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CODING OF TWO-DIMENSIONAL CONSTRAINTS OF FINITE TYPE BY SUBSTITUTIONS

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ABSTRACT

We give an automatic method to generate transition matrices associated with two-dimensional contraints of finite type by using squared substitutions of constant dimension.

Keywords: Two-dimensional Fibonacci constraints, two-dimensional subshift of finite type, transition matrices, two-dimensional substitutions, finite automata.

1. Introduction

The goal of this paper is to study two-dimensional codes with horizontal and vertical constraints of finite type, and to construct new tools in order to investigate the entropy or capacity of such codes. In current storage devices – magnetic disks and tape drives, and optical disk drives – the recording medium is considered as having one dimension [3, 18, 24]. Hence information is a sequence stored on a track. For instance, for binary information, some practical constraints like this one – bit transitions must not occur too closely – are encountered. A well studied class of codes is the $(d, k) - RLL$ run length-limited codes, with $d \leq k$, where there are at least d 0's, but no more than k 0's, between successive 1's. The $(2, 7) - RLL$ code is classical for coding information for example on a magnetic tape, see [3, 18], and the $(1, \infty) - RLL$ is important from a theoretical point of view because it is related to the Fibonacci sequence and to the golden number [15]. These topics are also related to number representation in irrational base, see [16, Chap. 7].

The research in future storage technologies considers recording medium as having two dimensions: they are arrays stored on surfaces. They are encountered for instance for holographic memories and 2D photon optical memories. Two-dimensional constraints of finite type have been considered in terms of transition matrices [7, 23], by bit-stuffing methods i. e. maps of unconstrained words into an array that satisfy $(1,\infty)-RLL$ in row and column [22], or from an ergodic point of view [6]. In most of the articles, authors investigate the case $(1,\infty)-RLL$ in row and column by using transition matrices constructed with square or rectangular blocks of various sizes. The value of the entropy for this constraint is very close to $0.587891\ldots$, see [7, 11, 10], but to give an algebraic characterization of it is an open problem.

Our purpose is to describe two-dimensional constraints of finite type with tools from combinatorics on words and automata theory. We consider general finite type constraints in row and in column (not necessarily the same) and, in a first step, we transform this problem into sets of minimal forbidden horizontal and vertical words. Minimal forbidden words have been proved to be very useful for the description of properties of symbolic dynamical systems, see in particular [4], and for the multidimensional case [5].

In a second step, we construct a sequence of transition matrices in order to compute the capacity of these two-dimensional codes. These transition matrices are built by using two-dimensional squared substitutions of constant size and automata. The link between automata and substitutions is also studied in [1] based on automatic sequences introduced in [8]. In a different context, two-dimensional substitutions with rectangle blocks have been considered in [17].

The organization of this article is the following: Section 2 is dedicated to definitions and properties of words, automata and finite type constraints. Section 3 deals with the construction of transition matrices by two-dimensional substitutions in the case that the substitution has a fixed point. Two illustrations of the construction are given: Section 4 presents the construction of transition matrices for the Fibonacci constraint, that is to say the constraint $(1,\infty)-RLL$ in row and in column, and in Section 5 we give a less peculiar example. In Section 6, the construction is shown for the general case where the substitution has no fixed point. In the last section we consider different types of constraint – such as diagonal ones – and by different methods we obtain transition matrices for first-order checkerboards constraints by substitutions.

2. Preliminaries

2.1. Words

An *alphabet* A is a finite set. A finite sequence of elements of A is called a *word*, and the set of words on A is the free monoid A^*. The *length* of a word v is equal to the number of its letters, and is denoted by $|v|$. The *empty word* is denoted by ε. Let v be a word of A^*, denote by v^n the concatenation of n times the word v, and by A^n the set of words on A of length n. A word u is a *factor* of a word v if $v = xuy$. If x (resp. y) is the empty word, u is a *prefix* (resp. *suffix*) of v. A factor u of v is *strict* if it is not equal to the entire word v.

In this work we consider a two-dimensional generalization. A *two-dimensional* word of height m and length n is an array of letters of A of dimension $m \times n$.

2.2. Automata

We refer the reader to [9, 20]. An *automaton over* A, $\mathcal{A} = (Q, A, E, I, T)$, is a directed graph labelled by elements of A. The set of vertices, traditionally called *states*, is denoted by Q, $I \subset Q$ is the set of *initial* states, $T \subset Q$ is the set of *terminal* states and $E \subset Q \times A \times Q$ is the set of labelled *edges*. If $(p, a, q) \in E$, we note $p \xrightarrow{a} q$. The automaton is *finite* if Q is finite. A subset L of A^* is said to be *recognizable by a finite automaton* if there exists a finite automaton $\mathcal{A}$ such that L is equal to the set of labels of paths starting in an initial state and ending in a terminal state.

We recall some classical notions we will use in the sequel. The *right congruence* modulo a language $L \subset A^*$ is defined by

$$u \sim_L v \ \text{ iff } \ (\forall w \in A^*, \ uw \in L \iff vw \in L).$$

It is known that L is recognizable by a finite automaton if and only if $\sim_L$ has finite index. In that case, the *minimal* finite automaton recognizing L has for set of states the set of equivalence classes modulo L. There is an edge $[u]_L \xrightarrow{a} [ua]_L$ for every a in A. The initial state is equal to $[\varepsilon]_L$. The set of terminal states is equal to $\{[u]_L \mid u \in L\}$. There is a *sink*, which is the class of words not in L.

