Distributed Low Tree-Depth Decompositions

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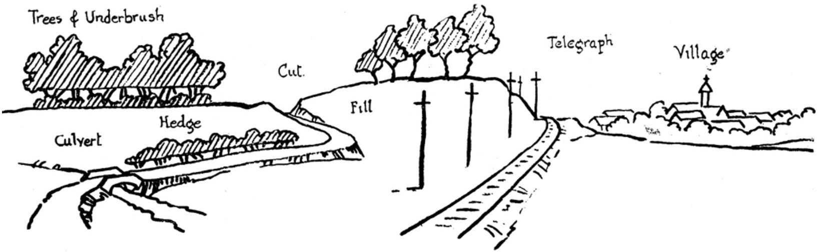
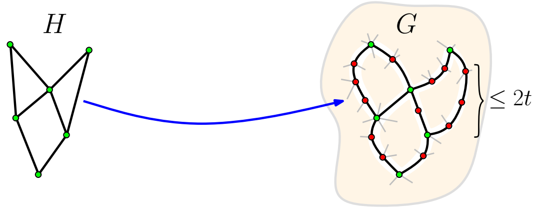
Landscape Sketch



Topological resolution of a class $\mathcal{C}$

$G \mathbin{\widetilde{\nabla}} t$ = set of *shallow topological minors* at depth t:



$$\mathcal{C} \mathbin{\widetilde{\nabla}} t = \bigcup_{G \in \mathcal{C}} G \mathbin{\widetilde{\nabla}} t.$$

Topological resolution:

$$\mathcal{C} \subseteq \mathcal{C} \mathbin{\widetilde{\nabla}} 0 \subseteq \mathcal{C} \mathbin{\widetilde{\nabla}} 1 \subseteq \dots \subseteq \mathcal{C} \mathbin{\widetilde{\nabla}} t \subseteq \dots \subseteq \mathcal{C} \mathbin{\widetilde{\nabla}} \infty$$

$\xrightarrow{\hspace{3em}}$ *time*

Taxonomy of Classes

A class $\mathcal{C}$ is *nowhere dense* if

$$\forall t \in \mathbb{N} : \quad \omega(\mathcal{C} \,\widetilde{\nabla}\, t) < \infty$$

... otherwise $\mathcal{C}$ is *somewhere dense*

$\mathcal{C}$ has *bounded expansion* if

$$\forall t \in \mathbb{N} : \quad \bar{\mathrm{d}}(\mathcal{C} \,\widetilde{\nabla}\, t) < \infty$$

Remark: bounded expansion $\Longrightarrow$ nowhere dense.

Notation: $\widetilde{\nabla}_t(G) = \frac{1}{2}\,\bar{\mathrm{d}}(G \,\widetilde{\nabla}\, t) = \max\{ \frac{\|H\|}{|H|} : \; H \in G \,\widetilde{\nabla}\, t \}.$

Other choices, other rooms?

	$\bar{\mathrm{d}}$	χ	ω
Minors			
Topological minors	Bounded expansion		Nowhere dense
Immersions			

