Voronoi Cells of Beta-Integers

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Abstract. In this paper are considered one-dimensional tilings arising from some Pisot numbers encountered in quasicrystallography as the quadratic Pisot units and the cubic Pisot unit associated with 7-fold symmetry, and also the Tribonacci number. We give characterizations of the Voronoi cells of such tilings, using word combinatorics and substitutions.

1 Introduction

Word combinatorics has been proved to be very useful in the solution of problems arising from the modelization of metallic alloys called *quasicrystals*. The first quasicrystal was discovered in 1984: it is a solid structure presenting a local symmetry of order 5, *i.e.* a local invariance under rotation of $\pi/5$, and it is linked to the golden mean and to the Fibonacci substitution. The Fibonacci substitution, given by

$$L \mapsto LS, \ S \mapsto L,$$

defines a quasiperiodic selfsimilar tiling of the positive real line, and is a historical model of a one-dimensional mathematical quasicrystal. The fixed point of the substitution is the infinite word

$$LSLLSLSLSL \cdots$$

Each letter L or S is considered as a tile. The vertices of the tiles are labelled by algebraic integers, the so-called β-integers, where β is equal to $\frac{1+\sqrt{5}}{2}$. The description and the properties of those β-integers use a base β number system.

A more general theory has been elaborated with Pisot numbers[1] for base, see [3, 6]. Note that so far, all the quasicrystals discovered by physicists present local symmetry of order 5, 8, 10, or 12, and are modelized using some quadratic Pisot units, namely $\frac{1+\sqrt{5}}{2}$, $1+\sqrt{2}$, and $2+\sqrt{3}$. More generally, a substitution can be associated with any Pisot number giving a selfsimilar quasiperiodic tiling of the positive real line [19].

[1] A *Pisot number* is an algebraic integer >1 such that the other roots of its minimal polynomial have a modulus less than 1. The golden mean and the natural integers are Pisot numbers

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The construction of quasiperiodic point sets involves a method called *cut and projection* [12, 13]. The determination of the quasicrystal depends on a set called *window*. For instance, when β is the Tribonacci number, the window of the set of β-integers is the well known Rauzy fractal, see [11] for instance.

The purpose of this work is to give a combinatorial characterization of the geometry of tilings associated with sets of beta-integers. More precisely we show that local geometrical configurations of beta-integers, given by their *Voronoi cells*, are characterized by their beta-expansions. This allows to give a fine partition of the window associated with positive beta-integers according to the combinatorial properties of the underlying numeration system.

It is worthwile to mention that the fixed point u_β of the substitutions associated with the Pisot numbers β considered here enjoys the following properties. When β is a quadratic Pisot unit, u_β is a Sturmian sequence [6], that is to say, the number $\mathcal{C}(n)$ of factors of length n, is equal to $n+1$. When β is the Tribonacci number, u_β is an Arnoux-Rauzy sequence [1], of complexity $\mathcal{C}(n) = 2n+1$. When β is the cubic Pisot unit associated with 7-fold symmetry, u_β has complexity $\mathcal{C}(n) = 2n+1$, but is not an Arnoux-Rauzy sequence [7].

The paper is organized as follows: after some definitions, we give characterizations of the Voronoi cells of the tilings associated with quadratic Pisot units, with the Tribonacci number and, with the cubic Pisot unit associated with 7-fold symmetry. These results are given in terms of the properties of the beta-expansions as words, and by the belonging of the conjugates of beta-integers to some connected region, the window. For the Tribonacci number, our results allow to give a nice combinatorial interpretation of the domain exchange defined by Rauzy on the Rauzy fractal, see [11, 15, 16].

2 Preliminaries

2.1 Words

Let A be a finite set of symbols called the *alphabet*. We denote by A^* the set of finite *words* over A, and by ε the empty word. A *factor* of a word x is a word z such that $x = yzt$. If $y = \varepsilon$, z is said to be a *prefix* of x; if $t = \varepsilon$, the word z is a *suffix* of x. A prefix (or a suffix) z of y is *proper* if it is different of the entire word y. If v is a word, the concatenation of v k times is denoted by v^k, with the convention that if $k = 0$, v^k is the empty word ε.

A function $f : A^* \to B^*$ is a *morphism* if $f(xy) = f(x)f(y)$, for all $x, y \in A^*$. A morphism is a *substitution* if for each a in A, $f(a) \neq \varepsilon$.

The *radix order* for finite words over an ordered alphabet is defined by $x \leqslant y$ if $|x| < |y|$, or $|x| = |y|$ and their exist factorizations $x = uax'$ and $y = uby'$, for some word $u \in A^*$, $a, b \in A$ such that $a \leqslant b$, and $x', y' \in A^*$.

The set of infinite words over A is denoted by $A^{\mathbb{N}}$. It is the set of sequences of symbols of A indexed by non-negative integers. Denote by $v^\omega = vvvv\ldots$ the word obtained by the infinite concatenation of the word v. A word of the form uv^ω is called *eventually periodic* if $u \neq \varepsilon$, *periodic* otherwise.

The *lexicographic order* for infinite words over an ordered alphabet is defined by $x <_{\text{lex}} y$ if their exist factorizations $x = uax'$ and $y = uby'$, for some word $u \in A^*$, $a, b \in A$ such that $a < b$, and $x', y' \in A^{\mathbb{N}}$.

