^{\mathrm{St}}\bar{m}\,\exists u\,\varphi$ holds, which itself is equivalent to $\exists u\,\bar{Q}^{\mathrm{St}}\bar{m}\,\varphi$ using Proposition 6. Hence it is reduced to the emptiness of a $\bar{Q}^{\mathrm{St}}\bar{m}$-language definable in $\bar{Q}^{\mathrm{St}}\bar{m}$-monadic logic.

## References


[1] C. C. Elgot, “Decision problems of finite automata design and related arithmetics,” *Trans. Amer. Math. Soc.*, vol. 98, pp. 21–52, 1961.

[2] B. A. Trakhtenbrot, “Finite automata and monadic second order logic (russian),” *Siberian Math. J*, vol. 3, pp. 103–131, 1962.

[3] J. R. Büchi, “On a decision method in restricted second order arithmetic,” in *Proceedings of the International Congress on Logic, Methodology and Philosophy of Science*. Stanford Univ. Press, 1962, pp. 1–11.

[4] M. O. Rabin, “Decidability of second-order theories and automata on infinite trees,” *Trans. Amer. Math. soc.*, vol. 141, pp. 1–35, 1969.

[5] I. Walukiewicz, “Monadic second-order logic on tree-like structures,” *Theoretical Computer Science*, vol. 275, no. 1–2, pp. 311–346, 2002.

[6] K. Hashiguchi, “Limitedness theorem on finite automata with distance functions,” *J. Comput. Syst. Sci.*, vol. 24, no. 2, pp. 233–244, 1982.

[7] H. Leung, “An algebraic method for solving decision problems in finite automata theory,” Ph.D. dissertation, Pennsylvania State University, Department of Computer Science, 1987.

[8] I. Simon, “On semigroups of matrices over the tropical semiring,” *RAIRO ITA*, vol. 28, no. 3-4, pp. 277–294, 1994.

[9] D. Kirsten, “Distance desert automata and the star height problem,” *RAIRO*, vol. 3, no. 39, pp. 455–509, 2005.

[10] T. Colcombet and C. Löding, “The nesting-depth of disjunctive $\mu$-calculus for tree languages and the limitedness problem,” in *Computer science logic*, ser. Lecture Notes in Comput. Sci. Berlin: Springer, 2008, vol. 5213, pp. 416–430.

[11] K. Hashiguchi, “Relative star height, star height and finite automata with distance functions,” in *Formal Properties of Finite Automata and Applications*, 1988, pp. 74–88.

[12] A. Blumensath, M. Otto, and M. Weyer, “Boundedness of monadic second-order formulae over finite words,” in *36th ICALP*, ser. Lecture Notes in Computer Science. Springer, Jul. 2009, pp. 67–78.

[13] M. Bojańczyk and T. Colcombet, “Bounds in $\omega$-regularity,” in *LICS 06*, 2006, pp. 285–296.

[14] T. Colcombet, “The theory of stabilisation monoids and regular cost functions,” in *Automata, languages and programming. Part II*, ser. Lecture Notes in Comput. Sci. Berlin: Springer, 2009, vol. 5556, pp. 139–150.

[15] ——, “Regular cost functions, part I: logic and algebra over words,” *Log. Methods Comput. Sci.*, 2013, special issue of ICALP09.

[16] T. Colcombet and C. Löding, “Regular cost functions over finite trees,” in *LICS*, 2010, pp. 70–79.

[17] T. Colcombet and C. Löding, “The non-deterministic Mostowski hierarchy and distance-parity automata,” in *Automata, languages and programming. Part II*, ser. Lecture Notes in Comput. Sci. Berlin: Springer, 2008, vol. 5126, pp. 398–409.

[18] M. V. Boom, “Weak cost monadic logic over infinite trees,” in *MFCS*, 2011, pp. 580–591.

[19] D. Kuperberg and M. V. Boom, “Quasi-weak cost automata: A new variant of weakness,” in *FSTTCS*, 2011, pp. 66–77.

[20] Y. Péreire, “Théorie relative des ensembles internes.”

[21] S. Toruńczyk, “Languages of profinite words and the limitedness problem,” Ph.D. dissertation, Warsaw University, 2011.

[22] S. Toruńczyk, “Languages of profinite words and the limitedness problem,” in *ICALP (2)*, ser. Lecture Notes in Comput. Sci., vol. 7392, 2012, pp. 377–389.

[23] T. Wilke, “An Eilenberg theorem for $\infty$-languages,” in *Automata, Languages and Programming*, ser. Lecture Notes in Computer Science, vol. 510. Springer, 1991, pp. 588–599.

[24] O. Carton, T. Colcombet, and G. Puppis, “Regular languages of words over countable linear orderings,” in *ICALP (2)*, ser. Lecture Notes in Computer Science, L. Aceto, M. Henzinger, and J. Sgall, Eds., vol. 6756. Springer, 2011, pp. 125–136.

[25] T. Colcombet, “Green’s relations and their use in automata theory,” in *LATA*, ser. Lecture Notes in Computer Science, A. H. Dediu, S. Inenaga, and C. Martín-Vide, Eds., vol. 6638. Springer, 2011, pp. 1–21, invited lecture.

[26] G. Lallement, *Semigroups and Combinatorial Applications*. Wiley, 1979.

[27] J.-E. Pin, *Varieties of Formal Languages*. North Oxford Academic, London and Plenum, New York, 1986.

[28] P. A. Grillet, *Semigroups. An introduction to the structure theory.* Pure and Applied Mathematics, Marcel Dekker. 193. New York, NY: Marcel Dekker, Inc. ix, 398 p, 1995.

[29] I. Simon, “Factorization forests of finite height,” *Theoretical Computer Science*, vol. 72, pp. 65–94, 1990.

[30] ——, "A short proof of the factorization forest theorem," *Tree Automata and Languages*, pp. 433–438, 92.

[31] M. Kufleitner, "The height of factorization forests," in *MFCS*, vol. 5162, 2008, pp. 443–454.

[32] T. Colcombet, "Factorization forests for infinite words and applications to countable scattered linear orderings," *Theoret. Comput. Sci.*, vol. 411, no. 4-5, pp. 751–764, 2010.

## VII. Proof of the existence of computation

In this section of the appendix, we prove the existence of computations:

**Theorem 2** (existence of computations)**.** *For all words $u$ over a magnitude semigroup $\mathcal{S}$ and all $\bar{n}$, there exists an $\bar{n}$-computation for $u$ of height at most $3|S|$.*

This result generalizes Theorem 3.3 in [15] which holds only for stabilisation monoids. Though it requires to be redone, the proof is exactly the same (and the proof below is almost a copy-paste of the proof in [15]).

