On multiplicatively dependent linear numeration systems, and periodic points

Christiane Frougny

[1] L.I.A.F.A.
Case 7014, 2 place Jussieu, 75251 Paris Cedex 05, France
Christiane.Frougny@liafa.jussieu.fr
[2] Université Paris 8

Abstract. Two linear numeration systems, with characteristic polynomial equal to the minimal polynomial of two Pisot numbers β and γ respectively, such that β and γ are multiplicatively dependent, are considered. It is shown that the conversion between one system and the other one is computable by a finite automaton. We also define a sequence of integers which is equal to the number of periodic points of a sofic dynamical system associated with some Parry number.

1 Introduction

This work is about the conversion of integers represented in two different numeration systems, linked in a certain sense. Recall that the conversion between base 4 and base 2 is computable by a finite automaton, but that conversion between base 3 and base 2 is not. More generally, two numbers $p > 1$ and $q > 1$ are said to be *multiplicatively dependent* if there exist positive integers k and ℓ such that $p^k = q^\ell$. A set of natural numbers is said to be *p-recognizable* if the set of representations in base p of its elements is recognizable by a finite automaton. Bűchi has shown that the set $\{q^n \mid n \geq 0\}$ is p-recognizable only if p and q are multiplicatively dependent integers [5]. In contrast, the famous theorem of Cobham [7] states that the only sets of natural numbers that are both p- and q-recognizable, when p and q are two multiplicatively independent integers > 1, are unions of arithmetic progressions, and thus are k-recognizable for any integer $k > 1$. Several generalizations of Cobham's Theorem have been given, see for instance [25, 10, 6, 17, 8]. In particular this result has been extended by Bès [4] to non-standard numeration systems.

The most popular non-standard numeration system is probably the Fibonacci numeration system. Recall that every non-negative integer can be represented as a sum of Fibonacci numbers, which can be chosen non-consecutive. It is also possible to represent an integer as a sum of Lucas numbers. Since Fibonacci and Lucas numbers satisfy the same recurrence relation, the question of the conversion between Lucas representations and Fibonacci representations is very natural. In [22] and [23], the relation between the Fibonacci sequence and the Lucas sequence is examined from another point of view. A sequence of non-negative integers $(v_n)_{n\geq 0}$ is said to be *exactly realizable* if there exists a dynamical system (S, σ), where S is a compact metric space and $\sigma : S \to S$ is an homeomorphism, for which for all $n \geq 1$, v_n is the number of periodic points of period n, that is,

$$v_n = \#\{s \in S \mid \sigma^n(s) = s\}.$$

The authors give a necessary and sufficient condition for a sequence to be exactly realizable in certain cases. In particular, they prove that amongst the sequences satisfying the Fibonacci recurrence $u_n = u_{n-1} + u_{n-2}$, the unique (up to scalar multiples) exactly realizable sequence is the one of Lucas numbers, and the dynamical system is the golden mean shift, that is to say, the set of bi-infinite sequences on the alphabet $\{0, 1\}$ such that a 1 is always followed by a 0.

A *linear numeration system* is defined by an increasing sequence of integers satisfying a linear recurrence relation. The generalization of the Cobham's Theorem by Bès [4] is the following one : let two linear numeration systems such that their characteristic polynomials are the minimal polynomials of two multiplicatively independent Pisot numbers[1]; the only sets of natural numbers such that their representations in these two systems are recognizable by a finite automaton are unions of arithmetic progressions.

From the result of Bès follows that the conversion between two linear numeration systems U and Y linked to two multiplicatively independent Pisot numbers cannot be realized by a finite automaton. In this paper, we prove that the conversion between two linear numeration systems U and Y such that their characteristic polynomials are the minimal polynomials of two multiplicatively *dependent* Pisot numbers is computable by a finite automaton. This implies that a set of integers which is U-recognizable is then also Y-recognizable. Note that in [6] it is proved that if U and V are two linear numeration systems with the same characteristic polynomial which is the minimal polynomial of a Pisot number, then a U-recognizable set is also V-recognizable.

[1] A Pisot number is an algebraic integer such that its algebraic conjugates are strictly less than 1 in modulus. The golden mean and the natural numbers are Pisot numbers.

The paper is organized as follows. First we recall several results which will be of use in this paper. In particular, the normalization in a linear numeration system consists in converting a representation on a "big" alphabet onto the so-called *normal* representation, obtained by a greedy algorithm. Here the system U is fixed. It is shown in [15] that, basically, when the sequence U is linked to a Pisot number, like the Fibonacci numbers are linked to the golden mean, then normalization is computable by a finite automaton on any alphabet of digits. In the present work we first construct a finite automaton realizing the conversion from Lucas representations to Fibonacci representations. Then we consider two sequences of integers U and V. If the elements of V can be linearly expressed (with rational coefficients) in those of U, and if the normalization in the system U is computable by a finite automaton, then so it is for the conversion from V-representations to U-representations. From this result we deduce that if U and V have for characteristic polynomial the same minimal polynomial of a Pisot number, with different initial conditions, then the conversion from V-representations to U-representations is computable by a finite automaton.

Next we introduce two different linear numeration systems associated with a Pisot number β of degree m. The first one, U_β, is defined from the point of view of the symbolic dynamical system defined by β. We call it *Fibonacci-like*, because when β is equal to the golden mean, it is the Fibonacci numeration system. The second one, V_β, is defined from the algebraic properties of β. More precisely, for $n \geq 1$, the n-th term of V_β is $v_n = \beta^n + \beta_2^n + \cdots + \beta_m^n$, where $\beta_2, \ldots, \beta_m$ are the algebraic conjugates of β. We call it *Lucas-like*, because when β is equal to the golden mean, it is the Lucas numeration system. The conversion from V_β to U_β (or any sequence with characteristic polynomial equal to the minimal polynomial of β) is shown to be computable by a finite automaton.

Then we consider two linear numeration systems, U and Y, such that their characteristic polynomial is equal to the minimal polynomial of a Pisot number β, or γ respectively, where β and γ are multiplicatively dependent. Then the conversion from Y to U is shown to be computable by a finite automaton (Theorem 2).

The Lucas-like sequence V_β plays a central role in the proof of Theorem 2. In fact, it is also closely related to the number of periodic points of the symbolic dynamical system S_β associated with β. Here we do not need the assumption that β is a Pisot number. A *Parry number* is a real number β such that the beta-expansion of 1 (see Section 2.4) is eventually periodic or finite. Such numbers are usually called *beta-numbers* after Parry [21]. Note that a Pisot number is a Parry number [2]. From now on β is a Parry number, and the Fibonacci-like sequence and the Lucas-like sequence are defined as in the Pisot case. If the symbolic dynamical system S_β associated with β is of finite type, that is to say if the beta-expansion of 1 is finite, then the sequence V_β is exactly realized by S_β. This is no more the case when the symbolic dynamical system associated with β is not of finite type, but is sofic, *i.e.* the beta-expansion of 1 is infinite eventually periodic. We define a sequence R_β which is exactly realized by S_β in the sofic case. It is shown that the set of greedy representations of the natural numbers in the linear numeration system defined by R_β is not recognizable by a finite automaton, and consequently the conversion between R_β and V_β cannot be realized by a finite automaton, even if β is a Pisot number.

