Exercises – map enumeration
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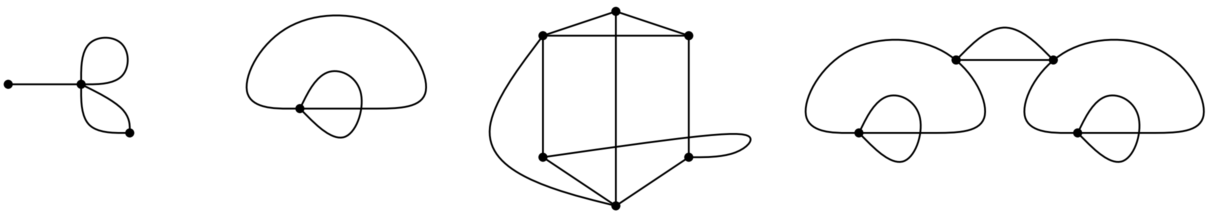
Exercise 0 – Warmup

1. How many faces do the following maps have? What is their genus?



2. Can you add one edge (and no vertex) to the first map on the left to create a map of genus 1? a map with 4 faces? a map of genus 1 with 4 faces?

3. Can you add one edge to the second map to create a map with 2 faces? a map of genus 2?

4. Draw the 2 (resp. 9) rooted planar maps with 1 (resp. 2) edges.

Exercise 1 – Tree-rooted maps

In this exercise all maps will be rooted and planar (genus 0*). A* tree-rooted *map is a map equipped with a distinguished spanning tree.*

1. Let $\mathbf{m}$ be a planar map equipped with a spanning tree $\mathbf{t}$. Let $\mathbf{m}'$ be the dual map of $\mathbf{m}$ and let $\mathbf{t}'$ be the submap of $\mathbf{m}'$ formed by duals of edge not in $\mathbf{t}$. Show that $\mathbf{t}'$ is a spanning tree of $\mathbf{m}'$ (the dual spanning tree). Hint*: A plane tree is a planar map with one face.*

2. Let $\mathbf{t}, \mathbf{t}'$ be two fixed rooted plane trees, respectively with k and $n-k$ edges. Show that the number of tree-rooted maps $\mathbf{m}$ you can build such that $\mathbf{t}$ and $\mathbf{t}'$ play the roles of the previous question is $\binom{2n}{2k}$.

3. Deduce from 2. a simple calculation that the the number of tree-rooted maps with n edges is equal to the product of two consecutive Catalan numbers $\mathrm{Cat}(n)\mathrm{Cat}(n+1)$.

4. (subjective question). Are tree-rooted maps in the "universality class" of planar maps?

> Historical note*: This result is due to Mullin (1976). Giving a bijective proof is difficult and was done by Bernardi (2006).*

Exercise 2 – The counting exponent of maps with fixed genus and number of faces

Recall that the generating function $T \equiv T(t)$ of rooted plane trees (that is, genus 0 maps with 1 face) by the number of edges is given by:

$$T = 1 + tT^2 = \frac{1 - \sqrt{1-4t}}{2t}.$$

This series has a unique dominant singularity at $t = 1/4$ of the form $T = 2 + c(1-4t)^{1/2} + O(1-4t)$. (in this exercise, c denotes a positive constant that varies among occurrences), and the number of rooted plane trees with n edges is asymptotically equivalent to $c \cdot 4^n n^{-3/2}$.

The goal of this exercise is to show that more generally, for fixed $k \geq 1$ and $g \geq 0$, the number of rooted maps of genus g with k faces is asymptotic to

$$c \cdot n^{\frac{6(g-1)+3k}{2}} 4^n$$

when n tends to infinity (for some constant $c = c(g,k)$). We now assume that $(g,k) \neq (0,1)$.