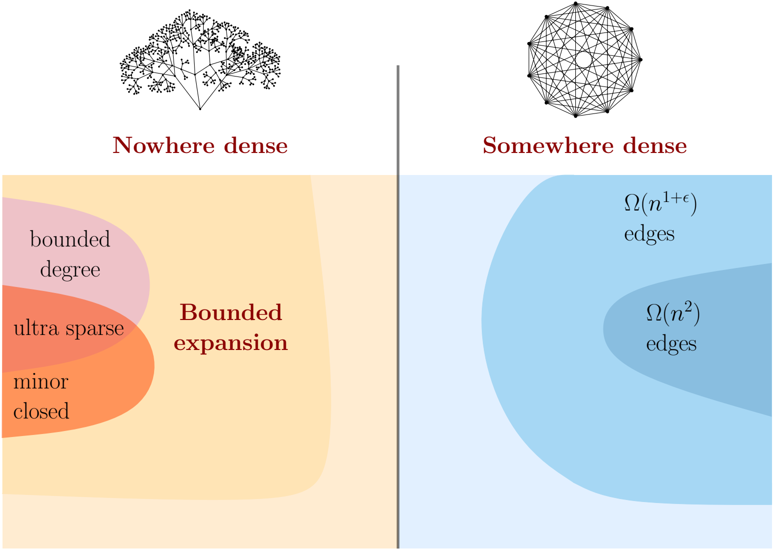
Definition

Other choices, other rooms?

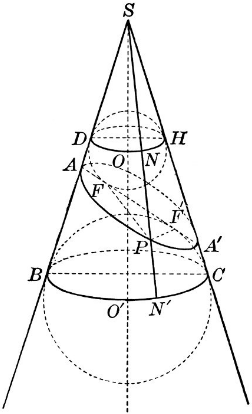
	$\bar{\mathrm{d}}$	χ	ω
Minors	Bounded expansion	Bounded expansion	Nowhere dense
Topological minors	Bounded expansion	Bounded expansion	Nowhere dense
Immersions	Bounded expansion	Bounded expansion	Nowhere dense

Theorem (Nešetřil, POM 2012)

Taxonomy of Classes

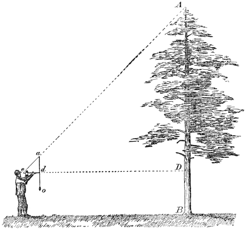


Low Tree-depth Decomposition



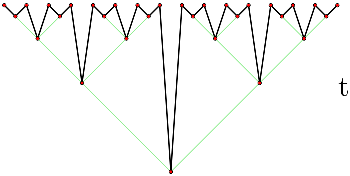
Tree-depth

Definition



The *tree-depth* td(G) of a graph G is the minimum height of a rooted forest Y s.t.

$$G \subseteq \text{Closure}(Y).$$



$$\text{td}(P_n) = \log_2(n+1)$$

Low tree-depth decompositions

Chromatic numbers $\chi_p(G)$

$\chi_p(G)$ is the minimum of colors such that any subset I of $\leq p$ colors induce a subgraph G_I so that $\mathrm{td}(G_I) \leq |I|$.

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$$\chi(G) = \chi_1(G) \leq \chi_2(G) \leq \cdots \leq \chi_p(G) \leq \cdots \leq \chi_{|G|}(G) = \text{td}(G).$$

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Stronger

$(p+1)$-*centered coloring*, s.t. in any connected subgraph:

- either $\geq p+1$ distinct colors appear,
- or some color appears exactly once.

Low tree-depth decompositions

Theorem (Nešetřil and POM; 2006, 2010)

$$\forall p,\ \sup_{G\in\mathcal{C}} \chi_p(G) < \infty \iff \mathcal{C} \text{ has } \textit{bounded expansion}.$$

$$\forall p,\ \limsup_{G\in\mathcal{C}} \frac{\log \chi_p(G)}{\log |G|} = 0 \iff \mathcal{C} \text{ is } \textit{nowhere dense}.$$

(extends DeVos, Ding, Oporowski, Sanders, Reed, Seymour, Vertigan on low tree-width decomposition of proper minor closed classes, 2004)

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Remark

Similar results for $(p+1)$-centered coloring.

