Graph polynomials determined by graph homomorphisms

Delia Garijo[1] **Andrew Goodall**[2] Jarik Nešetřil[2]

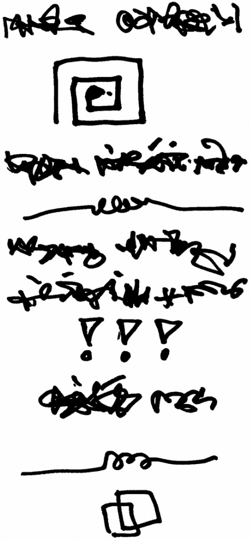
[1]University of Seville, Spain

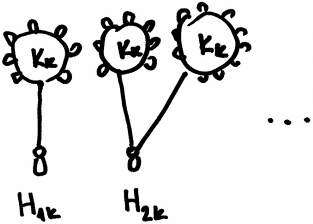
[2]Charles University, Prague

STRUCO Meeting, Abbaye des Prémontrés
15 November 2013, Pont-á-Mousson

Overview

1. Graph polynomials
2. Sequences giving graph polynomials
3. Constructions
4. A new construction
5. Open problems




K_k
K_k
K_k
...
H_{1k}
H_{2k}

Graph polynomials with a name for themselves...

- **chromatic polynomial**, $P(G;k) = P(G\backslash uv;k) - P(G/uv;k)$

Graph polynomials with a name for themselves...

- **chromatic polynomial**, $P(G;k) = P(G\backslash uv;k) - P(G/uv;k)$
- **Tutte polynomial** (universal for recurrence in $\backslash uv$ and $/uv$)

Graph polynomials with a name for themselves...

- **chromatic polynomial**, $P(G;k) = P(G\backslash uv;k) - P(G/uv;k)$
- **Tutte polynomial** (universal for recurrence in $\backslash uv$ and $/uv$)
- **Averbouch–Godlin–Makowsky polynomial** (recurrence in $\backslash uv$, $/uv$ and $-u-v$), includes **matching polynomial**

Graph polynomials with a name for themselves...

- **chromatic polynomial**, $P(G;k) = P(G\backslash uv;k) - P(G/uv;k)$
- **Tutte polynomial** (universal for recurrence in $\backslash uv$ and $/uv$)
- **Averbouch–Godlin–Makowsky polynomial** (recurrence in $\backslash uv$, $/uv$ and $-u-v$), includes **matching polynomial**
- **Tittmann–Averbouch–Makowsky** polynomial (recurrence in $\backslash v$, $/v$ and $-N[v]$), includes **independence polynomial**

Graph polynomials with a name for themselves...

- **chromatic polynomial**, $P(G;k) = P(G\backslash uv;k) - P(G/uv;k)$
- **Tutte polynomial** (universal for recurrence in $\backslash uv$ and $/uv$)
- **Averbouch–Godlin–Makowsky polynomial** (recurrence in $\backslash uv$, $/uv$ and $-u-v$), includes **matching polynomial**
- **Tittmann–Averbouch–Makowsky** polynomial (recurrence in $\backslash v$, $/v$ and $-N[v]$), includes **independence polynomial**

... polynomials all determined by counting H_k-colourings of a graph for a sequence of (multi)graphs $(H_k : k = 1, 2, \ldots)$
e.g. for $k \in \mathbb{N}$, $P(G;k)$ counts K_k-colourings

Definition

Graphs G, H.
$f : V(G) \to V(H)$ is a *homomorphism* from G to H if
$uv \in E(G) \Rightarrow f(u)f(v) \in E(H)$.

Definition

Graphs G, H.
$f : V(G) \to V(H)$ is a *homomorphism* from G to H if
$uv \in E(G) \Rightarrow f(u)f(v) \in E(H)$.

Definition

H with adjacency matrix $(a_{s,t})$, weight $a_{s,t}$ on $st \in E(H)$,

$$\hom(G, H) = \sum_{f:V(G)\to V(H)} \prod_{uv\in E(G)} a_{f(u),f(v)}.$$

Definition

Graphs G, H.
$f : V(G) \to V(H)$ is a *homomorphism* from G to H if
$uv \in E(G) \Rightarrow f(u)f(v) \in E(H)$.

Definition

H with adjacency matrix $(a_{s,t})$, weight $a_{s,t}$ on $st \in E(H)$,

$$\mathrm{hom}(G,H) = \sum_{f:V(G)\to V(H)} \prod_{uv\in E(G)} a_{f(u),f(v)}.$$

$$\begin{aligned}\mathrm{hom}(G,H) &= \#\{\text{homomorphisms from } G \text{ to } H\}\\ &= \#\{H\text{-colourings of } G\}\end{aligned}$$

when H simple ($a_{s,t} \in \{0,1\}$) or multigraph ($a_{s,t} \in \mathbb{N}$)

The main question

For sequence $(H_{k,\ell,\dots})$, when is, for all graphs G,

$$\hom(G, H_{k,\ell,\dots}) = p(G; k, \ell, \dots)$$

for polynomial $p(G)$?

Related work

Definition (by example)

"Generalized k-colourings" include proper k-colourings and:

- harmonious proper k-col'gs (pair of colours appears at most once)
- connected k-colourings (colour classes connected)
- k-colourings with small monochromatic components

Related work

Definition (by example)

"Generalized k-colourings" include proper k-colourings and:

- harmonious proper k-col'gs (pair of colours appears at most once)
- connected k-colourings (colour classes connected)
- k-colourings with small monochromatic components

Any permutation of the k colours preserves the property of being a generalized colouring.

Related work

Definition (by example)

"Generalized k-colourings" include proper k-colourings and:

- harmonious proper k-col'gs (pair of colours appears at most once)
- connected k-colourings (colour classes connected)
- k-colourings with small monochromatic components

Any permutation of the k colours preserves the property of being a generalized colouring.

Theorem (Kotek, Makowsky, Zilber, 2008)

Generalized k-colourings are counted by polynomials in k.