Section 9 is devoted to the quadratic case study as an example for the general case. We end this paper by exploring the connection between the Lucas-like sequence V_β and the base β-representations for the case where β is a Pisot quadratic unit. Note that in [14] we have proved that the conversion from U_β-representations to folded β-representations is computable by a finite automaton, and in [16], that this is possible only if β is a quadratic Pisot unit.

Part of this work has been presented in [13].

2 Preliminaries

2.1 Words

An *alphabet* A is a finite set. A finite sequence of elements of A is called a *word*, and the set of words on A is the free monoid A^*. The *empty word* is denoted by ε. The set of infinite sequences or infinite words on A is denoted by $A^{\mathbb{N}}$. Let v be a non-empty word of A^*, denote by v^n the concatenation of v to itself n times, and by v^ω the infinite concatenation $vvv\cdots$. An infinite word of the form uv^ω is said to be *eventually periodic*. A *factor* of a (finite or infinite) word w is a finite word f such that $w = ufv$.

2.2 U-representations

The definitions recalled below and related results can be found in the survey [20, Chap. 7]. We consider a generalization of the usual notion of numeration system, which yields a representation of the natural numbers.

The base is replaced by an infinite increasing sequence of integers. The basic example is the well-known Fibonacci numeration system.

Let $U = (u_n)_{n\geq 0}$ be a strictly increasing sequence of integers with $u_0 = 1$. A *U-representation* of a non-negative integer N is a finite sequence of integers $(d_i)_{k\geq i\geq 0}$ such that $N = \sum_{i=0}^{k} d_i u_i$. Such a representation will be written $(N)_U = d_k \cdots d_0$, most significant digit first.

Among all possible U-representations of a given non-negative integer N one is distinguished and called the *normal U-representation* of N; it is also called the *greedy* representation, since it can be obtained by the following greedy algorithm [11]: given integers m and p let us denote by $q(m,p)$ and $r(m,p)$ the quotient and the remainder of the Euclidean division of m by p. Let $k \geq 0$ such that $u_k \leq N < u_{k+1}$ and let $d_k = q(N, u_k)$ and $r_k = r(N, u_k)$, and, for $i = k-1, \ldots, 0$, $d_i = q(r_{i+1}, u_i)$ and $r_i = r(r_{i+1}, u_i)$. Then $N = d_k u_k + \cdots + d_0 u_0$. The normal U-representation of N is denoted by $\langle N\rangle_U$. The normal U-representation of 0 is the empty word ε. The set of greedy or normal U-representations of all the non-negative integers is denoted by $G(U)$. In this work, we consider only the case where the sequence U is linearly recurrent. Then the numeration system associated with U is said to be a *linear numeration system*. The digits of a normal U-representation are contained in a *canonical* finite alphabet A_U associated with U.

Let D be a finite alphabet of integers and let $w = d_k \cdots d_0$ be a word of D^*. Denote by $\pi_U(w)$ the numerical value of w in the system U, that is, $\pi_U(w) = \sum_{i=0}^{k} d_i u_i$. The *normalization* in the system U on D^* is the partial function $\nu_{U,D^*} : D^* \to A_U^*$ that maps a word w of D^* such that $N = \pi_U(w)$ is non-negative onto the normal U-representation of N.

Let U and V be two sequences of integers, and let D be a finite alphabet of integers. The *conversion* from the numeration system V to the numeration system U on D^* is the partial function $\chi : D^* \to A_U^*$ that maps a V-representation $d_k \cdots d_0$ in D^* of a non-negative integer $N = \sum_{i=0}^{k} d_i v_i$ onto the normal U-representation of N. In fact the alphabet D plays no peculiar role, and we will simply speak of the conversion from V to U.

2.3 Beta-expansions

We now consider numeration systems where the base is a real number $\beta > 1$. Representations of real numbers in such systems were introduced by Rényi [24] under the name of *beta-expansions*. Let the base $\beta > 1$ be a real number. First let x be a real number in the interval $[0,1]$. A *representation in base* β of x is an infinite sequence of integers $(x_i)_{i\geq 1}$ such that $x = \sum_{i \geq 1} x_i \beta^{-i}$. A particular beta-representation, called the *beta-expansion*, can be computed by the "greedy algorithm" : denote by $\lfloor y \rfloor$ and $\{y\}$ the integer part and the fractional part of a number y. Set $r_0 = x$ and let for $i \geq 1$, $x_i = \lfloor \beta r_{i-1} \rfloor$, $r_i = \{\beta r_{i-1}\}$. Then $x = \sum_{i \geq 1} x_i \beta^{-i}$, where the x_i's are elements of the canonical alphabet $A_\beta = \{0, \ldots, \lfloor \beta \rfloor\}$ if β is not an integer, or $A_\beta = \{0, \ldots, \beta - 1\}$ if β is an integer. The beta-expansion of x is denoted by $d_\beta(x)$.

Let D be a finite alphabet of integers. The *normalization* in base β on $D^{\mathbb{N}}$ is the partial function $\nu_{\beta, D^{\mathbb{N}}} : D^{\mathbb{N}} \to A_\beta^{\mathbb{N}}$ that maps a word $(x_i)_{i \geq 1}$ of $D^{\mathbb{N}}$ such that $x = \sum_{i \geq 1} x_i \beta^{-i} \in [0, 1[$ onto the β-expansion of x.

Secondly, we consider a real number x greater than 1. There exists $k \in \mathbb{N}$ such that $\beta^k \leq x < \beta^{k+1}$. Hence $0 \leq x/\beta^{k+1} < 1$, thus it is enough to represent numbers from the interval $[0, 1]$, since by shifting we will get the representation of any positive real number. A β-representation of an $x = \sum_{k \leq i \leq -\infty} x_i \beta^i$ will be denoted by $(x)_\beta = x_k \cdots x_0 {\boldsymbol .} x_{-1} x_{-2} \cdots$.

If a representation ends in infinitely many zeros, like $v0^\omega$, the ending zeros are omitted and the representation is said to be *finite*.

A *Pisot number* is an algebraic integer such that its algebraic conjugates are strictly less than 1 in modulus. It is known that if β is a Pisot number then $d_\beta(1)$ is finite or infinite eventually periodic [2].

2.4 Symbolic dynamical systems

The reader may consult [19] for more details on these topics. Let A be a finite alphabet, recall that $A^{\mathbb{N}}$ is endowed with the product topology and the *shift* σ defined by $\sigma((x_i)_{i \geq 1}) = (x_{i+1})_{i \geq 1}$. It is a compact metric space and σ is a homeomorphism. A *symbolic dynamical system* is a closed shift-invariant subset of $A^{\mathbb{N}}$. It is said to be a *system of finite type* if it is defined by the interdiction of a finite set of factors. It is said to be *sofic* if the set of its finite factors is recognizable by a finite automaton. Note that a system of finite type is sofic. The same notions can be defined for bi-infinite sequences and subsets of $A^{\mathbb{Z}}$.

Denote by D_β the set of β-expansions of numbers of $[0, 1[$. The closure of D_β in $A_\beta^{\mathbb{N}}$ is a symbolic dynamical system, called the *beta-shift* S_β. The following results are known: the beta-shift is of finite type if and only if if the β-expansion of 1, $d_\beta(1)$, is finite, and the beta-shift is sofic if and only if $d_\beta(1)$ is eventually periodic [2].