2.3. System of Finite Type

This notion is traditionally defined for biinfinite sequences, see [15] for more details on these topics. Here we introduce the same notion for finite words. Let A be a finite alphabet, and let H be a finite subset of A^*, the *constraint.* A *language of finite type* is a subset S_H of A^* such that no word in S_H contains a strict factor in H. A word v of A^* is said to be H*-admissible* if it does not contain a strict factor in H. In the following we assume that the set H is the set of *minimal forbidden words*, that is, no strict factor of H is in H. Clearly, if a word v is in S_H, any factor of v is in S_H as well. A language of finite type is recognizable by a finite automaton, where all the states are initial and terminal. Let $\mathcal{A}_H$ be a deterministic finite automaton recognizing S_H. The *transition matrix* M_H of the automaton is defined by $M_H[p, q] = k$ where k is the number of edges from state p to state q in the automaton $\mathcal{A}_H$. The *subshift of finite type* $\mathcal{S}_H$ defined by H is thus the set of biinfinite sequences that are labels of biinfinite paths in the automaton $\mathcal{A}_H$. Equivalently, any finite factor of $\mathcal{S}_H$ is in the language S_H.

Denote by $p_H(n)$ the number of admissible words of length n in S_H. The *entropy* (also called the *capacity*) of S_H is defined as

$$h(S_H) = \lim_{n \to \infty} \frac{1}{n} \log p_H(n).$$

We recall some results from Perron-Frobenius Theory. Let $\lambda_H > 0$ be the largest eigenvalue of M_H. The entropy of the subshift of finite type S_H is equal to $\log \lambda_H$, see [15, Chap. 4].

We now consider two-dimensional constraints. A two-dimensional subshift of finite type is usually defined as a set of two-dimensional arrays that avoid a finite number of patterns, see [5] for instance. In this work we consider a different sort of two-dimensional constraint. Take two finite sets of constraints on an alphabet A, the horizontal one H and the vertical one V. A two-dimensional word is said to be (H, V)-*admissible* if each row is H-admissible and each column is V-admissible. The two-dimensional language of finite type $S_{H,V}$ is the set of (H, V)-admissible two-dimensional words. Let $P_{H,V}(m, n)$ be the number of admissible words $m \times n$ with height m and length n under the constraints H and V. The *entropy* or capacity of $S_{H,V}$ is defined as

$$h(S_{H,V}) = \lim_{m,n\to\infty} \frac{1}{mn} \log P_{H,V}(m, n).$$

The value of $\lim_{n\to\infty} P_{H,V}(n, n)^{n^{-2}}$ is called the *entropy constant* by certain authors, see [10].

2.4. Substitutions

A *substitution* σ on the alphabet A is a morphism $\sigma : A \to A^*$. The image by σ of a word is the concatenation of the images of its letters, that is, if $\sigma(a) = w_0 \dots w_{n-1}$ then $\sigma^2(a) = \sigma(w_0) \dots \sigma(w_{n-1})$. If for each a in A, the length of $\sigma(a)$ is the same, the substitution is said to be of *constant length*. The link between substitutions and finite automata is explicited in [8].

When there is a letter a such that $\sigma(a)$ begins with a, the substitution has a *fixed point* $w = (w_j)_{j\geq 0}$ defined as the limit when p goes to infinity of $\sigma^p(a)$.

A two-dimensional substitution Σ maps a letter of A onto an array of letters of A

$$\Sigma(a) = \begin{matrix} w_{(0,0)} & \cdots & w_{(0,n-1)} \\ \vdots & & \vdots \\ w_{(m-1,0)} & \cdots & w_{(m-1,n-1)} \end{matrix} .$$

The image of such an array is a block-matrix image, that is,

$$\Sigma^2(a) = \begin{matrix} \Sigma(w_{(0,0)}) & \cdots & \Sigma(w_{(0,n-1)}) \\ \vdots & & \vdots \\ \Sigma(w_{(m-1,0)}) & \cdots & \Sigma(w_{(m-1,n-1)}) \end{matrix} .$$

If for each a in A, the dimension of $\Sigma(a)$ is always equal to $m \times n$, the substitution is said to be of *constant dimension*.

When there is a letter a such that in the array $\Sigma(a)$ the letter $w_{(0,0)}$ is equal to a, then the substitution Σ has a *fixed point* which is a semi-infinite matrix defined as the limit when p goes to infinity of $\Sigma^p(a)$.

3. Substitutions with Fixed Point

In this section we assume that H and V do not contain words beginning with 00. The general case will be handled in Section 5. One can always suppose that words in the constraints H and V have length ≥ 2.

The construction can be followed on two examples, the two-dimensional Fibonacci case in Section 4 and another example in Section 5.

3.1. Horizontal Constraint

Let A be a finite alphabet, which can always be taken as a set of contiguous digits $\{0, 1, \ldots\}$. Let H be a finite set of minimal forbidden words on A. We construct a *characteristic* automaton $\mathcal{C}_H$ associated with the horizontal constraint H as follows.

- The set of states of $\mathcal{C}_H$ is $Q_H = A \cup P(H)$ where $P(H)$ is the set of strict prefixes of H of length ≥ 2. All states are initial and terminal.
- There is an edge between states p and q labelled by a if and only if pa is H-admissible and $q = ua$, where ua is the largest word in Q_H which is a suffix of pa. Clearly the automaton is deterministic. Remark that, by construction, in $\mathcal{C}_H$ every edge arriving in a state of name ua is labelled by a.

