A first intermediate class with limit object

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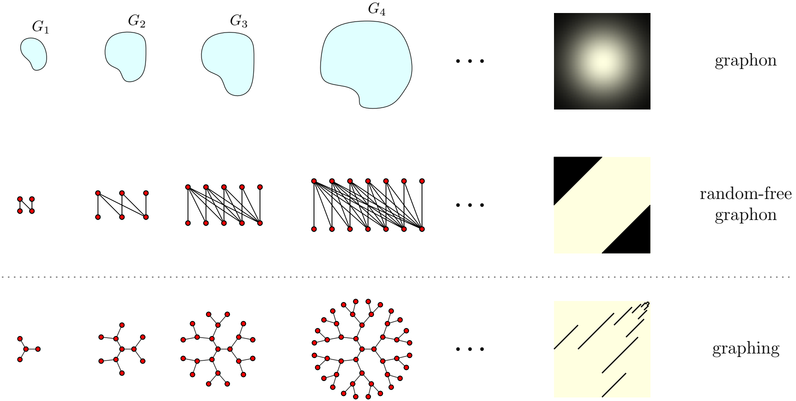
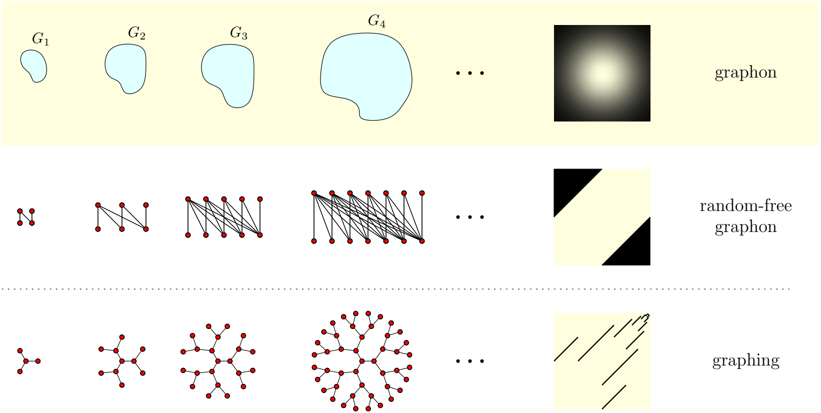
State of the art

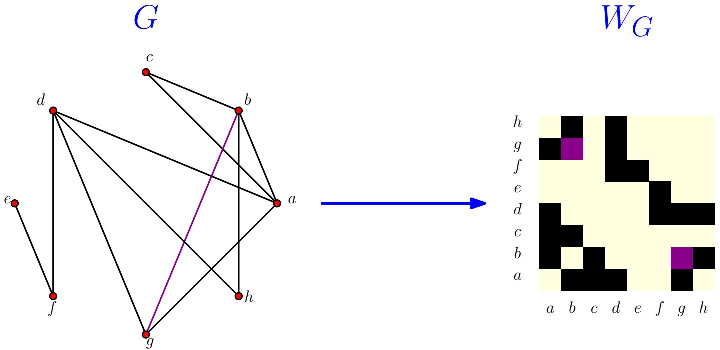
Limit objects



Limit objects: dense case



Graphons



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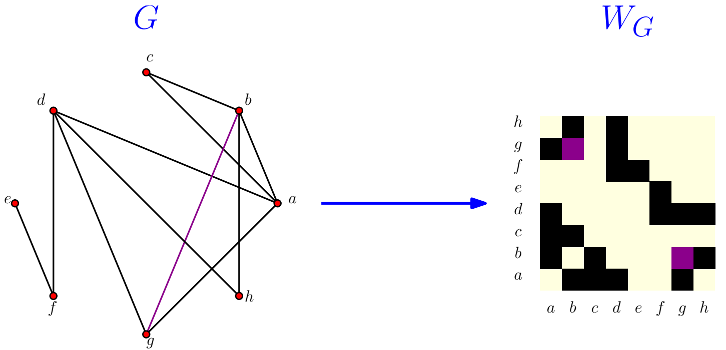
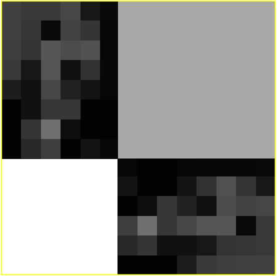