2.2 Beta-Expansions

For definitions and results on beta-expansions the reader may consult [10, Chapter 7]. Let $\beta > 1$ be a real number. A representation in base β, or a β-*representation*, of a real number $x > 0$ is an infinite sequence of integers $(x_i)_{i \leqslant N}$ such that $x = \sum_{i \leqslant N} x_i \beta^i$, for some N. A particular β-representation, called β-*expansion*, is computed by the "greedy algorithm" [17]. Denote by $\lfloor y \rfloor$ and by $\{y\}$ the integer part and the fractional part of the real number y, respectively. There exists $N \in \mathbb{Z}$ such that $\beta^N \leqslant x < \beta^{N+1}$. Let $x_N = \lfloor x/\beta^N \rfloor$, and let $r_N = \{x/\beta^N\}$. Then for $i < N$, $x_i = \lfloor \beta r_{i+1} \rfloor$, and $r_i = \{\beta r_{i+1}\}$. If $x < 1$, then $N < 0$ and we set $x_{-1} = \cdots = x_{N+1} = 0$. The β-expansion of x is denoted by

$$\langle x \rangle_\beta = x_N x_{N-1} \cdots x_1 x_0 \cdot x_{-1} x_{-2} \cdots ,$$

most significant digits first. The dot between x_0 and x_{-1} symbolizes the separation between positive and negative powers of the base. By abuse we refer to the word $x_N \cdots x_0$ as the β-*integer part*, and to the word $x_{-1} x_{-2} \cdots$ as the β-*fractional part* of x in base β. The digits x_i obtained by the greedy algorithm belong to the set $\mathbb{B} = \{0, 1, \dots, \lfloor \beta \rfloor\}$, called the *canonical* alphabet associated with β, if β is not an integer. If β is an integer, then $\mathbb{B} = \{0, 1, \dots, \beta - 1\}$, and the β-expansion is just the standard representation in base β. If a β-representation ends with infinitely many 0's it is said to be *finite* and the ending 0's are omitted.

A word (finite or infinite) is said to be *admissible* if it is the β-expansion of some number of $[0, 1[$. Let us introduce the so called *Rényi β-expansion of* 1, denoted by $d_\beta(1)$. It is computed as follows: let the β-*transform* of the unit interval be defined by $T_\beta(y) = \beta y \mod 1$. Then $d_\beta(1) = (t_i)_{i \geqslant 1}$, where $t_i = \lfloor \beta T_\beta^{i-1}(1) \rfloor$. Note that $d_\beta(1)$ belongs to $\mathbb{B}^{\mathbb{N}}$. A number β such that $d_\beta(1)$ is eventually periodic is called a *beta-number*, or a *Parry number*. When $d_\beta(1)$ is finite, β is said to be a *simple* Parry number. Set $d_\beta^*(1) = d_\beta(1)$ if $d_\beta(1)$ is eventually periodic, and $d_\beta^*(1) = (t_1 \cdots t_{m-1}(t_m - 1))^\omega$ if $d_\beta(1) = t_1 \cdots t_{m-1} t_m$ is finite. Let us recall the following result from [14]: An infinite sequence of non-negative integers $\xi = (\xi_i)_{i \geqslant 1}$ is admissible if and only if for every $p \geqslant 0$, $(\xi_{i+p})_{i \geqslant 1} <_{\text{lex}} d_\beta^*(1)$. We can now define the set of β-integers.

Definition 1 *The set of* β-integers *is the set of real numbers such that the* β-*expansion of their absolute value has a* β-*fractional part equal to* 0^ω

$$\mathbb{Z}_\beta = \{x \in \mathbb{R} \mid \langle |x| \rangle_\beta = x_N x_{N-1} \cdots x_1 x_0,\ N \geqslant 0\} . \tag{1}$$

Denote $\mathbb{Z}_\beta^+$ the set of non-negative β-integers. Note that $\mathbb{Z}_\beta = \mathbb{Z}_\beta^+ \cup (-\mathbb{Z}_\beta^+)$ and that $\beta \mathbb{Z}_\beta \subset \mathbb{Z}_\beta$. The set $\mathbb{Z}_\beta^+$ is ordered by the radix order on the (finite) β-expansions of its elements; its n-th element is denoted b_n.

An *algebraic integer* is a root of a monic polynomial with integer coefficients. A *Pisot number* is an algebraic integer greater than 1 such that the other roots of its minimal polynomial have a modulus smaller than 1. When the constant term of the minimal polynomial is equal to ± 1, β is said to be a *unit*. Recall that any Pisot number is a Parry number [2, 18].

2.3 Substitution Tilings

Let β be a Parry number. To such a number a substitution σ_β can be associated with β in a canonical way. Its fixed point u_β is written on an alphabet $\mathbb{A}$ of letters that are considered as tiles. This defines a tiling of the positive real line with a finite number of tiles. Each tile U is given a length $\ell(U)$, see [5, 19]. Each vertex of the positive real line is labelled by the length in tiles of the prefix of u_β ending in that vertex.

In the frame of this study, we restrict ourselves to a subclass of Parry numbers, namely the quadratic Pisot units and two examples of cubic Pisot units.

Quadratic Pisot Units. They are of two types.