Related work

Definition (by example)

"Generalized k-colourings" include proper k-colourings and:

- harmonious proper k-col'gs (pair of colours appears at most once)
- connected k-colourings (colour classes connected)
- k-colourings with small monochromatic components

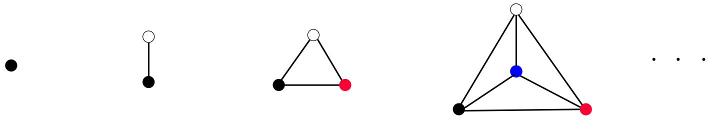
Any permutation of the k colours preserves the property of being a generalized colouring.

Theorem (Kotek, Makowsky, Zilber, 2008)

Generalized k-colourings are counted by polynomials in k.

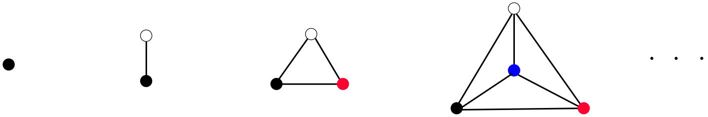
We look rather at colourings with a local restriction specified by the homomorphism target graph H_k.

Examples



(K_k)

Examples

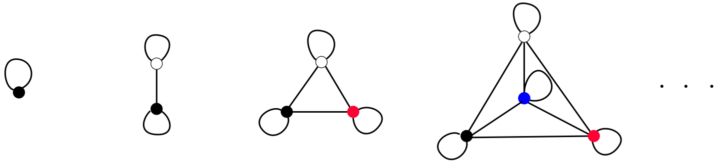


$$(K_k)$$

$$\mathrm{hom}(G, K_k) = P(G; k)$$

chromatic polynomial

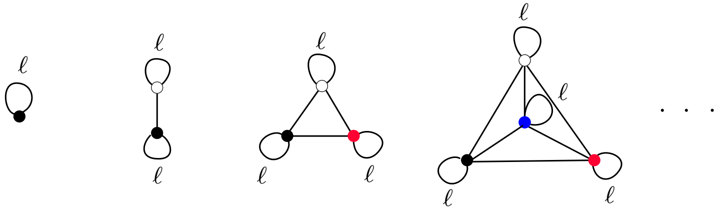
Examples



$$(K_k^1)$$

$$\hom(G, K_k^1) = k^{|V(G)|}$$

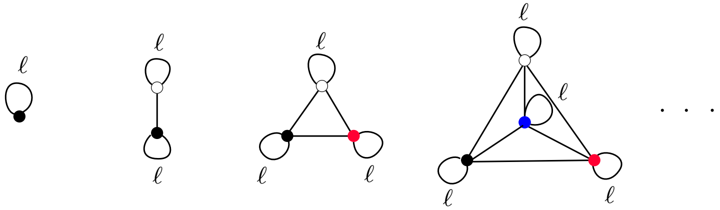
Examples



(K_k^ℓ)

$$\hom(G, K_k^\ell) = \sum_{f:V(G)\to[k]} \ell^{\#\{uv\in E(G)\,|\,f(u)=f(v)\}}$$

Examples

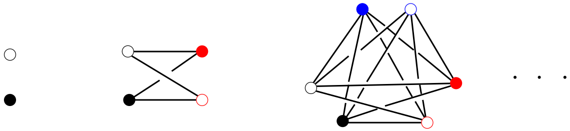


$$(K_k^\ell)$$

$$\hom(G, K_k^\ell) = \sum_{f:V(G)\to[k]} \ell^{\#\{uv\in E(G)\,|\,f(u)=f(v)\}}$$

$$= k^{c(G)}(\ell-1)^{r(G)}\,T(G; \tfrac{\ell-1+k}{\ell-1}, \ell)$$

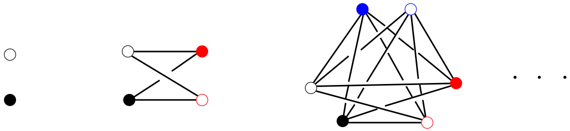
It has something to do with automorphisms...



$$(\overline{kK_2}) = (K_{2,\ldots,2})$$

$$\mathrm{Aut}(K_{2,\ldots,2}) \cong \mathrm{Sym}_k[\mathrm{Sym}_2]$$

It has something to do with automorphisms...

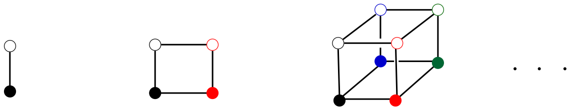


$$(\overline{kK_2}) = (K_{2,\ldots,2})$$

$$\mathrm{Aut}(K_{2,\ldots,2}) \cong \mathrm{Sym}_k[\mathrm{Sym}_2]$$

$$\mathrm{hom}(G, K_{2,\ldots,,2}) = 2^{|V(G)|}P(G;k)$$

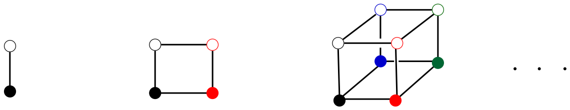
... but what precisely?



$$(K_2^{\square k}) = (Q_k) \text{ (hypercubes)}$$

$$\mathrm{Aut}(Q_k) \cong \mathrm{Sym}_k[\mathrm{Sym}_2]$$

... but what precisely?



$$(K_2^{\square k}) = (Q_k) \text{ (hypercubes)}$$

$$\mathrm{Aut}(Q_k) \cong \mathrm{Sym}_k[\mathrm{Sym}_2]$$

Proposition (Garijo, G., Nešetřil, 2013+)

$\hom(G, Q_k) = p(G; k, 2^k)$ *for bivariate polynomial* $p(G)$

Definition

(H_k) is *strongly polynomial* (in k) if $\forall G$ $\exists$ polynomial $p(G)$ such that $\hom(G, H_k) = p(G; k)$ for all $k \in \mathbb{N}$.

Definition

(H_k) is *strongly polynomial* (in k) if $\forall G$ $\exists$ polynomial $p(G)$ such that $\hom(G, H_k) = p(G; k)$ for all $k \in \mathbb{N}$.