By abuse, we will keep the same name of beta-shift for the set of bi-infinite sequences such that each right tail is in the one-sided beta-shift. We denote by $\mathrm{Per}_n(S_\beta)$ the number of periodic elements of period n under the shift of S_β .

Following [22, 23] we say that a sequence of non-negative integers $V = (v_n)_{n\geq 0}$ is *exactly realizable* if there exists a beta-shift S_β such that for every $n \geq 1$, $v_n = \mathrm{Per}_n(S_\beta)$.

2.5 Automata

We refer the reader to [9]. An *automaton over* A, $\mathcal{A} = (Q, A, E, I, T)$, is a directed graph labelled by elements of A. The set of vertices, traditionally called *states*, is denoted by Q, $I \subset Q$ is the set of *initial* states, $T \subset Q$ is the set of *terminal* states and $E \subset Q \times A \times Q$ is the set of labelled *edges*. If $(p, a, q) \in E$, we denote $p \xrightarrow{a} q$. The automaton is *finite* if Q is finite. A subset H of A^* is said to be *recognizable by a finite automaton* if there exists a finite automaton $\mathcal{A}$ such that H is equal to the set of labels of paths starting in an initial state and ending in a terminal state. A *2-tape automaton* with input alphabet A and output alphabet B is an automaton over the non-free monoid $A^* \times B^*$: $\mathcal{A} = (Q, A^* \times B^*, E, I, T)$ is a directed graph the edges of which are labelled by elements of $A^* \times B^*$. The automaton is finite if Q and E are finite. The finite 2-tape automata are also known as *transducers*. A relation R of $A^* \times B^*$ is said to be *computable by a finite automaton* if there exists a finite 2-tape automaton $\mathcal{A}$ such that R is equal to the set of labels of paths starting in an initial state and ending in a terminal state. A function is computable by a finite automaton if its graph is computable by a finite 2-tape automaton. These definitions extend to relations (and functions) of infinite words as follows: a relation R of infinite words is computable by a finite automaton if there exists a finite 2-tape automaton such that R is equal to the set of labels of infinite paths starting in an initial state and going infinitely often through a terminal state. Recall that the set of relations computable by a finite automaton is closed under composition and inverse.

2.6 Previous results

In this work we will make use of the following results. Let U be a linearly recurrent sequence of integers such that its characteristic polynomial is exactly the minimal polynomial of a Pisot number. Then the set $G(U)$ of normal U-representations of non-negative integers is recognizable by a finite automaton, and, for every alphabet of positive or negative integers D, normalization ν_{U,D^*} is computable by a finite automaton [15]. Normalization in base β, when β is a Pisot number, is computable by a finite automaton on any alphabet D [12]. Addition and multiplication by a fixed positive integer constant are particular cases of normalization, and thus are computable by a finite automaton, in the system U and in base β. These results on normalization do not extend to the case that β is a Parry number which is not a Pisot number.

3 Fibonacci and Lucas

Let us recall that the *Fibonacci numeration system* is defined by the sequence F of Fibonacci numbers

$$F = \{1, 2, 3, 5, 8, 13, \dots\}.$$

The canonical alphabet is $A_F = \{0, 1\}$ and the set of normal representations is equal to $G(F) = 1\{0, 1\}^* \setminus \{0, 1\}^* 11\{0, 1\}^* \cup \varepsilon$. Words containing a factor 11 are forbidden.

The *Lucas numeration system* is defined by the sequence L of Lucas numbers

$$L = \{1, 3, 4, 7, 11, 18, \dots\}.$$

The canonical alphabet is $A_L = \{0, 1, 2\}$ and the set of normal representations is equal to $G(L) = G(F) \cup (G(F) \setminus \varepsilon)\{02\} \cup \{2\}$. We give in Table 1 below the normal Fibonacci and Lucas representations of the first natural numbers.

N	Fibonacci	Lucas
1	1	1
2	10	2
3	100	10
4	101	100
5	1000	101
6	1001	102
7	1010	1000
8	10000	1001
9	10001	1002
10	10010	1010
11	10100	10000

Table 1. Normal Fibonacci and Lucas representations of the 11 first integers

The Fibonacci and the Lucas sequences both have for characteristic polynomial

$$P(X) = X^2 - X - 1.$$

The root > 1 of P is denoted by φ, the golden mean, and its algebraic conjugate by φ'. Since $\varphi + \varphi' = 1$, for coherence of notations with the general case, we denote $F = (F_n)_{n \geq 0}$ and $L = (L_n)_{n \geq 1}$. Recall that for every $n \geq 1$, $L_n = \varphi^n + {\varphi'}^n$. The associated dynamical system is the golden mean shift, which is the set of bi-infinite sequences on $\{0, 1\}$ having no factor 11.

Although the following result is a consequence of the more general one below (Theorem 1), we give here a direct construction.

Proposition 1. *The conversion from a Lucas representation of an integer to the normal Fibonacci representation of that integer is computable by a finite automaton.*

Proof. First, for every $n \geq 3$, we get $L_n = F_{n-1} + F_{n-3}$. Take N a positive integer and a L-representation $(N)_L = d_k \cdots d_1$, where the d_i's are in an alphabet $B \supseteq \{0, 1, 2\}$, and $k \geq 4$. Then $N = d_k L_k + \cdots + d_1 L_1$, thus $N = d_k F_{k-1} + d_{k-1} F_{k-2} + (d_{k-2} + d_k) F_{k-3} + \cdots + (d_3 + d_5) F_2 + (d_2 + d_4) F_1 + (d_1 + d_2 + d_3) F_0$, hence the word $d_k d_{k-1} (d_{k-2} + d_k) \cdots (d_3 + d_5)(d_2 + d_4)(d_1 + d_2 + d_3)$ is a Fibonacci representation of N on a certain finite alphabet of digits D.

The conversion from a word of the form $d_k \cdots d_1$ in B^*, where $k \geq 4$, onto a word of the form $d_k d_{k-1} (d_{k-2} + d_k) \cdots (d_3 + d_5)(d_2 + d_4)(d_1 + d_2 + d_3)$ on D^* is computable by a finite automaton $\mathcal{A} = (Q, B \times C, E, \{\varepsilon\}, \{t\})$: the set of states is $Q = \{\varepsilon\} \cup B \cup (B \times B) \cup \{t\}$ where $\{t\}$ is the unique terminal state. The initial state is ε. For each d in B, there is an edge $\varepsilon \overset{d/d}{\to} d$. For each d and c in B, there is an edge $d \overset{c/c}{\to} (d, c)$. For each $(d, c) \in B \times B$ and a in B, there is an edge $(d, c) \overset{a/a+d}{\to} (c, a)$. For each $(d, c) \in B \times B$ and a in B, there is a terminal edge $(d, c) \overset{a/a+c+d}{\to} t$. Words of length less than 4 are handled directly.

