Colouring weighted hexagonal graphs

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Definitions

weighted graph $=$ pair (G, p) where

- G is a graph;
- $p : V(G) \to \mathbb{N}$ weight function.

k-colouring of (G, p): $C : V(G) \to \mathcal{P}(\{1, \dots, k\})$ such that

- $|C(v)| = p(v)$ for all $v \in V(G)$;
- $C(u) \cap C(v) = \emptyset$ for all $e \in E(G)$.

chromatic number of (G, p):

$$\chi(G, p) = \min\{k \mid (G, p) \text{ admits a } k\text{-colouring}\}$$

Clique number and chromatic number

clique number of (G, p):

$$\omega(G,p) = \max\{p(C) \mid C \text{ clique of } G\}, \text{ where } p(C) = \sum_{v \in C} p(v)$$

.

$$\omega(G,p) \leq \chi(G,p)$$

Bipartite graphs

Proposition: If G is bipartite, then $\omega(G,p) = \chi(G,p)$.
<u>*Proof:*</u> Assign to v

- $\{1, 2, \ldots, p(v)\}$ if v is in A,
- $\{\omega(G,p), \ldots, \omega(G,p) - p(v) + 1\}$ if v is in B. □

Linear-time algorithm finding optimal colouring of a weighted bipartite graph:

- Compute $\omega(G,p)$.
  $\omega(G,p) = \max\left\{\max_{v \in V(G)} p(v) \;;\max_{uv \in E(G)} p(u) + p(v)\right\}$
- Assign as above.

BUT NOT DISTRIBUTED

Bipartite graphs: 1-local algorithm

k-local algorithm: to choose its colours each vertex knows only:

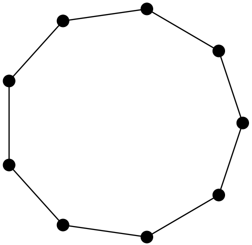
- the vertices at distance at most k from it (and their weights) .
- some precomputed fixed information independent from the weights.

- For each $a \in A$, assign $\{1, 2, \ldots, p(a)\}$ to a.
- For each vertex $b \in B$,
  Compute $\omega_1(b) = \max_{bv \in E(G)}(p(b) + p(v))$;
  Assign $\{\omega_1(b), \ldots, \omega_1(b) - p(b) + 1\}$ to b.

Odd cycles

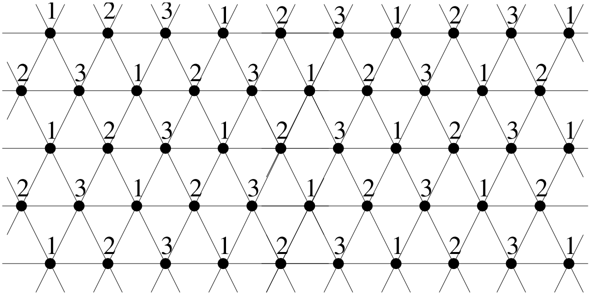
Proposition: $\chi(C_{2\ell+1}, \mathbf{k}) = \left\lceil \frac{(2\ell+1)k}{\ell} \right\rceil$

If $\ell \geq 2$, then $\omega(C_{2\ell+1}, \mathbf{k}) = 2k$.



Hexagonal graphs

hexagonal graph: induced subgraph of the triangular lattice TL.



Colouring weighted hexagonal graphs

H hexagonal graph.
$\chi(H) \leq \chi(TL) \leq 3$, so

$$\chi(H,p) \leq 3\max\{p(v) \mid v \in V(H)\} \leq 3\omega(H,p)$$

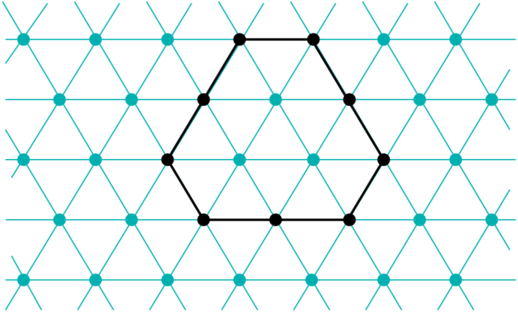
Theorem (McDiarmid and Reed): $\chi(H,p) \leq \frac{4\omega(H,p)+1}{3}$

Deciding whether $\chi(H,p) = 3$ or 4 is NP-complete.