(H_k) is *polynomial* (in k) if $\forall G$ $\exists$ polynomial $p(G)$ such that $\hom(G, H_k) = p(G; k)$ for sufficiently large k ($k \geq k_0(G)$)

Definition

(H_k) is *strongly polynomial* (in k) if $\forall G$ $\exists$ polynomial $p(G)$ such that $\hom(G, H_k) = p(G; k)$ for all $k \in \mathbb{N}$.
(H_k) is *polynomial* (in k) if $\forall G$ $\exists$ polynomial $p(G)$ such that $\hom(G, H_k) = p(G; k)$ for sufficiently large k ($k \geq k_0(G)$)

Since $\hom(G_1 \cup G_2, H) = \hom(G_1, H)\hom(G_2, H)$, suffices to consider *connected* G.

Definition

(H_k) is *strongly polynomial* (in k) if $\forall G$ $\exists$ polynomial $p(G)$ such that $\hom(G, H_k) = p(G; k)$ for all $k \in \mathbb{N}$.
(H_k) is *polynomial* (in k) if $\forall G$ $\exists$ polynomial $p(G)$ such that $\hom(G, H_k) = p(G; k)$ for sufficiently large k ($k \geq k_0(G)$)

Since $\hom(G_1 \cup G_2, H) = \hom(G_1, H)\hom(G_2, H)$, suffices to consider *connected* G.

Example

- (K_k), (K_k^1). $(\overline{kK_2})$ strongly polynomial in k
- (K_k^ℓ) strongly polynomial in k, ℓ

Definition

(H_k) is *strongly polynomial* (in k) if $\forall G$ $\exists$ polynomial $p(G)$ such that $\hom(G, H_k) = p(G; k)$ for all $k \in \mathbb{N}$.
(H_k) is *polynomial* (in k) if $\forall G$ $\exists$ polynomial $p(G)$ such that $\hom(G, H_k) = p(G; k)$ for sufficiently large k ($k \geq k_0(G)$)

Since $\hom(G_1 \cup G_2, H) = \hom(G_1, H)\hom(G_2, H)$, suffices to consider *connected* G.

Example

- (K_k), (K_k^1). $(\overline{kK_2})$ strongly polynomial in k
- (K_k^ℓ) strongly polynomial in k, ℓ
- (Q_k) not polynomial in k (but in k and 2^k)

Definition

(H_k) is *strongly polynomial* (in k) if $\forall G$ $\exists$ polynomial $p(G)$ such that $\hom(G, H_k) = p(G; k)$ for all $k \in \mathbb{N}$.
(H_k) is *polynomial* (in k) if $\forall G$ $\exists$ polynomial $p(G)$ such that $\hom(G, H_k) = p(G; k)$ for sufficiently large k ($k \geq k_0(G)$)

Since $\hom(G_1 \cup G_2, H) = \hom(G_1, H)\hom(G_2, H)$, suffices to consider *connected* G.

Example

- (K_k), (K_k^1). $(\overline{kK_2})$ strongly polynomial in k
- (K_k^ℓ) strongly polynomial in k, ℓ
- (Q_k) not polynomial in k (but in k and 2^k)
- (C_k), (P_k) polynomial in k

Subgraph criterion for strongly polynomial

$$\hom(G, H_k) = \sum_{\substack{S \subseteq H_k \\ |V(S)| \leq |V(G)|}} \mathrm{sur}_{\mathsf{V,E}}(G, S)$$

$$= \sum_{S/\cong} \mathrm{sur}_{\mathsf{V,E}}(G, S)\ \#\{\text{copies of } S \text{ in } H_k\}$$

Assuming G connected, homomorphic image S also connected

Subgraph criterion for strongly polynomial

$$\hom(G, H_k) = \sum_{\substack{S \subseteq H_k \\ |V(S)| \leq |V(G)|}} \mathrm{sur}_{\mathsf{V,E}}(G, S)$$

$$= \sum_{S/\cong} \mathrm{sur}_{\mathsf{V,E}}(G, S) \; \#\{\text{copies of } S \text{ in } H_k\}$$

Assuming G connected, homomorphic image S also connected

Proposition (de la Harpe & Jaeger, 1995)

- (H_k) *strongly polynomial in* k $\Leftrightarrow$ $\forall \text{connected } S \; \#\{\text{subgraphs} \cong S \text{ in } H_k\}$ *is polynomial in* k

Subgraph criterion for strongly polynomial

$$\hom(G, H_k) = \sum_{\substack{S \subseteq_{\text{ind}} H_k \\ |V(S)| \leq |V(G)|}} \mathrm{sur}_V(G, S)$$

$$= \sum_{S/\cong} \mathrm{sur}_V(G, S) \, \#\{\text{induced copies of } S \text{ in } H_k\}$$

when H_k simple.

Proposition (de la Harpe & Jaeger 1995)

- (H_k) *strongly polynomial in* $k \Leftrightarrow$ $\forall \text{connected } S \ \#\{\text{subgraphs} \cong S \text{ in } H_k\}$ *polynomial in* k *for all* $k \in \mathbb{N}$
- *can replace subgraphs* $\cong S$ *by induced subgraphs* $\cong S$ *when* (H_k) *simple graphs*

Subgraph criterion for strongly polynomial

$$\hom(G,H_k) = \sum_{\substack{S\subseteq_{\mathrm{ind}} H_k \\ |V(S)|\le|V(G)|}} \mathrm{sur}_{\mathsf{V}}(G,S)$$

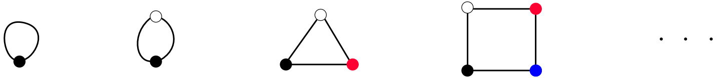
$$= \sum_{S/\cong} \mathrm{sur}_{\mathsf{V}}(G,S)\ \#\{\text{induced copies of } S \text{ in } H_k\}$$

when H_k simple. (for each S want this polynomial in k)

Proposition (de la Harpe & Jaeger 1995)

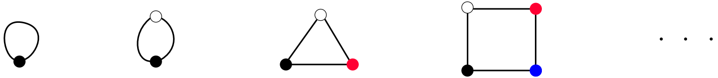
- *(H_k) strongly polynomial in k ⇔ $\forall$connected S $\#\{\text{subgraphs} \cong S \text{ in } H_k\}$ polynomial in k for all $k \in \mathbb{N}$*
- *can replace subgraphs $\cong S$ by induced subgraphs $\cong S$ when (H_k) simple graphs*

