Independent sets in triangle-free planar graphs

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Independent sets in planar graphs

Theorem (AH; RSST)

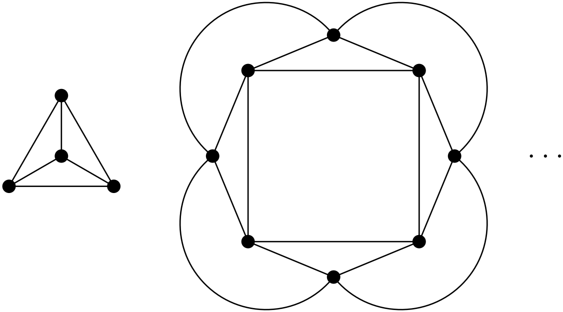
Every planar graph is 4-colorable.

Corollary

A planar graph G on n vertices has

$$\alpha(G) \geq n/4.$$

Tightness



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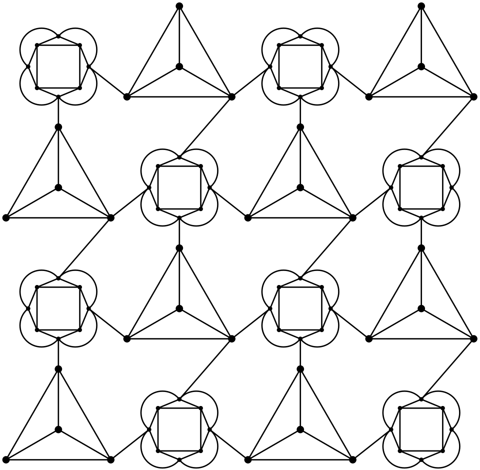


Larger independent sets

Largest independent set: NP-complete.

Problem

Decide whether a planar graph G on n vertices has an independent set of size at least

$$\frac{n+k}{4},$$

in time

$$f(k)\mathrm{poly}(n).$$

Open even for $k = 1$.

Difficulties

- Complicated structure of tight examples.
- No proof avoiding 4-color theorem.
  - Albertson: $\alpha(G) \geq n/4.5$
  - Can be strengthened, but things get complicated.
- 4-colorings do not absorb local changes.

Triangle-free planar graphs

Theorem (Grötzsch)

Every triangle-free planar graph is 3-colorable.

Corollary

A triangle-free planar graph G on n vertices has

$$\alpha(G) \geq n/3.$$

Non-tightness

Theorem (Steinberg and Tovey)

A triangle-free planar graph G on n vertices has

$$\alpha(G) \geq (n+1)/3.$$

Proof.

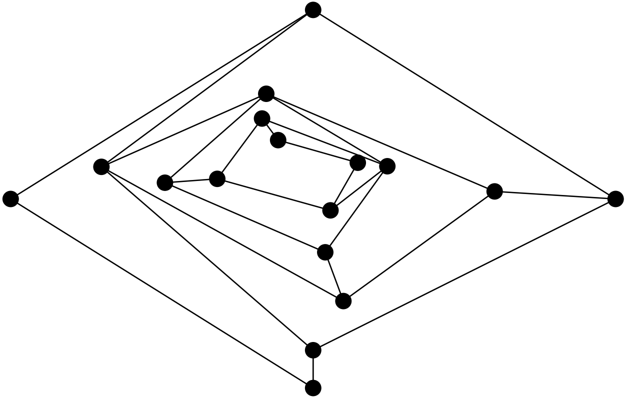
- G contains a vertex v of degree at most three.
- G has a 3-coloring φ s.t. $(\forall u \in N(v))\, \varphi(u) = 1$
  - Gimbel and Thomassen
- Let $I_1 = \varphi^{-1}(1)$, $I_2 = \varphi^{-1}(2) \cup \{v\}$, $I_3 = \varphi^{-1}(3) \cup \{v\}$
- $|I_1| + |I_2| + |I_3| = n + 1$, hence

$$\alpha(G) \geq \max(|I_1|, |I_2|, |I_3|) \geq \frac{n+1}{3}.$$

Tightness

Lemma (Jones)

For every $n \equiv 2 \pmod 3$, there exists a triangle-free planar graph G on n vertices with $\alpha(G) = (n+1)/3$.