Proposition 1 *The characteristic automaton $\mathcal{C}_H$ recognizes the set S_H. The entries of its transition matrix M_H are equal to 0 or 1.*

Proof. By construction there is no transition outgoing from a state of Q_H arriving in an element of H, so every word which is recognized by $\mathcal{C}_H$ is H-admissible.

Conversely let w be a word with no factor in H. There exists a factorization of w into elements of Q_H of maximal length, so w is the label of a path in $\mathcal{C}_H$. □

In general the characteristic automaton is not minimal.

Let κ be the cardinality of Q_H, and let $K = \{0, 1, \ldots, \kappa - 1\}$. To each state p of Q_H is associated an integer $\rho(p)$ which is its rank in the lexicographical order

$$\begin{aligned} \rho : Q_H &\to K \\ p &\mapsto \rho(p) \end{aligned}$$

3.2. Vertical Constraint

Let $V = \{v_1, \ldots, v_r\}$ be the vertical constraint on A, chosen minimal. For w a non-empty word, we denote by $\psi(w)$ the last letter of w. For each word v_i of V, of length $|v_i|$, we consider the set of stacks of states of Q_H of height $|v_i|$ such that the vertical word formed by the last letter of each state of the stack is equal to the word v_i, and let X be the set of all such stacks corresponding to all the words v_i of V. Formally let

$$X = \left\{ \begin{matrix} q_0 \\ \vdots \\ q_{|v_i|-1} \end{matrix} \;\middle|\; 0 \leq j \leq |v_i| - 1,\ q_j \in Q_H;\ v_i = \psi(q_{|v_i|-1}) \ldots \psi(q_0),\ \forall v_i \in V \right\}.$$

Let

$$F = \left\{ \rho(q_s)\dots\rho(q_0) \;\middle|\; \begin{matrix} q_0 \\ \vdots \\ q_s \end{matrix} \in X \right\}$$

be the set of forbidden stacks of states for the vertical constraint V.

We construct the minimal automaton $\mathcal{M}_F$ recognizing the set of F-admissible words on K. We use a sink, denoted by the letter z, because we need that the automaton be complete, i. e., from each state there is a transition labelled by each letter of the alphabet.

Let Q_F be the set of states of $\mathcal{M}_F$. We define a substitution σ_V of constant length κ on Q_F as follows. For each p in $Q_F \setminus \{z\}$, there is a rule $p \to p^{(0)} \dots p^{(\kappa-1)}$ with $p^{(j)} = q$ such that the edge $p \xrightarrow{j} q$, $0 \le j \le \kappa - 1$, is in $\mathcal{M}_F$. The sink rule is $z \to z^{\kappa-1}$. Set $\mathrm{Card}(Q_F) = \delta$, we order $Q_F = \{a_0, \dots, a_{\delta-1}\}$ with a_0 such that $\sigma_V(a_0)$ begins with a_0, and $a_{\delta-1} = z$.

Proposition 2 *Let $w = (w_j)_{j \ge 0}$ be the fixed point of σ_V. Then $w_j = z$ if and only if the base κ expansion of j contains a forbidden factor in F.*

Proof. This is a particular case of Corollary 7 below. □

3.3. Two-Dimensional Substitution

First we define the cartesian product $\mathcal{C}_H \times \mathcal{C}_H$ as follows. Its transition matrix is equal to the tensorial product $M_H \otimes M_H$, obtained by replacing in M_H each 1 by M_H and each 0 by the zero matrix of same dimensions. More generally, for each $m \ge 1$, we consider $\mathcal{C}_H^m$ with adjacency matrix $M_H \otimes^m$.

For $0 \le i \le 2^{m-1}$ let $<i>_\kappa = i_{m-1} \dots i_0$ be the base κ expansion of i and let

$$\hat{i} = \begin{matrix} i_0 \\ \vdots \\ i_{m-1} \end{matrix} \qquad \text{and by abuse} \qquad \rho^{-1}(\hat{i}) = \begin{matrix} \rho^{-1}(i_0) \\ \vdots \\ \rho^{-1}(i_{m-1}) \end{matrix}$$

Each $\rho^{-1}(i_k)$, for $0 \le k \le m-1$, is a state in Q_H. The following result is then straightforward.

Lemma 3 *For $0 \le i, j \le 2^{m-1}$, $M_H \otimes^m [i,j] = 1$ if and only if, for every $0 \le k \le m-1$, there is an edge between states $\rho^{-1}(i_k)$ and $\rho^{-1}(j_k)$ in the characteristic automaton $\mathcal{C}_H$.*

Now, we build a two-dimensional substitution $\Sigma_{H,V}$ from the substitution σ_V and the matrix M_H: $\Sigma_{H,V} = \sigma_V \wedge M_H$ is defined as follows. A rule $a_i \to a_i^{(0)} \dots a_i^{(\kappa-1)}$ of σ_V gives birth to a rule $a_i \to W_{a_i}$ in the two-dimensional substitution $\Sigma_{H,V}$, where W_{a_i} is a $\kappa \times \kappa$-matrix defined by $W_{a_i}[p,q] = a_i^{(q)}$ if $M_H[p,q] = 1$, and $W_{a_i}[p,q] = z$ otherwise.