Theorem (McDiarmid and Reed): There is a constant C s. t.

$$\chi(H,p) \leq \frac{9}{8}\omega(H,p) + C$$

Induced C_9 in the triangular lattice



$$\chi(C_9, \mathbf{k}) = \left\lceil \frac{9k}{4} \right\rceil$$

$$\omega(C_9, \mathbf{k}) = 2k$$

Proof of $\chi(H,p) \leq \frac{4\omega(H,p)+1}{3}$

Set $k = \left\lfloor \frac{\omega(H,p)+1}{3} \right\rfloor$.   3-colouring of TL: c_T.

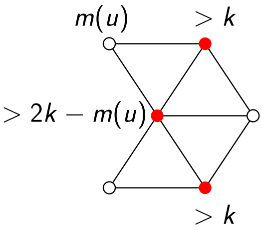
1. We use $3k$ colours: (i,j) for $i = 1,2,3$ and $j = 1,\dots,k$.
   - Assign to v the colours corresponding to its lattice colour $(c_T(v),1),\dots,(c_T(v),\min\{k,p(v)\})$.
   - $m(v) := \max\{\min\{k,p(u)\} \mid u \in N(v) \text{ and } c_T(u) = c_T(v)+1\}$
     $r(v) = \min\{p(v)-k, k-m(v)\}$.
     If $r(v) \geq 0$, then assign to v the unused colours on its right, leftup, and leftdown neighbours $(c_T(v)+1, k-r(v)+1),\dots,(c_T(v)+1,k)$.

2. U set of vertices whose demand is not yet fulfilled. For $u \in U$, $p'(u) = p(u) - 2k + m(u)$.
   Colour $(TL[U], p')$ using $\omega(TL[U],p') \leq \omega(H,p) - 2k$ colours.
   Possible because $TL[U]$ is acyclic.

Proving $TL[U]$ is acyclic

Claim: Every vertex v has at most one neighbour to its right.

- $p(u) \geq k+1$ for all $u \in U$, $\Rightarrow$ $TL[U]$ is triangle-free.
-



First distributed algorithms for hexagonal graphs

Janssen et al. '00: k-local algorithms adapted from global algorithms.

0-local: 3-competitive (fixed assignment according to c_T)

1-local: 3/2-competitive derived from Janssen et al. '99

2-local: 17/12-competitive derived from Nayaranan and Schende '97

4-local: 4/3-competitive derived from Nayaranan and Schende '97

α-competitive: using at most $\alpha \cdot \chi(H, p) + \beta$ colours for all (H, p) and some fixed β.

1-local 3/2-competitive algorithm

Idea: Decomposing TL into 3 bipartite graphs (according to c_T).

$\omega_1(v) = \omega(H[N[v]], p)$ = max. weighted clique in the neighbourhood of v.

Algorithm: For each v

- compute $\omega_1(v)$. Set $s = \lceil \omega_1(v)/2 \rceil$.
  For $i = 1, 2, 3$, set $S_i = \{i, 3+i, \dots, 3s+i-3\}$.
- if $c_T(v) = i$, then assign to v, the $\lceil p(v)/2 \rceil$ first colours of S_i and the $\lfloor p(v)/2 \rfloor$ colours of S_{i+1}.

Validity: u, v adjacent, $c_T(u) = i-1$ and $c_T(v) = i$.
$p(u) + p(v) \leq \min\{\omega_1(u), \omega_1(v)\}$.
Number of colours of S_i at u or $v \leq \min\{\omega_1(u)/2, \omega_1(v)/2\}$.
No colours is assigned to both u and v.