Case 1. $\beta > 1$ is the root of the polynomial $X^2 - aX - 1$, $a \geqslant 1$. The canonical alphabet is $\mathbb{B} = \{0, 1, \ldots, a\}$, the β-expansion of 1 is finite and equal to $d_\beta(1) = a1$. Note that a word on $\mathbb{B}$ is a β-expansion if and only if it does not contain a factor $a1$. The substitution σ_β is defined on the alphabet $\mathbb{A} = \{L, S\}$ by

$$\sigma_\beta = \begin{cases} L \mapsto L^a S \\ S \mapsto L\,. \end{cases} \tag{2}$$

To each letter of $\mathbb{A}$ we associate a tile with the same name of length $\ell(L) = 1$, and $\ell(S) = T_\beta(1) = \beta - a = 1/\beta$.

Case 2. $\beta > 1$ is the root of the polynomial $X^2 - aX + 1$, $a \geqslant 3$. The canonical alphabet is $\mathbb{B} = \{0, 1, \ldots, a-1\}$, the β-expansion of 1 is eventually periodic and equal to $d_\beta(1) = (a-1)(a-2)^\omega$. The substitution σ_β is defined on the alphabet $\mathbb{A} = \{L, S\}$ by

$$\sigma_\beta = \begin{cases} L \mapsto L^{a-1} S \\ S \mapsto L^{a-2} S\,. \end{cases} \tag{3}$$

Here we have $\ell(L) = 1$, and $\ell(S) = T_\beta(1) = \beta - (a-1) = 1 - 1/\beta$.

Cubic Pisot Units. We consider two particular cases of cubic Pisot units, namely the roots of the polynomials

$$X^3 - X^2 - X - 1, \text{ Case 1}$$

$$X^3 - 2X^2 - X + 1, \text{ Case 2.}$$

The root $\beta > 1$ in Case 1 is the so-called Tribonacci number, see for instance [11]. The root β in Case 2 is a cyclotomic Pisot unit with a 7-fold symmetry, that is to say, the ring $\mathbb{Z}[e^{i2\pi/7}]$, which is invariant by rotation of $2\pi/7$, satisfies $\mathbb{Z}[e^{i2\pi/7}] = \mathbb{Z}[\beta] + \mathbb{Z}[\beta]e^{i2\pi/7}$, see [3].

Case 1. The Tribonacci number. The canonical alphabet is $\mathbb{B} = \{0, 1\}$, the β-expansion of 1 is finite and equal to $d_\beta(1) = 111$. A word on $\mathbb{B}$ is a β-expansion if and only if it does not contain a factor 111. The substitution σ_β is defined on the alphabet $\mathbb{A} = \{L, M, S\}$ by

$$\sigma_\beta = \begin{cases} L \mapsto LM \\ M \mapsto LS \\ S \mapsto L\,. \end{cases} \tag{4}$$

We have $\ell(L) = 1$, $\ell(M) = T_\beta(1) = \beta - 1$, and $\ell(S) = T_\beta^2(1) = \beta^2 - \beta - 1$.

Case 2. Symmetry of order 7. The canonical alphabet is $\mathbb{B} = \{0, 1, 2\}$, the β-expansion of 1 is eventually periodic and equal to $d_\beta(1) = 2(01)^\omega$. The substitution σ_β is defined on the alphabet $\mathbb{A} = \{L, M, S\}$ by

$$\sigma_\beta = \begin{cases} L \mapsto LLS \\ S \mapsto M \\ M \mapsto LS \end{cases} \tag{5}$$

Here we have $\ell(L) = 1$, $\ell(S) = T_\beta(1) = \beta - 2$, and $\ell(M) = T_\beta^2(1) = \beta^2 - 2\beta$.

For all the above cases, the infinite word $u_\beta = \sigma_\beta^\infty(L)$ is the fixed point of the substitution σ_β. The interval $[0, \beta^j]$ is tiled by the tiling associated with the word $\sigma_\beta^j(L)$. Consequently the tiling associated with $\sigma_\beta^\infty(L)$ is a selfsimilar tiling of the positive real line and positive β-integers are the labels of the vertices of this tiling, see [3, 6]. The substitution σ_β acts on the tiles as the multiplication by β acts on β-integers.

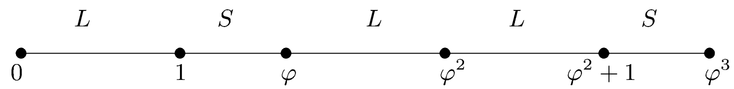
Example Let $\varphi = \frac{1+\sqrt{5}}{2}$. Then $d_\varphi(1) = 11$. The associated substitution is the Fibonacci substitution $L \mapsto LS$, $S \mapsto L$. We have $\ell(L) = 1$, and $\ell(S) = \varphi - 1$. The first non-negative φ-integers are

$$\begin{array}{ll} b_0 = 0 & \langle b_0 \rangle_\varphi = 0 \\ b_1 = 1 & \langle b_1 \rangle_\varphi = 1 \\ b_2 = \varphi & \langle b_2 \rangle_\varphi = 10 \\ b_3 = \varphi^2 & \langle b_3 \rangle_\varphi = 100 \\ b_4 = \varphi^2 + 1 & \langle b_4 \rangle_\varphi = 101 \\ b_5 = \varphi^3 & \langle b_5 \rangle_\varphi = 1000 \end{array}$$

The fixed point of the substitution is

$$u_\varphi = LSLLSLSLSL \cdots$$

Below is shown the beginning of the labelling of vertices of the Fibonacci tiling by φ-integers



2.4 Meyer Sets and Voronoi Cells

We recall here several definitions and results that can be found in [8, 9, 12, 13], see also [4] for a survey on these questions. Delaunay sets were introduced as a mathematical idealization of a solid-state structure. A set Λ in $\mathbb{R}^d$ is said to be *uniformly discrete* if there exists $r > 0$ such that every ball of radius r contains at most a point of Λ. A set Λ in $\mathbb{R}^d$ is said to be *relatively dense* if there exists $R > 0$ such that every ball of radius R contains at least a point of Λ. If both conditions are satisfied, Λ is said to be a *Delaunay set*.