1. Let $\mathcal{M}_{g,k}$ be the set of rooted maps of genus g with k faces having no vertices of degree 1 nor 2. Show that $\mathcal{M}_{g,k}$ is a finite set. Hint*: for $i \geq 3$, call d_i the number of vertices of degree i, and write Euler's formula in terms of the d_i's.*

2. Let $P(t)$ be the generating function of rooted plane trees with a marked leaf (different from the root vertex). Show (combinatorially!) that

$$P(t) = \frac{1}{1 - tT(t)^2} - 1.$$

3. Show that the generating function of rooted maps of genus g with k faces, by the number of edges, is given by the finite sum:

$$\sum_{\mathfrak{m} \in \mathcal{M}_{g,k}} tP'(t)P(t)^{e(\mathfrak{m})-1}.$$

(for $(g,k) = (0,2)$, use the following convention: there is a single element in $\mathcal{M}_{g,k}$ that has 0 edges – alternatively, just skip this case since it is elementary but a bit tricky).

Hint*: given a map, remove recursively all its vertices of degree 1 until no more is left; then, replace each maximal chain of vertices of degree 2 by an edge. What do you obtain?*

4. Show that an element of $\mathcal{M}_{g,k}$ has at most $3k + 6(g-1)$ edges and that this bound is realized. Deduce the form of the dominant singularity of the generating function and conclude.

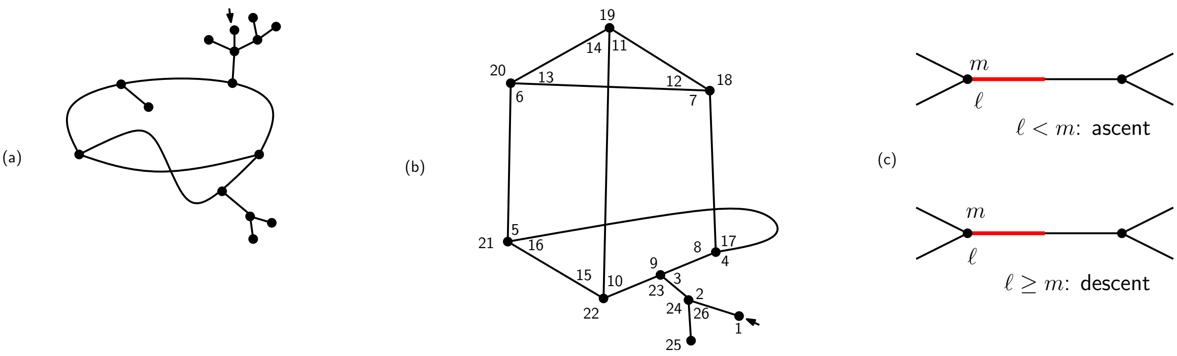
> Historical note*: Maps with a fixed genus and number of faces are very different from unconstrained maps (i.e. maps where only the genus is constrained). As shown by this exercise, the linear growth of the counting exponent with respect to the genus is $\frac{3}{2}g$ rather than $\frac{5}{2}g$ for unrestricted maps. They belong to different universality classes (for example, the diameter of the maps studied in this exercise is of order $O(\sqrt{n})$ when n tends to infinity, compared to $O(n^{1/4})$ for unconstrained maps)..*

Exercise 3 – (Pre)cubic one-face maps

In this exercise, for $g \geq 0$, we study the set $\mathcal{T}_g(m)$ of rooted maps with $n = 2m+1$ edges having the following properties:

- *all their vertices have degree* 1 *or* 3 *(such maps are called* precubic*);*
- *they have only one face;*
- *they have genus g;*
- *they are rooted on a vertex of degree* 1.

