OPTIMIZATION ⊕ ENUMERATION

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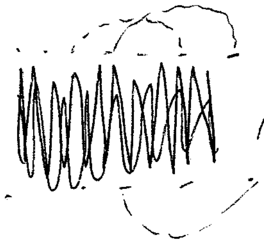
$G = (V, E)$ graph $w : E \to \mathbb{Q}$ edge-weights

MAX-CUT

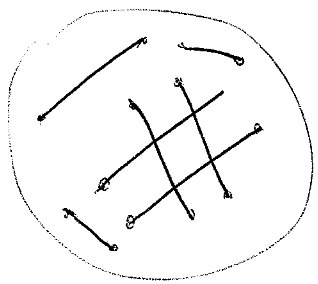
V_1



max

V_2

Perfect Matching



cover all vertices

Chromatic number

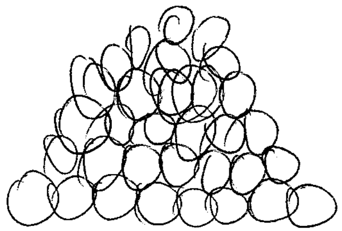
$f : V \to \{0, \dots, k-1\}$

$uv \in E \Rightarrow f(u) \neq f(v)$

$$\text{Algorithms} \mapsto \begin{cases} \text{local steps : blossom algorithm} \\ \boxed{\text{GLOBAL steps}} \end{cases}$$

Determinant calculation

$$A_{n \times n} \Rightarrow \det A = \sum_{\pi} \mathrm{sign}(\pi) \prod_{i=1}^{n} A_{i\pi(i)}$$



$\boxed{\text{Get BIGGEST STONE}}$!  !

I can count the number of biggest stones

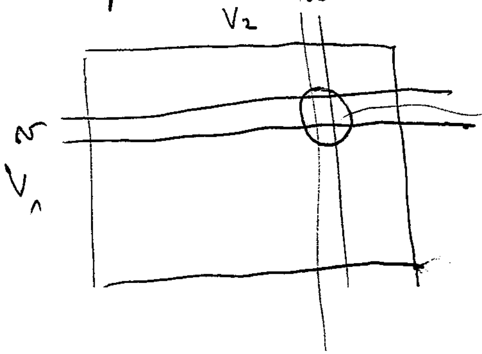
Functions : **Graph Polynomials** $\quad G=(V,E),\ \mathbb{X}=(x_e)_{e\in E}$ variables

$$\mathcal{C}(G,\mathbb{X}) = \sum_{\substack{E'\subseteq E \\ \text{edge-cut}}} \prod_{e\in E'} x_e \quad ; \quad \mathcal{E}(G,\mathbb{X}) = \sum_{\substack{E'\subseteq E \\ \text{even}}} \prod_{e\in E'} x_e$$

even $\rightsquigarrow$ (V,E') has all degrees even

$\mathcal{P}(G,\mathbb{X})$: perfect matchings

G bipartite $\Rightarrow \mathcal{P}(G,\mathbb{X})$ is $\boxed{\text{permanent}}$



$\to x_e,\ e=\{uv\}$

Typically

$$\mathbb{X}_e = x^{w_e}$$

Chromatic polynomial: coming from PIE

Number of proper colorings by k colors $= M(G, k) =$

$$\sum_{E' \subseteq E} (-1)^{|E'|} k^{c(E')} \quad ; \quad c(E') = \#\text{ components of } (V, E')$$

Why: $= k^{|V|} - \left|\bigcup_{e \in E} A_e\right|$; $A_{e=\{u,v\}} = \{ f : V \to \{0, \dots, k-1\} ; f(u) = f(v) \}$

$\boxed{\text{Tutte polynomial}}$ $\quad T(G, x, y) = \sum_{A \subseteq E} x^{|A|} y^{c(A)}$

$\boxed{\text{Multivariable Tutte polynomial}}$

Tutte specialises to $\mathcal{C}$, $\mathcal{E}$

$$\sum_{A \subseteq E} \left(\prod_{e \in A} x_e \right) y^{c(A)}$$

o

Is there a graph whose chromatic number determined first from chrom. polynomial?

Is there a graph whose max cut determined first from $\mathcal{C}(G, w)$?