Meyer introduced in [12, 13] the mathematical notion of *quasicrytals* as a generalization of ideal crystalline structures. They are now known as *Meyer sets*. A set $\Lambda \subset \mathbb{R}^d$ is said to be a *Meyer set* if it is a Delaunay set and if there exists a finite set F such that $\Lambda - \Lambda \subset \Lambda + F$. This is equivalent to $\Lambda - \Lambda$ being a Delaunay set [8]. The Meyer sets generalize the lattices of crystallography, that obey the relation $\Lambda - \Lambda \subset \Lambda$.

We now give the definition of Voronoi cells and of Voronoi tessellation.

Definition 2 *(i) Given a discrete set Λ in $\mathbb{R}^d$, the* Voronoi cell $\mathcal{V}(\lambda)$ *of $\lambda \in \Lambda$ is the closure of the set of all points in $\mathbb{R}^d$ closer to λ than to any other point of Λ*

$$\mathcal{V}(\lambda) = \{x \in \mathbb{R}^d \mid \delta(x-\lambda) \leqslant \delta(x-\lambda'),\ \lambda' \in \Lambda\}, \tag{6}$$

where δ is the Euclidean distance in $\mathbb{R}^d$.

(ii) The set of Voronoi cells of a discrete set Λ forms a tiling of $\mathbb{R}^d$ called the Voronoi tessellation *of $\mathbb{R}^d$ induced by Λ.*

Lagarias has proved in [9] that if Λ is a Meyer set, its Voronoi tessellation contains a finite number of tiles. It is proved in [3] that when β is a Pisot number, then the set $\mathbb{Z}_\beta$ of β-integers is a Meyer set.

There is a special class of Meyer sets, defined by Meyer [12, 13], called *model sets*, computed by the so called *cut and project algorithm* and in which arises the notion of *window*. In the frame of this article, we introduce the algebraic version of the cut and project algorithm in the particular cases we study.

Quadratic Pisot Units. Let $\beta > 1$ be a quadratic Pisot unit. Let now β' be the other root of the minimal polynomial associated with β, and let the *Galois conjugation automorphism* be the map $x = \sum_{0 \leqslant i \leqslant N} x_i \beta^i \mapsto x' = \sum_{0 \leqslant i \leqslant N} x_i \beta'^i$. We define the *window* of positive β-integers as the compact set Ω

$$\Omega = \overline{\{x' \mid x \in \mathbb{Z}_\beta^+\}} = \overline{(\mathbb{Z}_\beta^+)'}\,. \tag{7}$$

We know from [3] that a a number x of $\mathbb{Z}[\beta] \cap R^+$ is a positive β-integer if and only if its conjugate x' belongs to the window $\Omega = (-1, \beta)$ in Case 1, and $\Omega = (0, \beta)$ in Case 2.

Cubic Pisot Units. We have to consider our two cases separately.
Case 1. Let $\beta > 1$ be the Tribonacci number, and let α and α^c be its Galois conjugates (the symbol ^{c} denotes complex conjugation). The Galois conjugation automorphism is defined as $x = \sum_{0 \leqslant i \leqslant N} x_i \beta^i \mapsto x' = \sum_{0 \leqslant i \leqslant N} x_i \alpha^i$. Then the

window Ω of $\mathbb{Z}_\beta^+$ is a compact subset of $\mathbb{C}$ with a fractal boundary, see for instance Figure 3. This figure is called the Rauzy fractal [11, 16].
Case 2. Let $\beta > 1$ be the dominant root of the polynomial $X^3 - 2X^2 - X + 1$. The other roots of this polynomial are the real numbers $\alpha_1 = \beta^2 - 2\beta$ and $\alpha_2 = -\beta^2 + \beta + 2$. The Galois automorphism is $x = \sum_{0 \leqslant i \leqslant N} x_i \beta^i \mapsto x' = \sum_{0 \leqslant i \leqslant N} x_i (\alpha_1^i + \alpha_2^i e^{i4\pi/7})$.

The definition of the window Ω of positive β-integers is again given by Equation (7). Note that, unless for quadratic Pisot units, the determination of the window of positive β-integers is an open problem, see discussion in [6].

3 Beta-Integers Voronoi Cells

We shall now study the Voronoi tessellation of $\mathbb{Z}_\beta^+$, and characterize Voronoi cells of β-integers when β is a quadratic Pisot unit, the Tribonacci number, or a cubic Pisot unit associated with 7-fold symmetry.

When a β-integer is the common vertex of the generic tiles U and V, it is said to be an UV β-integer, and its Voronoi cell is consequently said to be an UV Voronoi cell. The window associated with positive UV β-integers is denoted by Ω_{UV}, and is given by

$$\Omega_{UV} = \overline{\{x' \mid x \in \mathbb{Z}_\beta^+,\, x \text{ is } UV\}}\,.$$

Since a negative β-integer b_{-n} is by definition equal to $-b_n$, by symmetry one obtains a tiling of the negative real line, and thus the beta-integer $b_0 = 0$ is always of type LL.