Szemerédi partitions



Regularity Lemma

$$\forall V_i' \subseteq V_i \quad \forall V_j' \subseteq V_j$$

$$|V_i'| > \epsilon |V_i| \text{ and } |V_j'| > \epsilon |V_j|$$

$$\Downarrow$$

$$\left|\mathrm{dens}(V_i', V_j') - \mathrm{dens}(V_i, V_j)\right| < \epsilon$$

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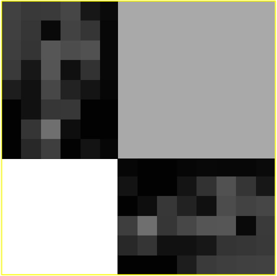

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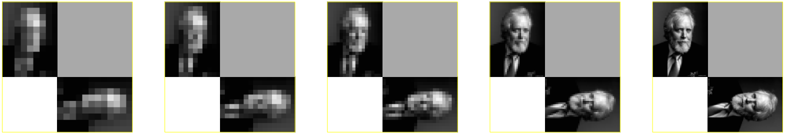
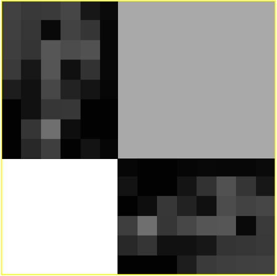


$$\epsilon \longrightarrow 0$$

Szemerédi partitions



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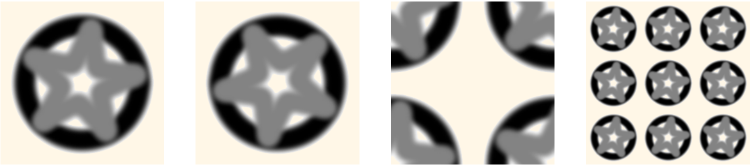


L-convergence

- Convergence of $\delta_\square$ or of *Lovász profile*

$$t(F, G_n) = \frac{\hom(F, G_n)}{|G_n|^{|F|}}.$$

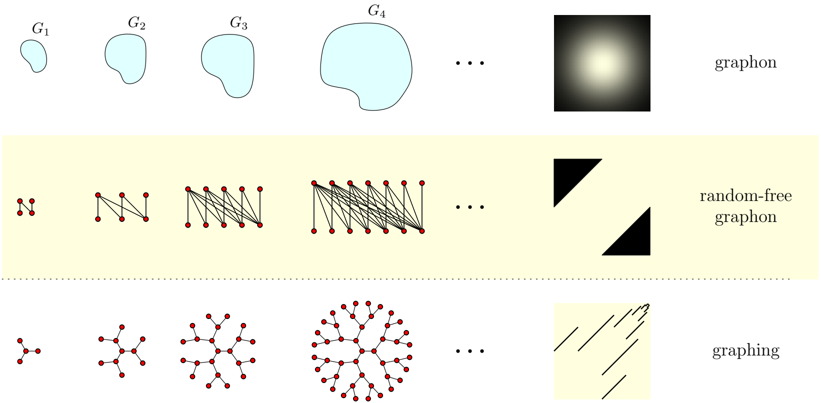
- Limit as a *graphon* (Lovász–Szegedy)

$$\text{symmetric } W : [0,1] \times [0,1] \to [0,1]$$

(up to weak-equivalence)

- Limit as an exchangeable random infinite graph (Aldous–Hoover–Kallenberg, Diaconis–Janson).