Algorithmic Version

Procedure A

for $k = 1$ **to** $2^{p-1} + 1$ **do**
  Compute a fraternal augmentation.
end for
Compute depth p transitivity
Greedily color vertices according to the augmented graph

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Theorem (Nešetřil and Ossona de Mendez 2006)

$\forall p \in \mathbb{N}\ \ \exists$ polynomial P_p (of degree about 2^{2^p}) such that $\forall G$ Procedure A computes a $(p+1)$-centered coloring of G with $N_p(G) \leq P_p(\widetilde{\nabla}_{2^{p-2}+\frac{1}{2}}(G))$ colors in time $O(N_p(G)n)$-time,

$$\text{where} \quad \widetilde{\nabla}_t(G) = \frac{1}{2}\,\bar{\mathrm{d}}(G\,\widetilde{\triangledown}\,t).$$

Model Checking

Theorem (Dvořák, Kráľ, and Thomas 2009; Grohe and Kreutzer 2011)

First-order properties may be checked in

- $O(n)$ time for G in a class with bounded expansion,
- $n^{1+o(1)}$ time for G in a class with locally bounded expansion.

Problem

Can first-order properties be checked in $O(n^c)$ time for G in a **nowhere dense class**?

Distributed Computing

The $\mathcal{LOCAL}$ model

Definition (Peleg 2000)

Synchronous message-passing model with ...

- fixed network = input graph of order n
- vertices: processors with unique id
- edges: communication links
- *running time*: # of rounds

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- edges: communication links
- *running time*: # of rounds

+ every vertex knows n

Orientation and Coloring of Degenerate Graphs

Theorem (Barenboim and Elkin 2008)

There are distributed procedures Partition and Arb-Color, such that for G with arboricity a and a positive parameter ϵ, $0 < \epsilon \leq 2$:

- Partition(a, ϵ) computes an acyclic orientation of G with maximum outdegree $\leq (2+\epsilon)a$ in time $O(\log n)$.
- Arb-Color(a, ϵ) computes an coloring of G into $(\lfloor (2+\epsilon)a \rfloor + 1)$ colors in time $O(a \log n)$;

Orientation and Coloring of Degenerate Graphs

Basic ideas: let $D = 2\widetilde{\nabla}_0(G)$ (= *maximum average degree*),

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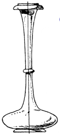

- Iteratively form parts $V_1, \ldots, V_k$ by removing vertices of degree $\leq (1+\epsilon)D$;

  Remark: $\Delta(G[V_i]) \leq (1+\epsilon)D$ and $k = O(\log n)$;

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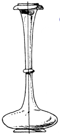

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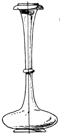

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- Orient vertices by lexicographic order of $(i, j, \mathrm{Id}(v))$ where $v \in V_i \cap S_j$;

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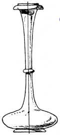

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- Greedily color with $\leq \lfloor (1+\epsilon)D \rfloor + 1$ colors.

(Note that $D < 2a(G)$.)

Distributed Low Tree-depth Decomposition

Procedure A

Orient the graph with indegree bounded by degeneracy
for $k = 1$ **to** $2^{p-1} + 1$ **do**
  Compute a *fraternal augmentation*
  Orient the added edges with indegree bounded by degeneracy
end for
Compute depth p transitivity arcs
Compute a coloring of the augmented graph.

Recall: $\text{degeneracy}(G) = \max_{H \subseteq G} \delta(H)$.

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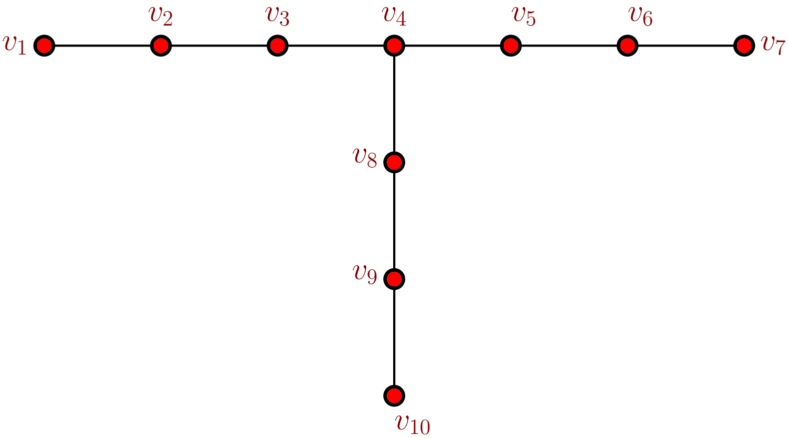
Problem

The graph is modified!