3.1 Quadratic Pisot Units

Recall that from the substitution σ_β we have only three possible tile-configurations, LL, LS and SL, since SS is excluded, so there are only three possible Voronoi cells. When a β-integer is SL or LS, the length of its Voronoi cell is $(1 + 1/\beta)/2$ in Case 1, and $(2 - 1/\beta)/2$ in Case 2. Figure 1 displays the case when the n^{th} β-integer is SL. When a β-integer is LL the length of its Voronoi cell is 1.

We will see in the following that it is possible to further differentiate Voronoi cells, from the analysis of the β-expansion of the β-integer they support.

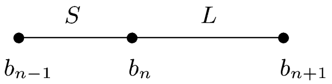


Fig. 1. Configuration where the n^{th} β-integer, b_n, is SL

Case 1. $\beta^2 = a\beta + 1$, $a \geqslant 1$

Proposition 1 *In each of the following assertions, (i), (ii) and (iii) are equivalent.*

1.1 (i) b_n *is SL; (ii)* $\langle b_n \rangle_\beta$ *ends in the suffix* $(0)^{2q+1}$, $q \in \mathbb{N}$ *maximal; (iii)* $b'_n \in \Omega_{SL} = (-1, 0)$.

1.2 (i) b_n *is LL; (ii) for* $n \geqslant 1$, $\langle b_n \rangle_\beta$ *ends in either the suffix* $(0)^{2q+2}$, $q \in \mathbb{N}$ *maximal, or by an* $h \in \{1, \ldots, a-1\}$*; (iii)* $b'_n \in \Omega_{LL} = [0, \beta - 1)$.

1.3 (i) b_n *is LS; (ii)* $\langle b_n \rangle_\beta$ *ends in a; (iii)* $b'_n \in \Omega_{LS} = (\beta - 1, \beta)$.

Proof We prove Proposition 1 in two steps. First we prove that $(i) \Leftrightarrow (ii)$ by recurrence. Recall that the set $\mathbb{Z}_\beta$ is symmetric with respect to the origin by definition, which makes the origin $b_0 = 0$ a LL β-integer. We have $b'_0 = 0$. The relations are clearly valid for small $n \geqslant 1$.

Suppose that b_n is an SL β-integer. In the fixed point of the substitution $u_\beta = \sigma_\beta^\infty(L)$, the factor SL can appear in two different configurations.

In the first configuration SL is a factor of $L^a S L^a S$. We thus have $b_n = \beta b_k$, where b_k is a LL β-integer,

$$\begin{array}{ccc} & b_n & \\ L^a S & | & L^a S \\ \uparrow & & \uparrow \\ & b_k & \\ L & | & L \end{array}$$

where the upright arrow symbolizes the action of the substitution σ_β on a given tile. By recurrence hypothesis, $\langle b_k \rangle_\beta$ ends either with an $h \in \{1, 2, \ldots, a-1\}$ and then $\langle b_n \rangle_\beta$ ends in $h0$, or $\langle b_k \rangle_\beta$ ends in $(0)^{2q+2}$, $q \in \mathbb{N}$ maximal and then $\langle b_n \rangle_\beta$ ends in $(0)^{2q+3}$, thus $\langle b_n \rangle_\beta$ ends with an odd number of 0's. Consequently, the β-expansions of $b_{n+1}, b_{n+2}, \ldots, b_{n+a-1}$, which are all LL β-integers, end in $1, 2, \ldots, (a-1)$ respectively, and the β-expansion of b_{n+a}, which is LS, ends in a.

In the second configuration SL is a factor of $L^a S L L^a S$ thus $b_n = \beta b_k$ where b_k is a LS β-integer. Since by recurrence hypothesis, $\langle b_k \rangle_\beta$ ends in a, then $\langle b_n \rangle_\beta$ ends with $a0$. Recall that $a1$ is not admissible for such a β. Thus the β-expansion of b_{n+1}, which is LL, ends in $j00$, where $j \neq 0$. Therefore, the β-expansion of $b_{n+2}, \ldots, b_{n+a}$, which are all LL, ends in $1, 2, \ldots, a-1$, respectively, and the β-expansion of b_{n+a+1}, which is LS, ends in a.

The cases where b_n is an LL or an LS β-integer have been already treated just above.

Let us now show that $(i) \Leftrightarrow (iii)$. Recall that a number x of $\mathbb{Z}[\beta] \cap R^+$ is a positive β-integer if and only if its conjugate x' belongs to $(-1, \beta)$. Let b_n be a SL β-integer. Then $b_{n-1} = b_n - \frac{1}{\beta}$ and $b_{n+1} = b_n + 1$ are β-integers, and $(b_{n-1})' = b'_n + \beta \in (-1, \beta)$ and $(b_{n+1})' = b'_n + 1 \in (-1, \beta)$. Therefore $b'_n \in (-1, 0)$.

Let b_n be a LL β-integer. Then $b_{n-1} = b_n - 1$ and $b_{n+1} = b_n + 1$ are β-integers, and $(b_{n-1})' = b'_n - 1 \in (-1, \beta)$ and $(b_{n+1})' = b'_n + 1 \in (-1, \beta)$. Since 0 is LL we have $b'_n \in [0, \beta - 1)$.