Then it is enough to normalize in the Fibonacci system on D^*, and it is known that this is realizable by a finite automaton, see Section 2.6. $\square$

On Figure 1 we give an automaton realizing the conversion from normal Lucas representations to Fibonacci representations on $\{0, 1, 2\}^* (\{\varepsilon\} \cup \{3\})$. States of the form (d, c) are denoted by dc. Note that this automaton is not deterministic on inputs. Since we are dealing with *normal* Lucas representations, the automaton has less states than the one constructed in the proof of Proposition 1 above. To decrease the complexity of the drawing, we introduce more than one terminal state. Terminal states are indicated by an outgoing arrow. The result must be normalized afterwards.

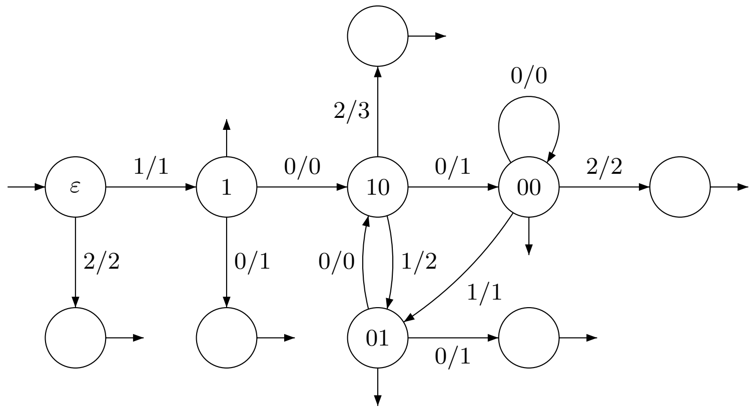

Fig. 1. Conversion from normal Lucas representations to Fibonacci representations

4 A technical result

We now consider two linearly recurrent sequences $U = (u_n)_{n\geq 0}$ and $V = (v_n)_{n\geq 0}$ of positive integers. The result below is the generalization of Proposition 1.

Proposition 2. *If there exist r rational constants λ_i's for $1 \leq i \leq r$ and $K \geq 0$ such that for every $n \geq K$, $v_n = \lambda_1 u_{n+r-1} + \cdots + \lambda_r u_n$, and if the normalization in the system U is computable by a finite automaton on any alphabet, then the conversion from a V-representation of an integer to the normal U-representation of that integer is computable by a finite automaton.*

Proof. One can assume that the λ_i's are all of the form p_i/q where the p_i's belong to $\mathbb{Z}$ and q belongs to $\mathbb{N}$, $q \neq 0$. Let N be a positive integer and consider a V-representation $(N)_V = b_j \cdots b_0$, where the b_i's are in an alphabet of digits $B \supseteq A_V$. Then $qN = b_j q v_j + \cdots + b_0 q v_0$. Since for $n \geq K$, $qv_n = p_1 u_{n+r-1} + \cdots + p_r u_n$, and $v_0, v_1, \ldots, v_{K-1}$ can be expressed in the system U, we get that qN is of the form $qN = d_{j+r-1} u_{j+r-1} + \cdots + d_0 u_0$. Since each digit d_i, for $0 \leq i \leq j+r-1$, is a linear combination of $q, p_1, \ldots, p_r$, the b_i's and the coefficients of the U-representation of the first terms $v_0, v_1, \ldots, v_{K-1}$, we get that d_i is an element of a finite alphabet of digits $D \supset A_U$. By assumption, ν_{U,D^*} is computable by a finite automaton. It remains to show that the function which maps $\nu_{U,D^*}(d_{j+r-1} \cdots d_0) =< qN >_U$ onto $< N >_U$ is computable by a finite automaton, and this is due to the fact that it is the inverse of the multiplication by the natural q, which is computable by a finite automaton in the system U, see Section 2.6. □

5 Common characteristic polynomial

The Fibonacci and the Lucas numeration systems are examples of different numeration systems having the same characteristic polynomial, but different initial conditions.

Theorem 1. *Let P be the minimal polynomial of a Pisot number of degree m. Let U and V be two sequences with common characteristic polynomial P and different initial conditions. The conversion from a V-representation of a positive integer to the normal U-representation of that integer is computable by a finite automaton.*

Proof. Since the polynomial P is the minimal polynomial of a Pisot number, normalization in the system U is computable by a finite automaton on any alphabet (see Section 2.6). On the other hand, the family $\{u_n, u_{n+1}, \ldots, u_{n+m-1} \mid n \geq 0\}$ is free, because the annihilator polynomial is the minimal polynomial. Since U and V have the same characteristic polynomial, it is known from standard results of linear algebra that there exist rational constants λ_i such that, for each $n \geq 0$, $v_n = \lambda_1 u_{n+m-1} + \cdots + \lambda_m u_n$. The result follows then from Proposition 2. □

6 Two numeration systems associated with a Parry number

Let β be a Parry number, *i.e.* the β-expansion of 1 is finite or eventually periodic. We define two numeration systems associated with β.

6.1 Fibonacci-like numeration system

First suppose that the β-expansion of 1 is finite, $d_\beta(1) = t_1 \cdots t_N$. A linear recurrent sequence $U_\beta = (u_n)_{n\geq 0}$ is canonically associated with β as follows

$$u_n = t_1 u_{n-1} + \cdots + t_N u_{n-N} \quad \text{for} \quad n \geq N$$

$$u_0 = 1, \quad \text{and for} \quad 1 \leq i \leq N-1, \quad u_i = t_1 u_{i-1} + \cdots + t_i u_0 + 1.$$

The characteristic polynomial of U_β is thus

$$K(X) = X^N - t_1 X^{N-1} - \cdots - t_N.$$

Suppose now that the β-expansion of 1 is infinite eventually periodic,

$$d_\beta(1) = t_1 \cdots t_N (t_{N+1} \cdots t_{N+p})^\omega$$

with N and p minimal. The sequence $U_\beta = (u_n)_{n \geq 0}$ is the following one

$$u_n = t_1 u_{n-1} + \cdots + t_{N+p} u_{n-N-p} + u_{n-p} - t_1 u_{n-p-1} - \cdots - t_N u_{n-N-p}$$

for $n \geq N + p$,

$$u_0 = 1, \text{ and for } 1 \leq i \leq N + p - 1, u_i = t_1 u_{i-1} + \cdots + t_i u_0 + 1.$$

The characteristic polynomial of U_β is now

$$K(X) = X^{N+p} - \sum_{i=1}^{N+p} t_i X^{N+p-i} - X^N + \sum_{i=1}^{N} t_i X^{N-i}.$$

Note that in general $K(X)$ may be reducible. Since $K(X)$ is defined from the beta-expansion of 1, we will say that it is the *beta-polynomial* of β.

The system U_β is said to be the *canonical* numeration system associated with β. In [3] it is shown that the set of normal representations of the integers $G(U_\beta)$ is exactly the set of finite factors of the beta-shift S_β. The numeration system U_β is the natural one from the point of view of symbolic dynamical systems. The set $G(U_\beta)$ is recognized by a finite automaton, see Section 8.

6.2 Lucas-like numeration system

Now we introduce another linear recurrent sequence $V_\beta = (v_n)_{n \geq 0}$ associated with β a Parry number of degree m as follows. Denote by $\beta_1 = \beta$, $\beta_2, \ldots, \beta_m$ the roots of the minimal polynomial $P(X) = X^m - a_1 X^{m-1} - \cdots a_m$ of β. Set

$$v_0 = 1, \text{ and for } n \geq 1, \ v_n = \beta_1^n + \cdots + \beta_m^n.$$

Then the characteristic polynomial of V_β is equal to $P(X)$. The set $G(V_\beta)$ is recognized by a finite automaton, [15].