Limit objects: random-free case



Random-free graphons & Borel graphs

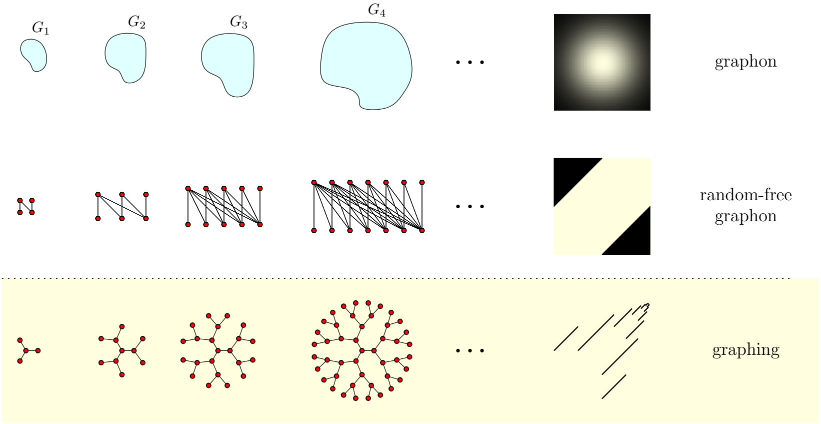
Definition

A graphon is *random-free* if it is a.e. $\{0, 1\}$-valued.
A *Borel graph* is a graph on a standard probability space, whose edge set is measurable.

Connections with ...

- Vapnik–Chervonenkis dimension (Lovász-Szegedy)
- δ_1-metric (Pikhurko)
- entropy (Aldous, Janson, Hatami & Norine)
- class speed (Chatterjee, Varadhan)

Limit objects: bounded degree case



Bounded degree graphs: BS-convergence

- Convergence of

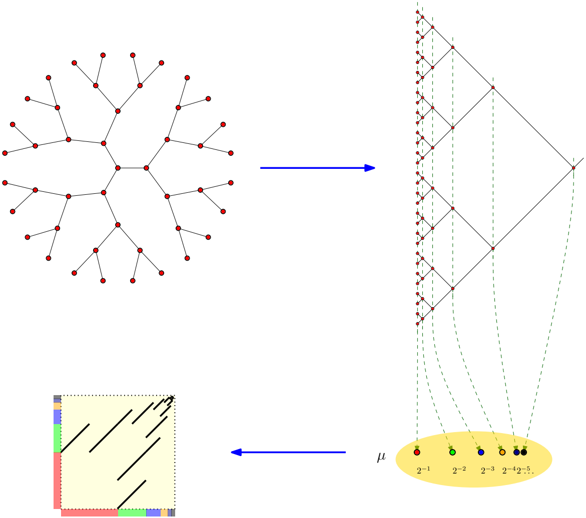
$$\frac{|\{v, B_d(G_n, v) \simeq (F, r)\}|}{|G_n|}.$$

- Limit as a *graphing* = Borel graph satisfying the *Mass Transport Principle* (Aldous–Lyons, Elek)

$$\forall A, B \in \Sigma \qquad \int_A \mathrm{d}_B(x)\,\mathrm{d}\nu(x) = \int_B \mathrm{d}_A(x)\,\mathrm{d}\nu(x).$$

- Limit as a unimodular distribution on rooted connected countable graphs (Benjamini–Schramm).

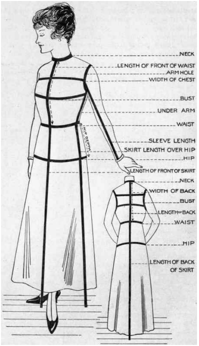
BS-convergence



Resume

Dense	Bounded degree
L-convergence	BS-convergence
conv. of $\frac{\hom(F,G_n)}{\lvert G_n\rvert^{\lvert F\rvert}}$	conv. of $\frac{\lvert\{v,\ B_d(G_n,v)\simeq(F,r)\}\rvert}{\lvert G_n\rvert}$
Graphon	Graphing
Exchangeable random infinite graph	Unimodular distribution of rooted connected countable graphs
edge density/regularity	structure

More statistics



Probabilistic approach of properties

Definition (Stone pairing)

Let ϕ be a first-order formula with p free variables and let $G = (V, E)$ be a graph.

The *Stone pairing* of ϕ and G is

$$\langle \phi, G \rangle = \Pr(G \models \phi(X_1, \ldots, X_p)),$$

for independently and uniformly distributed $X_i \in V$.

That is:

$$\langle \phi, G \rangle = \frac{\left|\{(v_1, \ldots, v_p) \in V^p :\ G \models \phi(v_1, \ldots, v_p)\}\right|}{|V|^p},$$

Structural Limits

Definition

A sequence (G_n) is FO-*convergent* if, for every $\phi \in \mathrm{FO}$, the sequence $\langle \phi, G_1 \rangle, \ldots, \langle \phi, G_n \rangle, \ldots$ is convergent.