Finally, let b_n be a LS β-integer. Then $b_{n-1} = b_n - 1$ and $b_{n+1} = b_n + \frac{1}{\beta}$ are β-integers, and $(b_{n-1})' = b'_n - 1 \in (-1, \beta)$ and $(b_{n+1})' = b'_n - \beta \in (-1, \beta)$. Therefore $b'_n \in (\beta - 1, \beta)$. □

For LL β-integers we can refine the characterization. We give a partition of the window of LL β-integers as

$$\Omega_{LL} = \bigcup_{0 \leqslant h \leqslant a-1} \Omega_{LL}(h)\,,$$

where $\Omega_{LL}(h)$ is the window associated with positive LL β-integers such that their β-expansions end in $h \in \{0, \cdots, a-1\}$.

Proposition 2 *Let b_n be a LL β-integer, then $\langle b_n \rangle_\beta$ ends in*

2.1 $(0)^{2q+2}$*, q maximal in $\mathbb{N}$, if and only if* $b'_n \in \Omega_{LL}(0) = [0, \frac{1}{\beta})$*,*

2.2 an $h \in \{1, \ldots, a-1\}$ if and only if $b'_n \in \Omega_{LL}(h) = (\frac{1}{\beta} + h - 1, \frac{1}{\beta} + h)$*.*

Proof Let us first prove *2.1*. Let b_n be a LL β-integer, such that $\langle b_n \rangle_\beta$ ends in $(0)^{2q+2}$, $q \in \mathbb{N}$. By Proposition 1, *1.1*, the β-integer b_n/β is SL, and $(b_n/\beta)' = -\beta b'_n \in (-1, 0)$. Therefore $b_n \in (0, 1/\beta)$.

We prove *2.2* in two steps. Suppose that b_n is a LL β-integer such that $\langle b_n \rangle_\beta$ ends in 1. There are two cases for $b_{n-1} = b_n - 1$.

• b_{n-1} is a LL β-integer such that $\langle b_{n-1} \rangle_\beta$ ends in $(0)^{2q+2}$, $q \in \mathbb{N}$. Thus $b'_{n-1} \in [0, \frac{1}{\beta})$, and $b'_n \in [1, 1 + \frac{1}{\beta})$.

• b_{n-1} is a SL β-integer such that $\langle b_{n-1} \rangle_\beta$ ends with $h0$, $h \neq a$. The conjugate b'_n of b_n lies in the window computed as follows. Let b_s be a SL β-integer such that $\langle b_s \rangle_\beta$ ends with $a0$. Then b_s/β ends with a, and by Proposition 1, *1.3*, $(b_s/\beta)' \in (\beta - 1, \beta)$. Thus $b'_s \in (-1, -1 + 1/\beta)$. It is obvious now that $(b_{n-1})' \in (-1 + 1/\beta, 0)$, and $b'_n \in (\frac{1}{\beta}, 1)$.

Putting the two cases together we get that $b'_n \in (\frac{1}{\beta}, 1 + \frac{1}{\beta})$. The end of the proof for any h in $\{1, \ldots, a-1\}$ follows easily. □

On Figure 2 we display the window Ω of $\mathbb{Z}_\beta^+$, when β is a quadratic Pisot unit of Case 1, and its decomposition into subwindows which correspond to the windows of β-integers having specific Voronoi cells.

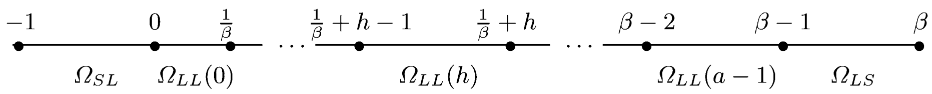


Fig. 2. Graphical representation of Proposition 1 *(iii)* and Proposition 2

Case 2. $\beta^2 = a\beta - 1$, $a \geqslant 3$

Let $\mathcal{M} = (a-1)(a-2)^*$ be the set of maximal words of each length in the radix order. Remark that any word in $\mathcal{M}$ is a prefix of $d_\beta(1) = (a-1)(a-2)^\omega$.

Proposition 3 *In each of the following assertions, (i), (ii) and (iii) are equivalent. For* $n \geqslant 1$

3.1 (i) b_n *is* SL*; (ii)* $\langle b_n \rangle_\beta$ *ends in* 0*; (iii)* $b'_n \in \Omega_{SL} = (0,1)$.

3.2 (i) b_n *is* LL*; (ii)* $\langle b_n \rangle_\beta$ *ends in a word* $w \notin \mathcal{M} \cup 0$*; (iii)* $b'_n \in \Omega_{LL} = [1, \beta - 1)$.

3.3 (i) b_n *is* LS*; (ii)* $\langle b_n \rangle_\beta$ *ends in a word* $w \in \mathcal{M}$*; (iii)* $b'_n \in \Omega_{LS} = (\beta - 1, \beta)$.

Proof. We prove $(i) \Leftrightarrow (ii)$ by recurrence. Suppose that b_n is SL. The factor SL can appear in three different configurations in the fixed point of the substitution u_β.

In the first configuration SL is a factor of $L^{a-1}SL^{a-1}S$ where b_n is issued from the LL β-integer b_k, $b_n = \beta b_k$.

$$\begin{array}{ccc} & b_n & \\ L^{a-1}S & | & L^{a-1}S \\ \uparrow & & \uparrow \\ & b_k & \\ L & | & L \end{array}$$

By recurrence hypothesis $\langle b_k \rangle_\beta$ ends in $w \notin \mathcal{M} \cup 0$. Then $\langle b_n \rangle_\beta$ ends in $w0$. Consequently, the β-expansions of $b_{n+1}, b_{n+2}, \ldots, b_{n+a-2}$, which are all LL, end respectively by $w1, w2, \ldots, w(a-2)$, and the β-expansion of b_{n+a-1}, which is LS, ends in $(a-1)$, which belongs to $\mathcal{M}$.

