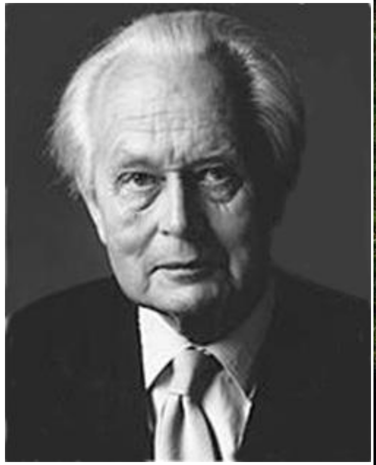
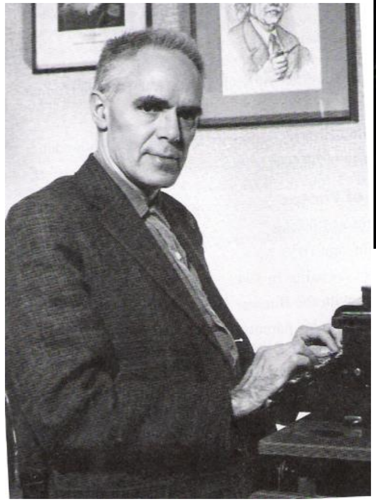
HEX & REX & T-REX & C-HEX

Piet Hein
1942

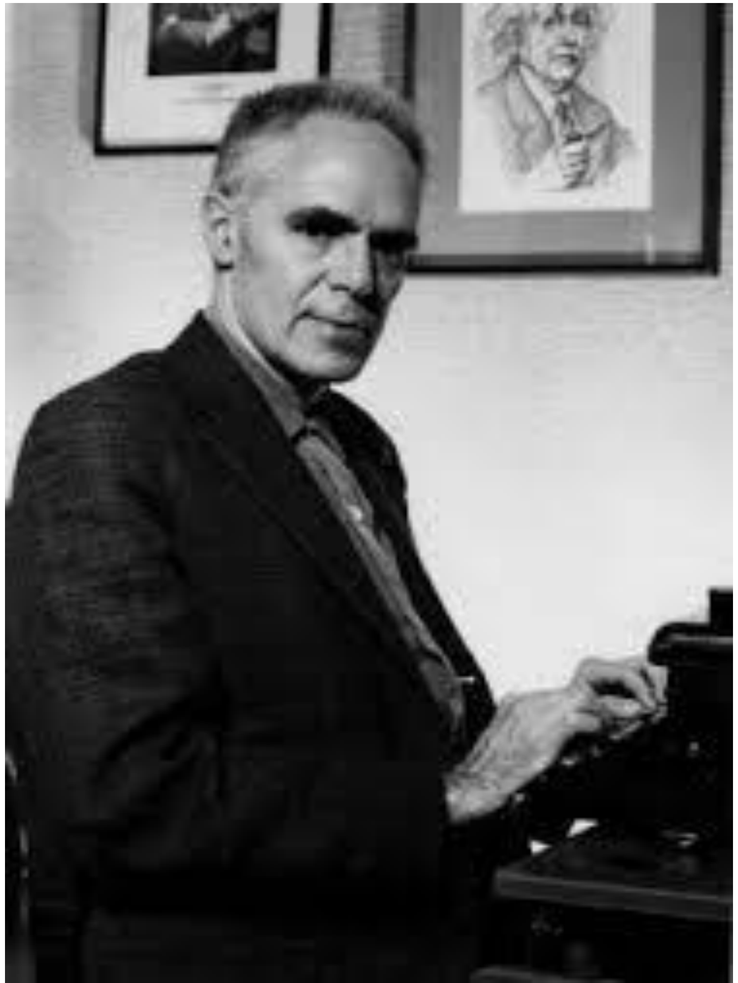


Martin
Gardner
1957



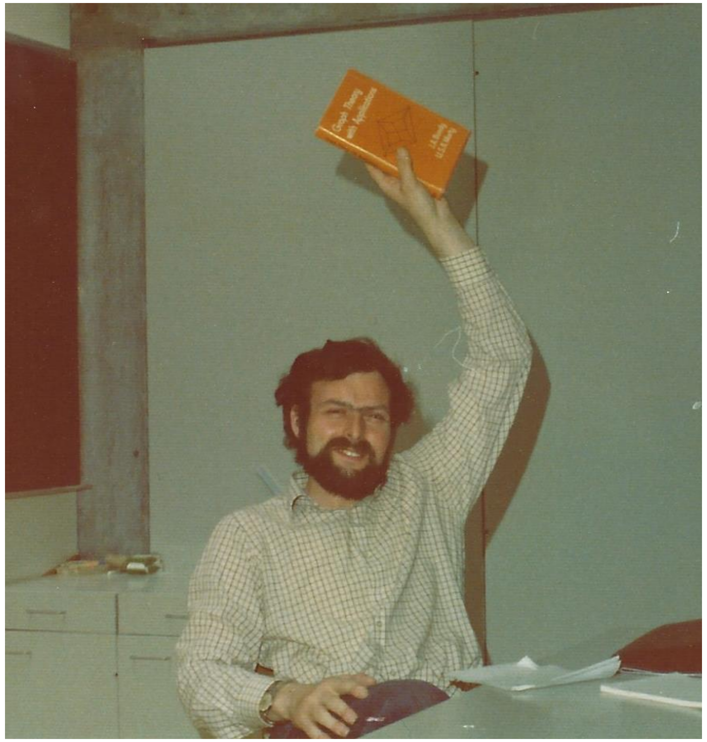
PROBLEM: FIND A FIRST WINNING MOVE FOR THE FIRST PLAYER!

Piet Hein discovered HEX in 1942, but it was only when Martin Gardner wrote about HEX in Scientific American in 1957 that it became widely known.

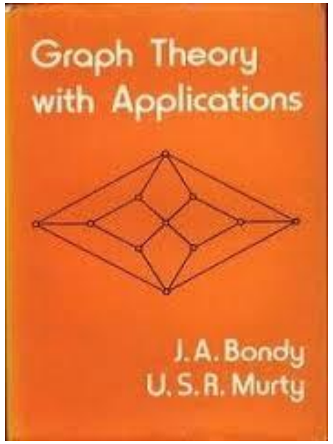


Three favorites:
Piet Hein - Martin Gardner - Adrian Bondy

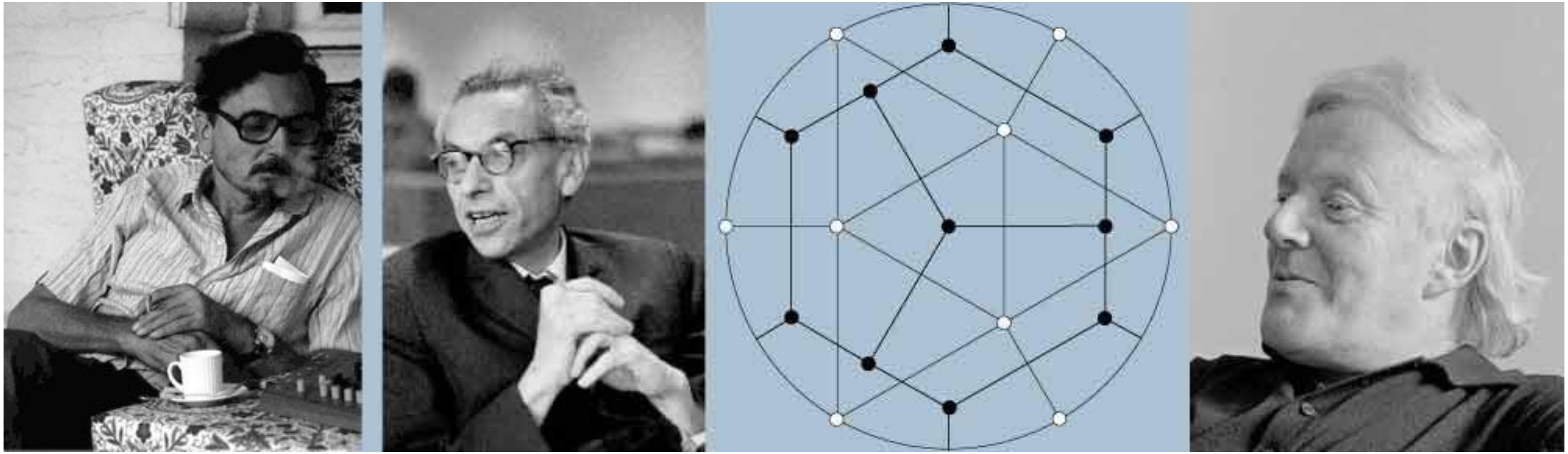
Talk by Bjarne Toft at the Bondy 70 conference in Paris 2014



- Wonderful ideas
- Beautiful arguments
- Catching presentations
- Seduction into mathematics!



From the front page of the blog for the new Bondy&Murty



Claude Berge playing Hex 1974

Claude Berge
Jean-Marie Pla
Neil Grabois
1974

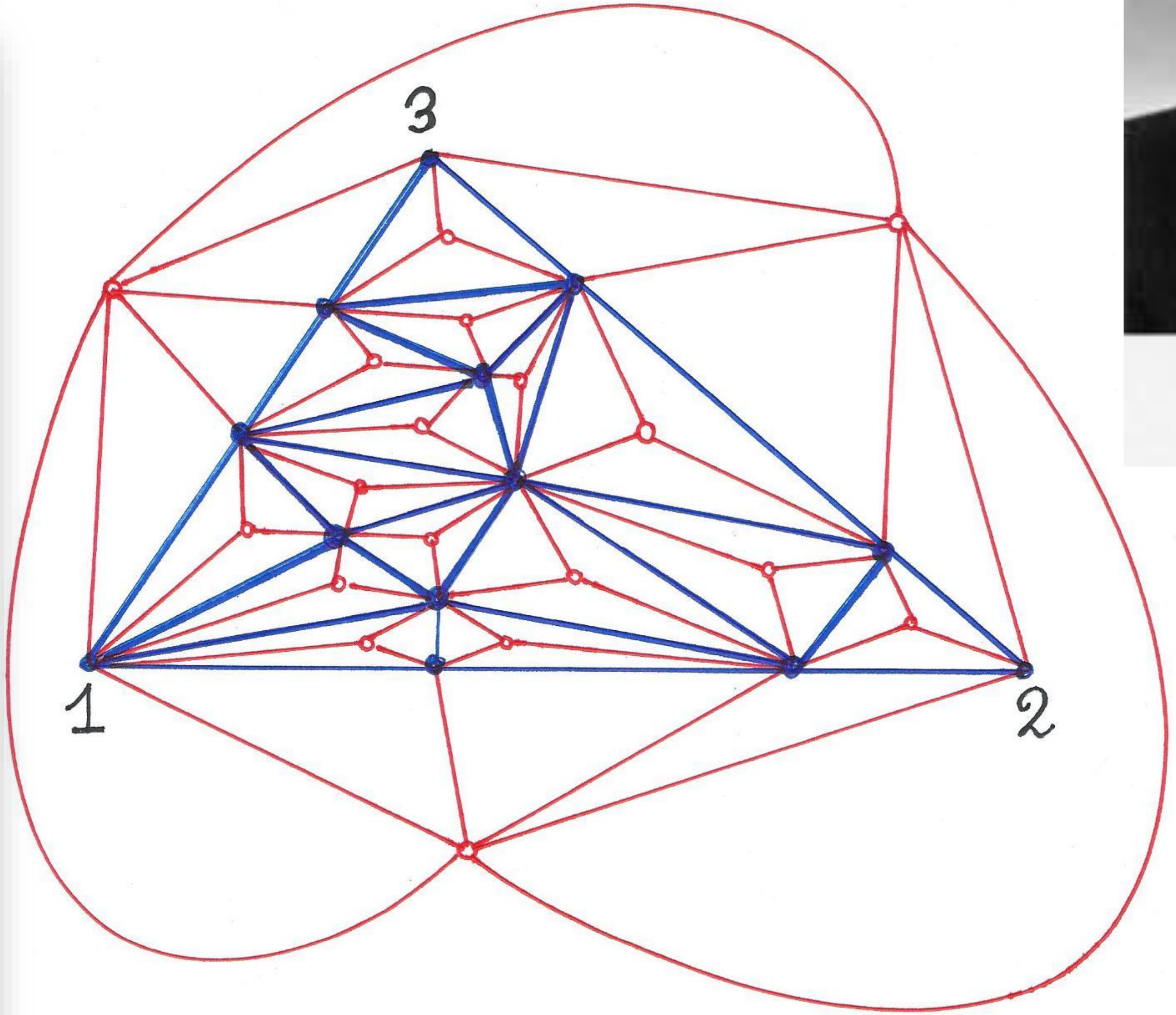
Claude Berge og Ryan Hayward Marseilles 1992