Fraternal augmentation



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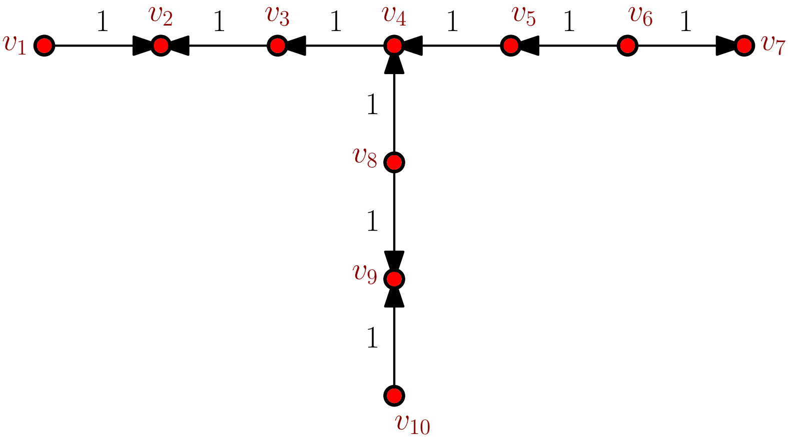

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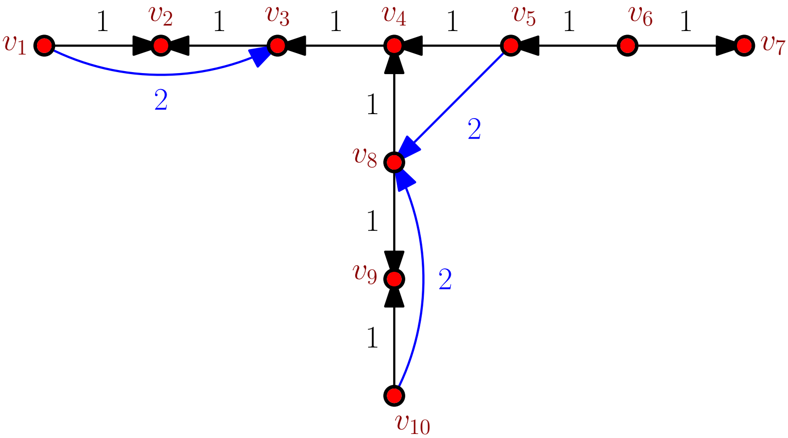

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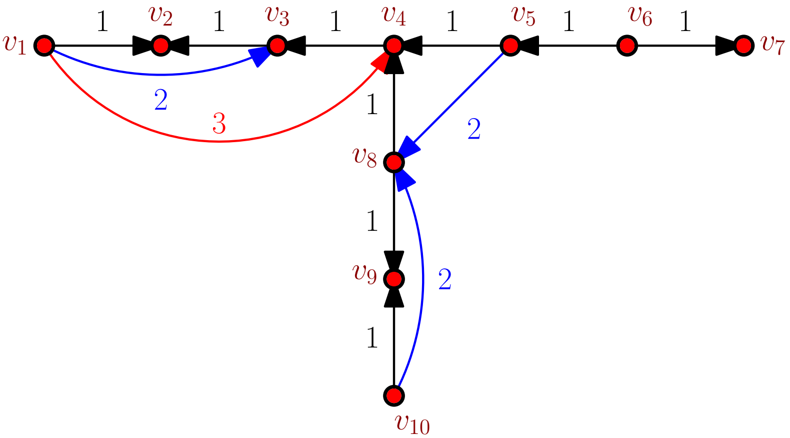