More precisely, we establish that for all words $u$ over a magnitude semigroup $\mathcal{S}$ and all $\bar{n}$, there exists an $n$-computation for $u$ of height at most $3|S|$. Hence, we are more precise that simply "of standard height". Remark that the convention in this context is to measure the height of a tree without counting the leaves. This result is a form of extension of the factorisation forest theorem due to Simon [29]:

**Theorem 6.** *[Simon [29], [30]] Define a* **Ramsey factorisation** *to be an $n$-computation in the pathological case all the stabilisation are the identity over idempotents.*
*For all non-empty words $u$ over a finite semigroup $\mathcal{S}$, there exists a Ramsey factorisation for $u$ of height[4] at most $3|\mathcal{S}|-1$.*

Some proofs of the factorisation forest theorem can be found in [31], [32], [25]. Our proof could follow similar lines as the above one. Instead of that, we try to reuse as much lemmas as possible from these constructions.

For proving Theorem 2, we will need one of Green's relations, namely the $\mathcal{J}$-relation (while there are five relations in general). Let us fix ourselves a semigroup $\mathcal{S}$. We denote by $\mathcal{S}^1$ the semigroup extended (if necessary) with a neutral element 1 (this transforms $\mathcal{S}$ into a monoid). Given two elements $a, b \in S$, $a \leq_{\mathcal{J}} b$ if $a = x \cdot b \cdot y$ for some $x, y \in S^1$. If $a \leq_{\mathcal{J}} b$ and $b \leq_{\mathcal{J}} b$, then $a\mathcal{J}b$. We write $a <_{\mathcal{J}} b$ to denote $a \leq_{\mathcal{J}} b$ and $b \not\leq_{\mathcal{J}} a$. The interested reader can see, *e.g.*, [25] for an introduction to the relations of Green (with a proof of the factorisation forest theorem), or monographs such as [26], [28] or [27] for deep presentations of this theory. Finally, let us call a **regular element** in a semigroup an element $a$ such that $a \cdot x \cdot a = a$ for some $x \in S^1$.

The next lemma gathers some classical results concerning finite semigroups.

**Lemma 15.** *Given a $\mathcal{J}$-class $J$ in a finite semigroup, the following facts are equivalent:*

- *$J$ contains an idempotent,*
- *$J$ contains a regular element,*
- *there exist $a, b \in J$ such that $a \cdot b \in J$,*
- *all elements in $J$ are regular,*
- *all elements in $J$ can be written as $e \cdot c$ for some idempotent $e \in J$,*
- *all elements in $J$ can be written as $c \cdot e$ for some idempotent $e \in J$.*

*Such $\mathcal{J}$-classes are called* **regular**.

[4]The exact bound of $3|S|-1$ is due to Kufleitner [31]. It is likely that the same bound could be achieved for Theorem 2. We prefer here a simpler proof with a bound of $3|S|$.

We will use the following technical lemma.

**Lemma 16.** *If $f = e \cdot x \cdot e$ for $e\mathcal{J}f$ two idempotents, then $e = f$.*

*Proof:* We use some standard results concerning finite semigroups. The interested reader can find the necessary material for instance in [27]. Let us just recall that the relations $\leq_{\mathcal{L}}$, $\leq_{\mathcal{R}}$ and $\mathcal{L}$ and $\mathcal{R}$ are the one-sided variants of $\leq_{\mathcal{J}}$ and $\mathcal{J}$ ($\mathcal{L}$ stands for "left" and $\mathcal{R}$ for "right"). Namely, $a \leq_{\mathcal{L}} b$ (*resp.* $a \leq_{\mathcal{R}} b$) holds if $a = x \cdot b$ for some $x \in S^1$ (*resp.* $a = b \cdot x$), and $\mathcal{L} = \leq_{\mathcal{L}} \cap \geq_{\mathcal{L}}$ (*resp.* $\mathcal{R} = \leq_{\mathcal{R}} \cap \geq_{\mathcal{R}}$). Finally, $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$.

The proof is very short. By definition $f \leq_{\mathcal{L}} e$ since $e \cdot x \cdot e = f$. Since by assumption $f\mathcal{J}e$, we obtain $f\mathcal{L}e$ (a classical result in *finite* semigroups). In a symmetric way $f\mathcal{R}e$. Thus $f\mathcal{H}e$. Since an $\mathcal{H}$-class contains at most one idempotent, $f = e$ (it is classical than any $\mathcal{H}$-class, when containing an idempotent, has a group structure; since groups contain exactly one idempotent element, this is the only one). ■

The next lemma shows that the stabilisation operation behaves in a very uniform way inside $\mathcal{J}$-classes (similar arguments can be found in the works of Leung, Simon and Kirsten).

**Lemma 17.** *If $e\mathcal{J}f$ are idempotents, then $e^{\sharp_i}\mathcal{J}f^{\sharp_i}$. Furthermore, if $e = x \cdot f \cdot y$ for some $x, y$, then $e^{\sharp_i} = x \cdot f^{\sharp_i} \cdot y$.*

*Proof:* For the second part, assume $e = x \cdot f \cdot y$ and $e\mathcal{J}f$. Let $f' = (f \cdot y \cdot e \cdot x \cdot f)$. We easily check $f' \cdot f' = f'$. Furthermore $f\mathcal{J}e = (x \cdot f \cdot y) \cdot e \cdot (x \cdot f \cdot y) \leq_{\mathcal{J}} f' \leq_{\mathcal{J}} f$. Hence $f\mathcal{J}f'$. It follows by Lemma 16 that $f' = f$. We now compute $e^{\sharp_i} = (x \cdot f \cdot f \cdot y)^{\sharp_i} = x \cdot f \cdot (f \cdot x \cdot y \cdot f)^{\sharp_i} \cdot f \cdot y = x \cdot f \cdot f^{\sharp_i} \cdot f \cdot y = x \cdot f^{\sharp_i} \cdot y$ (using consistency and $f = f'$).

This proves that $e\mathcal{J}f$ implies $e^{\sharp_i} \leq_{\mathcal{J}} f^{\sharp_i}$. Using symmetry, we obtain $e^{\sharp_i}\mathcal{J}f^{\sharp_i}$. ■

Hence, if $J$ is a regular $\mathcal{J}$-class, there exists a unique $\mathcal{J}$-class $J^{\sharp}$ which contains $e^{\sharp_i}$ for one/all idempotents $e \in J$. If $J = J^{\sharp_i}$, then $J$ is called ***i*-stable**, otherwise, it is called ***i*-unstable**. The following lemma shows that stabilisation of level $i$ is trivial over $i$-stable $\mathcal{J}$-classes.

**Lemma 18.** *If $J$ is an $i$-stable $\mathcal{J}$-class, then $e^{\sharp_i} = e$ for all idempotents $e \in J$.*

*Proof:* Indeed, we have $e^{\sharp_i} = e \cdot e^{\sharp_i} \cdot e$ and thus by Lemma 16, $e^{\sharp_i} = e$. ■

The situation is different for unstable $\mathcal{J}$-classes. In this case, the stabilisation always goes down in the $\mathcal{J}$-order.