Paris July 2004

Sperner's Simplex Lemma 1928

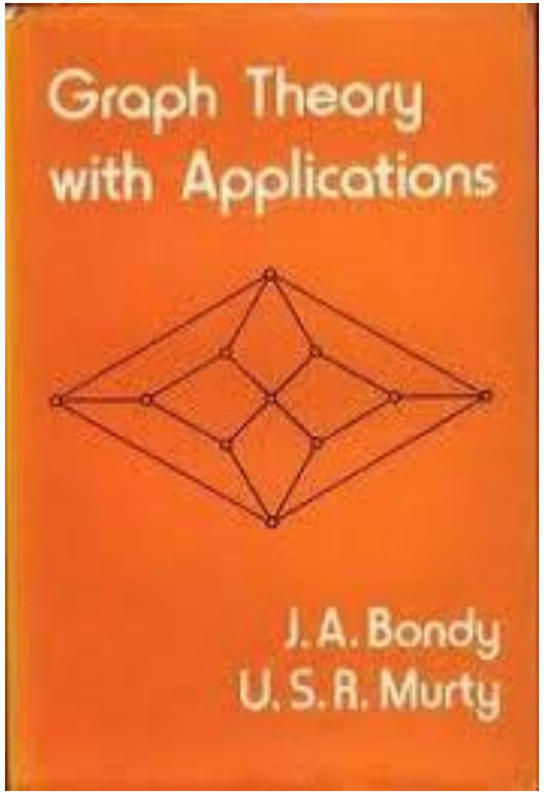
TIBOR GALLAI, 1912–1992

Photo: Ágnes Csánits



PROBLEM: Criticality in higher dmensions?

Bondy & Murty 1976



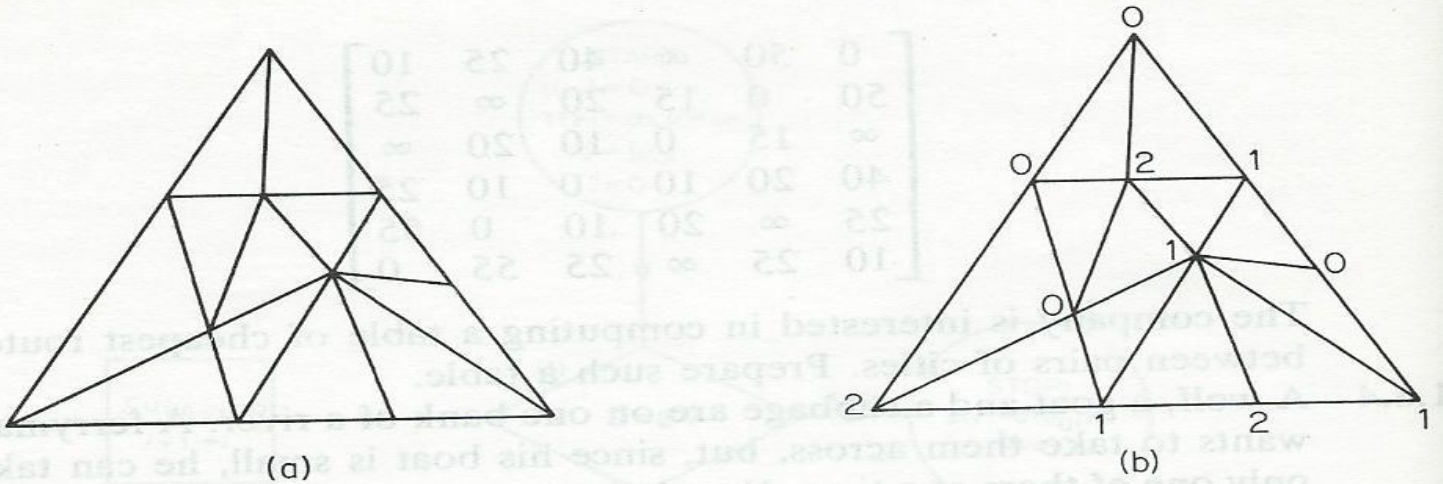
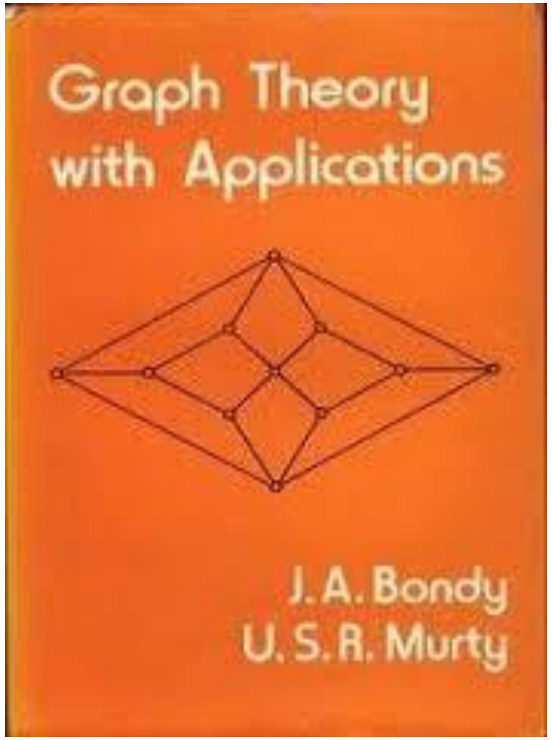


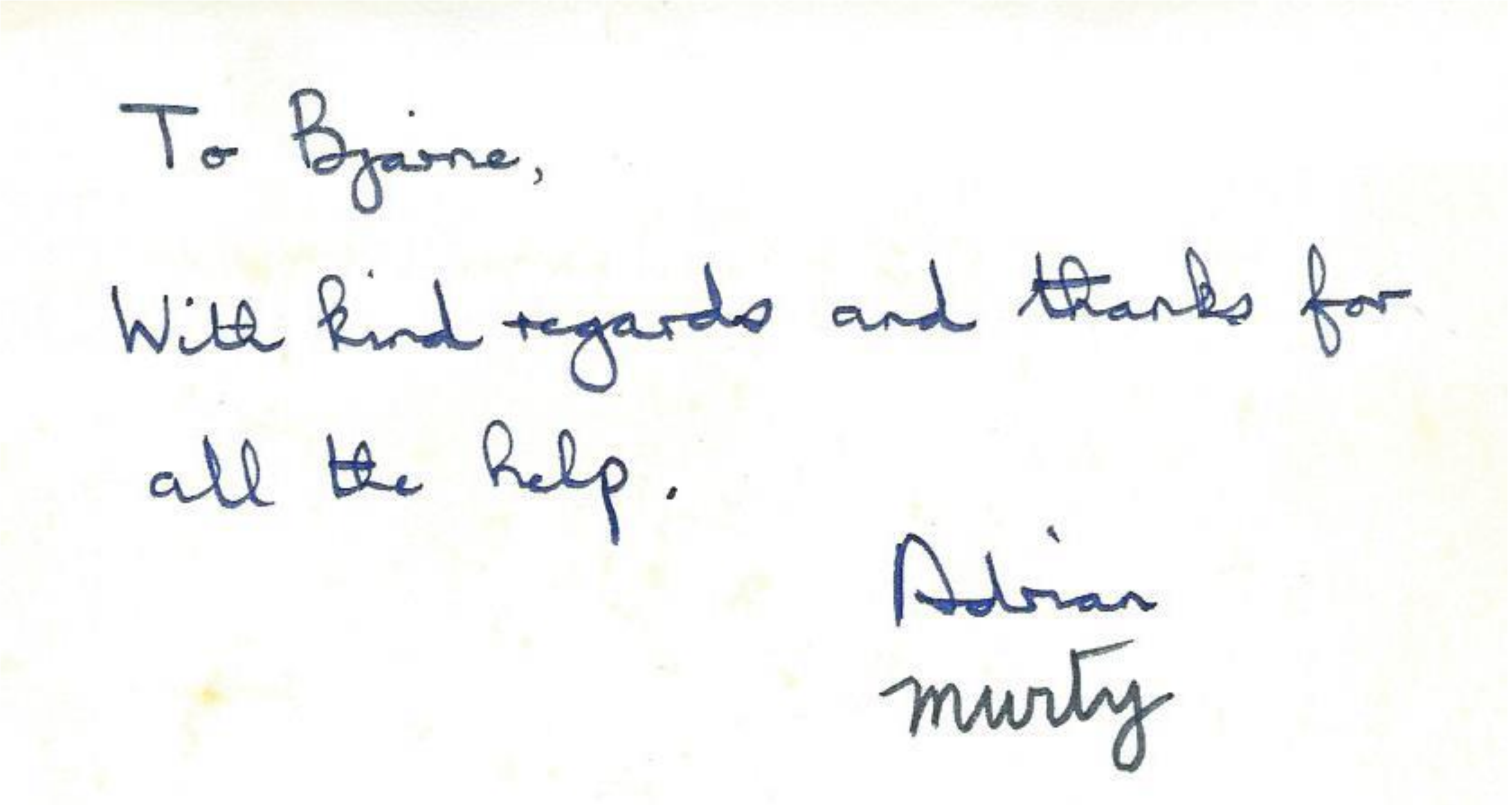
Figure 1.14. (*a*) A simplicial subdivision of a triangle; (*b*) a proper labelling of the subdivision

We call a triangle in the subdivision whose vertices receive all three labels a *distinguished* triangle. The proper labelling in figure 1.14*b* has three distinguished triangles.

Theorem 1.3 (Sperner's lemma) Every properly labelled simplicial subdivision of a triangle has an odd number of distinguished triangles.