Polynomial but not strongly polynomial



(C_k)

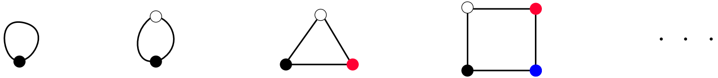
Polynomial but not strongly polynomial



$$(C_k)$$

$$\hom(G, C_k) = \sum_{1 \leq j \leq \min\{|V(G)|, k-1\}} \mathrm{sur}_{\mathsf{V}}(G, P_j)\ k + \mathrm{sur}_{\mathsf{V}}(G, C_k)$$

Polynomial but not strongly polynomial

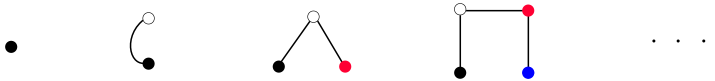


(C_k)

$$\hom(G, C_k) = \sum_{1 \le j \le \min\{|V(G)|, k-1\}} \mathrm{sur}_{\mathsf{V}}(G, P_j)\ k + \mathrm{sur}_{\mathsf{V}}(G, C_k)$$

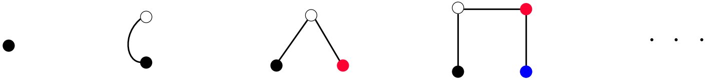
$\hom(C_3, C_3) = 6$, $\hom(C_3, C_k) = 0$ when $k = 2$ or $k \geq 4$

Polynomial but not strongly polynomial



(P_k)

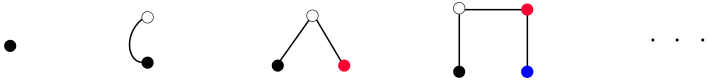
Polynomial but not strongly polynomial



(P_k)

$$\hom(G, P_k) = \sum_{1 \leq j \leq \min\{|V(G)|, k\}} \mathrm{sur}_{\mathsf{V}}(G, P_j)\,(k - j + 1)$$

Polynomial but not strongly polynomial



(P_k)

$$\hom(G, P_k) = \sum_{1 \leq j \leq \min\{|V(G)|, k\}} \mathrm{sur}_V(G, P_j)\,(k-j+1)$$

$\hom(P_4, P_2) = 2$, and $\hom(P_4, P_k) = 8k - 16$ for $k \geq 3$

Proposition (de la Harpe & Jaeger, 1995; Garijo, G., Nešetřil, 2013+)

If (H_k) *strongly polynomial,* H_k *simple, then*

- $(\overline{H_k})$
- $(L(H_k))$

strongly polynomial.
Also, (ℓH_k) *strongly polynomial in* k, ℓ*.*

Proposition (de la Harpe & Jaeger, 1995; Garijo, G., Nešetřil, 2013+)

If (H_k) strongly polynomial, H_k simple, then

- $(\overline{H_k})$
- $(L(H_k))$

strongly polynomial.
Also, (ℓH_k) strongly polynomial in k, ℓ.

Proposition (Garijo, G., Nešetřil, 2013+)

If (H_k) strongly polynomial, at most one loop each vertex of H_k, then

- (H_k^0) *(remove all loops)*
- (H_k^1) *(add loops to make 1 loop each vertex)*

strongly polynomial.
More generally, (H_k^ℓ) strongly polynomial in k, ℓ.

Proposition

If (F_j), (H_k) *strongly polynomial, then*

- $(F_j \cup H_k)$
- $(F_j + H_k)$

strongly polynomial in j, k.

Example

Beginning with trivial strongly polynomial sequence (K_1), following strongly polynomial:

Example

Beginning with trivial strongly polynomial sequence (K_1), following strongly polynomial:

- multiple: $(kK_1) = (\overline{K_k})$

Example

Beginning with trivial strongly polynomial sequence (K_1), following strongly polynomial:

- multiple: $(kK_1) = (\overline{K_k})$
- complement: (K_k) *(chromatic polynomial)*

Example

Beginning with trivial strongly polynomial sequence (K_1), following strongly polynomial:

- multiple: $(kK_1) = (\overline{K_k})$
- complement: (K_k) *(chromatic polynomial)*
- loop-addition: (K_k^ℓ) *(Tutte polynomial)*

Example

Beginning with trivial strongly polynomial sequence (K_1), following strongly polynomial:

- multiple: $(kK_1) = (\overline{K_k})$
- complement: (K_k) *(chromatic polynomial)*
- loop-addition: (K_k^ℓ) *(Tutte polynomial)*
- join: $(K_{k-j}^1 + K_j^\ell)$ *(Averbouch–Godlin–Makowsky polynomial)*

Example

Beginning with trivial strongly polynomial sequence (K_1), following strongly polynomial:

- multiple: $(kK_1) = (\overline{K_k})$
- complement: (K_k) *(chromatic polynomial)*
- loop-addition: (K_k^ℓ) *(Tutte polynomial)*
- join: $(K_{k-j}^1 + K_j^\ell)$ *(Averbouch–Godlin–Makowsky polynomial)*

$$\hom(G, K_{k-j}^1 + K_j^\ell) = \xi(G; k, \ell-1, -j(\ell-1))$$

Three-term recurrence: for $uv \in E(G)$,

$$\xi(G) = a\xi(G/uv) + b\xi(G\backslash uv) + c\xi(G - u - v)$$

Definition

Given simple graph H, set of graphs $\{F_v : v \in V(H)\}$, the *composition* $H[\{F_v : v \in V(H)\}]$ is formed by

- disjoint union of $\{F_v : v \in V(H)\}$,
- join F_u and F_v whenever $uv \in E(H)$

Definition

Given simple graph H, set of graphs $\{F_v : v \in V(H)\}$, the *composition* $H[\{F_v : v \in V(H)\}]$ is formed by

- disjoint union of $\{F_v : v \in V(H)\}$,
- join F_u and F_v whenever $uv \in E(H)$

Proposition (de la Harpe & Jaeger, 1995)

If $(F_{v;k_v})$ *strongly polynomial sequence in* k_v*, each* $v \in V(H)$*, then* $(H[\{F_{v;k_v}\}])$ *strongly polynomial in* $(k_v : v \in V(H))$.