As an example let us take $\beta = \varphi$ the golden mean. Then U_φ is the set of Fibonacci numbers, and V_φ is the set of Lucas numbers (for $n \geq 1$). If β is an integer, then the two systems U_β and V_β are the same, the standard β-ary numeration system.

6.3 Conversion in the Pisot case

Now we suppose that β is a Pisot number.

Proposition 3. *Let β be a Pisot number such that its beta-polynomial $K(X)$ is equal to its minimal polynomial. Let U be any linear sequence with characteristic polynomial equal to $K(X)$ (in particular U_β). The conversion from the linear numeration system V_β to the linear numeration system U (and conversely) is computable by a finite automaton.*

Proof. It comes from the fact that U and V_β have the same characteristic polynomial, which is the minimal polynomial of a Pisot number. Thus normalization in both systems is computable by a finite automaton on any alphabet, and the result follows by Theorem 1. □

7 Multiplicatively dependent numeration systems

First recall that if β is a Pisot number of degree m then, for any positive integer k, β^k is a Pisot number of degree m (see [1]). Two Pisot numbers β and γ are said to be *multiplicatively dependent* if there exist two positive integers k and ℓ such that $\beta^k = \gamma^\ell$. Then β and γ have the same degree m.

Theorem 2. *Let β and γ be two multiplicatively dependent Pisot numbers. Let U and Y be two linear sequences with characteristic polynomial equal to the minimal polynomial of β and γ respectively. Then the conversion from the Y-numeration system to the U-numeration system is computable by a finite automaton.*

Proof. Set $\delta = \beta^k = \gamma^\ell$. As above, let $V_\beta = (v_n)_{n\geq 0}$ with $v_0 = 1$ and $v_n = \beta_1^n + \cdots + \beta_m^n$ for $n \geq 1$. The conjugates of δ are of the form $\delta_i = \beta_i^k$, for $2 \leq i \leq m$. Set $W = (w_n)_{n\geq 0}$ with $w_n = \delta_1^n + \cdots + \delta_m^n$ for $n \geq 1$. Then W is the Lucas-like numeration system associated with δ. Now, for $n \geq 1$, $w_n = v_{kn}$. Thus any W-representation of an integer N of the form $(N)_W = d_k \cdots d_0$ gives a V_β-representation $(N)_{V_\beta} = d_k 0^{k-1} d_{k-1} 0^{k-1} \cdots d_1 0^{k-1} d_0$, and thus the conversion from W-representations to Lucas-like V_β-representations is computable by a finite automaton. The same is true for the conversion from W-representations to V_γ-representations. By Proposition 3 the conversion from Y to V_γ, and that from V_β to U are computable by a finite automaton, and the result follows. □

A set S of natural numbers is said to be *U-recognizable* if the set $\{< n >_U |\ n \in S\}$ of normal U-representations of the elements of S is recognizable by a finite automaton. The following result is an immediate consequence of Theorem 2.

Corollary 1. *Let β and γ be two multiplicatively dependent Pisot numbers. Let U and Y be two linear sequences with characteristic polynomial equal to the minimal polynomial of β and γ respectively. Then a set which is U-recognizable is Y-recognizable as well.*

8 Periodic points

Let β be a Parry number. The beta-shift S_β is sofic, *i.e.* the set of its finite factors is recognizable by a finite automaton, and periodic points of S_β are periodic bi-infinite words that are labels of bi-infinite paths in the automaton that recognizes it.

The determination of the number of periodic points of the beta-shift S_β is important, because the entropy of S_β is equal to

$$h(S_\beta) = \lim_{n\to\infty} \frac{1}{n} \log \mathrm{Per}_n(S_\beta) = \log \beta$$

see [19, Th. 4.3.6].

Note that, for any prime q, $\mathrm{Per}_q(S_\beta) \equiv \mathrm{Per}_1(S_\beta) \mod q$, see [23].

In the sequel, we assume that the minimal polynomial $P(X)$ of β and its beta-polynomial $K(X)$ are identical, of degree m. As above, let $V_\beta = (v_n)_{n\geq 0}$ with $v_n = \beta^n + \beta_2^n + \cdots + \beta_m^n$ for $n \geq 1$.

8.1 The finite type case

If $d_\beta(1) = t_1 \cdots t_N$, then S_β is a system of finite type. We construct an automaton $\mathcal{A}_\beta$ which recognizes the set of factors of S_β. There are N states $q_1, \ldots, q_N$. For each i, $1 \leq i < N$, there is an edge labelled t_i from q_i to q_{i+1}. For $1 \leq i \leq N$, there are edges labelled by $0, 1, \ldots, t_i - 1$ from q_i to q_1. The adjacency matrix of $\mathcal{A}_\beta$ is the companion matrix M of $K(X)$, defined by, for $1 \leq i \leq N$

$$M[i,1] = t_i$$
$$M[i,i+1] = 1$$

and other entries equal to 0.

Proposition 4. *Let β be a Parry number such that $d_\beta(1) = t_1 \cdots t_N$. Then for $n \geq 1$, $v_n = \mathrm{trace}\,(M^n) = \mathrm{Per}_n(S_\beta)$.*

Proof. Since M is the adjacency matrix of a system of finite type, the number of periodic points of period n in S_β is equal to trace (M^n), see for instance [19]. On the other hand, since M is the companion matrix of the minimal polynomial of β, we have that $\mathrm{trace}\,(M^n) = \beta_1^n + \cdots + \beta_N^n = v_n$ for $n \geq 1$. □

Corollary 2. *When $d_\beta(1)$ is finite, the Lucas-like sequence V_β is exactly realized by the beta-shift S_β.*

8.2 The infinite sofic case

This is the case when $d_\beta(1) = t_1 \cdots t_N (t_{N+1} \cdots t_{N+p})^\omega$. We construct an automaton $\mathcal{A}_\beta$ which recognizes the set of factors of S_β. There are $N+p$ states $q_1, \ldots, q_{N+p}$. For each i, $1 \leq i < N+p$, there is an edge labelled t_i from q_i to q_{i+1}. There is an edge labelled t_{N+p} from q_{N+p} to q_{N+1}. For $1 \leq i \leq N+p$, there are edges labelled by $0, 1, \ldots, t_i - 1$ from q_i to q_1. The adjacency matrix of $\mathcal{A}_\beta$ is the matrix M defined by for $1 \leq i \leq N+p$

$$\begin{aligned} M[i,1] &= t_i \\ M[i,i+1] &= 1 \text{ for } i \neq N+p \\ M[N+p,N+1] &= 1 \end{aligned}$$

and other entries equal to 0.

Proposition 5. *Let β be a Parry number such that*

$$d_\beta(1) = t_1 \cdots t_N (t_{N+1} \cdots t_{N+p})^\omega.$$

Then for $n \geq 1$, $v_n = \mathrm{trace}\,(M^n)$.