From my copy :

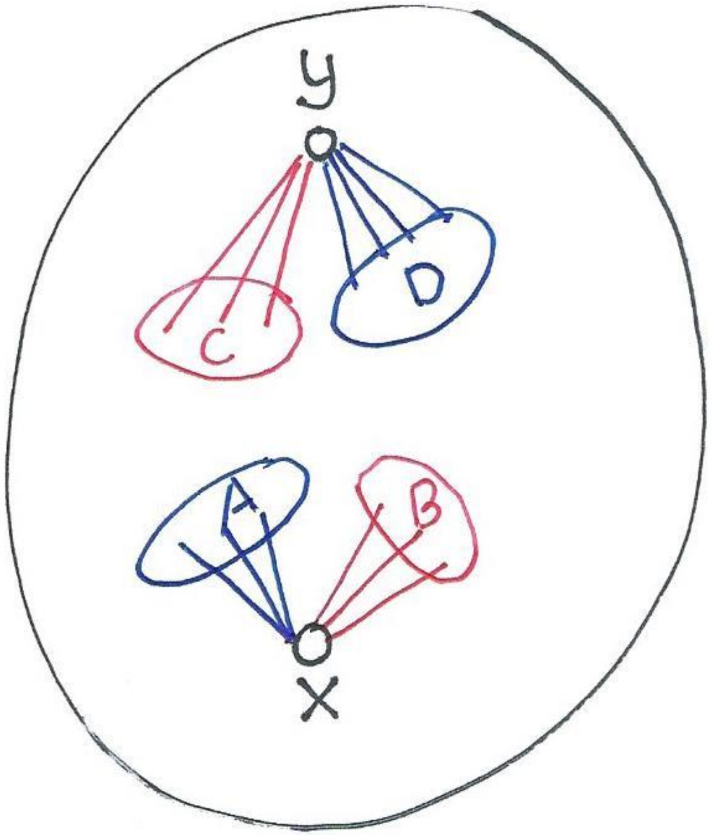



To Bjarne,

With kind regards and thanks for
all the help.

Adrian
Murty

Disjoint placements of graphs



G_1 G_2 both on the same n vertices

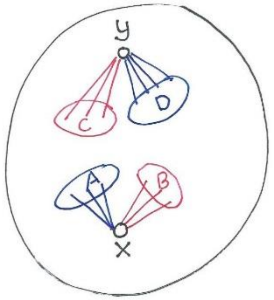
Assume

$A \cap C = \emptyset$

$B \cap D = \emptyset$

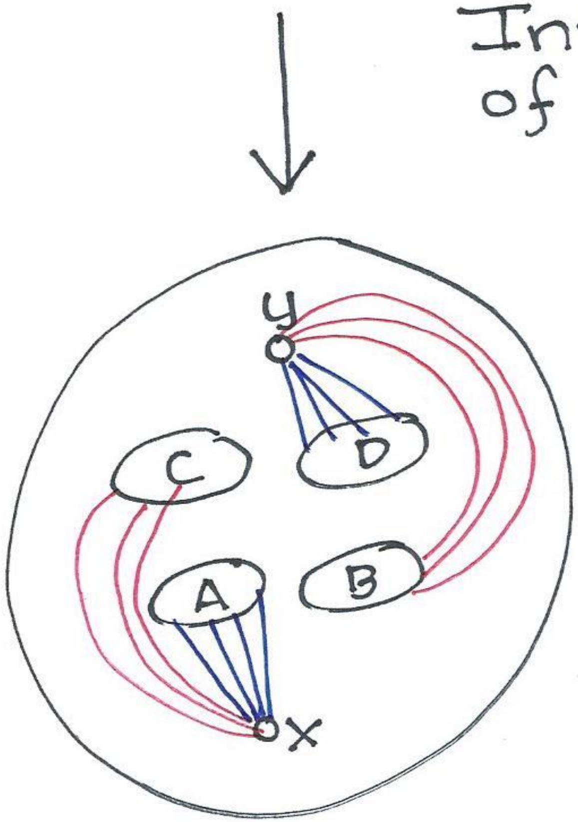
($x \in C \cup D$ possible;
$y \in A \cup B$ possible)

Interchange x and y in the red graph



G_1 G_2 both on the same n vertices

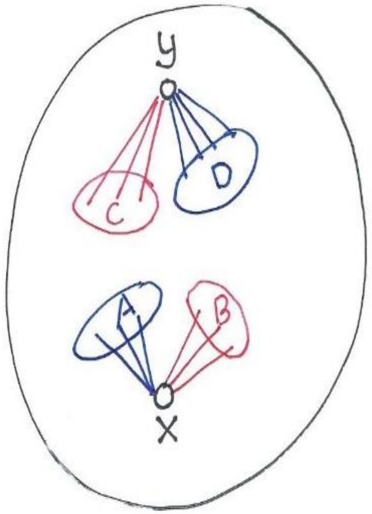
Assume
$A \cap C = \emptyset$
$B \cap D = \emptyset$
($x \in C \cup D$ possible; $y \in A \cup B$ possible)



Interchange the positions of x and y in the red graph

No overlap between red and blue edges at x nor at y

Conclusion



G_1 G_2 both on the same n vertices

Assume
$A \cap C = \emptyset$
$B \cap D = \emptyset$
($x \in C \cup D$ possible;
$y \in A \cup B$ possible)

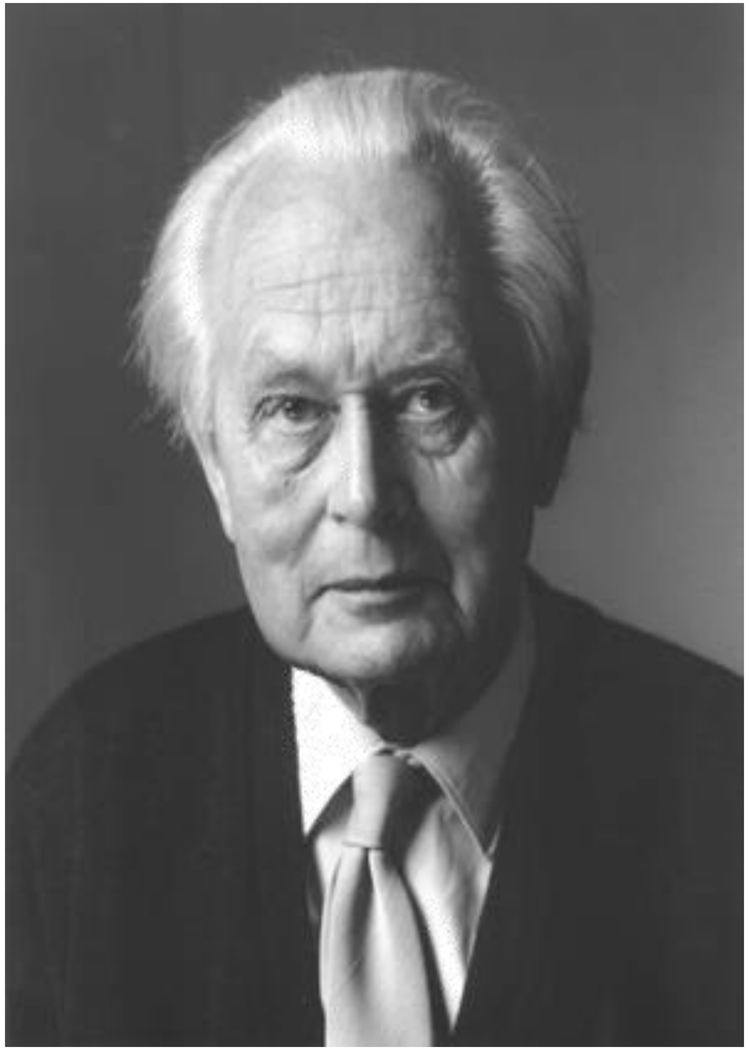
Assume $2 \cdot \Delta \cdot \Delta < n$

Then, given x, a y can always be found. Hence the red and the blue graph can be embedded with no edges overlapping.

PROBLEM: Can 2 be reduced? Is $(\Delta+1)(\Delta+1) < n+1$ enough?

Hindsgavl 1990

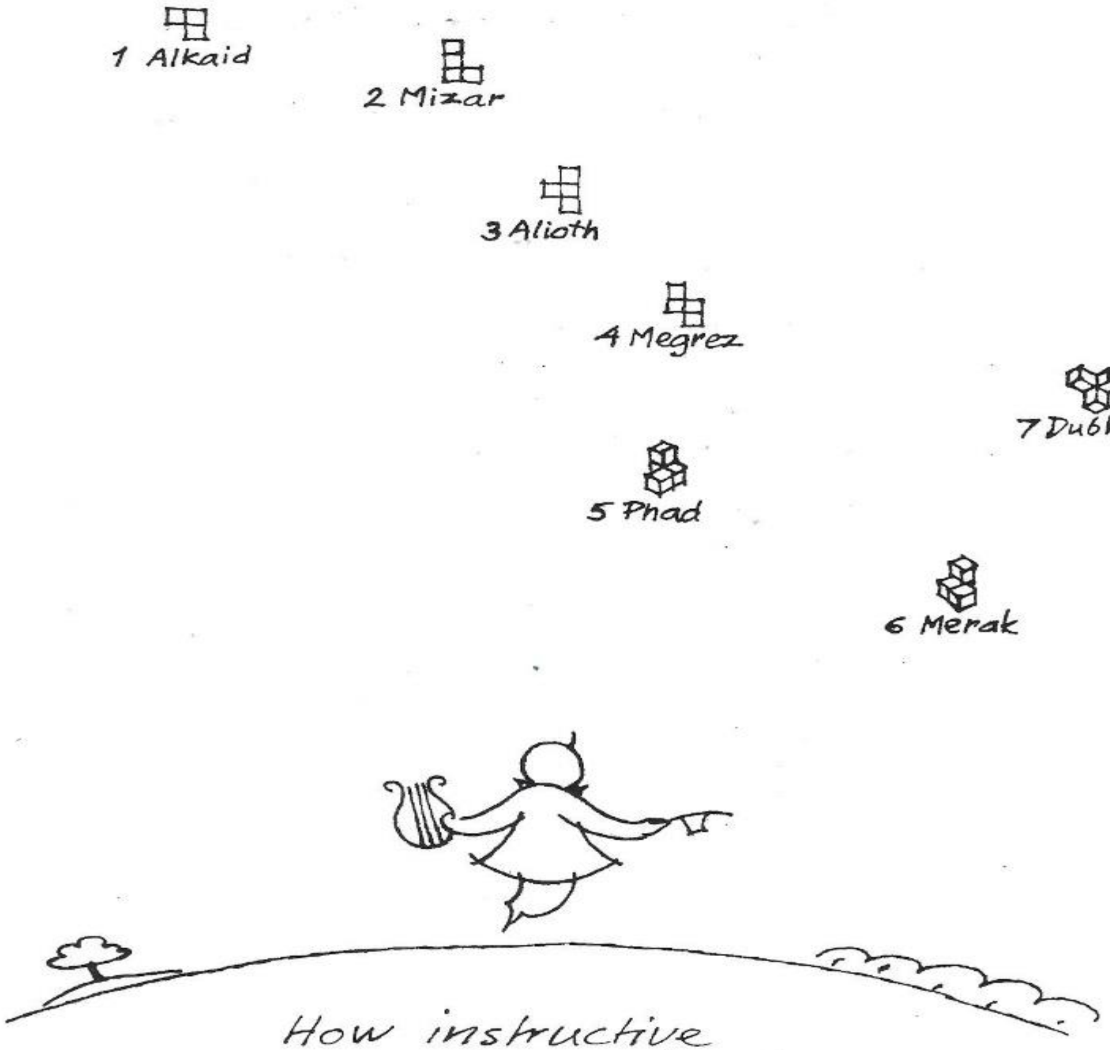
Piet Hein 1905-1996



Graduating from high school in 1924

Copenhagen Physics Conference 1932

Heisenberg, Werner Karl; Hein, Piet; Bohr, N.; Brillouin, Leon Nicolas; Rosenfeld, Leon; Delbrück, Max;
Heitler, Walter; Meitner, Lise; Ehrenfest, Paul; Bloch, Felix; Waller, Ivar; Solomon, Jacques; Fues, Erwin; Strømgren, Bengt; Kronig, Ralph de Laer; Gjelsvik, A; Steensholt, Gunnar; Kramers, Hendrik Anton; Weizsäcker, Carl Friedrich von; Ambrosen, J.P.; Beck, Guido; Nielsen, Harald Herborg; Buch-Andersen; Kalckar, Fritz; Nielsen, Jens Rud; Fowler, Ralph Howard; Hylleras, Egil Andersen; Lam, Ingeborg; Rindal, Eva; Dirac, Paul Adrian Maurice; N.N.; Darwin, Charles Galton;
Manneback, Charles; Lund, Gelius



How instructive
is a star!
It can teach us
from afar
just how small
each other are.