**Lemma 19.** *If $J$ is an $i$-unstable $\mathcal{J}$-class, then $e^{\sharp_i} <_{\mathcal{J}} e$ for all idempotents $e \in J$.*

*Proof:* Since $e^{\sharp_i} = e \cdot e^{\sharp_i}$, it is always the case that $e^{\sharp_i} \leq_{\mathcal{J}} e$. Assuming $J$ is $i$-unstable means that $e\mathcal{J}e^{\sharp_i}$ does not hold, which in turn implies $e^{\sharp_i} <_{\mathcal{J}} e$. ■

We say that a word $u = a_1 \ldots a_n$ in $S^+$ is ***J*-smooth**, for $J$ a $\mathcal{J}$-class, if $u \in J^+$, and $\pi(u) \in J$. It is equivalent to say that $\pi(a_i a_{i+1} \cdots a_j) \in J$ for all $1 \leq i < j \leq n$. Indeed for all $1 \leq i < j \leq n$, $a_i \mathcal{J} \pi(a_1 \ldots a_n) \leq_{\mathcal{J}} \pi(a_i a_{i+1} \cdots a_j) \leq_{\mathcal{J}} a_i \in J$. Remark that, according to Lemma 15, if $J$ is irregular, $J$-smooth words have length at most 1. We will use the following lemma from [25] as a black-box. This is an instance of the factorisation forest theorem, but restricted to a single $\mathcal{J}$-class.

**Lemma 20.** *[Lemma 14 in [25]] Given a finite semigroup $\mathcal{S}$, one of its $\mathcal{J}$-classes $J$, and a $J$-smooth word $u$, there exists a Ramsey factorisation for $u$ of height at most $3|J|-1$.*

Remark that Ramsey factorisations and $n$-computations do only differ on what is allowed for a node of large degree, *i.e.*, above $n$. That is why our construction makes use of Lemma 20 to produce Ramsey factorisations, and then based on the presence of nodes of large degree, constructs a computation by gluing pieces of Ramsey factorisations together.

**Lemma 21.** *Let $J$ be a $\mathcal{J}$-class, $u$ be a $J$-smooth word, and $\bar{n}$ be as above. Then one of the two following items holds:*

*1) there exists an $\bar{n}$-computation for $u$ of value $\pi(u)$ and height at most $3|J|-1$, or;*

*2) there exists an $\bar{n}$-computation for some non-empty prefix $w$ of $u$ of value[5] $a <_{\mathcal{J}} J$ and height at most $3|J|$.*

*Proof:* Remark that if $J$ is irregular, then $u$ has length 1 by Lemma 15, and the result is straightforward.

The case of $J$ unstable remains. Let us say that a node in a factorisation is **$\bar{n}$-incorrect** if its degree $d$ lies in $[n_i, n_{i+1})$ for some $i$ for which $i$-unstable. Indeed, this is the only case where a Ramsey factorisation and a computation differ. Our goal is to “correct” the value of incorrect nodes. If there is a Ramsey factorisation for $u$ which has no incorrect node, then it can be seen as an $n$-computation, and once more the first conclusion of the lemma holds.

Otherwise, consider the least non-empty prefix $u'$ of $u$ for which there is a Ramsey factorisation of height at most $3|J|-1$ which contains an $\bar{n}$-incorrect node. Let $F$ be such a factorisation and $x$ be a big node in $F$ which is maximal for the descendant relation (there are no other big nodes below). Let $F'$ be the subtree of $F$ rooted in $x$. This decomposes $u'$ into $vv'v''$ where $v'$ is the factor of $u'$ for which $F'$ is a Ramsey factorisation. For this $v'$, it is easy to transform $F'$ into an $\bar{n}$-computation $T'$ for $v'$: just replace the label $e$ of the root of $F'$ by $e^{\sharp_i}$ where $i$ is the one from the definition of $\bar{n}$-incorrectness. Indeed, since there are no other $\bar{n}$-incorrect nodes in $F'$ than the root, the root is the only place which prevents $F'$ from being an $\bar{n}$-computation. Remark that from Lemma 19, the value of $F'$ is $<_{\mathcal{J}} J$.

If $v$ is empty, then $v'$ is a prefix of $u$, and $F'$ an $n$-computation for it. The second conclusion of the lemma holds.

Otherwise, by the minimality assumption and Lemma 20, there exists a Ramsey factorisation $T$ for $v$ of height at most $3|J|-1$ which contains no big node. Both $T$ and $T'$ being $\bar{n}$-computations of height at most $3|J|-1$, it is easy to combine them into an $\bar{n}$-computation of height at most $3|J|$ for $vv'$. This is an $\bar{n}$-computation for $vv'$, which inherits from $F'$ the property that its value is $<_{\mathcal{J}} J$. It proves that the second conclusion of the lemma holds. ∎

We are now ready to establish Theorem 2.

[5]A closer inspection would reveal that $a \in J^\sharp$. This extra information is useless for our purpose.

*Proof:* The proof is by induction on the size of a left-right-ideal $Z \subseteq S$, *i.e.*, $S^1 \cdot Z \cdot S^1 \subseteq Z$ (remark that a left-right-ideal is a union of $\mathcal{J}$-classes). We establish by induction on the size of $Z$ the following induction hypothesis:

*IH:* for all words $u \in Z^+ + Z^*S$ there exists an $\bar{n}$-computation of height at most $3|Z|$ for $u$.

Of course, for $Z = S$, this proves Theorem 2.

The base case is when $Z$ is empty, then $u$ has length 1, and a single node tree establish the first conclusion of the induction hypothesis (recall that the convention is that the leaves do not count in the height, and as a consequence a single node tree has height 0).

Otherwise, assume $Z$ non-empty. There exists a maximal $\mathcal{J}$-class $J$ (maximal for $\leq_{\mathcal{J}}$) included in $Z$. From the maximality assumption, we can check that $Z' = Z \setminus J$ is again a left-right-ideal. Remark also that since $Z$ is a left-right-ideal, it is downward closed for $\leq_{\mathcal{J}}$. This means in particular that every element $a$ such that $a <_{\mathcal{J}} J$ belongs to $Z'$.

*Claim:* We claim ($\star$) that for all words $u \in Z^+ + Z^*S$,

1) either there exists an $\bar{n}$-computation of height $3|J|$ for $u$, or;
2) there exists an $\bar{n}$-computation of height at most $3|J|$ for some non-empty prefix of $u$ of value in $Z'$.

Let $w$ be the longest $J$-smooth prefix of $u$. If there exists no such non-empty prefix, this means that the first letter $a$ of $u$ does not belong to $J$. Two subcases can happen. If $u$ has length 1, this means that $u = a$, and thus $a$ is an $\bar{n}$-computation witnessing the first conclusion of ($\star$). Otherwise $u$ has length at least 2, and thus $a$ belongs to $Z$. Since furthermore it does not belong to $J$, it belongs to $Z'$. In this case, $a$ is an $\bar{n}$-computation witnessing the second conclusion of ($\star$).