> In other words, (G_n) is FO-convergent if, for every first-order formula $\phi \in \mathrm{FO}$, the probability that G_n satisfies ϕ for a random assignment of the free variables converges.

Structural Limits

Definition

Let X be a fragment of FO.
A sequence (G_n) is *X-convergent* if, for every $\phi \in X$, the sequence $\langle \phi, G_1 \rangle, \dots, \langle \phi, G_n \rangle, \dots$ is convergent.

> In other words, (G_n) is X-convergent if, for every first-order formula $\phi \in$ X, the probability that G_n satisfies ϕ for a random assignment of the free variables converges.

Special Fragments

QF	Quantifier free formulas	L-limits
FO_0	Sentences	Elementary limits
$\mathrm{FO}^{\mathrm{local}}$	Local formulas	(BS-limits)
FO	All first-order formulas	FO-limits

Structural Limits

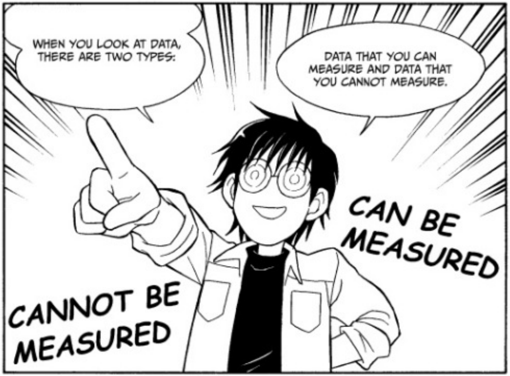
Boolean algebra $\mathcal{B}(X)$	Stone Space $S(\mathcal{B}(X))$
Formula ϕ	Continuous function f_ϕ
Vertices $v_1, \ldots, v_p, \ldots$	Type T of $v_1, \ldots, v_p, \ldots$
Graph G	statistics of types =probability measure μ_G
$\langle \phi, G \rangle$	$\int f_\phi(T) \, \mathrm{d}\mu_G(T)$
X-convergent (G_n)	weakly convergent μ_{G_n}
$\Gamma = \mathrm{Aut}(\mathcal{B}(X))$	Γ-invariant measure

Modelings

Modelings

Definition

A *modeling* **A** is a graph on a standard probability space s.t. every first-order definable set is measurable.



From "Manga Guide to Statistics", Shin Takahashi, 2008

Basic interpretations

$$G = (V, E) \;\mapsto\; I(G) = (V, E')$$
$$E' = \{(x, y) : G \models \theta(x, y)\}.$$

Examples

$$\begin{aligned} x \not\sim y \quad &\longrightarrow \quad I(G) = \overline{G} \\ (x \sim y) \vee (\exists z\ (x \sim z) \wedge (z \sim y)) \quad &\longrightarrow \quad I(G) = G^2 \end{aligned}$$

Basic interpretations

$$G = (V, E) \;\mapsto\; I(G) = (V, E')$$
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Properties

$$\exists I^\star : \text{FO} \to \text{FO}, \quad \langle \phi, I(G) \rangle = \langle I^\star(\phi), G \rangle$$

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G_n is FO-convergent $\implies$ $I(G_n)$ is FO-convergent.

Basic interpretations

$$G = (V, E) \;\mapsto\; I(G) = (V, E')$$

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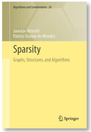
Properties

$$\exists I^\star : \mathrm{FO} \to \mathrm{FO}, \quad \langle \phi, I(G) \rangle = \langle I^\star(\phi), G \rangle$$

$$G_n \text{ is FO-convergent } \implies I(G_n) \text{ is FO-convergent.}$$

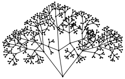
$$G_n \xrightarrow{\mathrm{FO}} \mathbf{A} \implies I(G_n) \xrightarrow{\mathrm{FO}} I(\mathbf{A}).$$

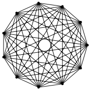
Modelings as FO-limits?