Piet Hein

POLITIKEN
TELEGRAM-ADR.: POLITIKEN, KØBENHAVN
POLITIKENS HUS
TLF.: CENTRAL 8511, RIGS 56, TELEX 2341
KØBENHAVN V
REDAKTIONEN

June 24, 57

Discription of
THE SOMA CUBE

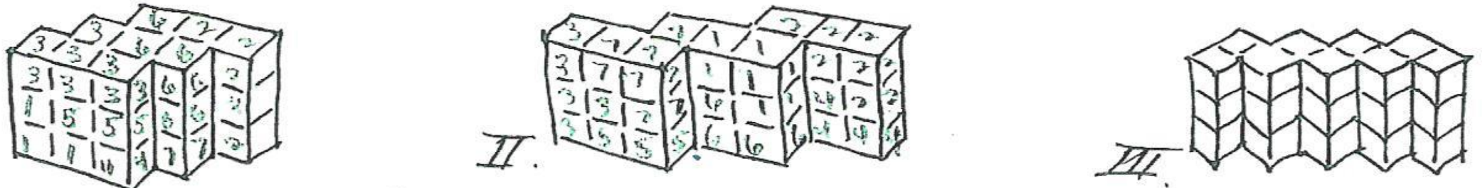
During a lecture in quantum physics by one very dry and systematisch middle European physisist – dealing with a space (6-dimensional at that) cut up in cubicles I fell asleep (please observe I don't pretend this to be a rare experience) — and had a revelation:

....

If you take all „elements of orthogonal connection" they can be put together to the orthogonal unit again. I woke up – and Heisenberg was still talking! – and tried the revelation on paper, and it proved true. – The variations of the unit – which combine to the unit again ... That is the smallest philosophical system of the World – and smallnes is no small quality in a philosophical system

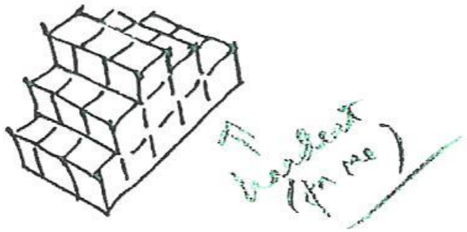
SOMA exercises

Besides there are lots of other shapes that can be build by the 7 pieces:



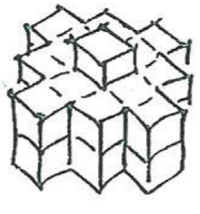
for inst. these 3 „variations of the cube"

And this staircase: which can be made in 8064 ways (84 if you count the positions of the pieces 1, 3, 4, 5, 6 & 7, which can pass into each other by turning the piece, identical.)



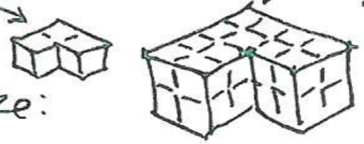
And scores of others. This one:

Problem:

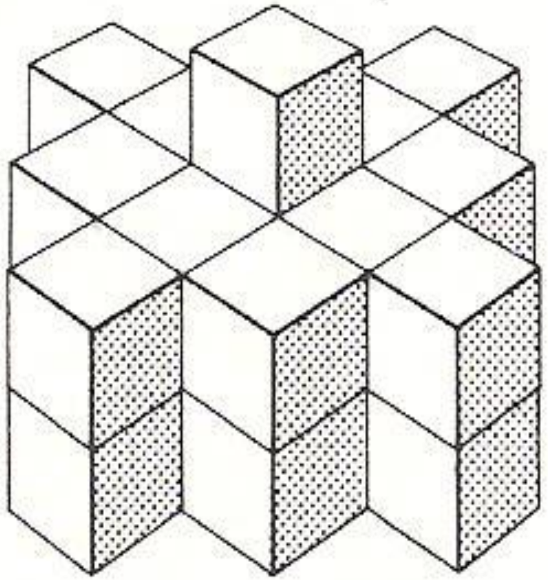


has never yet been built.

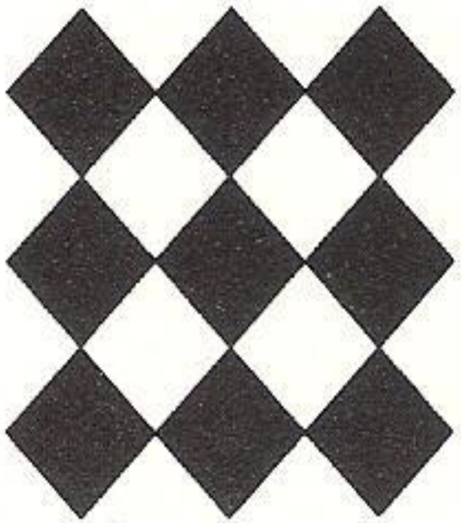
And then SOMA has the further pseudo-philosophical property that if you take away the only piece with 3 cubes only, nr. 1, then the remaining pieces can be assembled to the same shape:



Solution to the problem (count for each piece how many blacks it may cover – too few!)



An impossible Soma form.



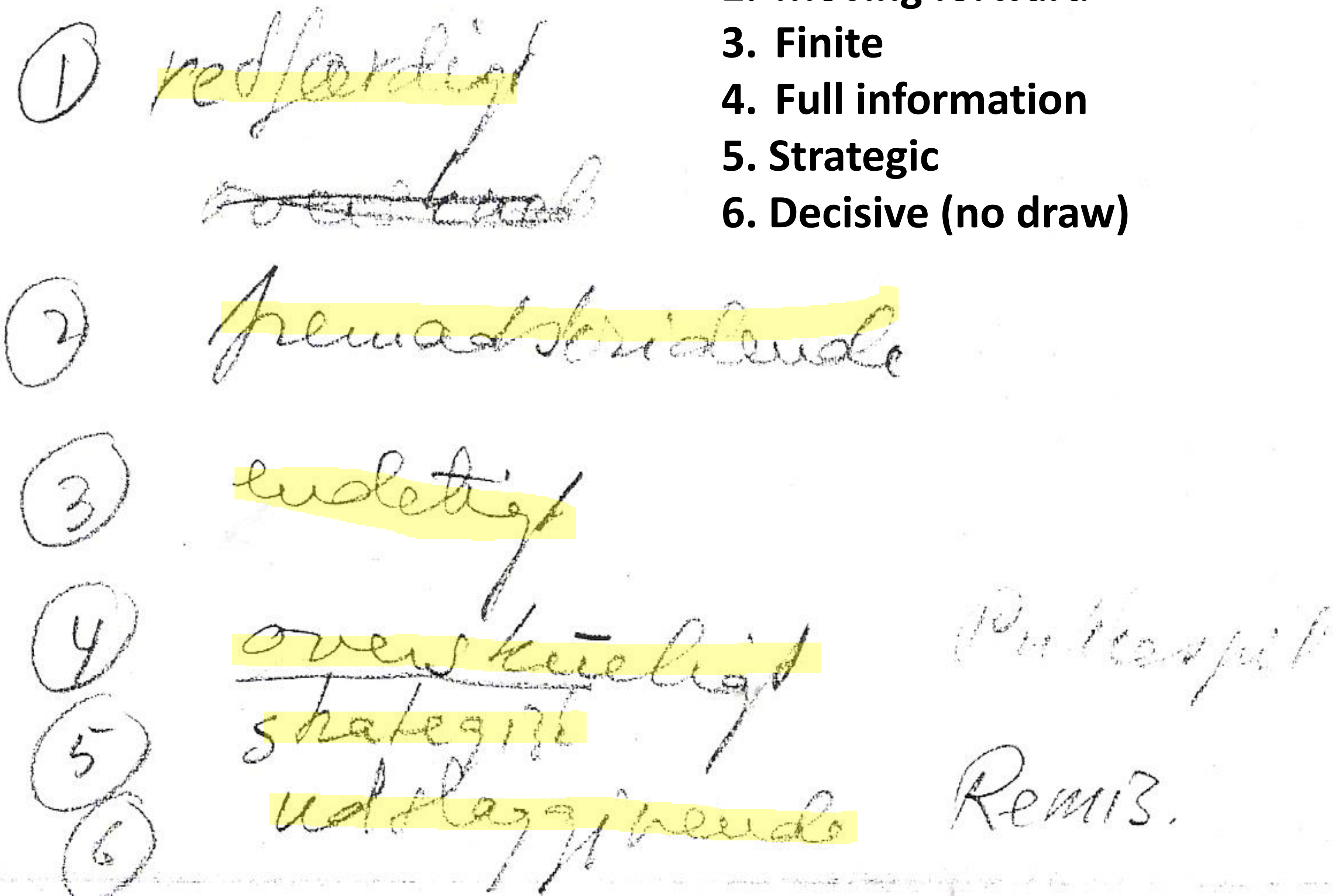
A means of labeling the form.

Piet Hein discovered Hex in 1942
Parentesen, Copenhagen University, December

Tanken er at se paa Matematikken som et Spil
Spillet er et Exempel paa at se paa Spil som Matematik.

Det jeg har at komme med i Aften er kun en Skitse til en Tanke som Indledning til et Spil. Jeg ved ikke, hvor meget aandelig Næring der er paa det for Dem, saa det vil berolige mig, hvis De vil fortsætte med at drikke og spise.

En litterær Anmelder af den Slags som - med Rette - ser deres egen Ophøjelse i at rakke ned paa den menneskelige Evne som kaldes Intelligens, srev for et par Aar siden i en Artikel om noget helt andet, at Matematik kan ikke give os andet end, hvad vi i Forvejen havde i Præmisserne. Det er jo rigtigt. Og det kaster et Skær over Matematikken af at være en ganske taabelig Virksomhed. Og i Artiklen fortsatte han da ogsaa som om Matematikken med denne Bemærkning en



1. Just
2. Moving forward
3. Finite
4. Full information
5. Strategic
6. Decisive (no draw)

Ikke krabbe – men Papir og Blyant

Kronologien osv. burde give et Spillebræt.

Den første kan vinde

bevis i Modsætning til Skak

Jeg troede den anden kunde vinde

The first can win

And this can be proved

Piet Hein Problems 1-46
from Politiken Dec.1942-June 1943

White to play.