Vertices of degree 1 *different from the root vertex are called* leaves. *Figure (a) and (b) show examples in genus 1 and* 2*, respectively.*



1. In genus 0, do you recognize these objects? Show (or recall) that $|\mathcal{T}_0(m)| = \mathrm{Cat}(m)$.

2. Draw an example in genus 3. Hint*: 1+2=3*

3. Show that an element of $\mathcal{T}_g(m)$ has $c = m + g$ vertices of degree 3 and $\ell = m + 1 - 3g$ leaves.

4. Let $\mathbf{m}$ be an element of $\mathcal{T}_g(m)$, and number its corners from 1 to $2n$ as on Figure (b) – i.e., in the polygonal representation, number corners from 1 to $2n$ clockwise, starting from the root. A half-edge is called an *ascent* (resp. *descent*) if the label on its sides increases (resp. weakly decreases) counterclockwise, see Figure (c). Show that $\mathfrak{m}$ has $n + 1$ descents and $n - 1$ ascents.

5. Deduce that $\mathfrak{m}$ has $2g$ *nontrivial* descents, where a descent is nontrivial its small label is not the minimal label around the vertex it belongs to.

6*. Show that elements of $\mathcal{T}_g(m)$ with a marked nontrivial descent are in bijection with elements of $\mathcal{T}_{g-1}(m)$ with three marked leaves.

7. Deduce that the cardinality $f_g(m)$ of $\mathcal{T}_g(m)$ satisfies the relation $2gf_g(m) = \binom{m+4-3g}{3} f_{g-1}(m)$, and finally that:

$$f_g(m) = \frac{1}{12^g g!} \frac{(m+1)!}{(m+1-3g)!} \mathrm{Cat}(m).$$

> Historical note*: This exercise illustrates the fact that* one-face *maps have a relatively simple structure, even in arbitrary genus. The formula proved in this exercise is due to Lehman and Walsh (1972), and the bijective proof presented here to G.C. (PTRF, 2009). The case of general one-face maps (not cubic) can be handled with a similar, yet more complicated approch, see G.C. (AAM, 2010) or G.C.-Féray-Fusy (JCTA, 2012). See also this paper for references, there are tons of results about one-face maps in the literature, in particular with the algebraic viewpoint (and for this, see also Problem B below).*

Exercise 4 – No short non-contractible cycles in random maps

In this exercise, we will show the following theorem. Let $g \geq 1$ and $K \geq 1$ be fixed, and let n be a (large) integer. Let M_n be a uniform random map of genus g with n edges. Then the probability that M_n has a non-contractible cycle of length $\leq K$ tends to 0 when n tends to infinity (in other words the length of the shortest non-contractible cycle tends to ∞ when n tends to infinity, in probability).

We recall that the generating function $F_g(t)$ of rooted maps of genus g by the number of edges has a unique dominant singularity at $t = \frac{1}{12}$ of the form:

$$F_g(t) \sim (\text{constant}) \frac{1}{(1-12t)^{\frac{5g-3}{2}}}.$$

1. Show that the number of maps of genus g with n edges and a *distinguished* cycle of length K is less than

$$\sum_{\substack{g_1+g_2=g \\ g_1,g_2>0}} \sum_{n_1+n_2=n+k} 2n_1 m_{g_1}(n_1) m_{g_2}(n_2) + (2n+2k)(2n+2k-1) m_{g-1}(n+k).$$

2. Conclude by analysing the singularity of the generating function.

> Historical note: *This exercise is taken from [Bender, Gao, Richmond, 1999]. This result and its variants have nice consequences, for example, that a random genus g map is 5-colorable with probability tending to 1 when n tends to infinity (from a nice result of Thomassen on the 5-colorability of fixed genus graphs). The same argument (and the same paper) shows that the shortest n.c. cycle is larger than $c \log n$ with high probability. However this bound is not sharp at all: by bijective methods (and by them only) one can show that the smallest n.c. cycle in a random map of fixed genus with n edges is of order $n^{1/4}$, which is much larger (this was proved by Bettinelli using the Marcus-Schaeffer bijection).*

Exercise 5 – Training with Tutte equations

A triangulation *is a map whose all faces have degree 3. A map is* nonseparable *if it is loopless and has no separating vertex, i.e. if it is not possible to disconnect it by removing a single vertex. The purpose of this exercise is to count planar nonseparable triangulations. We will also need to consider* pseudotriangulations, *defined as maps in which all faces different from the root face have degree 3.*