Let π be the projection defined on Q_F by $\pi(a_i) = 1$ if $a_i \ne z$, and $\pi(z) = 0$.

The previous construction does the following. The horizontal constraint for two-dimensional words of height m is controlled by the tensorial product $M_H \otimes^m$ of the matrix M_H. The fixed point of the substitution σ_V removes non-V-admissible states of the automaton. Thus the substitution $\Sigma_{H,V} = \sigma_V \wedge M_H$ replaces every non-V-admissible column by a zero column. We control only the admissibility of the indices of the columns because we are interested in the dominant eigenvalue, which means the cycles of the transition graph.

Theorem 4 *Let H and V be finite subsets of A^*, and let θ_m be the dominant eigenvalue of $T_m = \pi(\Sigma^m_{H,V}(a_0))$. The number $P_{H,V}(m,n)$ of (H,V)-admissible words of dimension $m \times n$ satisfies*

$$\lim_{m,n\to\infty} \frac{1}{mn} \log P_{H,V}(m,n) = \lim_{m\to\infty} \frac{1}{m} \log \theta_m.$$

Proof. For $0 \le i,j \le 2^{m-1}$, we have that $T_m[i,j] = 1$ if and only if, for every $0 \le k \le m-1$, there is an edge between states $\rho^{-1}(i_k)$ and $\rho^{-1}(j_k)$ in the characteristic automaton $\mathcal{C}_H$ (from Lemma 3), and if $\rho^{-1}(\hat{j})$ is V-admissible (by Proposition 2), hence both horizontal and vertical constraints are satisfied. Now fix m, and denote by $S_{H,V}(m)$ the set of bands of height m in $S_{H,V}$. Then the entropy of $S_{H,V}(m)$ is equal to

$$h(S_{H,V}(m)) = \lim_{p\to\infty} \frac{1}{p} \log \mathrm{Per}_p(S_{H,V}(m))$$

where $\mathrm{Per}_p(S_{H,V}(m))$ is the number of periodic points of period p of $S_{H,V}(m)$, see [15]. Since $S_{H,V}(m)$ is a system of finite type we have that $\mathrm{Per}_p(S_{H,V}(m)) = \mathrm{trace}(T_m)^p$. Thus the entropy of $S_{H,V}(m)$ is given by (see [12])

$$h(S_{H,V}(m)) = \lim_{n\to\infty} \frac{1}{mn} \log P_{H,V}(m,n) = \lim_{n\to\infty} \frac{1}{mn} \log \mathrm{trace}(T_m)^n = \frac{1}{m} \log \theta_m.$$

□

It is known that the entropy of the system $S_{H,V}$ exists, see [11, 14, 19].

Corollary 5 *The entropy θ of the system $S_{H,V}$ is given by $\lim_{m\to\infty} \frac{1}{m} \log \theta_m = \theta$ where θ_m is the dominant eigenvalue of T_m.*

Proof. Recall that $h(S_{H,V}) = \lim_{m,n\to\infty} \frac{1}{mn} \log P_{H,V}(m,n)$. By Theorem 4, $\lim_{m,n\to\infty} \frac{1}{mn} \log P_{H,V}(m,n) = \lim_{m\to\infty} \frac{1}{m} \log \theta_m$. And by sub-additivity argument (see [14]) $\lim_{m\to\infty} \frac{1}{m} \log \theta_m = \theta$. □

3.4. Finite Automaton

We now give the construction of the finite automaton which recognizes (H,V)-admissible words of fixed height m. The matrix $T_m = \pi(\Sigma^m_{H,V}(a_0))$ is the transition matrix of the automaton and is the transition matrix of $S_{H,V}(m)$. For each $0 \le i \le \kappa^m - 1$, $\rho^{-1}(\hat{\imath}) = \begin{matrix} p_0 \\ \vdots \\ p_{m-1} \end{matrix}$ is a state of the automaton.

Edges are of the form

$$\begin{array}{ccc} & d_0 & \\ p_0 & \vdots & q_0 \\ \vdots & \xrightarrow{d_{m-1}} & \vdots \\ p_{m-1} & & q_{m-1} \end{array}$$

where $\rho^{-1}(\hat{j}) = \begin{array}{c} q_0 \\ \vdots \\ q_{m-1} \end{array}$ and d_k is the last letter of the state q_k of Q_H for $0 \leq k \leq m-1$.

4. The two-Dimensional Fibonacci Case

The Fibonacci constraint in one dimension is classically defined for finite words on the alphabet $A = \{0, 1\}$: a word is Fibonacci-admissible if it does not contain two consecutive 1's as a subword. In the two-dimensional version, an array is admissible if it does not contain two consecutive 1's in row and in column.

This problem also appears in the literature with various denominations as hard-square model [22], diamond constraint [23], checkerboard constraint [7, 23], two-dimensional $(1, \infty) - RLL$ codes [21], two-dimensional golden subshift [15, 6]. The hard square entropy constant is equal to $1.50304808247533226\ldots$ Nothing is known about its arithmetic character, see [10].

Let $H = \{11\}$. The set of H-admissible words is recognizable by the following finite automaton $\mathcal{C}_H$ (see Figure 1). The set of states is $Q_H = \{0, 1\}$, where 0 and 1 are considered as letters. Every state is initial and terminal. Since there are two states, $\kappa = 2$ and $K = \{0, 1\}$ (here 0 and 1 are integers).