Better distributed algorithms for hexagonal graphs

0-local: 3-competitive (fixed assignment according to c_T)

1-local: 13/9-competitive — Chin, Zhang and Zhu. '13
17/12-competitive — Witowski '09
7/5-competitive — Witowski and Žerovnik. '10
33/24-competitive — Witowski and Žerovnik. '13

2-local: 4/3-competitive — Šparl and Žerovnik. '04

4-local: 4/3-competitive — derived from Nayaranan and Schende '97

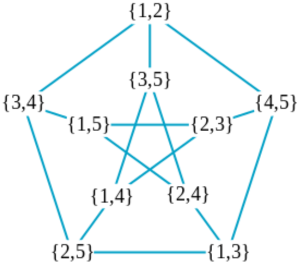
Triangle-free hexagonal graphs

H triangle-free hexagonal graph.

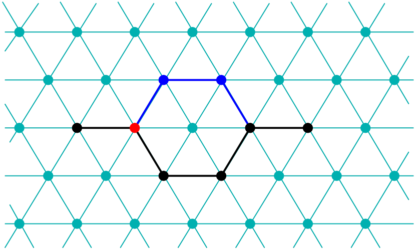
Proposition (H.): $\chi(H, \mathbf{2}) \leq 5$

5-colouring of $(H, \mathbf{2}) \equiv$ homomorphism of H into the Petersen graph $\mathcal{P}$.

Proof: By induction.



Consider the highest 3-vertex of H and the thread T going up.
A colouring of $(H - \dot{T}, \mathbf{2})$ can be extended to $(H, \mathbf{2})$.



If $length(T) = 3$, by symmetry.
If $length(T) \geq 4$, because two vertices are joined by a walk of any length at least 4 in $\mathcal{P}$.

Triangle-free hexagonal graphs

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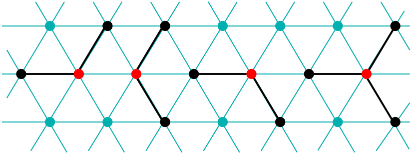
Corollary: $\chi(H,p) \leq \frac{5}{4}\omega(G,p) + 3$.

Proof:

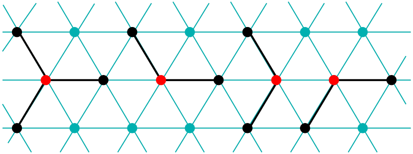
0. $U := V(H)$, $S := \emptyset$, $q = p$.
1. $S := S \cup \{u \in U : q(u) = 1\}$; $U := U \setminus \{u \in U : q(u) = 1\}$;
2. If $U \neq \emptyset$, take 5 new colours.
   a. Assign these colours to the set I of isolated vertices of $TL[U]$; for all $u \in I$, $q(u) := \max\{0, q(u) - 5\}$.
   b. Assign two of these colours to each vertex of $U \setminus I$ according to a 5-colouring of $(TL[U \setminus I], \mathbf{2})$. for all $u \in U$, $q(u) := q(u) - 2$.
   c. Go to 1.
3. Assign to all vertices of S a new colour according to c_T.

Different types of vertices in triangle-free hexagonal graphs

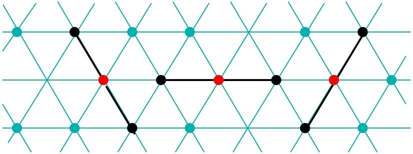
left corners



right corners

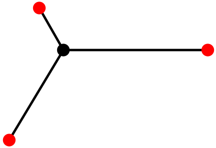


flat vertices



Triangle-free hexagonal graphs: distributed algorithm

1. Colour the left corners.
   - If $c_T(v) = 1$, then $C(v) = \{1, 2\}$.
   - If $c_T(v) = 2$, then $C(v) = \{2, 3\}$.
   - If $c_T(v) = 3$, then $C(v) = \{1, 5\}$.
2. Extend to the rest of the graph.
   Union of tristars.