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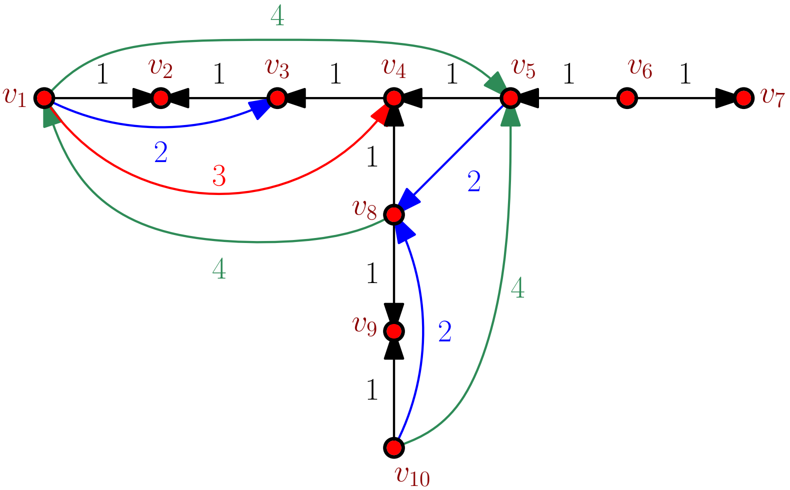

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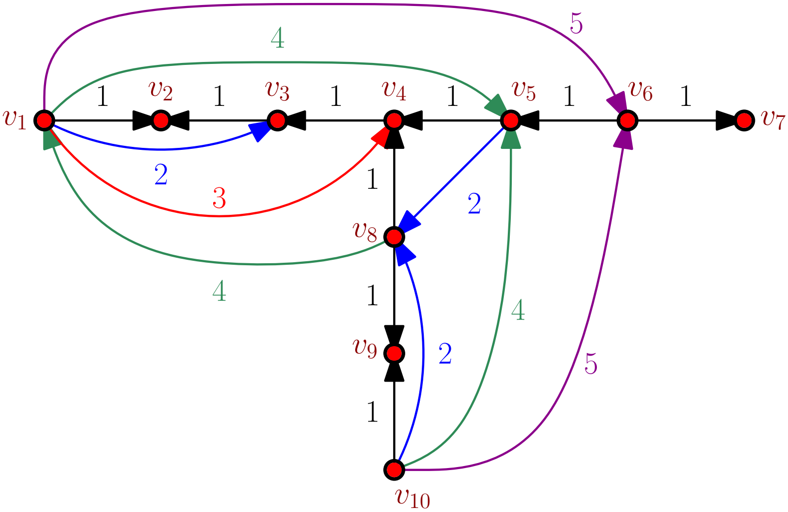