1. Show that a planar triangulation with $2n$ faces (and $3n$ edges) has $n+2$ vertices.

2. Let $F(t,u) \equiv F(u)$ be the generating function of (rooted) plane nonseparable pseudotriangulations where t marks the number of non-root faces, and u marks the degree of the root face. Show that F is solution of the following equation:

$$F(u) = u^2 + t\frac{F(u) - u^2 f}{u} + \frac{F(u)^2}{u}$$

where $f \equiv f(t)$ counts nonseparable pseudotriangulations with root-face degree equal to 2.

3. What kind of equation is that?

4. Either by hand or (better) with your favorite software, solve this equation using the ideas we used in the lectures. Hint: *You should be able to find that $tf(t) = T - tT^3$ where $T = \frac{t}{(1-2T)^2}$.*

5. Using Lagrange inversion, deduce that the number of rooted planar nonseparable triangulations with $n+2$ vertices is given by: $\frac{2^n(3n)!}{(n+1)!(2n+1)!}$.

6. Do the asymptotics and check that the counting exponent of rooted planar simple triangulations is $-\frac{5}{2}$, as for rooted planar maps (this is an example of a universality result).

Exercise 6 – Counting unrooted maps

In this (difficult) exercise we count unrooted *planar maps. Recall that a labelled map with n edges is a map whose half-edges are labelled from 1 to $2n$. The symmetric group $\mathbb{S}_{2n}$ acts on labelled maps by permutation of the labels, and orbits under this action are called unrooted maps. We let $m(n)$ be the number of rooted planar maps with n edges, given by Tutte's formula: $m(n) = \frac{2\cdot 3^n}{n+2}\mathrm{Cat}(n)$, and $l(n) = (2n-1)!m(n)$ be the number of labelled planar maps with n edges.*

1. Prove (or recall) "Burnside's lemma" saying that the number $u(n)$ of unrooted planar maps with n edges is given by:

$$u(n) = \frac{1}{(2n)!} \sum_{\sigma \in \mathfrak{S}_{2n}} \mathrm{fix}(\sigma) \tag{1}$$

where $\mathrm{fix}(\sigma)$ is the number of labelled planar maps with n edges that are fixed by σ.

2. Show that the contribution of the identity to (1) is $\frac{1}{2n}m(n)$.

3**. We admit the (intuitive) fact that each planar map with a distinguished symmetry can be embedded on the sphere $\mathbb{S}$ (viewed as the unit sphere in $\mathbb{R}^3$) in such a way that the automorphism is realized by an oriented isometry of $\mathbb{S}$, i.e., a rotation. Note that such a rotation fixes exactly two cells of $\mathfrak{m}$, where a cell is either a vertex, a face, or an edge.

We first consider rotations that fix no edges. Show that the contribution of such rotations to the sum (1) is given by:

$$\frac{1}{2n} \sum_{d|n} \phi\left(\frac{n}{d}\right) \binom{d+2}{2} m(d)$$

where $\phi(m)$ is the number of integers in $[1..m]$ relatively prime to m (Euler's totient function).

> Hint*: 1/Given a map $\mathfrak{m}$ fixed by a rotation σ, consider the quotient map $\mathfrak{m}/\sigma$ obtain by cutting a fundamental domain of the sphere and gluing its sides together. 2/By Euler's formula, a map with d edges has $d+2$ vertices and faces in total.*

4**. We now consider rotations that fix two edges. Show that for such rotations to exist n has to be even and that (in that case) the contribution of such rotations to (1) is:

$$\frac{1}{2n}\frac{n(n-1)}{2} m\left(\frac{n}{2}\right) = \frac{n-1}{4} m\left(\frac{n}{2}\right).$$