In the second configuration, SL is a factor of $L^{a-1}SL^{a-2}S$, and $b_n = \beta b_k$ with b_k a LS β-integer.

$$\begin{array}{ccc} & b_n & \\ L^{a-1}S & | & L^{a-2}S \\ \uparrow & & \uparrow \\ & b_k & \\ L & | & S \end{array}$$

By recurrence hypothesis, $\langle b_k \rangle_\beta$ ends in $(a-1)(a-2)^q$, with $q \in \mathbb{N}$. Then $\langle b_n \rangle_\beta$ ends in $(a-1)(a-2)^q 0$. Thus, the β-expansion of $b_{n+1}, b_{n+2}, \ldots, b_{n+a-3}$, which are LL, ends in $1, 2, \ldots, (a-3)$, respectively, and the β-expansion of b_{n+a-2}, which is LS, ends in $(a-1)(a-2)^{q+1} \in \mathcal{M}$.

In the third configuration, SL is a factor of $L^{a-2}SL^{a-1}S$, $b_n = \beta b_k$ with b_k a SL β-integer.

$$\begin{array}{ccc} & b_n & \\ L^{a-2}S & | & L^{a-1}S \\ \uparrow & & \uparrow \\ & b_k & \\ S & | & L \end{array}$$

By recurrence hypothesis $\langle b_k \rangle_\beta$ ends in 0, and $\langle b_n \rangle_\beta$ ends in 00. Therefore, the β-expansion of $b_{n+1}, b_{n+2}, \ldots, b_{n+a-2}$, which are LL, ends in $01, 02, \ldots, 0(a-2)$, and the β-expansion of b_{n+a-1}, which is LS, ends in $0(a-1)$.

The cases where b_n is an LL or an LS β-integer have been already treated. The case SS cannot occur.

Now let us prove $(i) \Leftrightarrow (iii)$. Recall that a number x of $\mathbb{Z}[\beta] \cap R^+$ is a positive β-integer if and only if its conjugate x' belongs to $(0, \beta)$. Let b_n be a SL β-integer. Then $b_{n-1} = b_n - (1 - 1/\beta)$ and $b_{n+1} = b_n + 1$ are β-integers, and $(b_{n-1})' = b'_n - 1 + \beta \in (0, \beta)$ and $(b_{n+1})' = b'_n + 1 \in (0, \beta)$. Therefore $b'_n \in (0, 1)$.

Let b_n be a LL β-integer. Then $b_{n-1} = b_n - 1$ and $b_{n+1} = b_n + 1$ are β-integers, and $(b_{n-1})' = b'_n - 1 \in (0, \beta)$ and $(b_{n+1})' = b'_n + 1 \in (0, \beta)$. Therefore $b'_n \in (1, \beta - 1)$. Since $b_1 = 1$ is LL, $b'_n \in [1, \beta - 1)$.

Finally, let b_n be a LS β-integer. Then $b_{n-1} = b_n - 1$ and $b_{n+1} = b_n + (1 - 1/\beta)$ are β-integers, and $(b_{n-1})' = b'_n - 1 \in (0, \beta)$ and $(b_{n+1})' = b'_n + 1 - \beta \in (0, \beta)$. Therefore $b'_n \in (\beta - 1, \beta)$. □

We now precise the characterization for LL β-integers.

Proposition 4 *Let b_n be a LL β-integer, then $\langle b_n \rangle_\beta$ ends in*

4.1 an $h \in \{1, \ldots, a-3\}$ if and only if $b'_n \in \Omega_{LL}(h) = [h, h+1)$

4.2 $(a-2)$ not prefixed by an element of $\mathcal{M}$ if and only if $b'_n \in \Omega_{LL}(a-2) = (a-2, \beta-1)$.

Proof. Let us first prove *4.1*. Let b_n be a LL β-integer such that $\langle b_n \rangle_\beta$ ends in 1. Then $b_n - 1 = b_{n-1}$ is SL, and $b'_n - 1 \in (0, 1)$. Then $b'_n \in (1, 2)$. Since $b_1 = 1$ is LL, $b'_n \in [1, 2)$. From the fact that $h' = h$ for $h \in \{1, \ldots, (a-3)\}$, we deduce *4.1*.

The proof of *4.2* is now straightforward. The only possibility for β-integers such that their β-expansion ends in $(a-2)$ not prefixed by an element of $\mathcal{M}$ is $\Omega_{LL}(a-2) = (a-2, \beta-1)$. □

3.2 Some Cubic Pisot Units

Case 1. The Tribonacci number: $\beta^3 = \beta^2 + \beta + 1$

The substitution σ_β allows only the following configurations (respectively Voronoi cells): LM, LS, ML, SL and LL in u_β.

Proposition 5 *In each of the following assertions (i) and (ii) are equivalent. For $n \geqslant 1$*

5.1 (i) b_n is LM; (ii) $\langle b_n \rangle_\beta$ ends in 01, or $n = 1$ and $\langle b_1 \rangle_\beta = 1$.

5.2 (i) b_n is LS; (ii) $\langle b_n \rangle_\beta$ ends in 011, or $n = 3$ and $\langle b_3 \rangle_\beta = 11$.