Proof. Remark that M is not the companion matrix of $P(X)$. The companion matrix C is in that case the following one

$$\begin{aligned} C[i,1] &= t_i \text{ for } 1 \leq i \leq p-1 \\ C[p,1] &= t_p + 1 \\ C[i,1] &= t_i - t_{i-p} \text{ for } p+1 \leq i \leq N+p \\ C[i,i+1] &= 1 \text{ for } 1 \leq i \leq N+p \end{aligned}$$

and other entries equal to 0. By a straightforward computation, it is possible to show that the matrices M and C are similar. More precisely, there exists a matrix Z such that $M = Z^{-1}CZ$, where Z is defined by, for $1 \leq i, j \leq N + p$

$$\begin{aligned} Z[i,j] &= 1 \ \text{ if } \ i \equiv j \mod p \ \text{ and } \ i \geq j \\ &= 0 \ \text{ otherwise} \end{aligned}$$

Therefore $\operatorname{trace}(M^n) = \operatorname{trace}(C^n) = \beta_1^n + \cdots + \beta_{N+p}^n = v_n$ for $n \geq 1$. □

Contrarily to what happens in the case where the system is of finite type, in the sofic case different loops in the automaton $\mathcal{A}_\beta$ may have the same label, see Section 9.2 for the quadratic case. So $\mathrm{Per}_n(S_\beta)$ is not equal to v_n.

Proposition 6. *Let β a Parry number such that*

$$d_\beta(1) = t_1 \cdots t_N (t_{N+1} \cdots t_{N+p})^\omega.$$

Then for $n \geq 1$,

$$\begin{aligned} &\mathrm{Per}_n(S_\beta) = v_n - p \text{ if } p \text{ divides } n \\ &\mathrm{Per}_n(S_\beta) = v_n \text{ otherwise.} \end{aligned}$$

Proof. Recall that $d_\beta(1)$ is strictly greater in the lexicographic order $<_{lex}$ than the shifted sequences $\sigma^i(d_\beta(1))$ for $i > 1$, [21].
First, suppose that for each i, $1 \leq i \leq p$, $t_{N+i} < t_1$. Then in the automaton $\mathcal{A}_\beta$ there are two loops with label $t_{N+1} \cdots t_{N+p}$, one starting from state q_1 and the other one from state q_{N+1}.
Second, suppose that there exists $1 \leq i < p$ maximum such that $t_1 \cdots t_i = t_{N+1} \cdots t_{N+i} = w$. Then necessarily $t_{N+i+1} < t_{i+1}$. Thus there is a path

$$q_1 \xrightarrow{w} q_{i+1} \xrightarrow{t_{N+i+1}} q_1$$

and since $t_{N+i+2} \cdots t_{N+p} <_{lex} t_1 \cdots t_{p-i-1}$, there is a loop with label $t_{N+i+2} \cdots t_{N+p}$ from q_1. Thus there are two loops with label $t_{N+1} \cdots t_{N+p}$. So there are p times two loops with same label, a circular permutation of the word $t_{N+1} \cdots t_{N+p}$. Thus when counting the periodic bi-infinite words in the automaton that are labels of loops, we must remove p of them each time the period is a multiple of p. □

Corollary 3. *The sequence $R_\beta = (r_n)_{n \geq 1}$ defined by $r_0 = 1$, and for $n \geq 1$, $r_n = v_n - p$ if p divides n and $r_n = v_n$ otherwise, is exactly realized by the sofic beta-shift S_β.*

Proposition 7. *The sequence R_β is a linear recurrent sequence, of characteristic polynomial $(X^p - 1)K(X)$.*

Proof. Let us rewrite the minimal polynomial of β as $K(X) = X^{N+p} - a_1 X^{N+p-1} - \cdots - a_{N+p}$. Hence, for $n \geq N + p + 1$,

$$v_n = a_1 v_{n-1} + \cdots + a_{N+p} v_{n-N-p}.$$

Suppose that p does not divide n. Then

$$r_n = v_n = \sum_{\substack{1 \leq i \leq N+p \\ p \nmid n-i}} a_i r_{n-i} + \sum_{\substack{1 \leq i \leq N+p \\ p \mid n-i}} a_i (r_{n-i} + p).$$

Thus

$$r_n = \sum_{1 \leq i \leq N+p} a_i r_{n-i} + \sum_{\substack{1 \leq i \leq N+p \\ p \mid n-i}} p. \tag{1}$$

Similarly

$$r_{n-p} = \sum_{1 \leq i \leq N+p} a_i r_{n-p-i} + \sum_{\substack{1 \leq i \leq N+p \\ p \mid n-p-i}} p.$$

Therefore, since the two last sums in r_n and r_{n-p} respectively are equal,

$$r_n = (\sum_{1 \leq i \leq N+p} a_i r_{n-i}) + r_{n-p} - \sum_{1 \leq i \leq N+p} a_i r_{n-p-i}.$$

If p divides n then

$$r_n = -p + \sum_{1 \leq i \leq N+p} a_i r_{n-i} + \sum_{\substack{1 \leq i \leq N+p \\ p \mid n-i}} p \tag{2}$$

and the result follows as above. Hence the characteristic polynomial of R_β is equal to $(X^p - 1)K(X)$. $\square$

Proposition 8. *The set $G(R_\beta)$ of normal R_β-representations of the natural numbers is not recognizable by a finite automaton.*

Proof. Suppose that $G(R_\beta)$ is recognizable by a finite automaton. Then the set

$$H = \{\langle r_n - 1 \rangle_{R_\beta} \mid n \geq 1\}$$

of words of $G(R_\beta)$ that are maximal for the lexicographic order is recognizable by a finite automaton as well, see [26]. It is also known, by [18], that the normal R_β-representation of $r_n - 1$, for n large enough, begins with a prefix of the form $t_1 \cdots t_N (t_{N+1} \cdots t_{N+p})^j$ for some integer j, because β is the dominant root of the characteristic polynomial $J(X) = (X^p - 1)K(X)$ of R_β, and $d_\beta(1) = t_1 \cdots t_N (t_{N+1} \cdots t_{N+p})^\omega$.

Denote by $K'(X)$ the opposite of the reciprocal polynomial of $K(X)$, $K'(X) = -1 + t_1 X + \cdots + t_{p-1} X^{p-1} + (t_p + 1) X^p + (t_{p+1} - t_1) X^{p+1} + \cdots + (t_{N+p} - t_N) X^{N+p}$. Similarly, let $J'(X) = K'(X) - X^p K'(X)$.