Definition

Given simple graph H, set of graphs $\{F_v : v \in V(H)\}$, the *composition* $H[\{F_v : v \in V(H)\}]$ is formed by

- disjoint union of $\{F_v : v \in V(H)\}$,
- join F_u and F_v whenever $uv \in E(H)$

Proposition (de la Harpe & Jaeger, 1995)

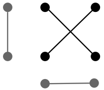
If $(F_{v;k_v})$ strongly polynomial sequence in k_v, each $v \in V(H)$, then $(H[\{F_{v;k_v}\}])$ strongly polynomial in $(k_v : v \in V(H))$.

Example

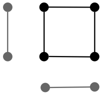
- $K_r[\{\overline{K_{k_1}}, \ldots, \overline{K_{k_r}}\}] \cong K_{k_1,\ldots,k_r}$ (*complete r-partite graph*)
- $F_{v;k_v} = F_k$ all $v \in V(H)$ gives lexicographic product $H[F_k]$

Graph products: direct, cartesian, lexicographic

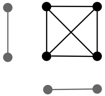
Graphs H, H', $u, v \in V(H)$, $u', v' \in V(H')$



$H \times H'$ $u \sim u'$ and $v \sim v'$



$H \,\square\, H'$ $u = v$ and $u' \sim v'$,
or $u \sim v$ and $u' = v'$



$H[H']$ $u \sim v$,
or $u = v$ and $u' \sim v'$

Proposition (Garijo, G., Nešetřil, 2013+)

If (F_j) *and* (H_k) *strongly polynomial, then*

- $(F_j \times H_k)$
- $(F_j[H_k])$

strongly polynomial in j, k.

Proposition (Garijo, G., Nešetřil, 2013+)

If (F_j) *and* (H_k) *strongly polynomial, then*

- $(F_j \times H_k)$
- $(F_j[H_k])$

strongly polynomial in j, k.

Question

Strongly polynomial:

- $(\overline{K_j} + \overline{K_k}) = (K_{j,k})$
- $(L(K_{j,k})) = (K_j \Box K_k)$ *(Rook's graph)*

Proposition (Garijo, G., Nešetřil, 2013+)

If (F_j) and (H_k) strongly polynomial, then

- $(F_j \times H_k)$
- $(F_j[H_k])$

strongly polynomial in j, k.

Question

Strongly polynomial:

- $(\overline{K_j} + \overline{K_k}) = (K_{j,k})$
- $(L(K_{j,k})) = (K_j \square K_k)$ *(Rook's graph)*

If (F_j), (H_k) strongly polynomial, is then $(F_j \square H_k)$ also?

Proposition (Garijo, G., Nešetřil, 2013+)

If (F_j) *and* (H_k) *strongly polynomial, then*

- $(F_j \times H_k)$
- $(F_j[H_k])$

strongly polynomial in j, k.

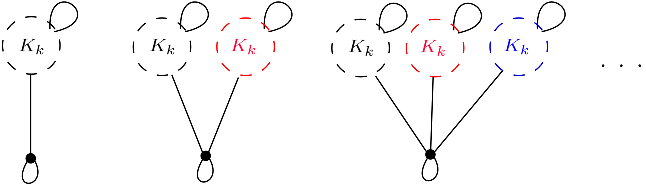
Question

Strongly polynomial:

- $(\overline{K_j} + \overline{K_k}) = (K_{j,k})$
- $(L(K_{j,k})) = (K_j \Box K_k)$ *(Rook's graph)*

If (F_j), (H_k) strongly polynomial, is then $(F_j \Box H_k)$ also?
(Yes – ask Patrice)

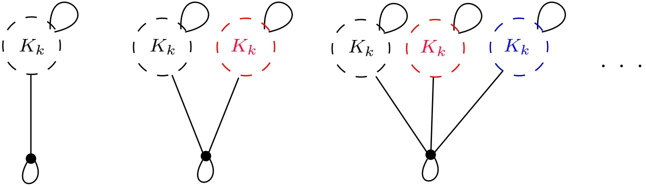
A new type of strongly polynomial sequence



$$H_{j,k} = K_{1,j}[\{K_1^1\} \cup \{K_k^1 \text{ on leaves}\}]$$

$$\hom(G, H_{j,k}) = \sum_{U \subseteq V(G)} j^{c(G[U])} k^{|U|}$$

A new type of strongly polynomial sequence



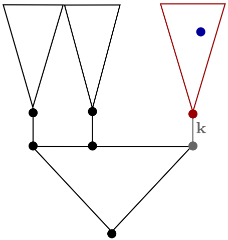
$$H_{j,k} = K_{1,j}[\{K_1^1\} \cup \{K_k^1 \text{ on leaves}\}]$$

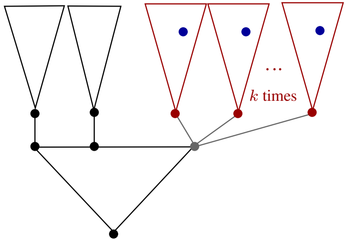
$$\hom(G, H_{j,k}) = \sum_{U \subseteq V(G)} j^{c(G[U])} k^{|U|}$$

Tittmann–Averbouch–Godlin polynomial

(includes independence polynomial, satisfies three-term recurrence)