5**. We finally consider rotations that fix exactly one edge. Show that for such rotations to exist n has to be odd and that (in that case) the contribution of such rotations to (1) is:

$$\frac{1}{2}\left(\frac{n-1}{2}+2\right) m\left(\frac{n-1}{2}\right) = \frac{n+3}{4} m\left(\frac{n-1}{2}\right).$$

6. Deduce the following formula for u_n:

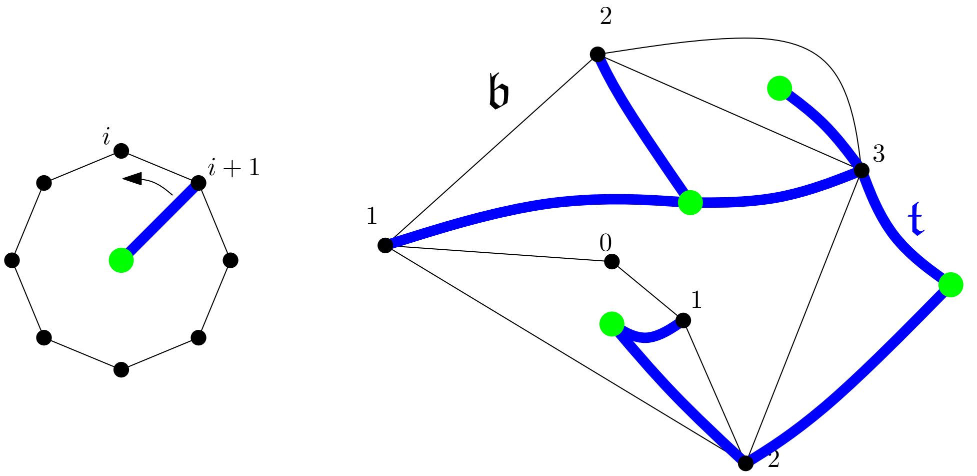
$$u_n = \frac{1}{2n}m(n) + \frac{1}{2n}\sum_{d|n} \phi\left(\frac{n}{d}\right)\binom{d+2}{2} m(d) + \left(\frac{n+1}{4} + \frac{(-1)^n}{2}\right) m\left(\left\lfloor \frac{n}{2}\right\rfloor\right).$$

> Historical note*: This result is due to Liskowets. The classification of symmetries in higher genus is more complicated see [Mednykh and Nedela, 2004].*

Problem A – The Bouttier-Di Francesco-Guitter bijection for bipartite maps

1. Let $\mathfrak{b}$ be a rooted bipartite map of genus g, pointed at a vertex. We construct a new map $\mathfrak{t}$ as follows:

   - We add a new vertex inside each face of $\mathfrak{b}$;
   - Going counterclockwise around each new vertex, we link the new vertex to every corner that is followed by a corner of smaller label, see the next figure;
   - We let $\mathfrak{t}$ be the map consisting of all the newly added vertices and all the original vertices of $\mathfrak{b}$ except the pointed one, and of all the newly added edges.



   Suppose $\mathfrak{b}$ has d_i faces of degree $2i$ for each $i \geq 1$. What is the number e of edges and v of vertices of $\mathfrak{b}$? Same question for $\mathfrak{t}$.

2. Consider the map $\mathfrak{m} = \mathfrak{b} \bigcup \mathfrak{t}$, and let $\mathfrak{a}$ be the dual submap of $\mathfrak{t}$ in $\mathfrak{m}$. Show that $\mathfrak{a}$ is a connected unicyclic graph, whose unique cycle encircles the pointed vertex.

   HINT*: orient edges as in the proof of the Marcus-Schaeffer construction, and reproduce the arguments from the lectures (the key point is: how does a face of* $\mathfrak{m}$ *look like?)*

3. Deduce that $\mathfrak{t}$ is a one-face map.

4. A one-face map is a *mobile* if:

   - it has two kind of vertices: labelled ones and unlabelled ones, and edges only link labelled to unlabelled vertices; the root vertex is a labelled vertex;
   - for each unlabelled vertex v, and for any two neighbours u and u' of v that are consecutive counterclockwise around v, their labels satisfy: $\ell(u') \geq \ell(u) - 1$.