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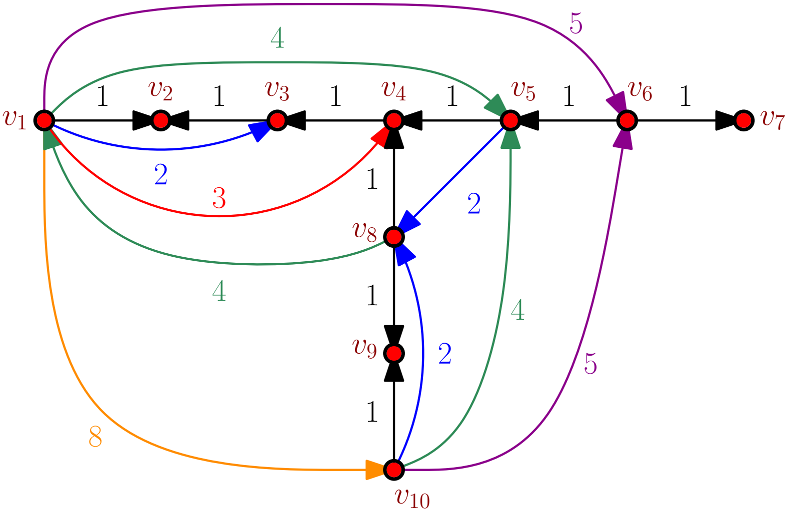
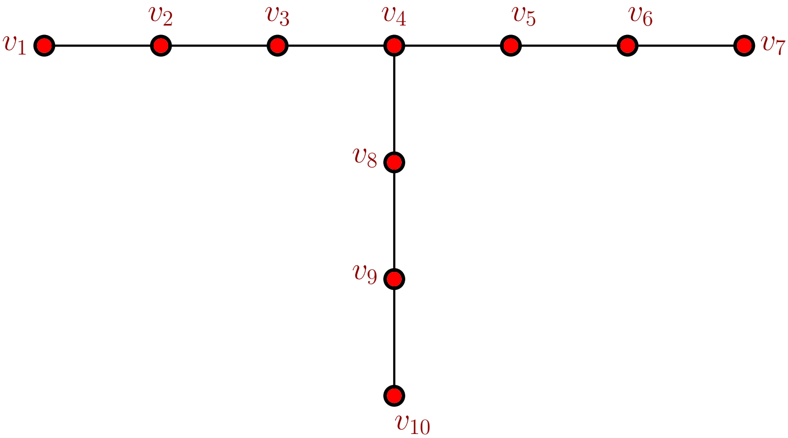


Fraternal augmentation (with routing)



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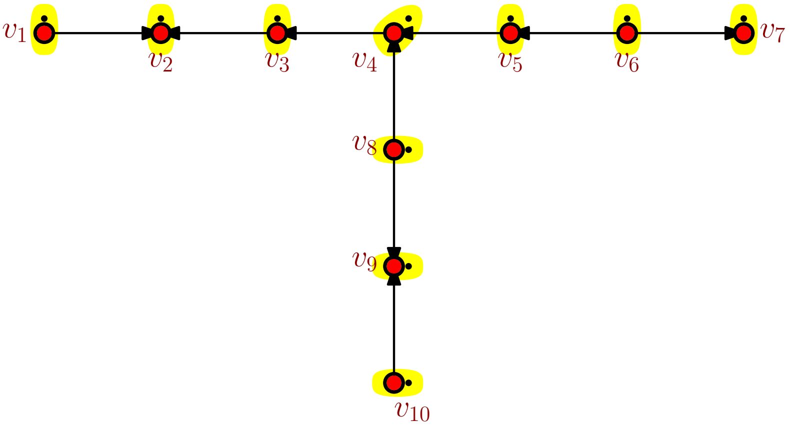

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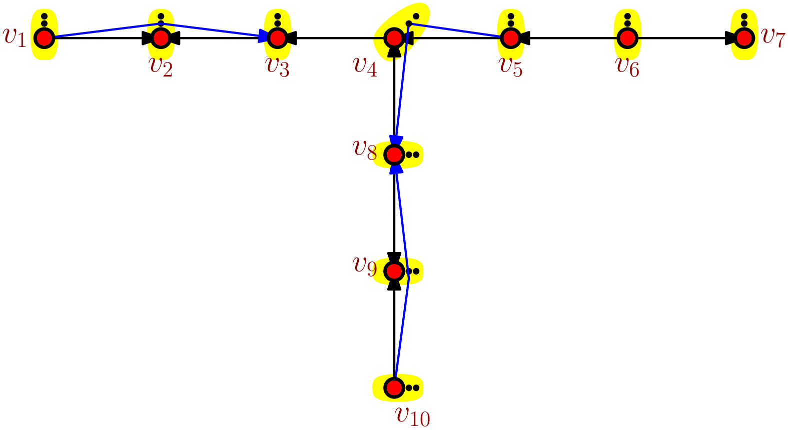