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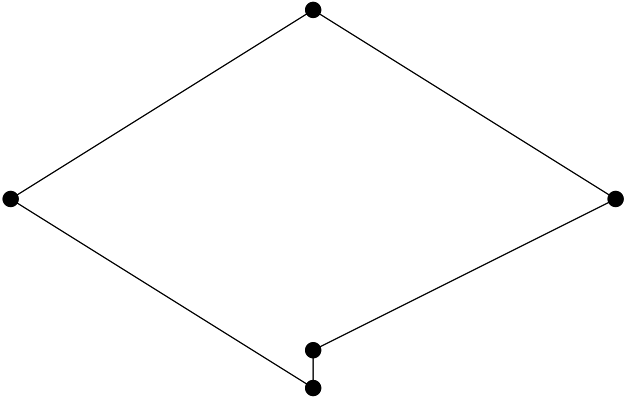

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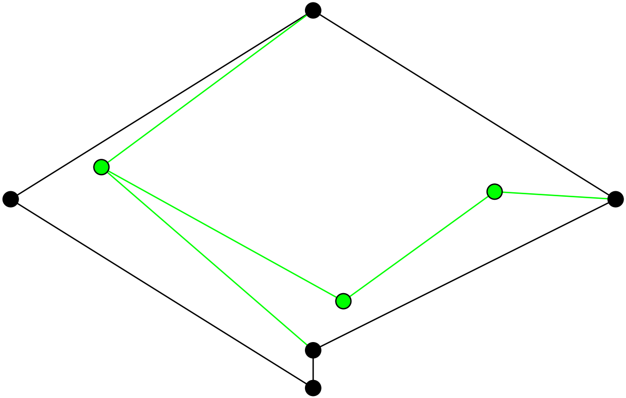

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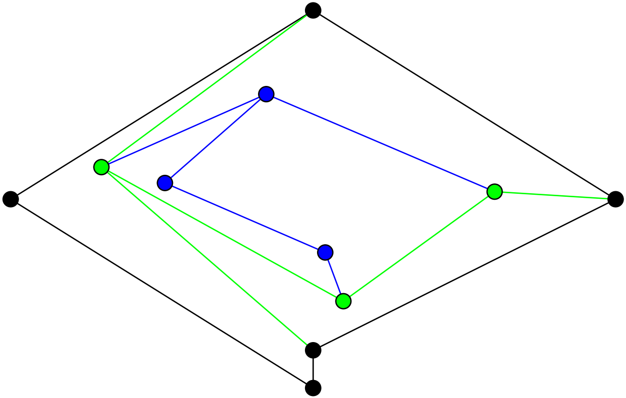

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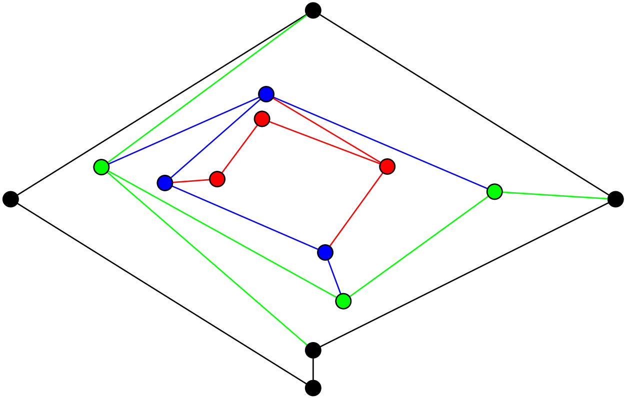

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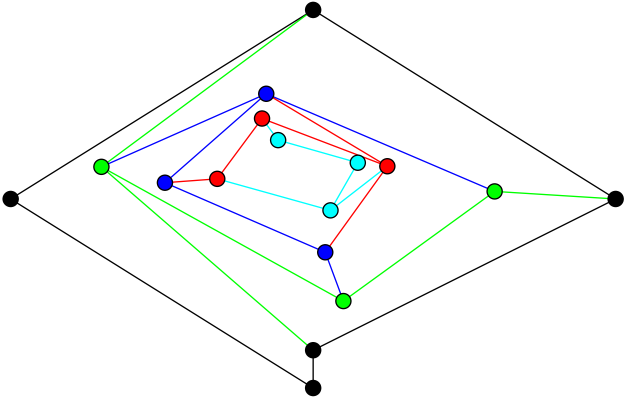