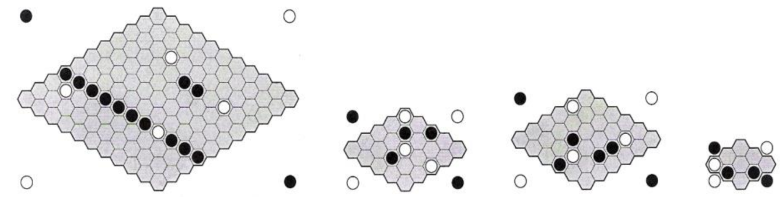


Fig. 1. Hein Puzzles 1–4

Fig. 2. Hein Puzzles 5–9

Fig. 3. Hein Puzzles 10–14

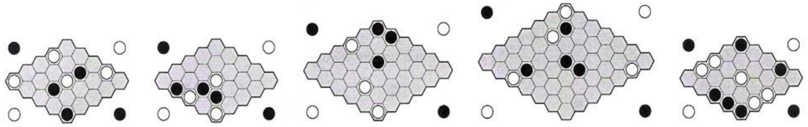


Fig. 4. Hein Puzzles 15–19

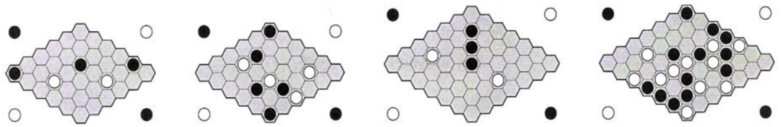


Fig. 5. Hein Puzzles 20–23

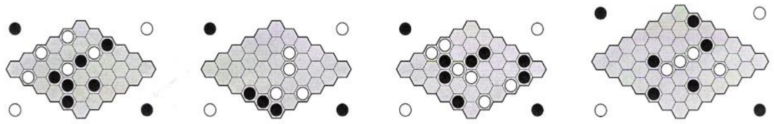


Fig. 6. Hein Puzzles 24–27

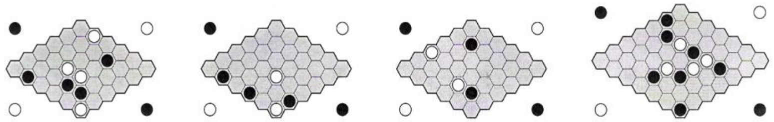


Fig. 7. Hein Puzzles 28–31

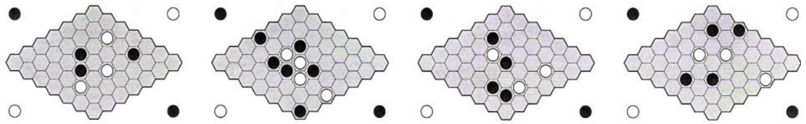


Fig. 8. Hein Puzzles 32–35

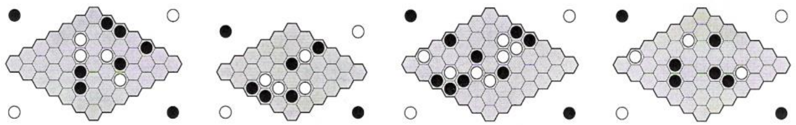


Fig. 9. Hein Puzzles 36–39

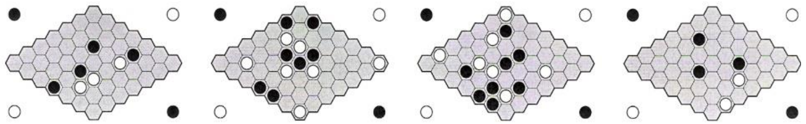


Fig. 10. Hein Puzzles 40–43

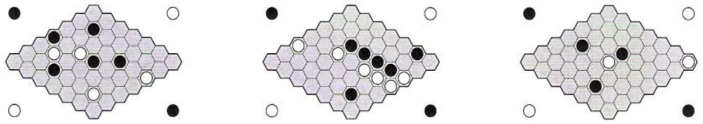
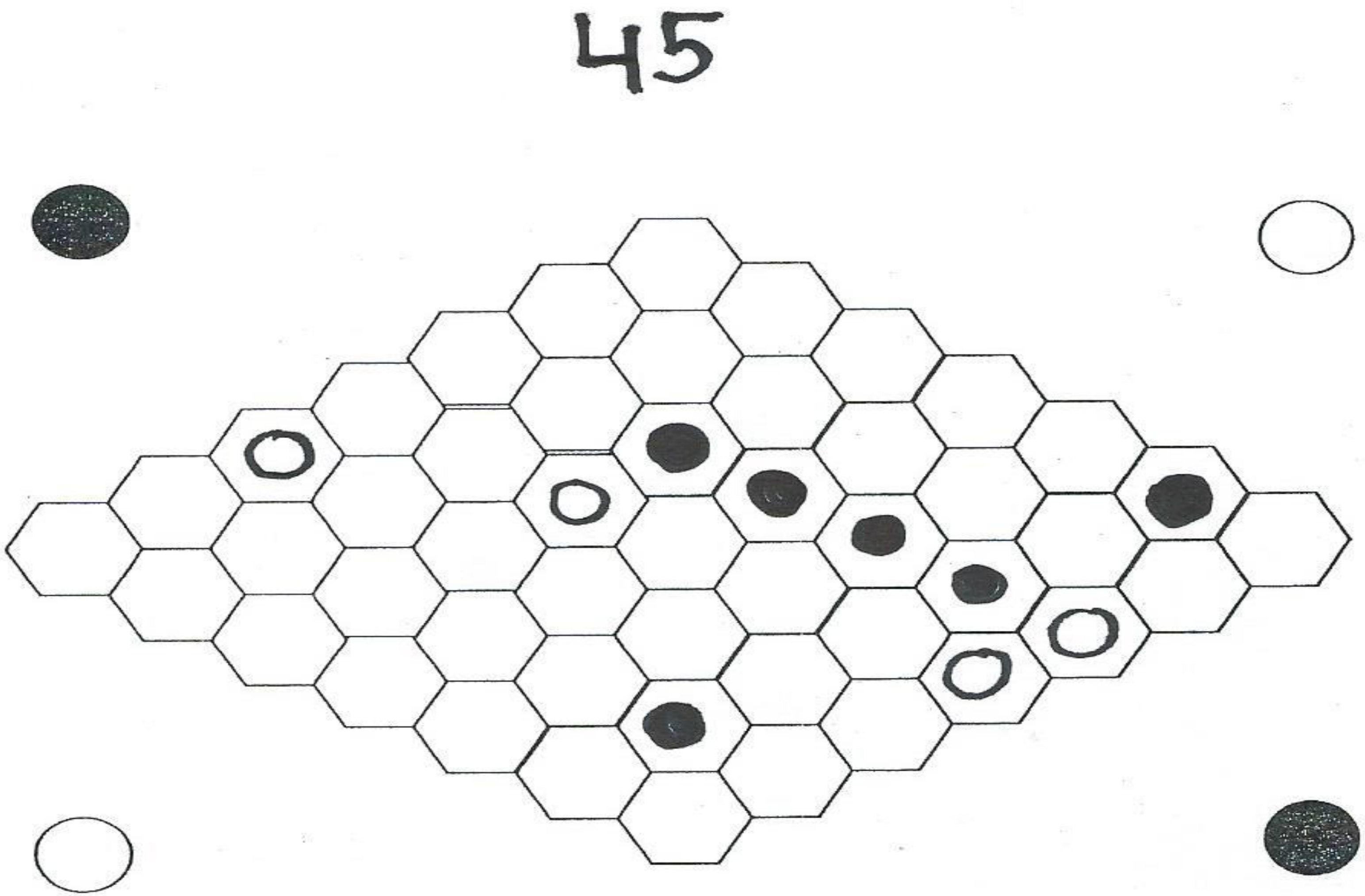


