Problem Session – STRUCO Meeting – June 2015

June 9th, 2015

Clique cover of claw-free graph

Communicated by P. Charbit.

A *claw* in a graph is an induced subgraph isomorphic to $K_{1,3}$. A graph is *claw-free* if it contains no claw.

$\mathrm{cc}(G)$ is the minimum number of cliques to cover all edges of a given graph.

Problem 1. Is it true that for every claw-free graph $\mathrm{cc}(G) \leq |V(G)|$?

It is not even known for graphs with stability 2, which are peculiar claw-free graphs.

Problem 2. Is it true that if $\alpha(G) \leq 2$, then $\mathrm{cc}(G) \leq |V(G)|$?

It is not difficult to prove that $\mathrm{cc}(G) \leq 2|V(G)|$ when $\alpha(G) \leq 2$. One can then derive $\mathrm{cc}(G) \leq 3|V(G)|$ when G is claw-free.

A *quasi-line graph* is a graph such that the neighborhood of every vertex can be covered by two cliques. Quasi-line graph are claw-free. For such graphs the answer is known.

Theorem 3. *If G is a quasi-line graph, then* $\mathrm{cc}(G) \leq |V(G)|$.

Subdivision with large girth and average degree in nowhere dense classes of graphs

Communiqué par P. Ossona de Mendes.

Let $\mathcal{C}$ be a class of graph. It is *monotone* if for any graph $G \in \mathcal{C}$ the pth subdivision of G is in $\mathcal{C}$.

It has *bounded expansion* if there exists f such that for all $G \in \mathcal{C}$, if the pth subdivision of G is in $\mathcal{C}$, then $\mathrm{Ad}(G) \leq f(p)$. (We denote by $\mathrm{Ad}(G)$ the average degree of G).

It is *nowhere dense* if there exists f such that for all $G \in \mathcal{C}$, if the pth subdivision of G is in $\mathcal{C}$, then $\omega(G) \leq f(p)$.

Conjecture 4. Let $\mathcal{C}$ be a nowhere dense class of graphs that does not have bounded expansion. Then there exists p such that for all k, g, there exists a graph $G \in \mathcal{C}$ such that

(i) the pth subdivision of G is in $\mathcal{C}$;

(ii) the girth if G is at least g;

(iii) $Ad(G) \geq k$.

Condition (iii) can be replaced by $\chi(G) \geq d$.
This conjecture would be implied by the two following conjectures.

Conjecture 5 (Erdős and Hajnal)**.** For all positive integers k and g, there exists $C(k, g)$ such that if $\chi(G) \geq C(k, g)$, then G has a subgraph H with girth at least g and chromatic number at least k.

The case $g = 4$ has been proved by Rödl.

Conjecture 6 (Thomassen)**.** For all positive integers k and g, there exists $D(k, g)$ such that if $\delta(G) \geq D(k, g)$, then G has a subgraph H with girth at least g and minimum degree at least k.

The case $g = 6$ has been solved by Kühn and Osthus.

Homomorphism and chromatic number of some Cayley graphs

Communicated by R. Naserasr.

$PC(2k) = Cay(\mathbb{Z}_2^{2k}, \{e_1, \dots, e_{2k}\})$
For $k \geq r$, there is an homomorphism from $PC(2k)$ into $PC(2r)$.

Conjecture 7. Every such homomorphism is surjective.

Conjecture 8. $\chi(PC(2k)^{2r-1}) = 2^{2r}$.

This conjecture would be implied by the following one.

Conjecture 9. The neighborhood simplicial complex of $PC(2k)^{(2r-1)}$ is 2^{2r}-connected.

Max Cut and Min Bisection of Random Graphs

Communicated by Lenka Zdeborova

If (A, B) is a partition of a vertex set of G, the size of the cut is the number of edges between A and B. $MaxC(G)$ is then defined as the maximum size of a cut of G. and $MinB(G)$ (for minimum bissection) is defined as the minimum size of a cut of G when we restrict to almost equals sides, i.e. $||A| - |B|| \leq 1$.

If G_N is now a random d-regular graph on N vertices, one can define

$$MaxC_d = \lim_{N\to\infty} \frac{1}{N} MaxC(G_N)$$

$$MinC_d = \lim_{N\to\infty} \frac{1}{N} MaxB(G_N)$$

The conjecture (supported by various results in Statistical Physics) is the following

Conjecture 10. $MaxC_d + MinB_d = \frac{d}{2}$.

Proper connectivity of edge-colored graphs

Communiqué par L. Montero.

A *proper path* is a path such that adjacent edges have different colors.

G is *properly connected* if for every two vertices v and w, there is a proper path from v to w.

The *proper connection number*, denoted $\mathrm{pc}(G)$, is the minimum ℓ such that there is an ℓ-edge-coloring of G that makes it properly connected.

We have the following results:

Theorem 11.
- *If G bipartite and 2-connected, then* $\mathrm{pc}(G) = 2$.
- *If G is 2-connected, then* $\mathrm{pc}(G) \leq 3$.

Problem 12. What is the complexity of deciding if $\mathrm{pc}(G) = 2$ for a 2-connected graph G ?

G is *k-properly connected* if for every two vertices v and w, there are k internally disjoint proper paths from v to w.

$pc_k(G)$, is the minimum ℓ such that there is an ℓ-edge-colouring of G that makes it k-properly connected.

I am not sure this are the conjectures Leandro gave. I was too slow.

Conjecture 13. If G bipartite and k-connected, then $\mathrm{pc}_k(G) = 2$.

Conjecture 14. If G is $(k+1)$-connected, then $\mathrm{pc}_k(G) \leq 3$.

Partitioning a graph into two subgraphs with large average degree

Communicated by P. Charbit.

Problem 15 (K. Edwards)**.** Let s and t be two positive integers, and let G be a graph with average degree at least $s+t+2$. Is it true that there is a partition of (A, B) of $V(G)$ such that the average degree of $G[A]$ is at least s and the average degree of $G[B]$ is at least t ?

Large acyclic subgraphs in digraphs

Communicated by A. Harutyunyan.

Conjecture 16 (Erdős-Hajnal – Tournament version)**.** For every tournament H, there exists $\epsilon_H > 0$ such that if T is a H-free tournament, then T has an acyclic subtournament of order at least n^{ϵ_H}.

Problem 17. Does there exist $\epsilon > 0$ such that every digraph with no directed 3-cycle contains an acyclic subdigraph of oder at least n^ϵ ?

If true ϵ would be at most $2/3$, as shown by random graph $G_{n,p}$ with $p = n^{2/3}$.

Problem 18. Let D be an oriented graph. If all directed cycles have length at most s, does there exist an acyclic subdigraph of order at least $\frac{n \log s}{s}$?

Getting an acyclic (even independent) set of order at least $\frac{n}{s}$ can be obtained using Bondy's Theorem stating that in a strong digraph there is a directed cycle of length at least the chromatic number.