METHODS

* Determinants (signed generating functions)

$$\operatorname{Det} A = \sum_{\pi} \operatorname{sign}\pi \prod A_{i\pi(i)}$$

$$\operatorname{Per} A = \sum_{\pi} \prod A_{i\pi(i)}$$

* Dualities

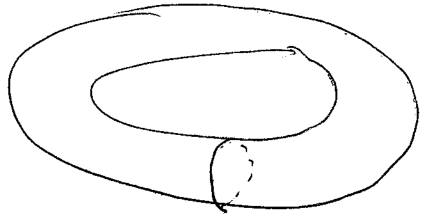
Mac Williams Theorem:

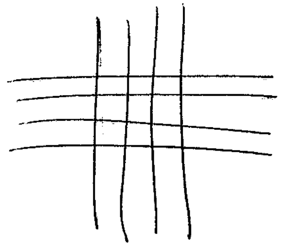
$$\mathcal{C}(G, x) \equiv \mathcal{E}(G, x)$$

* Quantum Computing

* Nature is a model for graph polynomials: physics methods, intuition

Max-Cut Problem for toroidal graphs





Input: $G=(V,E)$ on torus, $\gamma \in \mathbb{Z}$

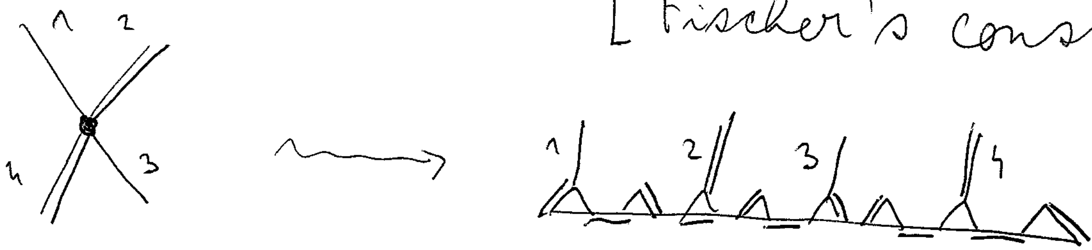
$w: E \to \mathbb{Z}$, $|w(e)| \le pol(|G|)$

strongly polynomial
OPEN

Output: Is there an edge-cut of weight $\ge \gamma$?

Algorithm.

1. Suffices to calculate $\mathcal{C}(G,w)$
2. Suffices to calculate $\mathcal{E}(G,w)$ [MacWilliams thm]
3. Suffices to calculate $\mathcal{P}(G_\Delta, w_\Delta)$ [Fischer's construction]



G

weight 0 on new edges

locally planar construction

Hence, suffices to calculate, for $\boxed{G \text{ toroidal}}$

$$\mathcal{P}(G, w) = \sum_{\substack{E' \text{ perf.} \\ \text{matching}}} x^{\sum_{e \in E'} w(e)} = \sum_{E' \text{ p.m.}} x^{w(E')} .$$

Determinantal method (Kasteleyn):

D orientation of G, M perfect matching of G

$$\mathcal{P}(D, M, w) = \sum_{P \text{ p.m.}} \operatorname{sign}(D, M, P)\, x^{w(P)}$$

quadratic forms, Arf-invariant fla

⊛ $\mathcal{P}(D, M, w)$ is determinant-type expression (Pfaffian)
can calculate efficiently

⊛ There are 4 orientations D_1, D_2, D_3, D_4 :

$$\mathcal{P}(G, w) = \frac{1}{2}\Big(\mathcal{P}(D_1) + \mathcal{P}(D_2) + \mathcal{P}(D_3) - \mathcal{P}(D_4)\Big).$$

Kasteleyn, Galluccio, Loebl ...

Natural aspect

1. $\mathcal{E}(G, w)$ is $\boxed{\text{Ising Partition Function}}$

$$Z(G, w) = \sum_{s: V \to \{1,-1\}} x^{\sum_{uv \in E} w(uv)\, s(u)\, s(v)} \qquad \left[x := e^{\frac{\beta k}{T}} \right]$$

2. $T(G, x, y)$ is $\boxed{\text{Potts Partition Function}}$

$$(*) \quad \sum_{s: V \to \{0, \dots, k-1\}} e^{H(P^k)(s)}, \qquad H(P^k)(s) = \sum_{e = uv \in E} J_e \Delta(s(u), s(v))$$

3. $\boxed{\text{Potts with magnetic field}}$

$$(*) \quad \sum_{s: V \to \{0, \dots, k-1\}} e^{f\left(\sum_V s(v)\right)} \cdot e^{H(P^k)(s)}$$

example : $f\left(\sum s(v)\right) = \overset{\text{const.}}{h} \cdot \left(\sum s(v)\right)$

limits,
random weights,

criticality
important
parameters
...

- Many ways to study graph polynomials
- Graph polynomials associated to natural processes on graphs.

* ? Is "using graph polynomials" to study graphs robust or (only) clever ?

? Can graph polynomials detect different graphs ?