Fig. 11. Hein Puzzles 44–46

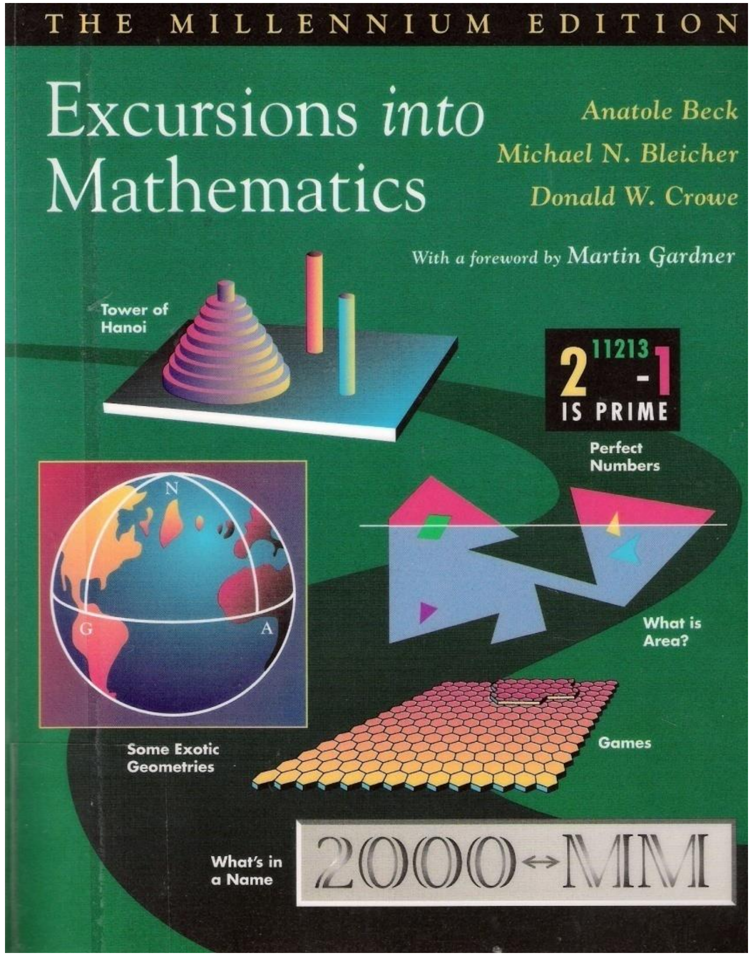
Piet Hein: Problem 45



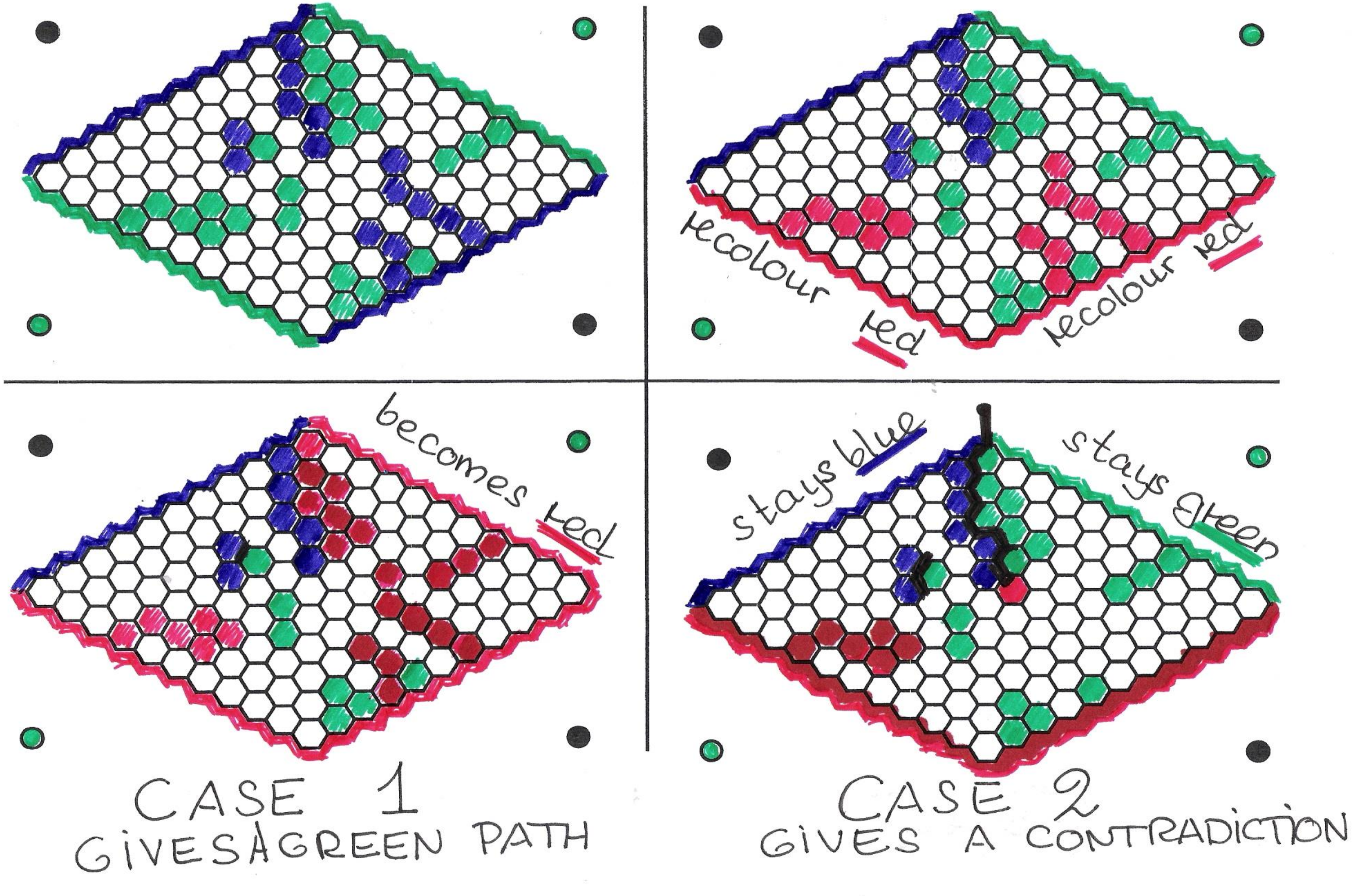
Theme 1: ORDINARY HEX

- THEOREM 1 (Piet Hein 1942, John Nash 1948, Anatole Beck 1969, David Gale 1979)
- HEX cannot end in a draw:
- Not both players can end without a chain!
- On a completely filled board either there is a yellow chain between the two yellow sides or there is a blue chain between the two blue sides.

NOT BOTH CAN END WITHOUT A CHAIN



- **First detailled proof by Anatole Beck et al in** *Excursions into Mathematics* 1969. Rather long verbal proof
- **David Gale** 1979
- Now a better version of Gale's proof (with an idea from Sperner's Simplex Lemma)



The contradiction follows also from
SPERNER's SIMPLEX LEMMA

Nash to Gardner 1957

(1) When the board is ~~field~~ filled one ~~or~~ the other of the players will have connected but not both.

(2) ~~One~~ Either the first player or the second will have a winning strategy.

(3) Suppose the second player could force a win.

(4) Consider a defensive strategy by first player imitating the winning second player strategy assumed in (3). The first move could be arbitrary. If the strategy ever called for a play where the arbitrary move was made another one could be made.

(5) Since an extra piece on the board is always an asset, never a handicap in connecting, at the end of the game first player will be better off using the adapted (assumed) second player strategy than he would have been if simply playing as second player. So he will win.

(6) Since this contradicts the hypothesis (3) that second player can win it follows that second player <u>cannot</u> win. Therefore ~~second~~ first player can always win <u>by correct play</u>.

John Nash

173 Bleecker St.
GR 5 4712

NOT BOTH CAN END WITHOUT A CHAIN is equivalent to Brouwer's Fix-point Theorem

THE GAME OF HEX AND THE BROUWER FIXED-POINT THEOREM

DAVID GALE

1. Introduction. The application of mathematics to games of strategy is now represented by a voluminous literature. Recently there has also been some work which goes in the other direction, using known facts about games to obtain mathematical results in other areas. The present paper is in this latter spirit. Our main purpose is to show that a classical result of topology, the celebrated Brouwer Fixed-Point Theorem, is an easy consequence of the fact that Hex, a game which is probably familiar to many mathematicians, cannot end in a draw. This fact is of some practical as well as theoretical interest, for it turns out that the two-player, two-dimensional game of Hex has a natural generalization to a game of n players and n dimensions, and the proof that this game must always have a winner leads to a simple algorithm for finding approximate fixed points of continuous mappings. This latter subject is one of considerable current interest, especially in the area of mathematical economics. This paper has therefore the dual purpose of, first, showing the equivalence of the Hex and Brouwer Theorems and, second, introducing the reader to the subject of fixed-point computations.

I should say that over the years I have heard it asserted in "cocktail conversation" that the Hex and Brouwer Theorems were equivalent, and my colleague John Stallings has shown me an argument which derives the Hex Theorem from familiar topological facts which are equivalent to the Brouwer Theorem. The proof going in the other direction only occurred to me recently, but in view of its simplicity it may well be that others have been aware of it. The generalization to n dimensions may, however, be new.

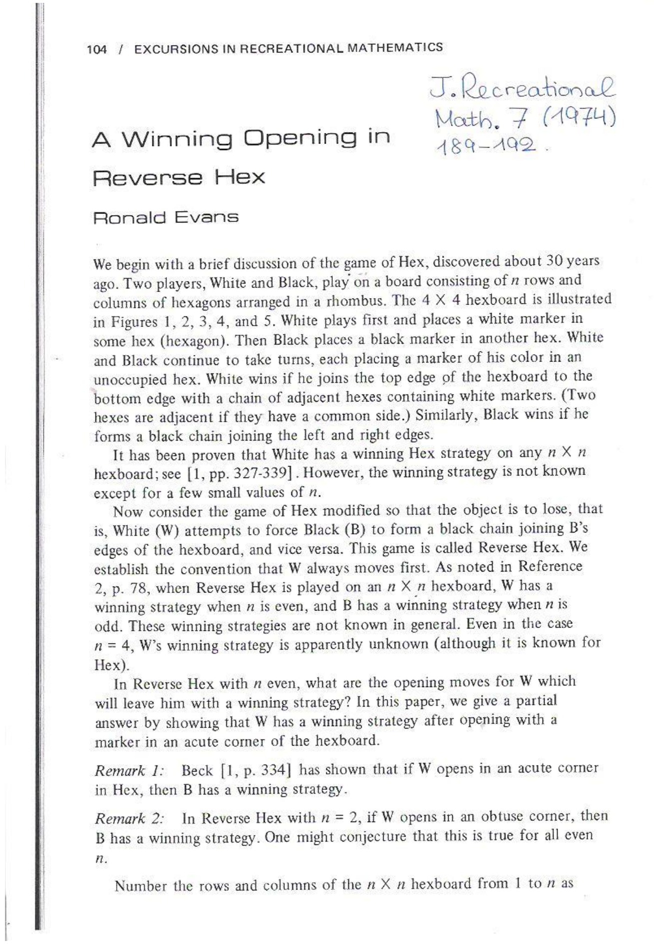
In the next section we will present the relevant facts about Hex. In Section 3 we prove the equivalence of the Hex and Brouwer Theorems. The general Hex Theorem and fixed-point algorithm are presented in the final section.

- **Brouwer's Fix-point Theorem** was an important tool for John Nash when the theory of the Nash equilibrium was developed.
- **David Gale proved in 1979 that NOT BOTH CAN END WITHOUT A CHAIN is equivalent to Brouwer's Fixpoint Theorem.**
- **Perhaps better to say: NOT BOTH CAN END WITHOUT A CHAIN is equivalent to Sperner's Simplex Lemma.**

Theme 2: REVERSE HEX (REX) – avoid a chain!

- THEOREM 2 (Robert O. Winder 1957, Ronald Evans 1974, Lagarians and Sleator 1999, Hayward, T. and Henderson 2010)
- Reverse HEX has a winning strategy for the first player when n is even, and for the second player when n is odd
- At optimal play the game is finished only when the whole board is full, and the loser is the player placing the last piece.
- First winning moves are known in all cases
- Problem: Find first losing moves

Evans 1974:
For Rex and n even, the acute corner is a winning first move for the first player



104 / EXCURSIONS IN RECREATIONAL MATHEMATICS

J. Recreational Math. 7 (1974) 189–192.

A Winning Opening in Reverse Hex

Ronald Evans

We begin with a brief discussion of the game of Hex, discovered about 30 years ago. Two players, White and Black, play on a board consisting of n rows and columns of hexagons arranged in a rhombus. The 4×4 hexboard is illustrated in Figures 1, 2, 3, 4, and 5. White plays first and places a white marker in some hex (hexagon). Then Black places a black marker in another hex. White and Black continue to take turns, each placing a marker of his color in an unoccupied hex. White wins if he joins the top edge of the hexboard to the bottom edge with a chain of adjacent hexes containing white markers. (Two hexes are adjacent if they have a common side.) Similarly, Black wins if he forms a black chain joining the left and right edges.

It has been proven that White has a winning Hex strategy on any $n \times n$ hexboard; see [1, pp. 327-339]. However, the winning strategy is not known except for a few small values of n.

Now consider the game of Hex modified so that the object is to lose, that is, White (W) attempts to force Black (B) to form a black chain joining B's edges of the hexboard, and vice versa. This game is called Reverse Hex. We establish the convention that W always moves first. As noted in Reference 2, p. 78, when Reverse Hex is played on an $n \times n$ hexboard, W has a winning strategy when n is even, and B has a winning strategy when n is odd. These winning strategies are not known in general. Even in the case $n = 4$, W's winning strategy is apparently unknown (although it is known for Hex).

In Reverse Hex with n even, what are the opening moves for W which will leave him with a winning strategy? In this paper, we give a partial answer by showing that W has a winning strategy after opening with a marker in an acute corner of the hexboard.

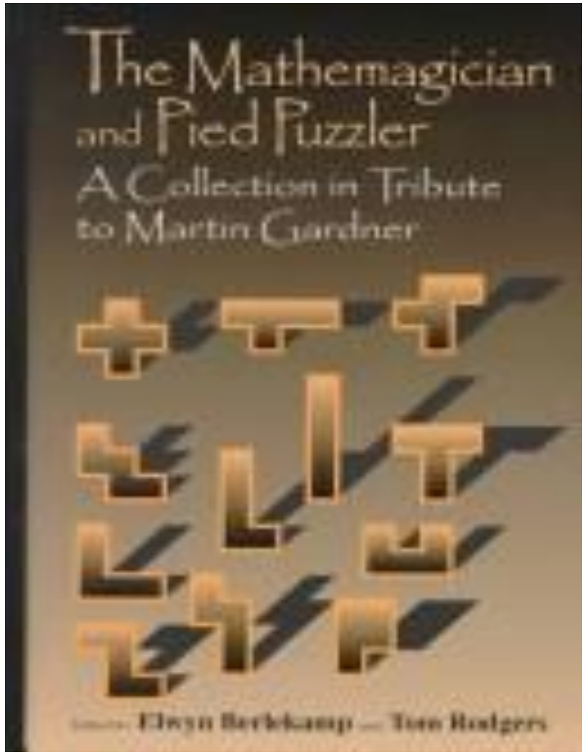
Remark 1: Beck [1, p. 334] has shown that if W opens in an acute corner in Hex, then B has a winning strategy.