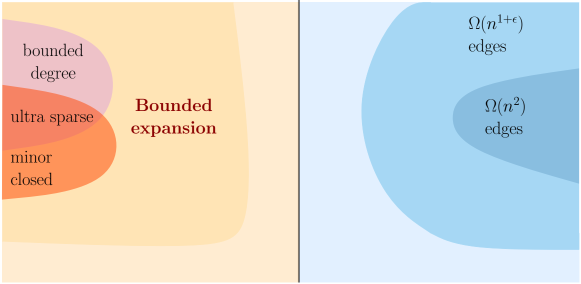
Theorem (Nešetřil, POM 2013)

If a monotone class $\mathcal{C}$ has modeling FO-limits then the class $\mathcal{C}$ is nowhere dense.



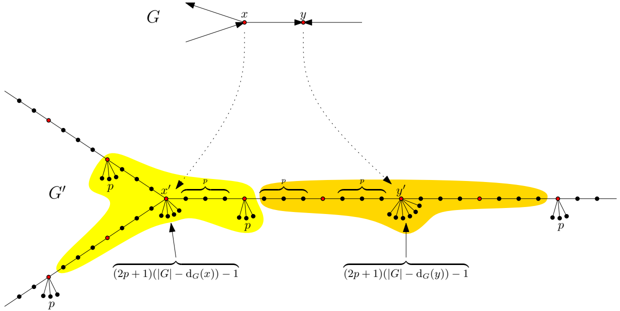


Nowhere dense | **Somewhere dense**



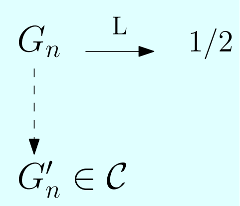
Proof (sketch)

- Assume $\mathcal{C}$ is somewhere dense. There exists $p \geq 1$ such that $\mathrm{Sub}_p(K_n) \in \mathcal{C}$ for all n;
- For an oriented graph G, define $G' \in \mathcal{C}$:



- ∃ basic interpretation I, such that for every graph G, $I(G') \cong G[k(G)] \stackrel{\text{def}}{=} G^+$, where $k(G) = (2p+1)|G|$.

Proof (sketch)



Proof (sketch)

$$G_n \xrightarrow{\text{L}} 1/2$$

$$G'_n \xrightarrow{\text{FO}} \mathbf{A}$$

Proof (sketch)

$$\begin{array}{ccc} G_n & \xrightarrow{\text{L}} & 1/2 \\ & & \\ G'_n & \xrightarrow{\text{FO}} & \mathbf{A} \\ I \downarrow & & \downarrow I \\ G_n^+ & \xrightarrow{\text{FO}} & I(\mathbf{A}) \end{array}$$

Proof (sketch)

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\begin{array}{ccc}
G_n & \xrightarrow{\text{L}} & 1/2 \\
 & & \\
G'_n & \xrightarrow{\text{FO}} & \mathbf{A} \\
I \big\downarrow & & \big\downarrow I \\
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 & \Downarrow & \\
G_n^+ & \xrightarrow{\text{L}} & W_{I(\mathbf{A})}
\end{array}
$$

Proof (sketch)

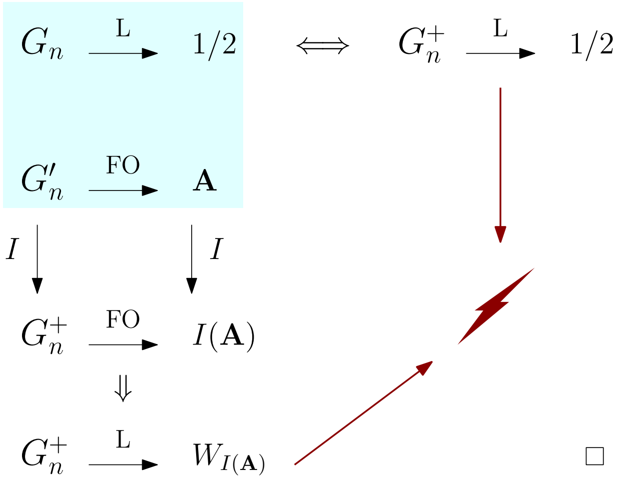
$$G_n \xrightarrow{\text{L}} 1/2 \quad \Longleftrightarrow \quad G_n^+ \xrightarrow{\text{L}} 1/2$$

$$\begin{array}{ccc} G_n' & \xrightarrow{\text{FO}} & \mathbf{A} \\ I \downarrow & & \downarrow I \\ G_n^+ & \xrightarrow{\text{FO}} & I(\mathbf{A}) \end{array}$$

$$\Downarrow$$

$$G_n^+ \xrightarrow{\text{L}} W_{I(\mathbf{A})}$$

Proof (sketch)



Modelings as FO-limits?