On each direction of TL, every fifth vertex is special.
Cut tristars along special vertices.
Colour each piece separately in a distributed way.

$\Longrightarrow$ 8-local algorithm.

Can be improved to 2-local. (Šparl, Žerovnik)

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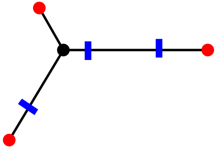

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k-local 17/12-competitive algorithm for hexagonal graphs

Fisrt phase: $\equiv$ 1-local version of first phase of McDiarmid-Reed.
For each vertex v.

- Compute $w_1(v)$. Set $k = \lceil w_1(v)/3 \rceil$.
- Assign to v the colours corresponding to its lattice colour
  $$(c_T(v), 1), \dots, (c_T(v), \min\{k, p(v)\}).$$
- $m(v) := \max\{\min\{k, p(u)\} \mid u \in N(v) \text{ and } c_T(u) = c_T(v) + 1\}$; $r(v) = \min\{p(v) - k, k - m(v)\}$.
  If $r(v) \geq 0$, then assign to v
  $$(c_T(v) + 1, k - r(v) + 1), \dots, (c_T(v) + 1, k).$$

2nd phase: k-local 5/4-comp. algo. for triangle-free graph on $(TL[U], p')$.

Uses $\omega(G, p) + \frac{5}{4}\omega(TL[U], p') + \beta \leq \frac{17}{12}\omega(G, p) + \beta'$.

Triangle-free hexagonal graphs

H triangle-free hexagonal graph.

Theorem (H.): $\chi(H, \mathbf{3}) \leq 7$

Corollary: $\chi(H, p) \leq \frac{7}{6}\omega(G, p) + 5$.

k-good: $\exists\ f : V \to \{1, \ldots, k\}$ s.t. every odd cycle has a vertex assigned i for all $1 \leq i \leq k$.

Lemma: If H is $k+1$-good, then $\chi(H, \mathbf{k}) \leq 2k + 2$.

Proof: For each $1 \leq i \leq k+1$, colour $G - f^{-1}(i)$ with 2 colours. Each vertex receives (at least) k colours. □

Sudeep & Vishwanathan: triangle-free hexagonal $\Rightarrow$ 7-good.

Conjecture: triangle-free hexagonal $\Rightarrow$ 9-good.

Triangle-free hexagonal graphs are 5-good

Lemma: (Sudeep & Vishwanathan)
Every odd cycle contains a flat vertex v s.t. $c_T(v) = i$ for all $1 \leq i \leq 3$.

5-good labelling f:

- If v is flat, then $f(v) = c_T(v)$.
- If v is right corner, then $f(v) = 4$.
- If v is left corner, then $f(v) = 5$.

Triangle-free hexagonal graphs are 7-good

Lemma: (Sudeep & Vishwanathan)
There is a partition (R_1, R_2) of the right corners s.t. every odd cycle intersects R_i, $i = 1, 2$.

7-good labelling f:

- If v is flat, then $f(v) = c_T(v)$.
- If $v \in R_i$, then $f(v) = 3 + i$.
- If $v \in L_i$, then $f(v) = 5 + i$.

Next problem to solve on hexagonal graphs

- Proving $\chi(H, p) \leq \alpha\omega(H, p) + \beta$ for $\alpha < 4/3$.

  McDiarmid-Reed Conjecture: $\alpha = 9/8$

- Finding an α-competitive distributed algorithm for colouring hexagonal graphs for $\alpha < 4/3$.

- Proving $\chi(H, p) \leq \alpha\omega(H, p) + \beta$ for $\alpha < 7/6$, when H is triangle-free.

- Finding a 7/6-competitive distributed algorithm for colouring triangle-free hexagonal graphs.