Remark 2: In Reverse Hex with $n = 2$, if W opens in an obtuse corner, then B has a winning strategy. One might conjecture that this is true for all even n.

Number the rows and columns of the $n \times n$ hexboard from 1 to n as

Lagarias and Sleator 1999

In Rex: if both play optimally then the whole board is filled before the game ends, i.e. the player playing the last move loses

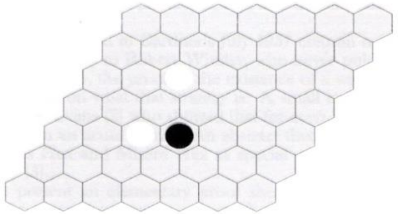


Who Wins Misère Hex?

Jeffrey Lagarias and Daniel Sleator

Hex is an elegant and fun game that was first popularized by Martin Gardner [4]. The game was invented by Piet Hein in 1942 and was rediscovered by John Nash at Princeton in 1948.

Two players alternate placing white and black stones onto the hexagons of an $N \times N$ rhombus-shaped board. A hexagon may contain at most one stone.



A game of 7×7 Hex after three moves.

White's goal is to put white stones in a set of hexagons that connect the top and bottom of the rhombus, and Black's goal is to put black stones in a set of hexagons that connect the left and right sides of the rhombus. Gardner credits Nash with the observation that there exists a winning strategy for the first player in a game of hex.

The proof goes as follows. First we observe that the game cannot end in a draw, for in any Hex board filled with white and black stones there must be either a winning path for white, or a winning path for black [1, 3]. (This fact is equivalent to a version of the Brouwer fixed point theorem, as shown by Gale [3].) Since the game is finite, there must be a winning strategy for either the first or the second player. Assume, for the sake of

237

Discrete Mathematics 312 (2012), 148-156

Discrete Mathematics 312 (2012) 148–156

Contents lists available at SciVerse ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

How to play Reverse Hex

Ryan B. Hayward [a,*], Bjarne Toft [c], Philip Henderson [b]

[a] Computing Science, University of Alberta, Canada
[b] Google, Mountain View, CA, United States
[c] Mathematics & Computer Science, University of Southern Denmark, Denmark

ARTICLE INFO

Article history:
Available online 27 July 2011

Keywords:
Hex
Reverse Hex
Rex
Misère Hex

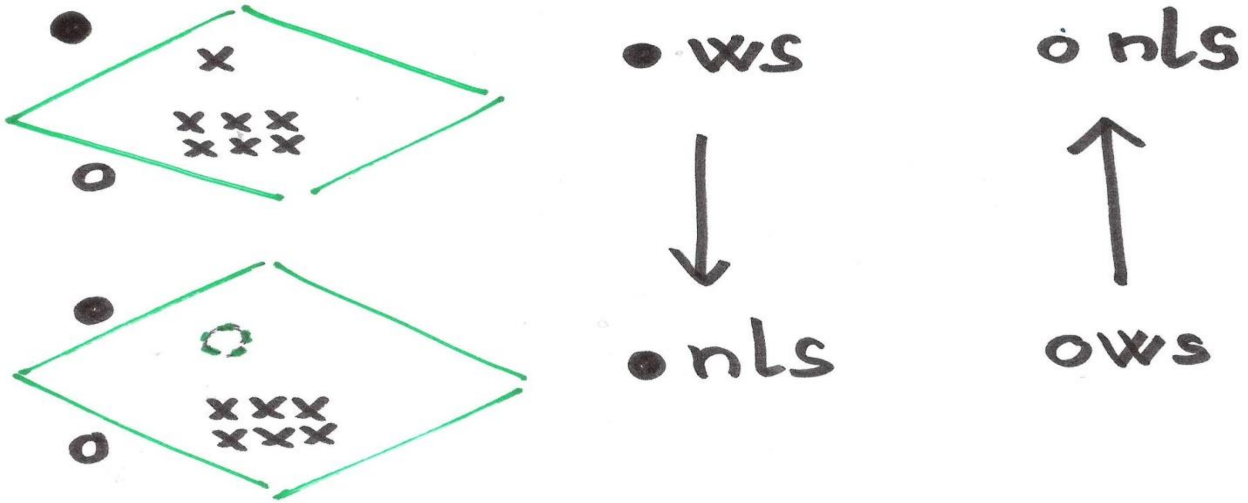
ABSTRACT

We present new results on how to play Reverse Hex, also known as Rex, or Misère Hex, on $n \times n$ boards. We give new proofs – and strengthened versions – of Lagarias and Sleator's theorem (for $n \times n$ boards, each player can prolong the game until the board is full, so the first/second player can always win if n is even/odd) and Evans's theorem (for even n, the acute corner is a winning opening move for the first player). Also, for even $n \geq 4$, we find another first-player winning opening (adjacent to the acute corner, on the first player's side), and for odd $n \geq 3$, and for each first-player opening, we find second-player winning replies. Finally, in response to comments by Martin Gardner, for each $n \leq 5$, we give a simple winning strategy for the $n \times n$ board.

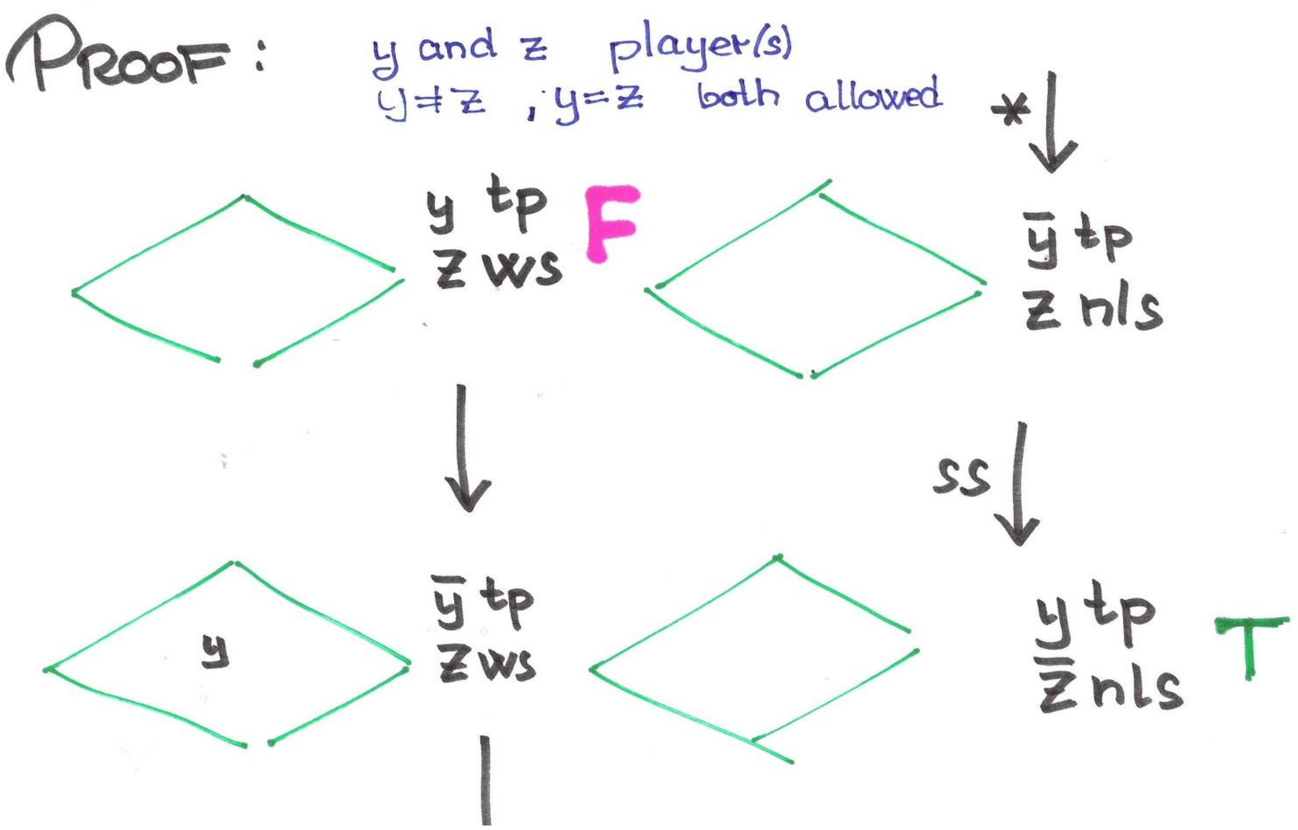
Terminated REX (T-REX) is Rex with the addition: the game stops when there is just one emty field left (i.e. there should always be a choice!)

LEMMA ✱

ADDING OR REMOVING A PIECE IN TRex CHANGES A WINNING STRATEGY (ws) INTO A NON-LOSING STRATEGY (nls)



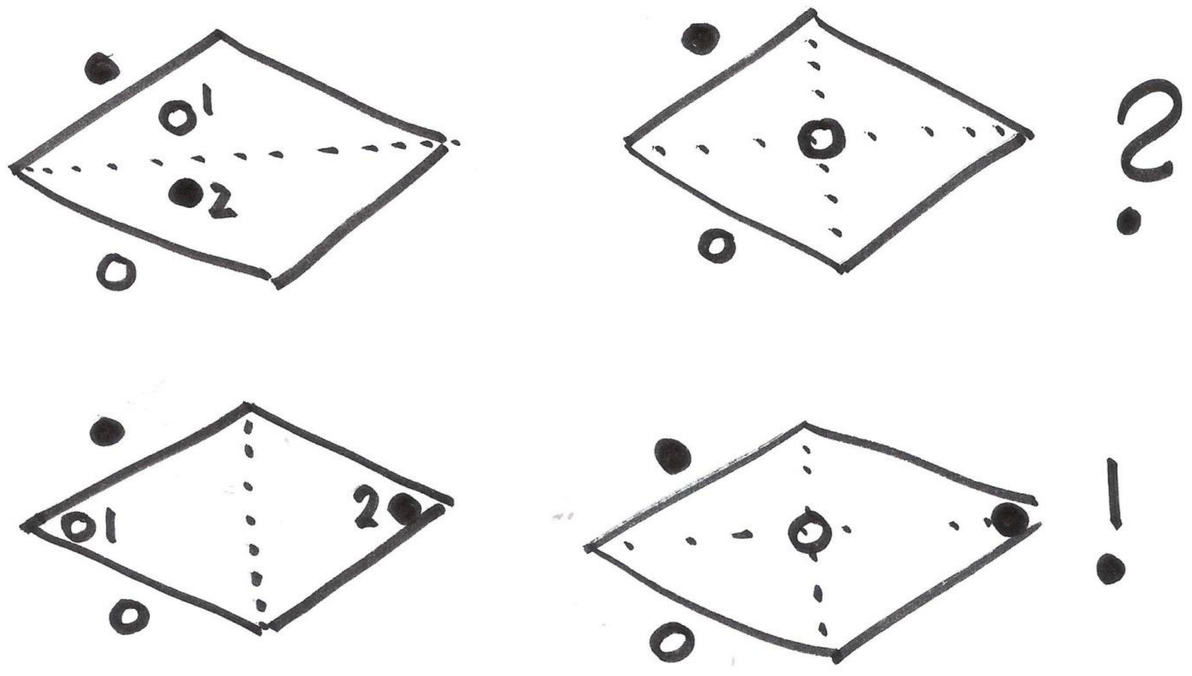
In TREX both players have non-losing strategies



REX on an nxn-board with n odd:

- Let the second player (Black) play the non-losing strategy from TRex. **THIS IS A WINNING STRATEGY FOR THE SECOND PLAYER IN REX:**
- *Either* the first player (White) creates a white chain *or* TRex ends in a draw with one emty field left. In the Rex game that field has to be chosen by White and a White chain is formed!
- If also White plays the non-loosing strategy from TRex, then the Rex game will be decided only when the board is full.