   Show that $\mathfrak{t}$ is a mobile.

5. Show that the previous construction is a 1-to-2 mapping between pointed bipartite maps and mobiles, that sends a bipartite map of genus g with d_i faces of degree $2i$ for each $i \geq 1$, and v vertices, to a mobile with d_i unlabelled vertices of degree i for each $i \geq 1$, and $v - 1$ labelled vertices.

   HINT*: design a closure operation similar to the Marcus-Schaeffer bijection.*

6. For $i \geq 1$, show that the number of sequences of integers $(0 = \ell_1, \ell_2, \dots, \ell_i)$ such that for $\ell_{j+1} \geq \ell_j - 1$ for all j (indices being understood modulo i) is $\binom{2i-1}{i}$.

7. Let $T = T(x; t_1, t_2, \dots)$ be the generating function of mobiles, where the exponent of the variable t_i records the number of unlabelled vertices of degree i, and where x records the

total number of unlabelled vertices. Show that

$$T = x \sum_{i \geq 1} \binom{2i-1}{i} t_i (1+T)^{i-1}.$$

8. Show (in a few lines, you don't need more) that the coefficient of $x^k \prod_i t_i^{d_i}$ in T is:

$$\frac{e!}{(e-f+1)!} \prod_i \frac{1}{d_i!} \prod_i \binom{2i-1}{i}^{d_i}.$$

9. Deduce that the number of rooted planar bipartite maps having d_i faces of degree $2i$ for each $i \geq 1$ is given by:

$$\frac{2e!}{(e-f+2)!} \prod_i \frac{1}{d_i!} \prod_i \binom{2i-1}{i}^{d_i}.$$

10. Check that for $d_i = n\mathbf{1}_{i=2}$ you recover Tutte's formula for bipartite quadrangulations.

> HISTORICAL NOTE: *This formula for planar bipartite maps is (also) due to Tutte via a by-hand tour-de-force, and was recovered by Bender and Canfield by solving a refined Tutte equation. The first bijective proof was given by Schaeffer (1997) using blossoming trees, and is different from the one presented here due to BDFG. As we have seen the present bijection has the advantage to work in positive genus, and enables in particular to prove the universality of the counting exponent $\frac{5}{2}(g-1)$ for bipartite maps with controlled degrees (G.C). This bijection extends to the case of non-bipartite maps, but mobiles become more complicated, and no closed formulas exist anymore, even in the planar case.*

Problem B – Maps, the symmetric group, and the Harer-Zagier formula.

This problem is an introduction to the link between map enumeration and algebraic combinatorics. It requires knowledge in algebraic combinatorics of the symmetric group. We use algebra to prove the Harer-Zagier formula for the enumeration of one-face maps.

1. For $n, g \geq 0$ we let $\epsilon_g(n)$ be the number of rooted one-face maps of genus g with n edges. From the lecture we know that $(2n-1)!\epsilon_g(n)$ is the number of triples of permutations (σ, α, ϕ) in $\mathbb{S}_{2n}$ such that $\sigma\alpha = \phi$, and α has cycle type $[2^n]$, ϕ has cycle type $[2n]$, and σ has... how many cycles?

2. Recall Frobenius's formula from representation theory. If G is a finite group, and $\{\chi^\lambda, \lambda \in \Lambda\}$ is the list of its irreducible characters (where Λ is some indexing set), then for any conjugacy classes A, B, C in G, the number of triples $(\sigma_a, \sigma_b, \sigma_c) \in A \times B \times C$ such that $\sigma_a\sigma_b\sigma_c = 1$ is given by:

$$\frac{1}{|G|}\sum_{\lambda\in\Lambda}\frac{|A|\cdot|B|\cdot|C|}{\dim\lambda}\chi^\lambda(A)\chi^\lambda(B)\chi^\lambda(C).$$

Optional question: prove it (requires familiarity with representation theory).