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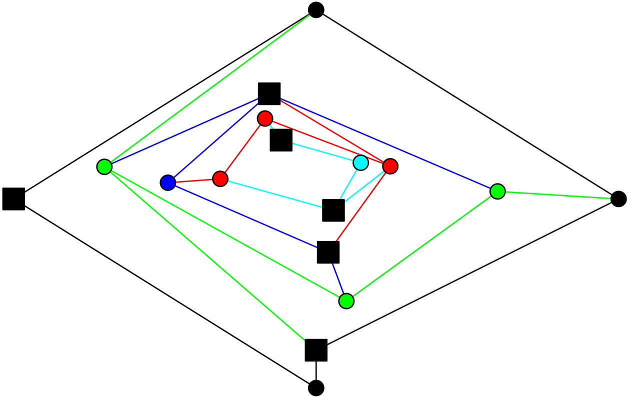

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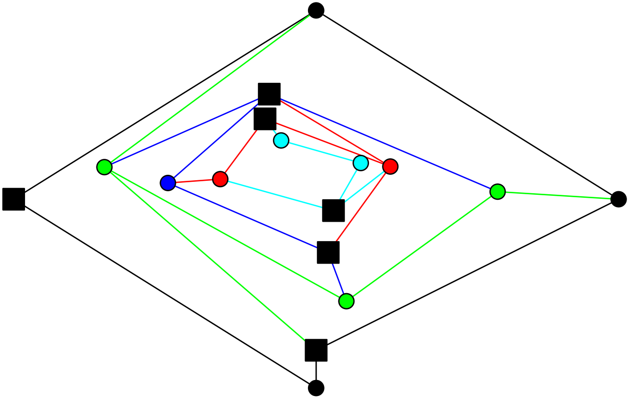

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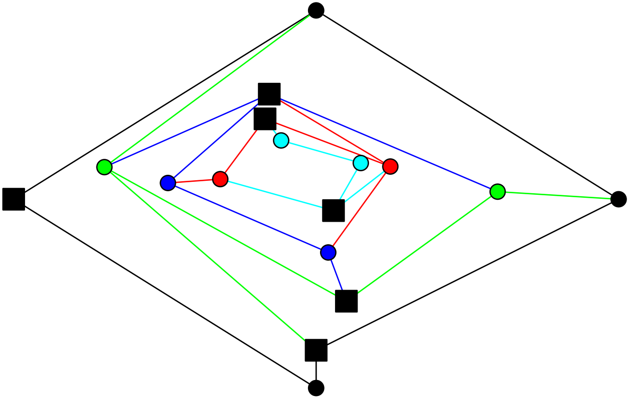


Results

Theorem

There exists an algorithm deciding whether a triangle-free planar graph G on n vertices satisfies

$$\alpha(G) \geq \frac{n+k}{3},$$

in time

$$2^{O(\sqrt{k})} n.$$

Theorem

There exists $\varepsilon > 0$ *such that every planar graph of girth at least 5 on n vertices has*

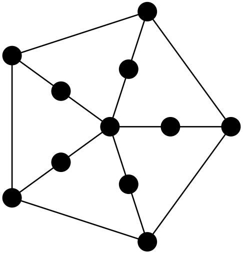
$$\alpha(G) \geq \frac{n}{3-\varepsilon}.$$

Open problem

Problem

Does there exist $\varepsilon > 0$ such that every planar graph of girth at least 5 has fractional chromatic number at most $3 - \varepsilon$?

False for circular chromatic number.



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There exists $\varepsilon > 0$ *such that every planar graph of girth at least 5 on n vertices has*

$$\alpha(G) \geq \frac{n}{3-\varepsilon}.$$

The main result

A subgraph H of a plane graph is nice if

- H has no separating 4-cycles, and
- each face of H either
  - is a face of G, or
  - has length 4.