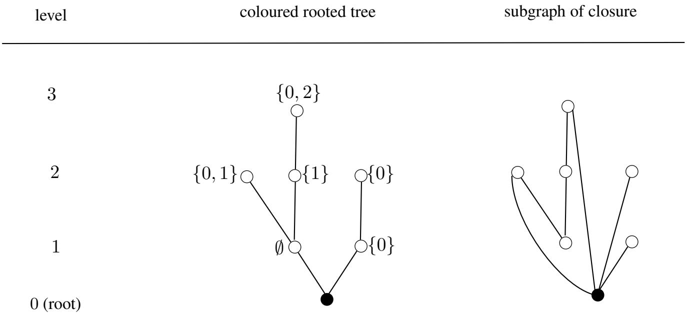
Branching coloured rooted trees



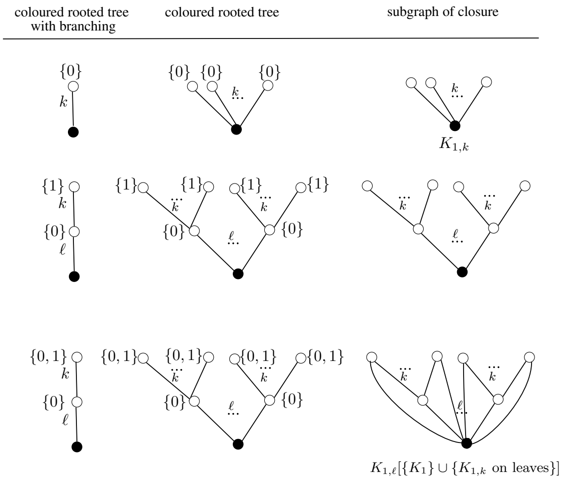


"*k*-branching" at edge of coloured rooted tree

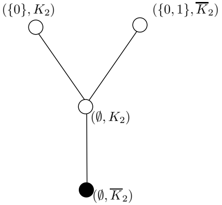
Colours encoding subgraph of closure of rooted tree



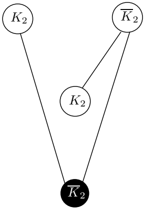
(1) Branching rooted tree encoding subgraph of closure



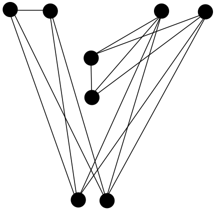
(2) Colours encoding subgraph along with ornaments



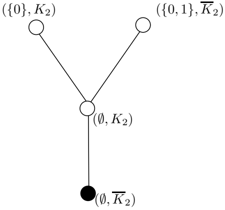
Ornamented graph



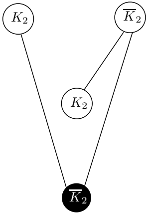
Composition



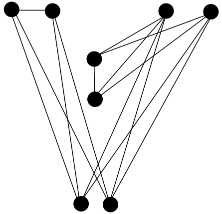
(2) Colours encoding subgraph along with ornaments

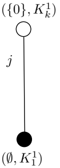


Ornamented graph

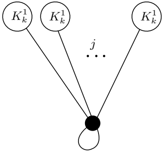


Composition

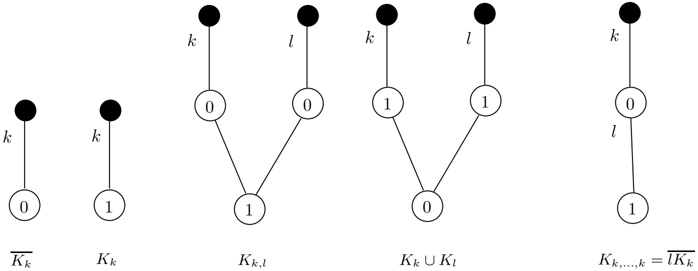




(Tittmann–Averbouch–Makowsky polynomial)



(3) Colours encoding cographs by cotrees



leaves = vertices of cograph

0 = disjoint union, 1 = join

Theorem (Garijo, G., Nešetřil, 2013+)

- *Coloured rooted tree* T *representing graph* H
- $k, \ell, \ldots$ *branching variables on edges of* T
- *after* k*-branching,* ℓ*-branching, ..., obtain coloured rooted tree representing graph* $H_{k,\ell,\ldots}$

Theorem (Garijo, G., Nešetřil, 2013+)

- *Coloured rooted tree* T *representing graph* H
- $k, \ell, \dots$ *branching variables on edges of* T
- *after* k*-branching,* ℓ*-branching, …, obtain coloured rooted tree representing graph* $H_{k,\ell,\dots}$

Then $(H_{k,\ell,\dots})$ *strongly polynomial in* $k, \ell, \dots$.

Theorem (Garijo, G., Nešetřil, 2013+)

- *Coloured rooted tree T representing graph H*
- $k, \ell, \ldots$ *branching variables on edges of T*
- *after k-branching, ℓ-branching, $\ldots$, obtain coloured rooted tree representing graph $H_{k,\ell,\ldots}$*

Then $(H_{k,\ell,\ldots})$ strongly polynomial in $k, \ell, \ldots$.

Example

(1) H as a subgraph of closure of T,
colour $s \in V(T) = V(H)$ subset of $\{0, 1, \ldots, \mathrm{height}(T)\}$

Theorem (Garijo, G., Nešetřil, 2013+)

- *Coloured rooted tree T representing graph H*
- *$k, \ell, \ldots$ branching variables on edges of T*
- *after k-branching, ℓ-branching, $\ldots$, obtain coloured rooted tree representing graph $H_{k,\ell,\ldots}$*

Then $(H_{k,\ell,\ldots})$ strongly polynomial in $k, \ell, \ldots$.

Example

(1) H as a subgraph of closure of T, colour $s \in V(T) = V(H)$ subset of $\{0, 1, \ldots, \mathrm{height}(T)\}$

(2) ornamented version of (1), strongly poly'l seq. $(F_{s;j_s})$ each vertex $s \in V(H)$, colour as in (1) paired with $F_{s;j_s}$

Theorem (Garijo, G., Nešetřil, 2013+)

- *Coloured rooted tree* T *representing graph* H
- $k, \ell, \ldots$ *branching variables on edges of* T
- *after* k*-branching,* ℓ*-branching,* $\ldots$*, obtain coloured rooted tree representing graph* $H_{k,\ell,\ldots}$

Then $(H_{k,\ell,\ldots})$ *strongly polynomial in* $k, \ell, \ldots$.