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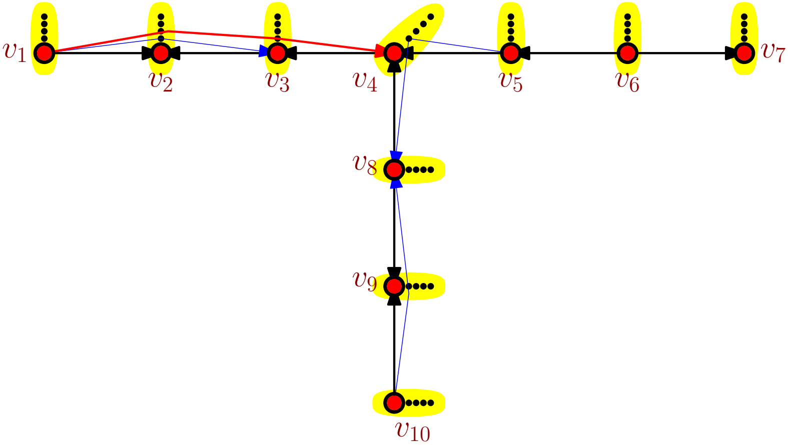

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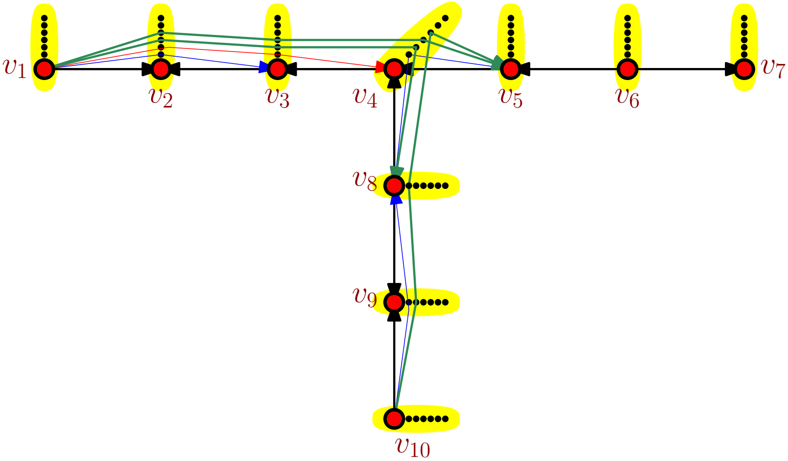

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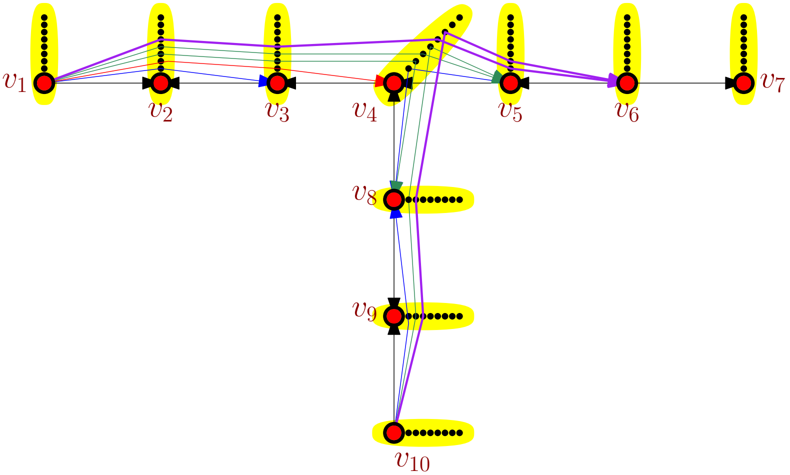