Theorem

There exists $\varepsilon > 0$ such that every a plane triangle-free graph on n vertices containing a nice subgraph on p vertices has

$$\alpha(G) \geq \frac{n + \varepsilon p}{3}.$$

The algorithm

Proposition

If a planar graph G has no nice subgraph with p vertices, then G has tree-width $O(\sqrt{p})$.

To decide whether G satisfies $\alpha(G) \geq \frac{n+k}{3}$:

- Approximate tree-width within a constant factor.
- If $\mathrm{tw}(G) = \Omega(\sqrt{k})$, then answer “yes”.
- Otherwise, use dynamic programming.

The basic idea

- Find a large set of vertices $S \subseteq V(G)$ and a 3-coloring φ of G s.t. the neighborhood of each vertex of S is monochromatic.
- For $i \in \{1, 2, 3\}$, let

$$I_i = \varphi^{-1}(i) \cup \{v \in S : \text{neighbors of } S \text{ do not have color } i\}.$$

- $|I_1| + |I_2| + |I_3| \geq n + |S|$, hence $\alpha(G) \geq \frac{n+|S|}{3}$.

How to choose S?

- Small degrees (say ≤ 4).
- The neighborhoods should not influence each other.
  - The vertices in S should be pairwise far apart.
  - Not always possible (e.g., if $G = K_{1,n-1}$).

Theorem (Atserias, Dawar and Kolaitis; NOdM)

For every d, m, there exists n such that for every planar graph G and every $R \subseteq V(G)$ with $|R| \geq n$, there exist $S \subseteq R$ and $X \subseteq V(G) \setminus S$ such that

- $|S| = m$, $|X| \leq 3$
- *the distance between vertices of S in $G - X$ is at least d.*

The basic idea, version 2

- Find a large $S \subseteq V(G)$, a small $X \subseteq V(G) \setminus S$ and a 3-coloring φ of $G - X$ s.t. the neighborhood of each vertex of S is monochromatic.
- For $i \in \{1, 2, 3\}$, let

$$I_i = \varphi^{-1}(i) \cup \{v \in S : \text{neighbors of } S \text{ do not have color } i\}.$$

- $|I_1| + |I_2| + |I_3| \geq n - |X| + |S|$, hence $\alpha(G) \geq \frac{n-|X|+|S|}{3}$.

Choosing S

Theorem (ADK; NOdM)

For every d, m, there exists n such that for every planar graph G and every $R \subseteq V(G)$ with $|R| \geq n$, there exist $S \subseteq R$ and $X \subseteq V(G) \setminus S$ with

- $|S| = m$, $|X| \leq 3$
- *the distance between vertices of S in $G - X$ is at least d.*

- We need $|S| = \Omega(|R|)$.
- This is false if $|X| = O(1)$, e.g. in $\sqrt{n} \cdot K_{1,\sqrt{n}}$

Choosing S, version 2

- For a small $\delta > 0$, we can choose $|S| = \Omega(|R|)$ and $|X| \le \delta|S|$.

Theorem (D., Mnich)

For every class $\mathcal{G}$ with bounded expansion and every $\delta > 0, d$, there exists $\varepsilon > 0$ such that for every graph $G \in \mathcal{G}$ and $R \subseteq V(G)$, there exist $S \subseteq R$ and $X \subseteq V(G) \setminus S$ with

- *$|S| \ge \varepsilon|R|$, $|X| \le \delta|S|$, and*
- *the distance between vertices of S in $G - X$ is at least d.*

Coloring

Theorem (D., Král', Thomas)

There exists $d \geq 3$ such that if G is a planar triangle-free graph <u>without separating 4-cycles</u> and vertices of $S \subseteq V(G)$ are pairwise at distance at least d, then G has a 3-coloring such that the neighborhood of each vertex of S is monochromatic.

- The coloring of the nice subgraph extends to the whole graph.
    - Further complication: the extension can destroy monochromatic neighborhoods.
- We have a polynomial time (but not linear) algorithm to find the coloring.
- Nothing like this holds for 4-coloring.

Thank you for the attention.

Questions?