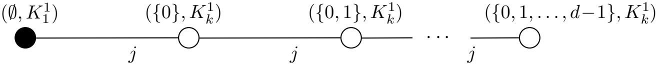
Example

(1) H as a subgraph of closure of T,
colour $s \in V(T) = V(H)$ subset of $\{0, 1, \ldots, \mathrm{height}(T)\}$

(2) ornamented version of (1), strongly poly'l seq. $(F_{s;j_s})$ each vertex $s \in V(H)$, colour as in (1) paired with $F_{s;j_s}$

(3) cotree T encoding of cograph H,
colour non-leaf of T from $\{\cup, +\}$, leaves of $T = V(H)$

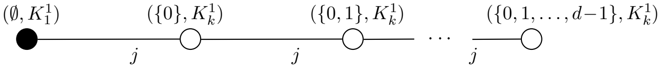
coloured rooted tree encoding graph $H_{j,k}$



(ornamented closure of perfect j-ary tree of depth d)

$$\hom(G, H_{j,k}) = \sum_{\emptyset \subseteq W_1 \subseteq W_2 \subseteq \cdots \subseteq W_d \subseteq V} j^{|W_d|} k^{\sum_{1 \le \ell \le d} c(G[W_\ell])}$$

coloured rooted tree encoding graph $H_{j,k}$



(ornamented closure of perfect j-ary tree of depth d)

$$\hom(G, H_{j,k}) = \sum_{\emptyset \subseteq W_1 \subseteq W_2 \subseteq \cdots \subseteq W_d \subseteq V} j^{|W_d|} k^{\sum_{1 \leq \ell \leq d} c(G[W_\ell])}$$

Question

This bivariate polynomial generalizes the Tittmann– Averbouch–Makowsky polynomial (case $d = 1$).
Properties? Evaluations?

Definition

Generalized Johnson graph $J_{k,\ell,D}$, $D \subseteq \{0, 1, \dots, \ell\}$
vertices $\binom{[k]}{\ell}$, edge uv when $|u \cap v| \in D$

Definition

Generalized Johnson graph $J_{k,\ell,D}$, $D \subseteq \{0, 1, \ldots, \ell\}$
vertices $\binom{[k]}{\ell}$, edge uv when $|u \cap v| \in D$

- Johnson graphs $D = \{k - 1\}$
- Kneser graphs $D = \{0\}$

Definition

Generalized Johnson graph $J_{k,\ell,D}$, $D \subseteq \{0, 1, \ldots, \ell\}$
vertices $\binom{[k]}{\ell}$, edge uv when $|u \cap v| \in D$

- Johnson graphs $D = \{k - 1\}$
- Kneser graphs $D = \{0\}$

Proposition (de la Harpe & Jaeger, 1995; Garijo, G., Nešetřil, 2013+)

For every ℓ, D, sequence $(J_{k,\ell,D})$ is strongly polynomial in k.

Definition

Generalized Johnson graph $J_{k,\ell,D}$, $D \subseteq \{0, 1, \ldots, \ell\}$
vertices $\binom{[k]}{\ell}$, edge uv when $|u \cap v| \in D$

- Johnson graphs $D = \{k - 1\}$
- Kneser graphs $D = \{0\}$

Proposition (de la Harpe & Jaeger, 1995; Garijo, G., Nešetřil, 2013+)

For every ℓ, D, sequence $(J_{k,\ell,D})$ is strongly polynomial in k.

Question

Can generalized Johnson graphs be generated from simpler sequences by branching coloured rooted trees?

Prime power $q = p^d \equiv 1 \pmod 4$
Payley graph P_q =Cayley($\mathbb{F}_q$, non-zero squares),
automorphism group order $dq(q-1)$

Prime power $q = p^d \equiv 1 \pmod 4$
Payley graph P_q =Cayley($\mathbb{F}_q$, non-zero squares),
automorphism group order $dq(q-1)$
Quasi-random graphs: $\mathrm{hom}(G, P_q)/\mathrm{hom}(G, G_{q,\frac{1}{2}}) \to 1$ as $q \to \infty$.

Prime power $q = p^d \equiv 1 \pmod 4$
Payley graph P_q =Cayley($\mathbb{F}_q, \text{non-zero squares}$),
automorphism group order $dq(q-1)$
Quasi-random graphs: $\hom(G, P_q)/\hom(G, G_{q,\frac{1}{2}}) \to 1$ as $q \to \infty$.

Proposition (Corollary to result of de la Harpe & Jaeger, 1995)

$\hom(G, P_q)$ *is polynomial in* q *for series-parallel* G

Prime power $q = p^d \equiv 1 \pmod 4$
Payley graph P_q =Cayley($\mathbb{F}_q, \text{non-zero squares}$),
automorphism group order $dq(q-1)$
Quasi-random graphs: $\hom(G, P_q)/\hom(G, G_{q,\frac{1}{2}}) \to 1$ as $q \to \infty$.

Proposition (Corollary to result of de la Harpe & Jaeger, 1995)

$\hom(G, P_q)$ *is polynomial in q for series-parallel G*

Questions

- Is $\hom(G, P_q)$ polynomial in q for all graphs G?

Prime power $q = p^d \equiv 1 \pmod 4$
Payley graph P_q =Cayley($\mathbb{F}_q, \text{non-zero squares}$),
automorphism group order $dq(q-1)$
Quasi-random graphs: $\hom(G, P_q)/\hom(G, G_{q,\frac{1}{2}}) \to 1$ as $q \to \infty$.

Proposition (Corollary to result of de la Harpe & Jaeger, 1995)

$\hom(G, P_q)$ *is polynomial in q for series-parallel G*

Questions

- Is $\hom(G, P_q)$ polynomial in q for all graphs G?
- Which Cayley graph sequences are (strongly) polynomial?

Prime power $q = p^d \equiv 1 \pmod 4$
Payley graph P_q =Cayley($\mathbb{F}_q$, non-zero squares),
automorphism group order $dq(q-1)$
Quasi-random graphs: $\hom(G, P_q)/\hom(G, G_{q,\frac{1}{2}}) \to 1$ as $q \to \infty$.