3. Deduce that the number of one-face of genus g maps satisfies:

$$\epsilon_g(n) = \frac{1}{2^n n!}\sum_{k=0}^{2n-1}(-1)^k\chi^k([2^n])\sum_{\substack{\lambda\vdash 2n\\ \ell(\lambda)=n+1-2g}}|K_\lambda|\frac{\chi^k(\lambda)}{\dim\lambda},$$

where χ^k is the character of the symmetric group indexed by the hook partition $[1^k, 2n-k]$, and K_λ is the conjugacy class made by permutations of cycle-type λ. Hint*: Apply the Frobenius formula, and notice that the character $\chi^\lambda([2n])$ vanishes for most λ – use the Murnaghan-Nakayama rule.*

4. Show (using for example the Murnaghan-Nakayama rule) that the evaluation of a hook-character on an a fixed-point free involution is given by: $\chi^k([2^n]) = (-1)^{\lfloor\frac{k+1}{2}\rfloor}\binom{n-1}{\lfloor\frac{k}{2}\rfloor}$.

5. Recall (or even better: prove*) that one has: $\sum_{\lambda\vdash 2n}|K_\lambda|\frac{\chi^k(\lambda)}{\dim\lambda}u^{\ell(\lambda)} = n!\binom{2n-k-1+u}{n}$.
Hint*: Use Jucys Murphy elements – how do they act on irreducible representations?*

6. Deduce the formula:

$$\sum_{g\geq 0}\epsilon_g(n)u^{n+1-2g} = \frac{(2n)!}{2^n n!}\sum_{p=1}^{n}(-1)^{p-1}\binom{n-1}{p-1}\left(\binom{2(n-p)+1+u}{2n}+\binom{2(n-p)+u}{2n}\right) \tag{2}$$

7. Deduce the "Harer-Zagier formula":

$$\sum_{g\geq 0}\epsilon_g(n)u^{n+1-2g} = (2n-1)!!\sum_{q=1}^{n+1}\binom{u}{q}2^{q-1}\binom{n}{q-1}.$$

8. Deduce the even fancier form:

$$\sum_{\substack{g\geq 0\\ n\geq 0}}\frac{\epsilon_g(n)}{(2n-1)!!}u^{n+1-2g}y^{n+1} = \frac{1}{2}\left(\frac{1+y}{1-y}\right)^u - \frac{1}{2},$$

and the "Lehman-Walsh formula":

$$\epsilon_g(n) = \left(\sum_{\gamma\vdash g}\frac{(n+1)_{2g+\ell(\gamma)+1}}{2^{2g}\prod_i m_i(\gamma)!(2i+1)^{m_i(\gamma)}}\right)\mathrm{Cat}(n).$$

Historical note: *The HZ formula is independently due to Harer and Zagier (1986) and to Walsh and Lehman (1972!) who respectively proved it by a link with matrix integrals and by variants of the Tutte equation. The proof given here, is due independently to Jackson and Zagier (1986). This approach also enables to give refined formulas for one-face maps with control on the degrees (Goupil-Schaeffer, or Poulalhon-Schaeffer). The usefulness of the Frobenius formula goes well beyond the case of one-face maps. In particular it enables one to express the generating function of maps in terms of Schur function, and deduce a connection with integrable hierarchies with nice enumerative consequence (see e.g. Goulden Jackson 2010, or Carell-G.C.2014). back to one-face maps, the Harer-Zagier formula and its variants have long been mysterious. This gave rise to lots of combinatorial work, see [Lass], [Goulden-Nica], [C, PTRF], [Bernardi, AAM, 2012] or [C-Féray-Fusy,JCTA, 2013] (see also exercise 3).*