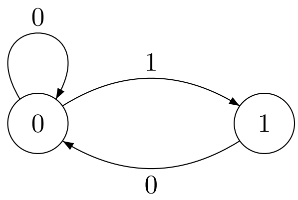

Figure 1: Automaton for the Fibonacci constraint

The transition matrix M_H is equal to

$$M_H = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

It is well known that, under this constraint, the entropy is equal to $h(S_H) = \lim_{n\to\infty} \frac{1}{n} \log p_H(n) = \log \frac{1+\sqrt{5}}{2}$.

The vertical constraint is $V = \{11\}$, thus the set X is equal to $X = \left\{ \begin{array}{c} 1 \\ 1 \end{array} \right\}$, and $F = \{11\}$.

The minimal automaton $\mathcal{M}_F$ recognizing the set of F-admissible words is the same as in Figure 1, with a sink denoted by z (see Figure 2). Every state excepted z is terminal.

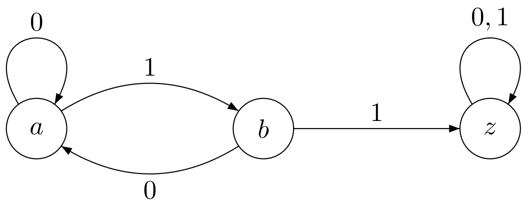

Figure 2: Minimal automaton for the vertical Fibonacci constraint

The associated substitution σ_V with constant length 2 is

$$\sigma_V : \begin{cases} a \to ab \\ b \to az \\ z \to zz \end{cases}$$

Since $\sigma_V(a)$ begins with an a, the substitution σ_V has a fixed point, denoted by $w = (w_j)_{j\geq 0} = abazabzzab\ldots$. We have that $w_j \neq z$ if and only if the 2-expansion of j, $<j>_2$, is without two consecutive ones. For instance, $w_3 = z$ and $<3>_2 = 11$.

The two-dimensional substitution $\Sigma = \Sigma_{H,V} = \sigma_V \wedge M_H$ is defined by

$$a \to \begin{matrix} a & b \\ a & z \end{matrix} \qquad b \to \begin{matrix} a & z \\ a & z \end{matrix} \qquad z \to \begin{matrix} z & z \\ z & z \end{matrix}.$$

As an example, for words of height two, we compute the matrix $\Sigma^2(a)$ replacing each letter by the corresponding 2×2 block

$$a \to \Sigma(a) = \begin{matrix} a & b \\ a & z \end{matrix} \to \Sigma^2(a) = \begin{matrix} a & b & a & z \\ a & z & a & z \\ a & b & z & z \\ a & z & z & z \end{matrix}.$$

Now, we project $\Sigma^2(a)$ by π

$$T_2 = \pi(\Sigma^2(a)) = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Remark that words of height 2 satisfying the horizontal constraint H only (no vertical constraint) are recognized by the cartesian product $\mathcal{C}_H \times \mathcal{C}_H$, which has for transition matrix the tensorial product $M_H \otimes M_H$,

$$M_H \otimes M_H = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Note that T_2 is just the matrix $M_H \otimes M_H$ in which the last column, of index 3, is replaced by a zero column. This is because $<3>_2 = 11$ is not Fibonacci-admissible.

The matrix T_2 is the transition matrix of the automaton recognizing Fibonacci-admissible (horizontally and vertically) words of height 2. Only the trimmed part is shown on Figure 3. The labelling of an edge here is just the name of the arrival state.

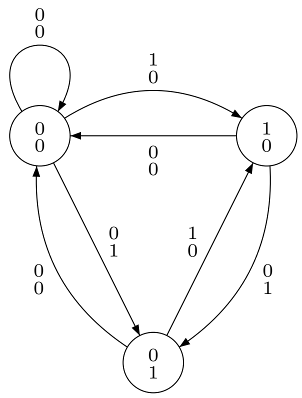

Figure 3: Automaton for Fibonacci-admissible words of height 2

The entropy for Fibonacci-admissible words of height 2 is equal to

$$\lim_{n\to\infty} \frac{1}{2n} \log P_{\{11\},\{11\}} = \frac{1}{2} \log(1+\sqrt{2})$$

because $1+\sqrt{2}$ is the dominant eigenvalue of T_2, the transition matrix of the system.

5. Another Example

Let A=$\{0,1,2\}$, $H = \{202, 212, 222\}$ and V= $\{22\}$. The characteristic automaton of S_H has six states $Q_H = \{0, 1, 2, 20, 21, 22\}$, ordered by lexicographic order. It is shown on Figure 4.

Its transition matrix is

$$M_H = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

The set K is equal to $K = \{0, \ldots, 5\}$ and $\kappa = 6$. Then $X = \left\{ \begin{matrix} 2 \\ 2 \end{matrix}, \begin{matrix} 2 \\ 22 \end{matrix}, \begin{matrix} 22 \\ 2 \end{matrix}, \begin{matrix} 22 \\ 22 \end{matrix} \right\}$ and $F = \{22, 52, 25, 55\}$.

The minimal automaton is shown on Figure 5. Every state is terminal excepted the sink z.