Graph Isomorphism

polynomial trees, planar, bonded degree, likely not NP-complete

Tutte pol. does not distinguish non-isomorphic trees

$\boxed{\text{Conjecture}}$ (Bollobas, Riordan)

$G \sim H$ if $T(G) = T(H)$

Almost all graphs have trivial equivalence class

$\boxed{\text{Mighty Graph Polynomials}}$

$$|\{e = uv \in E \,;\, s(u) = s(v)\}|$$

Stanley (95) $$XB(G, t, x_0, x_1, \dots) = \sum_{s : V \to \{0,1,\dots\}} (1+t)^{b(s)} \prod_{v \in V} x_{s(v)}$$

$\boxed{\text{Conjecture}}$ X distinguishes non-isomorphic trees

Noble, Welsh (99) $$U(G, z, x_1, \dots) = \sum_{A \subseteq E(G)} x_{\tau_A} (z-1)^{|A| - |V| + c(A)}$$

$\tau_A = (n_1 \geq \dots \geq n_e)$ partition of V given by components of A

$$x_{\tau_A} = x_{n_1} \cdots x_{n_e}$$

$\boxed{\text{Conjecture}}$ U distinguishes trees.

Noble, Welsh, Sarmiento:
$U \approx XB$
Thatte: Conjectures are equivalent

Chromatic Functions

Loebl (2007)
$$M_q(G,k) = \sum_{\substack{s: V \to \{0,\dots,k-1\} \\ \text{proper}}} q^{\sum_{v \in V} s(v)} = \sum_{A \subseteq E} (-1)^{|A|} \prod_{W \in C(A)} (k)_{q^{|W|}}$$

Klazar, Loebl, Moffatt (2013)
$$M_{r,q}(G,k) = \sum_{\substack{s: V \to \{0,\dots,k-1\} \\ \text{proper}}} r^{\sum_V q^{s(v)}} = \sum_{A \subseteq E(G)} (-1)^{|A|} \prod_{W \in C(A)} \sum_{i=0}^{k-1} r^{|C| q^i}$$

$$B_{r,q}(G,x,k) = \sum_{A \subseteq E(G)} x^{|A|} \prod_{W \in C(A)} \sum_{i=0}^{k-1} r^{|C| q^i} = \sum_{s: V \to \{0,\dots,k-1\}} e^{\beta \sum_{uv \in E} J \Delta(s(u), s(v))} r^{\sum_V q^{s(v)}}$$

Potts with magnetic field

1st volume ☺

<u>Theorem</u> [Klazar, Loebl, Moffatt 2013, Annales de l'Inst. Poincaré D
AIHPD: Combinatorics, Physics and their Interaction]

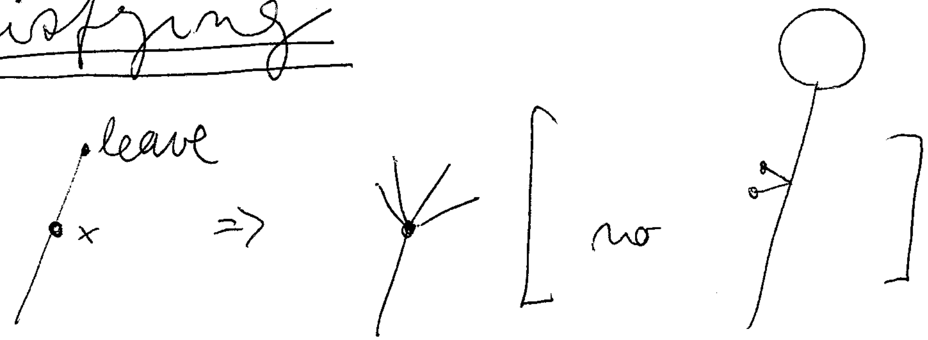
$B_{r,q}(G, x, k)$ equivalent to U.

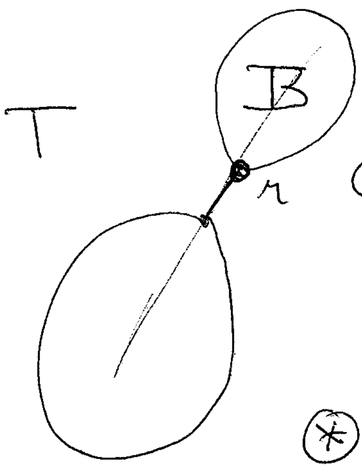
$\boxed{\text{Robustness Conjecture}}$ $B_{r,q}(G, x, k)$ distinguishes non-isomorphic chordal graphs.

If true then Potts ~~[illegible]~~ in magnetic field
+
straightforward preprocessing
solves graph isomorphism

Work in progress with Jean-Sebastien Sereni

U distinguishes non-isomorphic trees with vertex-weights, satisfying

- no degree 2 vertex
- 

- weights: 

⊛ $w(B) \le w(T)/2 \Rightarrow$

$$w(r) > \sum_{D \text{ branch of } B} w(D)$$

⊛ B, B' shapes, $w(B') < w(B) \le w(T)/2$

$$\Downarrow$$

$$w(B) > 2\,w(B')$$