REX on an nxn board with odd n: (second player has winning strategy)



Theme 3:
CYLINDRICAL HEX – play on cylinder!

- THEOREM 3 (Alpern and Belck 1991, Samuel Huneke 2012, Huneke, Hayward and T. 2014)
- Cylindrical HEX has a winning strategy for the up-down player when the circular dimension n is even (pairing strategy)
- Cylindrical HEX has a winning strategy for the up-down player when the circular dimension is 3
- Problem: Circular dimensions 5, 7, 9, ….?

C-Hex (Alpern & Belck 1991)
(Circular Hex or Cylindrical Hex or Annular Hex)

E.2.8 Cylindrical Hex

Other Names: Annular Hex.

Origin: Alpern and Beck [1991].

Rules: As for Hex, except that the board wraps or repeats periodically in one player's direction of play. The player in the wrapped direction must form a continuous chain of pieces that connects across the edge to win, whereas their opponent must simply connect the top and bottom edge of the cylinder.

Analysis: This game is similar to Modulo Hex except that the board repeats periodically only in one direction of play rather than both. This is equivalent to wrapping the game around a cylinder, which can then be squashed flat into an annulus for representational purposes.

The process that converts rectangular Hex to Cylindrical Hex preserves winning paths for the player joining the ends of the cylinder, but not for the player wrapping around the cylinder. Alpern and Beck prove that Cylindrical Hex cannot end in a tie, and that the player playing end-to-end can always win if the board dimension in their direction of play is even.

Figure E.8 shows a Hex board described as a skewed rectangular grid on the left (see Section 2.1). For the purposes of Cylindrical Hex, the board wraps in the horizontal direction. The cylindrical board mapped to the annulus is shown on the right.

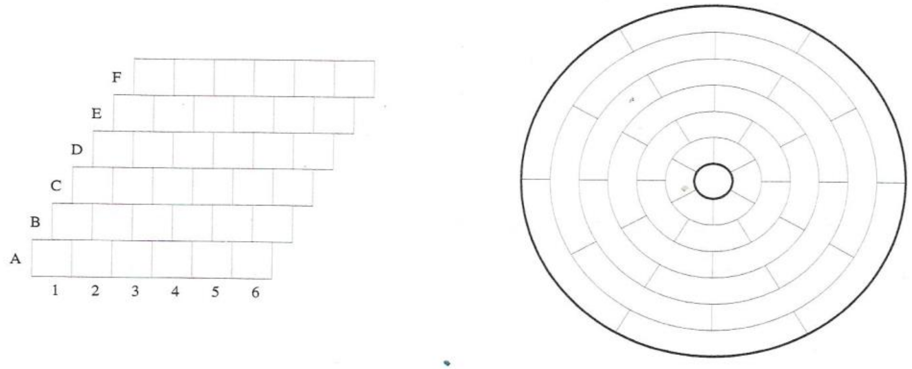
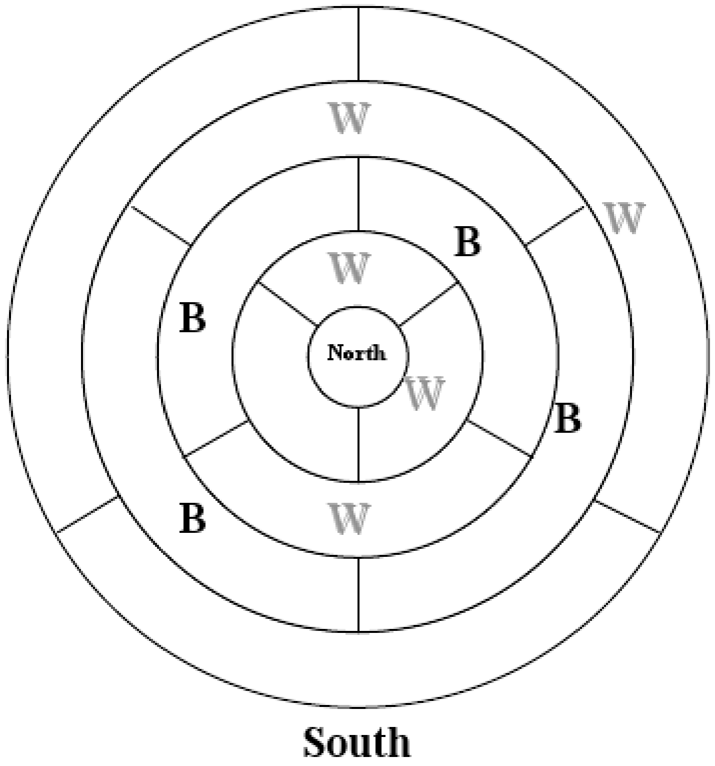
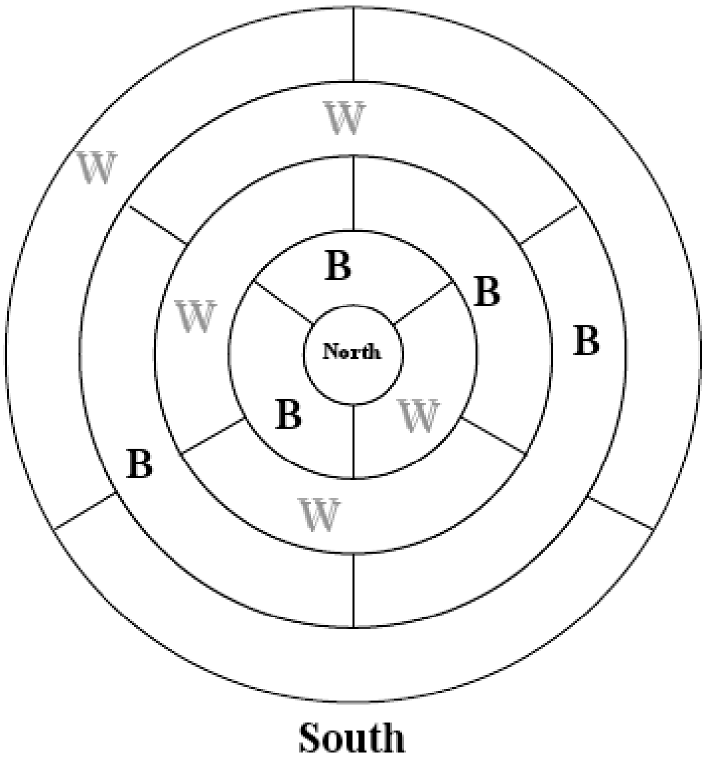


Figure E.8. Hex board described on a skewed rectangular grid and mapped to the annulus.

Circular dimension 3 – two examples





My mistakes (circular dimension 3):

- Describe the strategy by giving for any position in the game and any move by the oponent an answering move
- Consider only the ring the oponent plays and the two neighboring rings (some 3 to the power 8 possibilities)
- Use up-down symmetry

Samuel Huneke at LSE 2011-2012

- A New Result in Circular Hex
- A dissertation submitted to the Department of Mathematics of the London School of Economics and Political Science for the degree of Master of Science (London, September 2012)
- Paper by Huneke submitted to Discrete Mathematics 2013

Cylindrical HEX winning strategy for White (the up-down player) in the circular dimension 3 case

I. If Black has not played, then play anywhere.

II. If Black's previous move is at cell (j, i), play as follows:

1. in one of rings $i-1, i, i+1$ so that there is then a white cell in ring i touching a white cell in ring $i-1$ and a white cell in ring $i+1$ (See Fig. 3),
2. in ring i or $i+1$ so that there is then a white cell in ring i touching a white cell in ring $i+1$,
3. in ring i or $i-1$ so that there is then a white cell in ring i touching a white cell in ring $i-1$ (See Fig. 4),
4. in ring i,
5. anywhere.

Discrete Mathematics 331 (2014) 93-97

Discrete Mathematics 331 (2014) 93–97

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

A winning strategy for 3 × n Cylindrical Hex

Samuel Clowes Huneke [a,b,*], Ryan Hayward [c], Bjarne Toft [d]

[a] Department of Mathematics, London School of Economics and Political Science, United Kingdom
[b] Department of History, Stanford University, United States
[c] Department of Computer Science, University of Alberta, Canada
[d] IMADA, Syddansk Universitet, Denmark

ARTICLE INFO

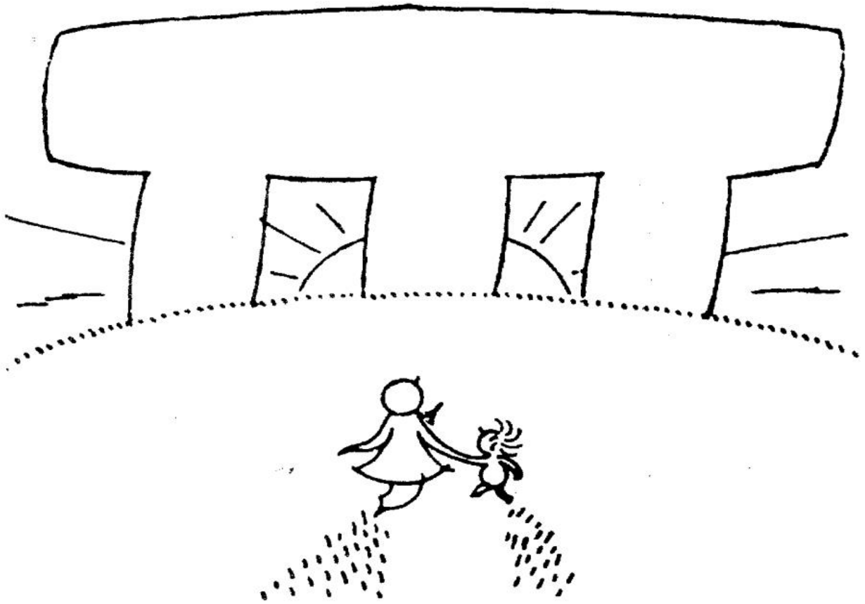
Article history:
Received 8 February 2013
Received in revised form 29 April 2014
Accepted 3 May 2014

Keywords:
Hex
Cylindrical Hex
Annular Hex
Steven Alpern
Anatole Beck
David Gale

ABSTRACT

For Cylindrical Hex on a board with circumference 3, we give a winning strategy for the end-to-end player. This is the first known winning strategy for odd circumference at least 3, answering a question of David Gale.

Time to wake up!



Mind these three:
T. T. T.
Hear their chime:
Things Take Time.

**Dear Adrian:
Congratulations, and thank you for more than 40 years of friendship!**

Husk de tre:
T. T. T.
Slid men vid:
Ting Tar Tid.

Thank you for your attention!