By a direct computation, one gets, for each $j \geq 1$

$$
\begin{aligned}
&J'(X) + 2X^p J'(X) + \cdots + (j+1) X^{pj} J'(X) = \\
&\quad K'(X) + X^p K'(X) + \cdots + X^{pj} K'(X) - (j+1) X^{p(j+1)} K'(X)
\end{aligned} \tag{3}
$$

We introduce a notation: if $w = w_0 \cdots w_n$ is a word, $\psi(w) = w_0 + w_1 X + \cdots + w_n X^n$ is the polynomial associated with w (with increasing powers). The signed digit $-d$ is denoted by $\bar{d}$. We then get, for each $j \geq 1$

$$
\begin{aligned}
&K'(X) + X^p K'(X) + \cdots + X^{pj} K'(X) = \\
&\quad \psi(\bar{1} t_1 \cdots t_N (t_{N+1} \cdots t_{N+p})^{j+1}) + X^{p(j+1)} \psi(1 \bar{t_1} \cdots \bar{t_N})
\end{aligned} \tag{4}
$$

Case 1. $p \geq N + 1$.
From Eq. (3) and (4) follows that, for $n = N + p(j+2) + \ell$, with $1 \leq \ell \leq p$, $r_n - 1$ has a R_β-representation of the form

$$(r_n - 1)_{R_\beta} = t_1 \cdots t_N (t_{N+1} \cdots t_{N+p})^j w^{(n)}$$

where $w^{(n)}$ is a word of length $2p + \ell$, corresponding to the polynomial

$$
\begin{aligned}
W^{(n)}(X) = {} & t_{N+1} + t_{N+2} X + \cdots t_{N+p} X^{p-1} \\
& + X^{p-N-1} - t_1 X^{p-N} - \cdots - t_N X^{p-1} \\
& - (j+1) X^{p-N-1} K'(X) - X^{2p+\ell-1}.
\end{aligned} \tag{5}
$$

The difference between $W^{(N+p(j+3)+\ell)}$ and $W^{(N+p(j+2)+\ell)}$ is equal to $-X^{p-N-1} K'(X)$. The word associated with $-X^{p-N-1} K'(X)$ is of the form $s = 0^{p-N-1} 1 \bar{t_1} \cdots \overline{t_{p-1}} (-t_p - 1)(t_1 - t_{p+1}) \cdots (t_N - t_{N+p}) 0^\ell$, and the value of s in the system R_β is equal to $\pi_{R_\beta}(s) = r_{N+p+\ell} - t_1 r_{N+p+\ell-1} - \cdots - t_{p-1} r_{N+\ell+1} - (t_p + 1) r_{N+\ell} + (t_1 - t_{p+1}) r_{N+\ell-1} + \cdots + (t_N - t_{N+p}) r_\ell$.

Suppose that $N + p + \ell$ is not divisible by p. From Eq. (1) follows that $\pi_{R_\beta}(s)$ is equal to the positive constant

$$C(\ell) = \sum_{\substack{1 \leq i \leq N+p \\ p \mid N+\ell-i}} p.$$

For $1 \leq \ell \leq p$ fixed such that $N+p+\ell$ is not divisible by p, let $I(\ell) = \{n \in \mathbb{N} \mid n = N+p(j+2)+\ell,\ j \geq 1\}$. Let $\kappa(n) = \pi_{R_\beta}(w^{(n)})$. The family $(\kappa(n))_{n\in I(\ell)}$ is thus strictly increasing. Remember that the length $|w^{(n)}|$ is equal to $2p+\ell$.
If $\kappa(n) < r_{2p+\ell}$, then the normal R_β-representation of $r_n - 1$ is of the form $\langle r_n - 1\rangle_{R_\beta} = t_1 \cdots t_N(t_{N+1}\cdots t_{N+p})^j z^{(n)}$ where $z^{(n)}$ is a word of length $2p+\ell$, equal to the normal R_β-representation of $w^{(n)}$, prefixed by an adequate number of 0's.
If $\kappa(n) \geq 2p+\ell$, then let h be the smallest positive integer such that $\pi_{R_\beta}((t_{N+1}\cdots t_{N+p})^h w^{(n)}) < r_{p(h+2)+\ell}$. Then

$$\langle r_n - 1\rangle_{R_\beta} = t_1 \cdots t_N(t_{N+1}\cdots t_{N+p})^{j-h} z^{(n)}$$

where $z^{(n)}$ is a word of length $p(h+2)+\ell$ that is the normal R_β-representation of $(t_{N+1}\cdots t_{N+p})^h w^{(n)}$. From this follows that the set $\{\langle r_n - 1\rangle_{R_\beta} \mid n \in I(\ell)\}$ is not recognizable by a finite automaton, and so it is for the set H itself.

Case 2. $p < N+1$.
Let k be the smallest integer ≥ 2 such that $N+1 \leq kp$. Then from Eq. (3) and (4) follows that, for $n = N+p(j+2)+\ell$, with $1 \leq \ell \leq p$, $r_n - 1$ has a R_β-representation of the form

$$(r_n - 1)_{R_\beta} = t_1 \cdots t_N(t_{N+1}\cdots t_{N+p})^{j+1-k} w^{(n)}$$

where $w^{(n)}$ is a word of length $p(k+1)+\ell$, corresponding to the polynomial

$$\begin{aligned} W^{(n)}(X) = {} & (t_{N+1} + t_{N+2}X + \cdots t_{N+p}X^{p-1})(1 + X + \cdots + X^k) \\ & + X^k(X^{p-N-1} - t_1X^{p-N} - \cdots - t_N X^{p-1}) \\ & - (j+1)K'(X)X^{pk-N-1} - X^{p(k+1)+\ell-1}. \end{aligned} \tag{6}$$

With the same reasoning as in Case 1, we show that H is not recognizable by a finite automaton. $\square$

9 Example : the quadratic case

Here we are interested only in the case where the root $\beta > 1$ of the polynomial $P(X) = X^2 - aX - b$, with a and b in $\mathbb{Z}$, is a Parry number, which is the case only if $a \geq b \geq 1$, or if $a \geq 3$ and $-a+2 \leq b \leq -1$. Note that β is in fact a Pisot number. We denote the conjugate of β by β', $|\beta'| < 1$.

9.1 The finite type case

Suppose that $a \geq b \geq 1$. Then the β-expansion of 1 is $d_\beta(1) = ab$, and the canonical alphabet is $A_\beta = \{0, \dots, a\}$. Forbidden words are those containing a factor in the finite set $I = \{ab, a(b+1), \dots, aa\}$, hence the dynamical system S_β associated with β is of finite type. It is the set of bi-infinite sequences in the automaton described in Figure 2.

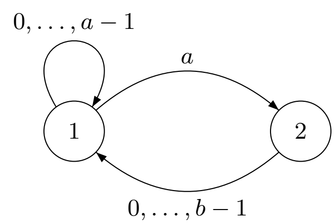

Fig. 2. Automaton in the finite type case

The matrix M of S_β is

$$M = \begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}$$

The Fibonacci-like sequence U_β is defined by $u_n = au_{n-1} + bu_{n-2}$ for $n \geq 2$, with $u_0 = 1$ and $u_1 = a + 1$.

The Lucas-like sequence V_β is defined by $v_n = av_{n-1} + bv_{n-2}$ for $n \geq 3$, with $v_0 = 1$, $v_1 = \beta + \beta' = a$ and $v_2 = \beta^2 + {\beta'}^2 = a^2 + 2b$. In the special case in which $a = b = 1$ (Fibonacci), this definition gives $v_0 = v_1 = 1$, which is not allowed, since the sequence must be stricly increasing. This case has been handled in Section 3.

Note that, for $n \geq 1$

$$v_n = \frac{a - 2b}{a - b + 1} u_n + \frac{2a + 2b - ab}{a - b + 1} u_{n-1}.$$

The sequence V_β is exactly realizable. It is proved in [23] that if a and b are in $\mathbb{N}$, if $\Delta = a^2 + 4b$ is not a square, and if a and $a^2 + 2b$ are relatively prime, then a sequence V satisfying the polynomial P is exactly realizable if and only if $\frac{v_2}{v_1} = \frac{a^2+2b}{a}$.

