Colouring graphs excluding fixed subgraphs

joint work
with S. Thomassé, M. Bonamy

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Formalization

- A class C of graphs is said to be *chi-bounded* if

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Now our question is : what families $\mathcal{F}$ are chi-bounding?

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Conjecture (Gyarfas–Sumner)

If F is a forest, the class of graphs excluding F as an induced subgraph is chi-bounded.

$\mathcal{F} = T$ tree

Little is really known :

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Scott proved the following very nice ”topological” version of the conjecture

- For every tree T, the class of graphs excluding all subdivisions of T is chi-bounded.

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- Graphs that do not contain any odd hole nor any complement of odd hole : Berge graphs.
  Strong Perfect Graph Theorem : $\chi = \omega$.
- No simple proof of any (even much worse) other chi-bounding function.

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Using Erdős Theorem construct a sequence F_i such that

- $\chi(F_i) \geqslant i$
- $\text{girth}(F_i) > |2^{F_{i-1}}|$.

Let $\mathcal{F}$ be the set of cycles that do not occur in any F_i.

Then $\mathcal{F}$ is NOT chi-bounding and is infinite (it contains at least all the $|F_i|$).

Even more it has upper density 1 since it contains every interval $[|F_i|, 2^{|F_i|}]$.

Conjecture (Scott-Seymour,2014)

If $I \subset \mathbf{N}$ has bounded gaps ($\exists k$ s.t. every k consecutive integers contains an element of F), then $\{C_i,\ i \in I\}$ is k-bounding.

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  Chudnovsky et al recently proved that $\chi > 4$ implies the existence of a $3k$ cycle as a (not necessarily induced) subgraph.
- The question originally came as a sub case of a more general question of Kalai and Meschulam.

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- Trinity changing paths : try to find vertices x and y such that many independent paths exist between the two.

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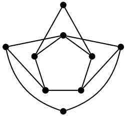

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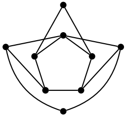

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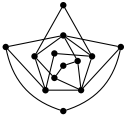

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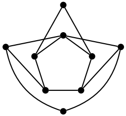

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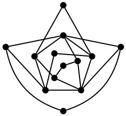

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Next

- $\{C_{3k}, k > 1\}$ is chi-bounding
- $\{C_{4k}\}$ is chi-bounding