Proposition (Corollary to result of de la Harpe & Jaeger, 1995)

$\hom(G, P_q)$ *is polynomial in* q *for series-parallel* G

Questions

- Is $\hom(G, P_q)$ polynomial in q for all graphs G?
- Which Cayley graph sequences are (strongly) polynomial?
  e.g. For $D \subset \mathbb{N}$, sequence $(\,\mathrm{Cayley}(\mathbb{Z}_k, \pm D)\,)$ is polynomial iff D is finite or cofinite. (de la Harpe & Jaeger, 1995)

Prime power $q = p^d \equiv 1 \pmod 4)$
Payley graph P_q =Cayley($\mathbb{F}_q, \text{non-zero squares}$),
automorphism group order $dq(q-1)$
Quasi-random graphs: $\hom(G, P_q)/\hom(G, G_{q,\frac{1}{2}}) \to 1$ as $q \to \infty$.

Proposition (Corollary to result of de la Harpe & Jaeger, 1995)

$\hom(G, P_q)$ *is polynomial in q for series-parallel G*

Questions

- Is $\hom(G, P_q)$ polynomial in q for all graphs G?
- Which Cayley graph sequences are (strongly) polynomial? e.g. For $D \subset \mathbb{N}$, sequence $(\,\mathrm{Cayley}(\mathbb{Z}_k, \pm D)\,)$ is polynomial iff D is finite or cofinite. (de la Harpe & Jaeger, 1995)
- Approximation of left-convergent graph sequences like $(G_{n,\frac{1}{2}})$ by (strongly) polynomial sequences? (Recall Jarik's talk)

Further problems

- Is there a characterization of strongly polynomial sequences (H_k) by the sequence of automorphism groups $(\mathrm{Aut}(H_k))$?

Further problems

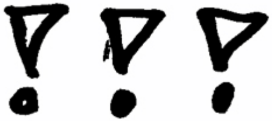
- Is there a characterization of strongly polynomial sequences (H_k) by the sequence of automorphism groups $(\mathrm{Aut}(H_k))$?
- Can (H_k) be verified to be strongly polynomial by testing $\hom(G, H_k)$ for G only in a restricted class of graphs? (yes, for connected graphs – but for a smaller class?)

Further problems

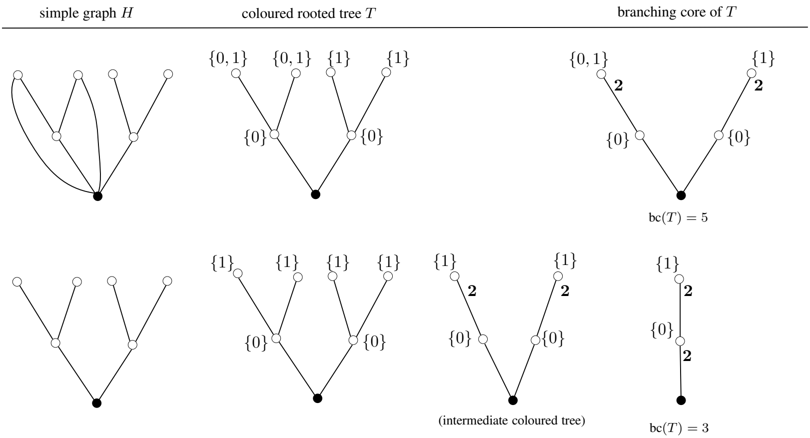
- Is there a characterization of strongly polynomial sequences (H_k) by the sequence of automorphism groups $(\mathrm{Aut}(H_k))$?
- Can (H_k) be verified to be strongly polynomial by testing $\hom(G, H_k)$ for G only in a restricted class of graphs? (yes, for connected graphs – but for a smaller class?)
- Which graph polynomials defined by strongly polynomial sequences of graphs satisfy a reduction formula (size-decreasing recurrence) like the chromatic polynomial?

Further problems

- Is there a characterization of strongly polynomial sequences (H_k) by the sequence of automorphism groups $(\mathrm{Aut}(H_k))$?
- Can (H_k) be verified to be strongly polynomial by testing $\hom(G, H_k)$ for G only in a restricted class of graphs? (yes, for connected graphs – but for a smaller class?)
- Which graph polynomials defined by strongly polynomial sequences of graphs satisfy a reduction formula (size-decreasing recurrence) like the chromatic polynomial?
- For input graph F, complexity of counting induced copies of F in H_k when (H_k) is a strongly polynomial sequence?



Branching core



Branching core size

Let $\gamma(H)$ be the smallest branching core size of a coloured rooted tree representing H.

Question

For given simple graph H, how to choose rooted tree representing H that has smallest branching core size $\gamma(H)$?

Branching core size

Let $\gamma(H)$ be the smallest branching core size of a coloured rooted tree representing H.

Question

For given simple graph H, how to choose rooted tree representing H that has smallest branching core size $\gamma(H)$?

Theorem (Garijo, G., Nešetřil, 2013+)

Let $\mathcal{H}$ be a family of graphs of bounded branching core size. Then $\mathcal{H}$ can be covered by finitely many strongly polynomial sequences (produced by branching).

Example

Kneser graph $J_{k,\ell,\{0\}}$.
$\mathrm{hom}(G, J_{k,\ell,\{0\}}) = p_\ell(G; k)$ polynomial.

Example

Kneser graph $J_{k,\ell,\{0\}}$.
$\hom(G, J_{k,\ell,\{0\}}) = p_\ell(G;k)$ polynomial.

$p_\ell(P_n;k) = \binom{k}{\ell}\binom{k-\ell}{\ell}^{n-1}$, and trees on n vertices.

Example

Kneser graph $J_{k,\ell,\{0\}}$.
$\hom(G, J_{k,\ell,\{0\}}) = p_\ell(G;k)$ polynomial.

$p_\ell(P_n;k) = \binom{k}{\ell}\binom{k-\ell}{\ell}^{n-1}$, and trees on n vertices.

$$p_\ell(C_n;k) = \binom{k-\ell}{\ell}^n + \sum_{j=1}^{\ell}(-1)^{jn}\left(\binom{k}{j} - \binom{k}{\ell-1}\right)\binom{k-\ell-j}{\ell-j}^n$$