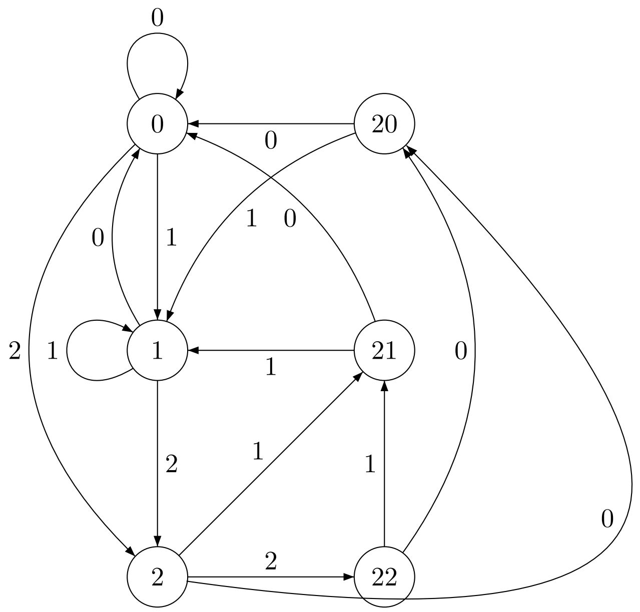

Figure 4: Characteristic automaton $\mathcal{C}_H$ for $H = \{202, 212, 222\}$

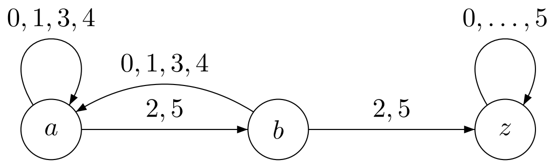

Figure 5: Minimal automaton $\mathcal{M}_F$ for $F = \{22, 25, 52, 55\}$ on $\{0, \ldots, 5\}$

The associated substitution σ_V is

$$\sigma_V : \begin{cases} a \to aabaab \\ b \to aazaaz \\ z \to zzzzzz \end{cases}$$

The two-dimensional substitution $\Sigma_{H,V} = \sigma_V \wedge M_H$ is

$$a \to \begin{array}{cccccc} a & a & b & z & z & z \\ a & a & b & z & z & z \\ z & z & z & a & a & b \\ a & a & z & z & z & z \\ a & a & z & z & z & z \\ z & z & z & a & a & z \end{array} \qquad b \to \begin{array}{cccccc} a & a & z & z & z & z \\ a & a & z & z & z & z \\ z & z & z & a & a & z \\ a & a & z & z & z & z \\ a & a & z & z & z & z \\ z & z & z & a & a & z \end{array} \qquad z \to \begin{array}{cccccc} z & z & z & z & z & z \\ z & z & z & z & z & z \\ z & z & z & z & z & z \\ z & z & z & z & z & z \\ z & z & z & z & z & z \\ z & z & z & z & z & z \end{array}.$$

Let $T_2 = \pi(\Sigma^2_{H,V}(a))$. Take $i = 2$ and $j = 11$, then $T_2[i,j] = 1$ because $<2>_6 = 02$, $<11>_6 = 15$, $\rho^{-1}(\hat{\imath}) = \begin{smallmatrix}2\\0\end{smallmatrix}$, $\rho^{-1}(\hat{\jmath}) = \begin{smallmatrix}22\\1\end{smallmatrix}$, $\rho^{-1}(\hat{\jmath})$ is V-admissible, and in the characteristic automaton $\mathcal{C}_H$, there is an edge between states 2 and 22, and between states 0 and 1.

6. General Case

Here we consider the case where H or V can contain words beginning with 00. The consequence of this fact is that it will not be possible to construct a substitution $\Sigma_{H,V}$ having a fixed point.

First the construction given in Section 3 is carried along the same way. Denote by L the language recognized by the automaton $\mathcal{M}_F$. The *start word* for σ_V is a word $s = s_0 \dots s_{\kappa-1} \in K^*$ such that for each $0 \le j \le \kappa - 1$, $s_j = [j]_L$ is the state of $\mathcal{M}_F$ denoting the right class of j modulo L (see Section 2.2 for definitions).

Let n and i be positive integers with $i < \kappa^n$, and denote by $(i)_{\kappa,n}$ the representation of i in base κ with n digits.

Proposition 6 *Let $\sigma_V^n(s) = y_0 \dots y_{\kappa^{n+1}-1}$. Then, for $0 \le j \le \kappa^{n+1} - 1$, the letter y_j is the state denoting the class $[(j)_{\kappa,n+1}]_L$.*

Proof. We have that $j = \kappa i + \ell$, with $0 \le \ell < \kappa$. Thus in the word $\sigma_V^{n-1}(s) = x_0 \dots x_{\kappa^n-1}$, the image of the letter x_i by the substitution σ_V is equal to

$$\sigma_V(x_i) = x_i^{(0)} \dots x_i^{(\kappa-1)}$$

with $y_j = x_i^{(\ell)}$. By recurrence hypothesis, x_i is the state denoting the class $[(i)_{\kappa,n}]_L$. By construction, there is an edge $x_i \xrightarrow{\ell} x_i^{(\ell)}$ in the automaton $\mathcal{M}_F$. Hence y_j is the state denoting the class modulo L of the word $(i)_{\kappa,n}\ell$, which is equal to $(j)_{\kappa,n+1}$. $\square$

Corollary 7 *In $\sigma_V^n(s) = y_0 \dots y_{\kappa^{n+1}-1}$ the letter y_j is equal to z if and only if the representation $(j)_{\kappa,n+1}$ is not F-admissible.*

If $\sigma_V(a_0)$ begins with a_0, then the start word is nothing else than $\sigma_V(a_0)$. Thus Proposition 2 is a consequence of Corollary 7.