Otherwise, according to Lemma 21 applied to $w$, two situations can occur. The first case is when there is an $\bar{n}$-computation $T$ for $w$ of value $\pi(w)$ and height at most $3|J|-1$. There are several sub-cases. If $u = w$, of course, the $\bar{n}$-computation $T$ is a witness that the first conclusion of ($\star$) holds. Otherwise, there is a letter $a$ such that $wa$ is a prefix of $u$. If $wa = u$, then $\pi(wa)[T, a]$ is an $\bar{n}$-computation for $wa$ of height at most $3|J|$, witnessing that the first conclusion of ($\star$) holds. Otherwise, $a$ has to belong to $Z$ (because all letters of $u$ have to belong to $Z$ except possibly the last one). But, by maximality of $w$ as a $J$-smooth prefix, either $a \in Z'$, or $\pi(wa) \in Z'$. Since $Z'$ is a left-right-ideal, $a \in Z'$ implies $\pi(wa) \in Z'$. Then, $\pi(wa)[T, a]$ is an $\bar{n}$-computation for $wa$ of height at most $3|J|$ and value $\pi(wa) \in Z'$. This time, the second conclusion of ($\star$) holds.

The second case according to Lemma 21 is when there exists a prefix $v$ of $w$ for which there is an $\bar{n}$-computation of height at most $3|J|$ of value $<_{\mathcal{J}} J$. In this case, $v$ is also a prefix of $u$, and the value of this computation is in $Z'$. Once more the second conclusion of ($\star$) holds. This concludes the proof of Claim ($\star$).

As long as the second conclusion of the claim ($\star$) applied on the word $u$ holds, this decomposes $u$ into $v_1u'$, and we can proceed with $u'$. In the end, we obtain that all words $u \in Z^+ + Z^*S$ can be decomposed into $u_1 \dots u_k$ such that there exist $\bar{n}$-computations $T_1,\dots,T_k$ of height at most $3|J|$ for $u_1,\dots,u_k$ respectively, and such that the values of $T_1,\dots,T_{k-1}$ all belong to $Z'$ (but not necessarily the value of $T_k$). Let $a_1,\dots,a_k$ be the values of $T_1,\dots,T_k$ respectively. The word $a_1\dots a_k$ belongs to $Z'^{+}+Z'^{*}S$. Let us apply the induction hypothesis to the word $a_1\dots a_k$. We obtain an $\bar{n}$-computation $T$ for $a_1\dots a_k$ of height at most $3|Z'|$. By simply substituting $T_1,\dots,T_k$ to the leaves of $T$, we obtain an $\bar{n}$-computation for $u$ of height at most $3|J|+3|Z'|=3|Z|$. (Remark once more here that the convention is to not count the leaves in the height. Hence the height after a substitution is bounded by the sum of the heights.) ■

## VIII. Unicity of computations

Let us recall the unicity theorem we want to prove.

**Theorem 3** (unicity)**.** *For all words $u\in S^+$, all $\bar{\iota}$-under-computations for $u$ of value $a$ and and all $\bar{\iota}$-over-computations for $u$ of value $b$,*

$$a\leq b\ .$$

One says that a word $u\in S^+$ $\bar{m}$**-under-evaluates to** $b$ if there exists an $\bar{m}$-under-computation for $u$ of value $b$. One will also say that $\varepsilon$ under-evaluates to $b$ for all $b\leq 1$. Of course, we can also say that $u$ $\bar{\iota}$-under-evaluates to $b$. The same definitions apply for $\bar{m}$-over-computations.

Given a sequence of words $u_1,\dots,u_\ell$, one says that $u_1,\dots,u_\ell$ $\bar{m}$-under-evaluates to $b_1,\dots,b_\ell$ if $u_i$ $\bar{m}$-under-evaluates to $b_i$ for all $i=1\dots\ell$.

Let us stress right now the following tricky point, which may seem obvious, but requires care.

**Lemma 22.** *Let $u_1,\dots,u_n$ be words. If there exists $a_i$ such that $u_i$ $\bar{\iota}$-under-evaluates to $a_i$ for all $i=1\dots n$, then there exists $a_1,\dots,a_n$ such that $u_1,\dots,u_n$ $\bar{\iota}$-under-evaluate to $a_1,\dots,a_n$.*

*Proof:* Indeed, since the definition of an $\bar{\iota}$-under-computation is external, there is no reason $a_1,\dots,a_n$ exists as an object (a sequence). However, using the Proposition 6 with the fact that being an $\bar{m}$-under-computation is a $\bar{Q}\bar{m}$-property, the property "for all $i$ there exists $a_i$ such that $u_i$ $\bar{\iota}$-under-evaluates to $a_i$" is equivalent to "for all $i$ there exists $a_i$ such that $u_i$ $\bar{m}$-under-evaluates to $a_i$" for some $\bar{m}\leq_{\bar{Q}}\bar{\iota}$. Hence, once $\bar{m}$ fixed, one can construct $a_1,\dots,a_n$ such that $u_1,\dots,u_n$ $\bar{m}$-under-evaluate to $a_1,\dots,a_n$. We now have the expected conclusion that $u_1,\dots,u_n$ $\bar{\iota}$-under-evaluate to $a_1,\dots,a_n$. ■

This witnesses something one should be very sensitive to when performing proofs in non-standard analysis. Whenever some external property of existence of elements (here the $a_i$'s) is proved for an uncontrolled number of items (here $n$ can be arbitrary large, and in particular non-standard), then it is not possible a priori to aggregate them (here build $a_1,\dots,a_n$) unless some extra argument is used. Being a $\bar{Q}\bar{m}$-formula gives such an argument.

We shall not mention anymore Lemma 22, and use it implicitly whenever needed.

**Lemma 23.** *Let $u=a_1\dots a_\ell\in S^*$ of standard length then $u$ $\bar{\iota}$-under-evaluates to $b$ if and only if $b\leq\pi(a_1\dots a_\ell)$*

Consider a computation (under or over) $T$ over a word $u$ that is not reduced to its leaf. We will say it **factorizes** $u$ **into** $u_1,\dots,u_\ell$ **with values** $b_1,\dots,b_m$ if $u=u_1\dots u_\ell$, the children of the root of $T$ are $T_1,\dots,T_\ell$ when read from left to right, $T_i$ is a computation (under or over) for $u_i$ for all $i=1\dots\ell$, and $b_i$ is the value of $T_i$ for all $i=1\dots\ell$.

**Lemma 24.** *Let $u_1u_2\in S^*$, then if $u_1u_2$ $\bar{\iota}$ under-evaluates to $c$, there exists $b_1b_2$ such that $u_1,u_2$ $\bar{\iota}$ under-evaluates to $b_1,b_2$ with $c\leq b_1\cdot b_2$.*

*Proof:* Let $u=u_1u_2$. Remark first that if $u_1$ or $u_2$ is $\varepsilon$, this is obvious. Otherwise, the proof is by induction on the height of the $\bar{\iota}$-under-computation $T$ for $u$. This is possible since $T$ is of standard height, according to Lemma 2. Note first that $T$ cannot be reduced to a leaf since $u$ has length at least 2. Otherwise $T$ decomposes $u$ into $v_1, \ldots, v_m$ with values $c_1, \ldots, c_m$. There are several cases.