Theorem (Nešetřil, POM 2013)

If a monotone class $\mathcal{C}$ has modeling FO-limits then the class $\mathcal{C}$ is nowhere dense.

Conjecture (Nešetřil, POM)

Every nowhere dense class has modeling FO-limits.

- true for bounded degree graphs (Nešetřil, POM 2012)
- true for colored bounded height trees (Nešetřil, POM 2013)
- true for bounded tree-depth graphs (Nešetřil, POM 2013)

Small trees, and more

Are star forests easy?

$$G_n = \overbrace{S_{2^{2^n}(2^{-1}+2^{-n})} + \cdots + S_{2^{2^n}(2^{-i}+2^{-n})} + \cdots + S_{2^{2^n}(2^{-2^n}+2^{-n})}}^{2^n \text{ stars}}$$

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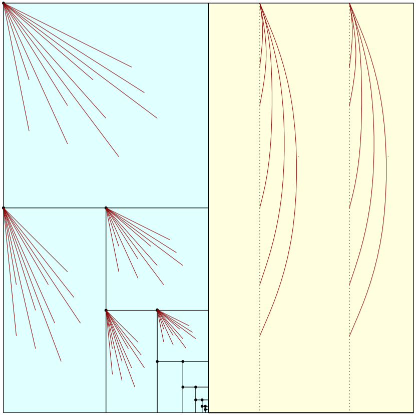

Rooted colored trees with height $\leq t$

(proof by induction on t)

1. Cut the tree into pieces via basic interpretation

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4. Consider the limit probability measure μ on $S(\mathcal{B}(\mathrm{FO}_1))$

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6. Use induction to handle big components and put everything together (Merging Lemma)

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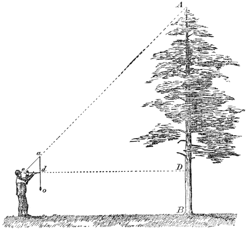
5. Pullback μ to a suitable universal measurable rooted tree and check that this actually defines a modeling FO-limit
6. Use induction to handle big components and put everything together (Merging Lemma)
7. Glue the components via basic interpretation

□

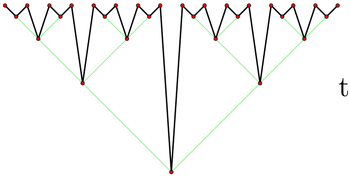
Tree-depth

Definition



The *tree-depth* td(G) of a graph G is the minimum height of a rooted forest Y s.t.

$$G \subseteq \text{Closure}(Y).$$



$$\text{td}(P_n) = \log_2(n+1)$$

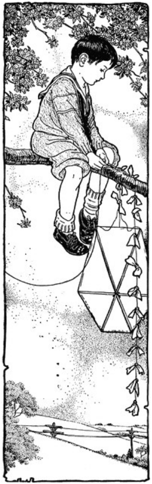
Tree-depth at most t

- Let (G_n) be an FO-convergent sequence of graphs with tree-depth $\leq t$.
- There is a basic interpretation I and rooted colored trees Y_n with height $\leq t+1$ such that $G_n = I(Y_n)$.
- By compactness, there is a subsequence $(Y_{f(n)})_{n\in\mathbb{N}}$, which is FO-convergent.
- Let $Y_{f(n)} \xrightarrow{\text{FO}} \mathbf{A}$.
- Then $G_n \xrightarrow{\text{FO}} I(\mathbf{A})$.

□

Colored Trees

- Reduction (mod countable) to countably many essentially connected sequences and a residual sequence, by cuting the trees and taking subsequence;
- For a residual sequence, construction via Stone space;
- For an essentially connected sequence, inductive construction of a modeling limit.



Thank you for your attention.