We then define the substitution $\Sigma_{H,V} = \Sigma$ as in Section 3.3. If H or V contains some words beginning with 00, there is no fixed point for Σ. Let $s = s_0 \dots s_{\kappa-1}$ be the start word for σ_V. The start matrix W for Σ is defined by $W[p,q] = s_q$ if $M_H[p,q] = 1$, $W[p,q] = z$ otherwise.

Example 1 Take $A = \{0,1\}$, $H = \{11\}$, and for vertical constraint the constraint $(d,k) = (1,2)$. Thus $V = \{000, 11\}$. Since the automaton for S_H has two states, 0 and 1 (see Figure 1), K is equal to $\{0,1\}$, and the constraint V consists in forbidding stacks of states having a factor in $F = \{000, 11\}$. On Figure 6 is the minimal automaton $\mathcal{M}_F$.

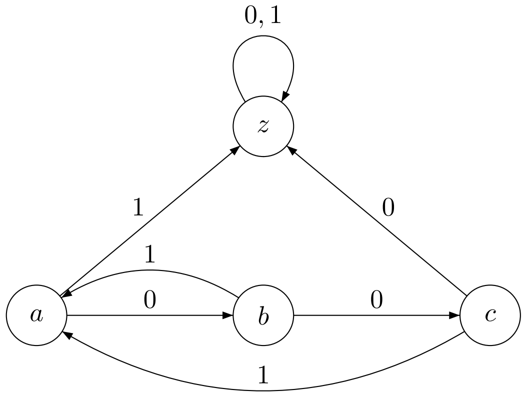

Figure 6: Minimal automaton for $F = \{000, 11\}$ on $\{0, 1\}$

We have that $a = [1]_L$, $b = [0]_L$, $c = [00]_L$, and $z = [000]_L = [11]_L$.
The associated substitution σ_V is

$$\sigma_V : \begin{cases} a \to bz \\ b \to ca \\ c \to za \\ z \to zz \end{cases}$$

The start word for σ_V is $s = ba$. Then $\sigma_V(s) = cabz$, $\sigma_V^2(s) = zabzcazz$, and so on.

The substitution $\Sigma = \Sigma_{H,V} = \sigma_V \wedge M_H$ is given by

$$a \to \begin{matrix} b\ z \\ b\ z \end{matrix} \qquad b \to \begin{matrix} c\ a \\ c\ z \end{matrix} \qquad c \to \begin{matrix} z\ a \\ z\ z \end{matrix} \qquad z \to \begin{matrix} z\ z \\ z\ z \end{matrix}.$$

The start matrix for Σ is equal to

$$W = \begin{matrix} b\ a \\ b\ z \end{matrix}.$$

Then

$$\Sigma(W) = \begin{matrix} c\ a\ b\ z \\ c\ z\ b\ z \\ c\ a\ z\ z \\ c\ z\ z\ z \end{matrix}.$$

$\diamond$

Example 2 A famous example where the entropy is exactly computed is the following one. Take $A = \{0, 1, 2\}$, and $H = V = \{00, 11, 22\}$. The value of the entropy of $S_{H,V}$ is equal to $\frac{3}{2} \log \frac{4}{3}$, see [2].

The associated substitution given by our method is $\Sigma = \Sigma_{H,V}$

$$a \to \begin{matrix} z\ b\ c \\ z\ z\ c \\ z\ b\ z \end{matrix} \qquad b \to \begin{matrix} z\ z\ c \\ a\ z\ c \\ a\ z\ z \end{matrix} \qquad c \to \begin{matrix} z\ b\ z \\ a\ z\ z \\ a\ b\ z \end{matrix} \qquad z \to \begin{matrix} z\ z\ z \\ z\ z\ z \\ z\ z\ z \end{matrix}.$$

The start matrix for Σ is equal to

$$W = \begin{matrix} z & b & c \\ a & z & c \\ a & b & z \end{matrix}. \qquad \diamond$$

As in Section 3, we have

Theorem 8 *Let θ_m be the dominant eigenvalue of $\pi(\Sigma^{m-1}(W))$. The entropy θ of $S_{H,V}$ exists and is given by*

$$\lim_{m,n\to\infty} \frac{1}{mn} \log P_{H,V} = \lim_{m\to\infty} \frac{1}{m} \log \theta_m = \theta.$$

7. Checkerboard Constraints

We now consider other types of constraints, the checkerboard constraints, see [23]. They are binary two-dimensional arrangements where a 1 is surrounded by 0's according to some constraints in rows, columns and diagonals. We consider only first-order constraints. A future work is to find a general construction for these kinds of constraints.

The Fibonacci constraint $H = V = \{11\}$ on the alphabet $A = \{0, 1\}$ presented in Section 4 can be seen as the following cross

$$\begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix}$$

which means that each 1 in a word $m \times n$ is surrounded by 0's in rows and columns.

Let m be the fixed height, and denote $<i>_2 = i_{m-1} \ldots i_0$, and

$$\hat{i} = \begin{matrix} i_0 \\ \vdots \\ i_{m-1} \end{matrix}.$$

As we have seen earlier, 11 is forbidden in row in the juxtaposition $\hat{i}\hat{j}$ if and only if $M_H \otimes^m [i, j] = 1$ where

$$M_H = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Remark that the matrix $M_H \otimes^m$ is equal to the matrix B_{2^m} of the Pascal triangle modulo 2 of dimension 2^m defined by $B_{2^m}[i, j] = B_{2^m}[i-1, j] + B_{2^m}[i, j-1] \mod 2$. Denote the golden number by φ and its conjugate by φ'. The set of eigenvalues of B_{2^m} is equal to $\{\varphi^k \varphi'^\ell \mid k + \ell = m\}$.