- It is a binary node, *i.e.*, $m = 2$. Then two symmetric subcases can happen: (a) $u_1 = v_1 w$ and $v_2 = w u_2$, or (b) $v_1 = u_1 w$ and $u_2 = w v_2$, for some $w \in S^*$. Let us treat the case (a). Since $v_2$ under-evaluates to $c_2$ and can be decomposed into $w, u_2$, we can apply the induction hypothesis. We get that $w, u_2$ $\bar{\iota}$-under-evaluate to $b, b_2$ such that $c_2 \leq b \cdot b_2$. Since furthermore $v_1$ $\bar{\iota}$-under-evaluates to $c_1$, this means $u_1, u_2$ $\bar{\iota}$-evaluate to $c_1 \cdot c, b_2$, and $(c_1 \cdot b) \cdot b_2 = c_1 \cdot (b \cdot b_2) \geq c_1 \cdot c_2 \geq c$.
- It is a stabilisation node of level $i$, *i.e.*, $m$ is strictly $i$-standard and $c_1 = \cdots = c_m = e$ which is an $i-1$-stable idempotent. In this case, there exists words $w_1, w_2$ and some $n$ among $1, \ldots, m$ such that

$$u_1 = v_1 \ldots v_{n-1} w_1 \,, \quad v_n = w_1 w_2 \quad \text{and } u_2 = w_2 v_{n_1} \ldots v_m \,.$$

One can apply the induction hypothesis over $v_n$ wich is decomposed into $w_1, w_2$. We get that $w_1, w_2$ $\bar{\iota}$-under-evaluate to $d_1, d_2$ such that $e = c_n \leq d_1 \cdot d_2$. Furthermore, since $m$ is strictly $i$-standard, both $n$ and $m-n$ are $i$-standard, and either $n$ or $m-n$ is strictly $i$-standard. Let us assume without loss of generality that $n$ is strictly $i$-standard.
Since $n$ is strictly $i$-standard (and thus $n-1$ too), $v_1 \ldots v_{n-1}$ $\bar{\iota}$-under-evaluates to $e^{\sharp_i}$. Hence $u_1 = v_1 \ldots v_m w_1$ $\bar{\iota}$-under-evaluates to $b_1 := e^{\sharp_i} \cdot d_1$. If $m-n$ is also strictly $i$-standard, then similarly $u_2$ $\bar{\iota}$-under-evaluates to $b_2 := d_2 \cdot e^{\sharp_i}$. We get that $u_1, u_2$ $\bar{\iota}$-under-evalutate to $b_1, b_2$ and $b_1 \cdot b_2 \geq e^{\sharp_i} \cdot d_1 \cdot d_2 \cdot e^{sharp_i} = e^{\sharp_i} \leq c$. Otherwise $m-n$ is $i-1$-standard. This means, since $e$ is $i-1$-stable, that $u_2$ $\bar{\iota}$-under-evaluates to $b_2 := d_2 \cdot e$. We get that $u_1, u_2$ $\bar{\iota}$-under-evalutate to $b_1, b_2$ and $b_1 \cdot b_2 \geq e^{\sharp_i} \cdot d_1 \cdot d_2 \cdot e = e^{\sharp_i} \geq c$.

■

From this lemma, we can generalise to decompositions of standard length as follows.

**Lemma 25.** *Given a word $u \in S^*$ decomposed into $u_1, \ldots, u_\ell \in S^*$ with $\ell$ standard, that $\bar{\iota}$-under-evaluates to $c$, then $u_1, \ldots, u_\ell$ $\bar{\iota}$-under-evaluate to $b_1, \ldots, b_\ell$ such that $b_1 \ldots b_\ell$ $\bar{\iota}$-under-evaluates to $c$.*

*Proof:* This is a simple induction on $\ell$. According to Lemma 2, and since $\ell$ is standard, this is correct. For $\ell = 0$ (this means $u = \varepsilon$), the result is obvious. Otherwise, applying Lemma 24, $u_1, u_2 u_3 \ldots u_\ell$ $\bar{\iota}$-under-evaluate to $b_1, d$ such that $b_1 \cdot d \geq c$. We now apply the induction hypothesis on $u_2 \ldots u_\ell$, and obtain that $u_2, \ldots, u_\ell$ $\bar{\iota}$-under-evaluate to $b_2, \ldots, b_\ell$ such that $b_2 \cdots b_\ell$ $\bar{\iota}$-under-evaluate to $d$. Thus $u_1, \ldots, u_\ell$ $\bar{\iota}$-under-evaluate to $b_1, \ldots, b_\ell$ and $b_1 \ldots b_\ell$ $\bar{\iota}$-under-evaluate to $b_1 \cdot d \geq c$. ■

The key lemma is the following.

**Lemma 26.** *Given a word $u \in S^*$ decomposed into $u_1, \ldots, u_\ell \in S^*$, that $\bar{\iota}$-under-evaluates to $c$, then $u_1, \ldots, u_\ell$ $\bar{\iota}$-under-evaluate to $b_1, \ldots, b_\ell$ such that $b_1 \ldots b_\ell$ $\bar{\iota}$-under-evaluates to $c$.*

Before proving this statement, let us show why it is sufficient for establishing Theorem 3.

*Proof:* Let $T$ be a $\bar{\iota}$-over-computation for some $u$ of value $b$. We prove, by induction (of standard length) on the height of $T$ that if $u$ $\bar{\iota}$-under-evalutates to $a$, $a \leq b$;

Of course, for $T$ reduced to a single leaf the result holds.

Otherwise $T$ has a binary root, and factorizes $u$ into $u_1, u2$ with values $b_1, b_2$. This means that $b \geq b_1 \cdot b_2$. Using 24, $u_1, u_2$ $\bar{\iota}$-under-evaluates to $a_1, a_2$ such that $a \leq a_1 \cdot a_2$. By inductin hypothesis, $a_1 \leq b_1$ and $a_2 \leq b_2$. Hence $a \leq a_1 \cdot a_2 \leq b_1 \cdot b_2 \leq b$.