9.2 The infinite sofic case

Suppose that $a \geq 3$ and $-a+2 \leq b \leq -1$. Then $d_\beta(1) = (a-1)(a+b-1)^\omega$ and the canonical alphabet is $A_\beta = \{0, \dots, a-1\}$. The dynamical system S_β associated with β is sofic : it is the set of bi-infinite sequences in the automaton described in Figure 3. A word is forbidden if and only if it contains a factor in the set $I = \{(a-1)(a+b-1)^n d \mid a+b \leq d \leq a-1,\ n \geq 0\}$, which is recognizable by a finite automaton.

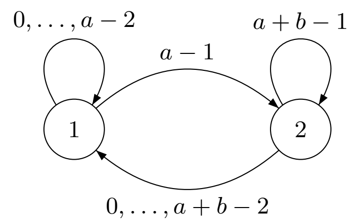

Fig. 3. Automaton in the sofic case

The matrix M of S_β is

$$M = \begin{pmatrix} a-1 & 1 \\ a+b-1 & 1 \end{pmatrix}$$

The companion matrix of β is

$$C = \begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}$$

The Fibonacci-like sequence U_β is defined by $u_n = au_{n-1} + bu_{n-2}$ for $n \geq 2$, with $u_0 = 1$ and $u_1 = a$.

The Lucas-like sequence V_β is defined by $v_n = av_{n-1} + bv_{n-2}$ for $n \geq 3$, with $v_0 = 1$, $v_1 = \beta + \beta' = a$ and $v_2 = \beta^2 + \beta'^2 = a^2 + 2b$.

Note that, for $n \geq 1$ we have

$$v_n = 2u_n - au_{n-1}.$$

We have that, for $n \geq 1$, $\mathrm{Per}_n(S_\beta) = v_n - 1$, since there are two different loops labelled by $(a+b-1)$ in the automaton of Figure 3, one from state 1 and the other one from state 2, because $0 < a+b-1 \leq a-2$.

The sequence $R_\beta = (r_n)_{n \geq 0}$ defined by

$$r_n = (a+1)r_{n-1} + (b-a)r_{n-2} - br_{n-3}$$

for $n \geq 3$, and $r_0 = 1$, $r_1 = a - 1$, $r_2 = a^2 + 2b - 1$ and $r_3 = a^3 + 3ab - 1$, exactly realizes the beta-shift.

Example 1. Take $a = 3$ and $b = -1$. Then $\beta = \frac{3+\sqrt{5}}{2}$, $d_\beta(1) = 21^\omega$, and $U_\beta = \{1, 3, 8, 21, 55, 144, 377, \ldots\}$ is the sequence of Fibonacci numbers of even index; $V_\beta = \{1, 3, 7, 18, 47, 123, 322, \ldots\}$ is the sequence of Lucas numbers of even index n for $n \geq 1$. The sequence which exactly realizes S_β is $R_\beta = \{1, 2, 6, 17, 46, 122, 321, \ldots\}$. The set $H = \{\langle r_n - 1\rangle_{R_\beta} \mid n \geq 1\}$ is equal to $H = \{1, 21, 220, 2121, 21200, 211201, 2111210, 21111211,$ $211111220, 2111112000, \ldots\}$. $\square$

10 Quadratic Pisot units

Here β is a quadratic Pisot unit, that is to say the root > 1 of the polynomial $P(X) = X^2 - aX - 1$, with $a \geq 1$, or of the polynomial $P(X) = X^2 - aX + 1$, with $a \geq 3$. In that case there are nice properties connecting the numeration in the systems U_β and V_β and in base β. It is known that, when β is a quadratic Pisot unit, every positive integer has a finite β-expansion [15], the conversion from U_β-representations to β-representations folded around the radix point is computable by a finite automaton [14], and this property is characteristic of quadratic Pisot units [16].

As an example, we give in Table 2 the φ-expansions of the first integers.

N	φ-expansions
1	1.
2	10.01
3	100.01
4	101.01
5	1000.1001
6	1010.0001
7	10000.0001
8	10001.0001
9	10010.0101
10	10100.0101
11	10101.0101

Table 2. φ-expansions of the 11 first integers

We now make the link with the Lucas-like numeration V_β.

10.1 Case $\beta^2 = a\beta + 1$

First suppose that $a \geq 2$. The following result is a simple consequence of the fact that for $n \geq 1$, $v_n = \beta^n + \beta'^n$ and that $\beta' = -\beta^{-1}$.

Lemma 1. *Let B be a finite alphabet of digits containing A_{V_β}. If $(N)_{V_\beta} = d_k \cdots d_0$, with $d_i \in B$, then $(N)_\beta = d_k \cdots d_0 . \bar{d}_1 d_2 \bar{d}_3 \cdots (-1)^k d_k$.*

Note that the digits in $(N)_\beta$ are elements of the alphabet $\tilde{B} = \{d, \bar{d} \mid d \in B\}$. Then the β-expansion of N is obtained by using the normalization $\nu_{\beta,\tilde{B}^{\mathbb{N}}}$ (which is computable by a finite automaton).

Now we treat the case $a = 1$. The connection between Lucas representations and representations in base the golden mean φ is the following one.

Lemma 2. *Let B be a finite alphabet of digits containing A_L. If $(N)_L = d_k \cdots d_1$, with $d_i \in B$, then $(N)_\varphi = d_k \cdots d_1 0 . \bar{d}_1 d_2 \cdots (-1)^k d_k$.*

As above, the φ-expansion of N is obtained by using the normalization $\nu_{\varphi,\tilde{B}^{\mathbb{N}}}$.

10.2 Case $\beta^2 = a\beta - 1$

Then $d_\beta(1) = (a-1)(a-2)^\omega$.

The following lemma is just a consequence of the fact that for $n \geq 1$, $v_n = \beta^n + \beta'^n$ and that $\beta' = \beta^{-1}$.

Lemma 3. *Let B be a finite alphabet of digits containing A_{V_β}. If $(N)_{V_\beta} = d_k \cdots d_0$, with $d_i \in B$, then $(N)_\beta = d_k \cdots d_0 . d_1 \cdots d_k$.*

Proposition 9. *If $d_k \cdots d_0$ is the normal V_β-representation of N then $d_k \cdots d_0 . d_1 \cdots d_k$ is the β-expansion of N.*

Proof. Note that $G(V_\beta) = \{w \in G(U_\beta) \mid w \neq w'(a-1)(a-2)^n,\ n \geq 1\}$. Now, it is enough to show that if $w = d_k \cdots d_0$ is in $G(V_\beta)$, then $d_k \cdots d_1 d_0 d_1 \cdots d_k$ contains no factor in $I = \{(a-1)(a-2)^n(a-1) \mid n \geq 0\}$. First, w has no factor in I since $G(V_\beta) \subset G(U_\beta)$. Second, $d_0 d_1 \cdots d_k$ has no factor in I either, because I is symmetrical. Third, suppose that $g = d_k \cdots d_1 d_0 d_1 \cdots d_k$ is of the form $g = g'(a-1)(a-2)^j(a-2)^{n-j}(a-1)g''$, with $w = g'(a-1)(a-2)^j$. Then $w \notin G(V_\beta)$, a contradiction. $\square$

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