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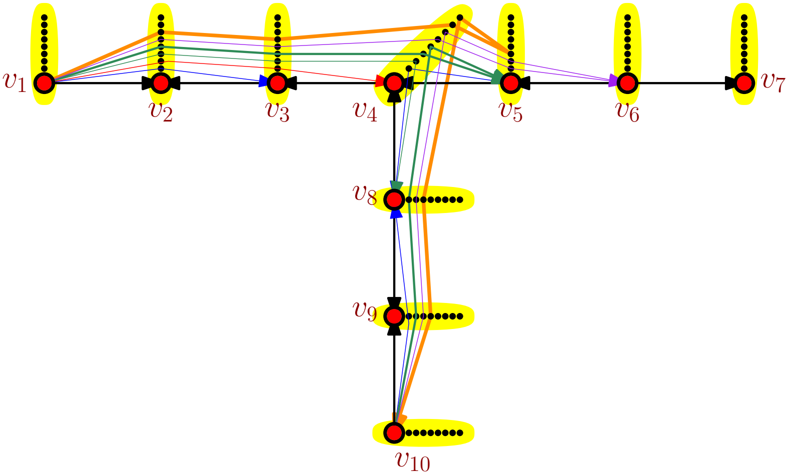

The Distributed Algorithm (sketch)

Procedure D

Orient the graph with indegree bounded by degeneracy (within a constant factor)

for $k = 1$ **to** $2^{p-1} + 1$ **do**

  Compute the edges of the set of fraternal edges to be added. These edges are routed as paths of length $k + 1$

  Orient these edges with indegree bounded by degeneracy (within a constant factor)

end for

Compute depth p transitivity paths

Compute a coloring of the augmented graph.

Bounding the Congestion

$$N(i) = \begin{cases} 0, & \text{if } i = 1; \\ \binom{\Delta_1^-}{2}, & \text{if } i = 2; \\ \sum_{j=2}^{i-1} N(j)\Delta_{i-j}^- & \\ \quad + \sum_{j=1}^{(i-1)/2} \Delta_j^- \Delta_{i-j}^-, & \text{if } i \equiv 1 \pmod 2; \\ \sum_{j=2}^{i-1} N(j)\Delta_{i-j}^- & \\ \quad + \sum_{j=1}^{i/2-1} \Delta_j^- \Delta_{i-j}^- + \binom{\Delta_{i/2}^-}{2}, & \text{if } i \equiv 0 \pmod 2. \end{cases}$$

$$\Delta_i^- \le (1+\epsilon)\big(N(i) + (N(i)+1)^2\, \widetilde{\nabla}_{(i-1)/2}(G)\big)$$

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where $\widetilde{\nabla}_t(G) = \frac{1}{2}\,\bar{\mathrm{d}}(G \,\widetilde{\triangledown}\, t) \le f_{\mathcal{C}}(t)$ OK

Distributed Low Tree-depth Decomposition

in a Bounded Expansion Class

Theorem (Nešetřil and Ossona de Mendez 2013)

For every graph G in a fixed bounded expansion class $\mathcal{C}$ and positive parameters $p \in \mathbb{N}$ and ϵ, $0 < \epsilon \leq 2$ the procedure $\mathrm{D}(\mathcal{C}, p, \epsilon)$ computes a $(p+1)$-centered coloring with $N(\mathcal{C}, p, \epsilon)$ colors in time $O(\log n)$.

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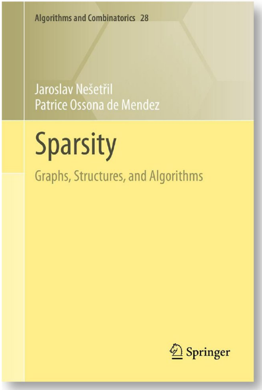
Corollary (Example 🙂)

There is a procedure such that for every graph G in a fixed bounded expansion class $\mathcal{C}$, the procedure computes in time $O(\log n)$ a coloring of the vertices of G with $f(\mathcal{C})$ colors, such that any two vertices of G at distance 3 get different colors.

Discussion



- What about non-sparse graphs?
- Stronger models of distributed computation?

Thank you for your attention.