The last case is when $T$ has an idempotent node as root. In this case $T$ factorizes $u$ into $u_1, \ldots, u_n$ with values $b_1 = b_2 = \cdots = b_n = e$ idempotents. We have $b \geq e^{\sharp_i}$ for $i$ such that $n$ is strictly $i$-standard. Applying Lemma 26, $u_1, \ldots, u_n$ $\bar{\iota}$-under-evaluate to $a_1, \ldots, a_n$ such that $a_1 \ldots a_n$ $\bar{\iota}$-under evaluate to $a$. Using the induction hypothesis, $a_i \leq e$ for all $i = 1 \ldots n$. We need now prove that $a \leq e^{\sharp_i}$. This is done by a simple induction (of standard length) on the height of an $\bar{\iota}$-under-computation. Hence $a \leq b$. ■

**Lemma 27.** *If $x_1, y_1, x_2, y_2 \ldots, x_m, y_m \in M$ ($m \geq 1$) and $e$ is an idempotent such that $x_h \cdot y_h \leq e$ for all $h = 1 \ldots m$, then $(y_1 \cdot x_2)(y_2 \cdot x_3) \cdots (y_{m-1} \cdot x_m)$ $\bar{\iota}$-under-evaluates to $z$ such that:*

$$e^{\sharp_g} \leq x_1 \cdot z \cdot y_m \,,$$

*where $g$ is such that $m$ is $g$-standard, $e^{\sharp_{g'}} \leq x_h \cdot y_h$ for some $g' \leq g$ for all $h$, and, either $m$ is strictly $i$-standard, or $e^{\sharp g} \leq x_h \cdot y_h$ for some $h$.*

*Proof:* The principle of the proof is to take some $\bar{\iota}$-over-computation for $(y_1 \cdot x_2)(y_2 \cdot x_3) \cdots (y_{m-1} \cdot x_m)$ of value $z$, and prove the statement for this $z$. This is done by induction and case distinction. The result then follows from Proposition 8 ■

Hence, let us concentrate now on the proof of Lemma 26. The proof is by induction on the height of the $\bar{\iota}$-under-evaluation $T$ for $u$ of value $a$. We slightly change the induction hypothesis for more ease. Consider a word $u$ that $\bar{\iota}$-under-evaluates to $a$ and is decomposed into $u_1, \ldots, u_n$. Our goal is to prove that $u_1, \ldots u_n$ $\bar{\iota}$-under-evaluate to $a_1, \ldots, a_n$ such that $a_2 \ldots a_{n-1}$ $\bar{\iota}$-under-evaluates to $c$ such that $a \leq a_1 \cdot c \cdot a_n$.

Assume $T$ factorizes $u$ into $v_1, \ldots, v_\ell$ with values $b_1, \ldots, b_\ell$. The proof proceeds with an analysis of the different situation of overlap that can happen between the $v_i$'s and the $u_j$'s.

**Case of a leaf:** $\ell = 1$. The result is obvious.

**Case of a binary node:** $\ell = 2$. Let $c_1, c_2$ be the values of the two children of the root. This means that the following decomposition holds:

$$v_1 = u_1 \ldots u_{m-1} w_1 \,, \qquad u_m = w_1 w_2 \,,$$
$$\text{and} v_2 = w_2 u_{m+1} \ldots u_n \,.$$

By induction hypothesis this means that $u_1, \ldots, u_{m-1}, w_1, w_2, u_{m+1}, \ldots, u_n$ $\bar{\iota}$-under-evaluates to $a_1, \ldots, a_{m-1}, b_1, b_2, a_{m+1}, \ldots, a_n$ such that

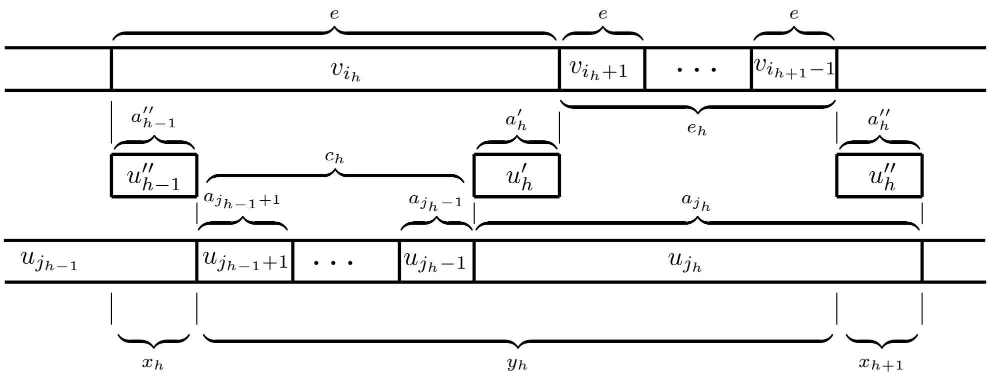



Fig. 1. Decomposition in the case of an idempotent node

$a_2 \dots a_{m-1}, a_{m+1} \dots a_{n-1}$ $\bar{\iota}$-under-evaluate to $d_1, d_2$. such that $c_1 \leq a_1 \cdot d_1 \cdot b_1$ and $c_2 \leq b_2 \cdot d_2 \cdot a_n$. It follows that $u_m$ $\bar{\iota}$-under-evaluates to $a_m := b_1 \cdot b_2$, and that $a_2 \cdots a_{n-1}$ $\bar{\iota}$-under-evaluates to $d := d_1 \cdot a_m \cdot d_2$. Furthermore we have $a_1 \cdot d \cdot a_n = a_1 \cdot d_1 \cdot b_1 \cdot b_2 \cdot d_2 \cdot a_n = c_1 \cdot c_2 \geq c$.

**Case of an idempotent node:** All the children of the root are labeled $e$, ***i.e.***, $v_1, \dots, v_\ell$ $\bar{\iota}$-under-evaluate to $e, \dots, e$, and the degree is $\ell$. We aim at proving that

- $u_1, \dots, u_n$ $\bar{\iota}$-under-evaluate to some $a_1, \dots, a_n$,
- $a_2 \dots a_{n-1}$ $\bar{\iota}$-under-evaluate to some $c$,
- and $a_1 \cdot c \cdot a_n \leq e^{\sharp_g}$ where $g$ is such that $n$ is strictly $g$-standard.

We rely on a suitable decomposition of the words: there exist $0 = i_0 < i_1 < \cdots < i_m < i_{m+1} = \ell + 1$ and $1 = j_0 < \cdots < j_m = n$, as well as words $\varepsilon = u'_0, u''_0, u'_1, u''_1, \dots, u'_m, u''_m = \varepsilon$ such that

$$v_{i_h} = u''_{h-1}\, u_{j_{h-1}+1} \dots u_{j_h-1}\, u'_h \quad \text{for all } h = 1 \dots m, \tag{$\star$}$$

$$\text{and} \quad u_{j_h} = u'_h\, v_{i_h+1} \dots v_{i_{h+1}-1}\, u''_h \quad \text{for all } h = 0 \dots m. \tag{$\star\star$}$$

The best is to present it through a drawing, as in Figure 1. It is annotated with all the variables that will be used during the proof. The two main rows represent the two possible decompositions of the word into $v_i$'s and $u_j$'s.

Such a decomposition is not unique. It is sufficient to guarantee that each separation between some $u_s$ and some $u_{s+1}$ fall in some $v_{i_h}$, and that each $v_{i_h}$ contains such a separation.