Note that $B_{2^m}[i, j] = 1$ if and only if the scalar product $< i, j >= i_0 j_0 + \cdots + i_{m-1} j_{m-1} = 0$. The associated two-dimensional substitution is of course defined by

$$a \to \begin{matrix} a & a \\ a & z \end{matrix}. \tag{1}$$

In this section, the rule

$$\begin{array}{ll} z \to & z\ z \\ & z\ z \end{array}$$

must be added for each substitution. Then each letter not equal to z is projected onto 1 and z is projected onto 0. We consider z as a "zero".

The vertical constraint

$$\begin{array}{c} 0 \\ 1 \\ 0 \end{array}$$

is generated by the two-dimensional substitution

$$\begin{array}{ll} a \to & a\ b \\ & a\ b \end{array} \qquad \begin{array}{ll} b \to & a\ z \\ & a\ z \end{array}. \tag{2}$$

We need to define the *cartesian product* of two two-dimensional substitutions. We do it only for dimension 2×2. Let Σ_1 and Σ_2 be defined on alphabets A_1 and A_2 respectively. The cartesian product $\Sigma = \Sigma_1 \times \Sigma_2$ is defined on the alphabet of couples $A = A_1 \times A_2$. If

$$\Sigma_1(a_1) = \begin{array}{cc} b_1 & c_1 \\ d_1 & e_1 \end{array}, \qquad \Sigma_2(a_2) = \begin{array}{cc} b_2 & c_2 \\ d_2 & e_2 \end{array}$$

then

$$\Sigma((a_1, a_2)) = \begin{array}{cc} (b_1, b_2) & (c_1, c_2) \\ (d_1, d_2) & (e_1, e_2) \end{array}$$

Then the 2-dimensional Fibonacci substitution

$$\begin{array}{ll} a \to & a\ b \\ & a\ z \end{array} \qquad \begin{array}{ll} b \to & a\ z \\ & a\ z \end{array} \qquad \begin{array}{ll} z \to & z\ z \\ & z\ z \end{array}$$

(see Section 4) can be obtained as the cartesian product of substitutions (1) and (2) with the additional convention that any couple of the form (a, z) or (z, a) must be considered as a zero z.

We now introduce diagonal constraints. The 1-diagonal constraint

$$\begin{array}{ccc} & & 0 \\ & 1 & \\ 0 & & \end{array}$$

is equivalent to the scalar product $i_1 j_0 + \cdots + i_{m-1} j_{m-2} = 0$. This is realized by the two-dimensional substitution

$$\begin{array}{ll} a \to & a\ a \\ & b\ b \end{array} \qquad \begin{array}{ll} b \to & a\ z \\ & b\ z \end{array}. \tag{3}$$

As the other 1-diagonal constraint

$$\begin{array}{ccc} 0 & & \\ & 1 & \\ & & 0 \end{array}$$

is equivalent to $i_0 j_1 + i_1 j_2 + \cdots + i_{m-2} j_{m-1} = 0$, it is given by

$$\begin{array}{ll} a \to a\ b & \quad b \to a\ b \\ \phantom{a \to{}} a\ b & \quad \phantom{b \to{}} z\ z \end{array}. \tag{4}$$

Thus the 2-diagonal constraint

$$\begin{array}{ccc} 0 & & 0 \\ & 1 & \\ 0 & & 0 \end{array}$$

is obtained by the product of the substitutions (3) and (4)

$$\begin{array}{llll} a \to a\ b & \quad b \to a\ b & \quad c \to a\ z & \quad d \to a\ z \\ \phantom{a \to{}} c\ d & \quad \phantom{b \to{}} z\ z & \quad \phantom{c \to{}} c\ z & \quad \phantom{d \to{}} z\ z \end{array}. \tag{5}$$

The oblique constraint is the following one

$$\begin{array}{ccc} & & 0 \\ 0 & 1 & 0 \\ 0 & & \end{array}.$$

The associated substitution is obtained by the product of the substitutions (1) and (3), thus

$$\begin{array}{ll} a \to a\ a & \quad b \to a\ z \\ \phantom{a \to{}} b\ z & \quad \phantom{b \to{}} b\ z \end{array}. \tag{6}$$

The oblique constraint has the same entropy as the 2-dimensional Fibonacci constraint.

The hexagonal constraint is

$$\begin{array}{ccc} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \end{array}.$$

The associated substitution is the product of the 2-dimensional Fibonacci substitution and the substitution (3)

$$\begin{array}{ll} a \to a\ b & \quad b \to a\ z \\ \phantom{a \to{}} b\ z & \quad \phantom{b \to{}} b\ z \end{array}. \tag{7}$$

For the hexagonal constraint it is known that the entropy constant is an algebraic number [13, 10].

The square constraint is

$$\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}.$$

The substitution is the product of the 2-dimensional Fibonacci substitution and the substitution (5)

$$a \to \begin{matrix} a & b \\ b & z \end{matrix} \qquad b \to \begin{matrix} a & z \\ z & z \end{matrix} . \tag{8}$$

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