We can apply the induction hypothesis on each equation ($\star\star$). Hence, it follows that $u''_{h-1}, u_{j_{h-1}+1}, \dots, u_{j_h-1}, u'_h$ $n$-under-evaluate to $a''_{h-1}, a_{j_{h-1}+1}, \dots, a_{j_h-1}, a'_h$ and $a_{j_{h-1}+1} \dots a_{j_h-1}$ $\bar{\iota}$-under-evaluates to $c_h$, such that $a''_{h-1} \cdot c_h \cdot a'_h \leq e$. Set furthermore $a'_0 = a''_m = 1$. We get that $u'_h, u''_h$ $n$-valuate to $a'_h, a''_h$ for all $h = 0 \dots m$. Define furthermore for all $h = 0 \dots m$, $e_h$ as

$$e_h = \begin{cases} 1 & \text{if } (i_{h+1} - i_h - 1) = 0 \\ e^{\sharp_r} & \text{if } 0 < (i_{h+1} - i_h - 1) \text{ is strictly } r\text{-standard.} \end{cases}$$

Since each $v_h$ $\bar{\iota}$-under-evaluate to $e$, each $v_h$ also $\bar{\iota}$-under-evaluates to $e$. Now $e_h$ has been chosen such that $v_{i_h+1} \dots v_{i_{h+1}-1}$ $\bar{\iota}$-under-evaluates to $e_h$. Thus from ($\star\star$), $u_{j_h}$ $\bar{\iota}$-under-evaluates to $a_{j_h}$ for all $h = 0 \dots m$ that we define as $a_{j_h} = a'_h \cdot e_h \cdot a''_h$. At this point, we have that

**C1** $u_1, \dots, u_n$ $\bar{\iota}$-under-evaluate to $a_1, \dots, a_n$.

To head toward the conclusion, we will use Lemma 27. Thus, let us set $x_h$ to be $a''_{h-1}$ and $y_h$ to be $c_h \cdot a'_h \cdot e_h$ for all $h = 1 \dots m$. We have

$$x_h \cdot y_h = (a''_{h-1} \cdot c_h \cdot a'_h) \cdot e_h = e \cdot e_h. \tag{$\dagger$}$$

It is clear that since $\ell$ is $g$-standard, the same holds for $m$. Furthermore, $e_h = e^{\sharp_r}$ for some $r \leq g$, for all $h$. Furthermore, since $\ell$ is strictly $g$ standard, then either $m$ is strictly $g$-standard, or some $e_h$ equals $e^{\sharp_g}$.

We can apply Lemma 27 to

$$x_1, y_1, x_2, \dots, x_m, y_m$$

and obtain that $(y_1 \cdot x_2) \dots (y_{m-1} \cdot x_m)$ $\bar{\iota}$-under-evaluates to some $z$ subject to the conclusion of the lemma.

Let us now establish the following claims C2 and C3.

**C2** $a_2 \dots a_{n-11}$ $\bar{\iota}$-under-evaluates to $(z \cdot c_m)$.

Indeed, for all $h = 1 \dots m$, $c_h$ is chosen such that $a_{j_{h-1}+1} \dots a_{j_h-1}$ $\bar{\iota}$-under-evaluates to $c_h$, thus $a_{j_{h-1}+1} \dots a_{j_h}$ $\bar{\iota}$-under-evaluates to:

$$c_h \cdot a_{j_h} = c_h \cdot a'_h \cdot e_h \cdot a''_h = y_h \cdot x_{h+1} \ ,$$

by just unfolding the definitions. Since furthermore $(y_1 \cdot x_2) \dots (y_{m-1} \cdot x_m)$ $\bar{\iota}$-under-evaluates to $z$, it follows that $a_2 \dots a_{j_{m-1}}$ $\bar{\iota}$-under-evaluates to $z$. Furthermore, by choice of $c_m$, $a_{j_{m-1}+1} \dots a_{n-1}$ $\bar{\iota}$-under-evaluates to $c_m$. Thus $a_2 \dots a_{n-1}$ $\bar{\iota}$-under-evaluates to $(z \cdot c_m)$ as claimed.

**C3** $e^{\sharp_g} \leq a_1 \cdot (z \cdot c_m) \cdot a_k$ from Lemma 27.

Gathering the claims C1, C2, C3, we get that $u_1, \dots, u_k$ $n$-evaluate to $a_1, \dots, a_k$, that $a_2 \dots a_{k-1}$ $\bar{\iota}$-under-evaluates to $c = z \cdot c_m$ and that $a_1 \cdot c \cdot a_k \leq e$. This is exactly the induction hypothesis for the idempotent node case.

## IX. PROJECTION

We shall prove the following lemma:

**Lemma 14.** *If $L$ is a recognizable $\bar{Q}^{\mathrm{St}}\bar{m}$-language over alphabet $\mathbb{A}$, and $h$ a mapping from $\mathbb{A}$ to $\mathbb{B}^*$, then $\tilde{h}(L)$ is a recognizable $\bar{Q}^{\mathrm{St}}\bar{m}$-language.*

This is a powerset construction; more precisely a construction of ideals. Let $\mathcal{M}, h, F$ be recognizing some $\bar{Q}^{\mathrm{St}}\bar{m}$-language $L \subseteq \mathbb{A}^*$. Let also $z$ be a mapping from $\mathbb{A}$ to $\mathbb{B}$.

An **ideal** is a subset of $M$ which is downward closed for $\leq$. The set of ideals of $M$ is simply denoted $\downarrow(M)$. Given some $A \subseteq M$, $A\downarrow$ is the least ideal that contains $A$, namely the downward closure of $A$.

We construct a new $\bar{Q}\bar{m}$-magnitude monoid $\mathcal{M}_\downarrow$ that has as set of elements $\downarrow(M)$, as order $\subseteq$, and such that

$$A \cdot B = \{a \cdot b \ : \ a \in A,\ b \in B\}\downarrow ,$$

and for all $i = 1 \dots k$ and all idempotent $E$,

$$E^{\sharp_i} = \{\rho_{\mathcal{M}}(a_1 \dots a_n) \ : \ n \text{ is strictly } i\text{-standard}, \text{ and } a_\ell \in E \text{ for all } \ell = 1 \dots n\}\downarrow .$$

Remark that $E^{\sharp_i}$ can be effectively computed thanks to Lemma 13.

One shall first prove:

**Lemma 28.** *$\mathcal{M}_\downarrow$ is a $\bar{Q}\bar{m}$-magnitude monoid.*

*Proof:* By case analysis. In fact, following [15] this can be derived from the subsequent results. ∎

The interesting part is to prove the correctness of this construction.

We will use the following notation. $U, V, \dots$ implicitly range over words over the alphabet $\downarrow(M)$, and $u, v, \dots$ over words over $M$. We also denote by $u \in U$ the componentwise membership, *i.e.*, that $|u| = |U|$ and for all $i = 1 \dots |u|$, $u_i = U_i$ (where $u_i$ and $U_i$ denote the respective $i$-th letters of $u$ and $U$ respectively).

**Lemma 29.** *For all $U$ that $\bar{\iota}$-under-evaluates to $A$ and all $a \in A$, there exists a word $u \in U$ that $\bar{\iota}$-under-evaluates to $a$.*

*Proof:* The proof is by induction on the height of the $\bar{\iota}$-under-computation $T$ for $U$ of value $A$.

If $T$ is a leaf, this is straightforward.

If $T$ has a binary root, of children $T', T''$ of values $A', A''$. Then $a \in A$ means that there exists $a' \in A$ and $a'' \in A''$ such that $a \leq a' \cdot a''$. Let us apply the induction hypothesis on $T', a'$ and $T'', a''$. We obtain that $U = U'U''$ such that there exists $u' \in U'$ that $\bar{\iota}$-under-evaluate to $a'$ and $u'' \in U''$ that $\bar{\iota}$-under-evaluates to $a''$. Hence $u'u''$ $\bar{\iota}$-under-evaluates to $a \leq a' \cdot a''$.

If the root of $T$ is an idempotent node, of children $T_1, \dots, T_n$ of values $A_1 = \dots = A_n = E$ idempotent, and decomposing $U$ into $U_1 \dots U_n$. Assume $n$ strictly $\ell$-standard. This means that $A \subseteq E^{\sharp_\ell}$. Hence, by definition of $E^{\sharp_\ell}$, $a \in A$ is the value of a word $a_1 \dots a_n$ with $a_1, \dots, a_n \in E$. One can apply the induction hypothesis for all $T_i, a_i$ yielding some word $u_i \in U_i$ that $\bar{\iota}$-under-evaluates to $a_i$ for all $i = 1 \dots n$. By plugging[6] on top of it the $\bar{\iota}$-under-computation witnessing that $a_1 \dots a_n$ $\bar{\iota}$-under-evaluates to $a$, we obtain that $u_1 u_2 \dots u_n \in U$ $\bar{\iota}$-under-evaluates to $a$.

However, be careful, hidden in the last argument is the use of the $\bar{Q}\bar{m}$-nature of $\bar{m}$-under-computations. Indeed, in general, this is not because we can, separately, choose $u_i$ for a all $i$, that we can aggregate them into a single word $u = u_1 \dots u_n$. This is because being an $\bar{\iota}$-under-computation is an external formula, and thus no set in general can constructed from this definition. In the present case, from the fact that there exists $u_i$ for all $i$ that $\bar{\iota}$-under-evaluates to $a_i$, you can deduce thanks to Proposition 6 that there exists $\bar{m} \leq \bar{\iota}$ such that for all $i$ there exists $u_i$ that $\bar{m}$-under-evaluates to $a_i$. These can be aggregated since the notion of $\bar{m}$-under-evaluation is internal. Using again Proposition 6 the result is recovered. ∎

**Lemma 30.** *For all $U$ that $\bar{\iota}$-over-evaluates to $A$, and all $u \in U$, $u$ $\bar{\iota}$-over-evaluates to some $a \in A$.*

*Proof:* The proof is by induction on the height of the $\bar{\iota}$-over-computation $T$ for $U$ of value $A$.

If $T$ is restricted to its root, it is once more obvious.

If $T$ has a binary root of children $T_1, T_2$ of values $A_1, A_2$, corresponding to $U = U_1U_2$ and $u = u_1u_2$. By induction hypothesis, $u_1$ $\bar{\iota}$-over-evaluates to some $a_1 \in A_1$ and $u_2$ $\bar{\iota}$-over-evaluates to some $a_2 \in A_2$. Since $A \supseteq A_1 \cdot A_2$, this mean $a = a_1 s \cdot a_2 \in A$. Furthermore $u = u_1u_2$ $\bar{\iota}$-over-evaluates to $a$.

If the root of $T$ is an idempotent node, of children $T_1, \dots, T_n$ of values $A_1 = \dots = A_n = E$ idempotent, and decomposing $U$ into $U_1 \dots U_n$ and $u$ into $u_1 \dots u_n$. Assume $n$ strictly $\ell$-standard. By induction hypotheses, $u_i$ $\bar{\iota}$-over-evaluates to $a_i \in A_i$ for all $i = 1 \dots n$. Now[7] $a_1 \dots a_n$ is a word over $E$ and strictly $\ell$-standard length $\ell$. Since $A \supseteq E^{\sharp_i}$ and by definition of $E^{\sharp_i}$ there exists an $\bar{\iota}$-over-computation over $a_1 \dots a_n$ and value $a \in A$. Plugging these over evaluations together, we obtain that $u_1 \dots u_n$ $\bar{\iota}$-over-evaluates to $a \in A$. ∎

It is now easy to conclude the proof of Lemma 14.

*Proof:* Assume $L$ recognized by $\mathcal{M}, h, F$, and $z$ from $\mathbb{A}$ to $\mathbb{B}$ given. Let $h'$ be mapping any $b \in \mathbb{B}$ to $(h(z^{-1}(b)))\downarrow$. Let finally $F' = \{I \in \downarrow \ : \ I \cap F \neq \emptyset\}$. We claim that $v \in \mathbb{B}^*$ is accepted by $\mathcal{M}_\downarrow, h', F'$ if and only if $v \in \tilde{z}(L)$.

Assume first that $v$ is accepted by $\mathcal{M}_\downarrow, h', F'$. This means that $U = \tilde{h}'(v)$ $\bar{\iota}$-under-evaluates to $A \in F'$. Hence there is some $a \in A \cap F$ according to the definition of $F'$. By Lemma 29 this mean that some $u \in U$ $\bar{\iota}$-under-evalutes to $a \in F$. Finally, by definition of $h'$, there exists some $v \in \mathbb{B}^*$ and some $w \in \mathbb{A}^*$ such that $\tilde{z}(w) = v$ and $\tilde{h}(w) = u$. This means that $w$ is accepted by $\mathcal{M}, h, F$. Thus $v \in \tilde{Z}(L)$.

Assume now that $v$ is not accepted by $\mathcal{M}_\downarrow, h', F'$. This means that there is an $\bar{\iota}$-over-computation for $V = \tilde{h}'(v)$ of value $A \notin F'$. This means in particular that $A \cap F = \emptyset$. Let us prove that for no $u \in L$, $v = \tilde{z}(u)$. For this, consider any $u$ such that $v = \tilde{z}(u)$. This means by definition of $h'$ that $\tilde{h}(u) \in V$. Hence, we can apply Lemma 30, and get that $\tilde{h}(u)$

[6] See the subtlety below.

[7] The same subtlety involving the ${}^d\bar{Q}\bar{m}$-nature of $\bar{m}$-over-computations need be used here.

$\bar{\iota}$-over-evaluates to some $a \in A$. But since $A \cap F = \emptyset$, this means that $a \notin F$. Hence $u$ is not accepted by $\mathcal{M}, h, F$. ■