5.3 (i) b_n is ML; (ii) $\langle b_n \rangle_\beta$ ends in $10(000)^q$, $q \in \mathbb{N}$.

5.4 (i) b_n is SL; (ii) $\langle b_n \rangle_\beta$ ends in $100(000)^q$, $q \in \mathbb{N}$.

5.5 (i) b_n is LL; (ii) $\langle b_n \rangle_\beta$ ends in $1000(000)^q$, $q \in \mathbb{N}$.

Proof. Suppose that b_n is ML. The word ML can be issued from the substitution of three configurations of letters. In the first configuration, ML is a factor of $LMLSLM$, and $b_n = \beta b_k$, where b_k is LM.

$$\begin{array}{ccccc} & b_n & b_{n+1} & b_{n+2} & b_{n+3} \\ LM & | & LS & | & LM \\ \uparrow & & \uparrow & & \uparrow \\ & b_k & & b_{k+1} & \\ L & | & M & | & L \end{array}$$

Since by recurrence hypothesis, $\langle b_k \rangle_\beta$ ends in 01, $\langle b_n \rangle_\beta$ ends with 010. Consequently the β-expansions of b_{n+1}, b_{n+2} and b_{n+3}, which are respectively LS, SL and LM, end with 011, 100 and 101, respectively.

In the second configuration ML is a factor of $LMLLM$, and $b_n = \beta b_k$, where b_k is LS.

$$\begin{array}{ccccc} & b_n & & b_{n+1} & b_{n+2} \\ LM & | & L & | & LM \\ \uparrow & & \uparrow & & \uparrow \\ & b_k & & b_{k+1} & \\ L & | & S & | & L \end{array}$$

By recurrence hypothesis, $\langle b_k \rangle_\beta$ ends in 011, then $\langle b_n \rangle_\beta$ ends with 0110. Thus, the β-expansions of b_{n+1}, b_{n+2} which are respectively LL and LM end with 1000 and 1001, respectively. Since M is always follwed by L, b_{n+3} is ML, and ends in 1010.

Finally, in the third configuration ML is a factor of $LMLM$, and $b_n = \beta b_k$, where b_k is LL.

$$\begin{array}{ccc} & b_n & b_{n+1} \\ LM & | & LM \\ \uparrow & & \uparrow \\ & b_k & \\ L & | & L \end{array}$$

By recurrence hypothesis, $\langle b_k \rangle_\beta$ ends in $1000(000)^q$, $q \in \mathbb{N}$, then $\langle b_n \rangle_\beta$ ends with $10(000)^{q+1}$. Then the β-expansions of b_{n+1} and b_{n+2}, which are respectively LM and ML, end with $10(000)^q001$ and $10(000)^q010$, respectively.

Let now b_n be SL, which appears as a factor of $LSLM$. Then $b_n = \beta b_k$, where b_k is ML.

$$\begin{array}{ccc} & b_n & b_{n+1} \\ LS & | & LM \\ \uparrow & & \uparrow \\ & b_k & \\ M & | & L \end{array}$$

Since by recurrence hypothesis $\langle b_k \rangle_\beta$ ends with $10(000)^q$, then $\langle b_n \rangle_\beta$ ends with $100(000)^q$, and consequently, the β-expansions of b_{n+1} and b_{n+2}, which are LM and ML, respectively, end in 001 and 010, respectively.

Eventually, let b_n be LL, which appears as a factor of LLM. Then $b_n = \beta b_k$, with b_k SL.

$$\begin{array}{ccc} & b_n \; b_{n+1} & \\ L & \mid & LM \\ \uparrow & & \uparrow \\ & b_k & \\ S & \mid & L \end{array}$$

By recurrence hypothesis $\langle b_k \rangle_\beta$ ends in $100(000)^q$, then $\langle b_n \rangle_\beta$ ends with $1000(000)^q$, and consequently, the β-expansions of b_{n+1} and b_{n+2}, which are LM and ML respectively, end in 001 and 010. Cases LM and LS are treated above. □

Let α be one of the complex roots of $X^3 - X^2 - X - 1$. On Figure 3 we display the Rauzy fractal, *i.e.* the set $\Omega = \overline{(\mathbb{Z}_\beta^+)'}$, which is the closure of the set $\{\sum_{0 \leqslant i \leqslant N} x_i \alpha^i \mid \text{there is no factor 111 in } x_N \cdots x_0,\ N \geqslant 0\}$, and its partition according to Voronoi cells of β-integers. Recall that, by definition, $b_0 = 0$ is LL. The origin 0, although it belongs to Ω_{LL}, lies at the intersection of Ω_{LL} with Ω_{ML} and Ω_{SL}.

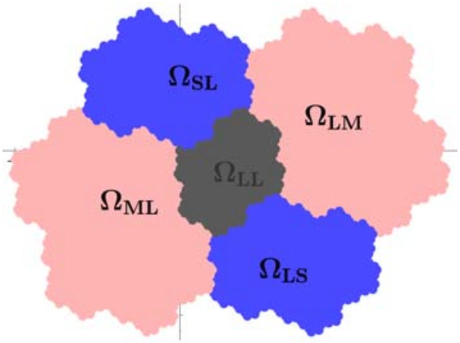


Fig. 3. The Rauzy fractal, obtained when β is the Tribonacci number

Usually the Rauzy fractal is divided into three basic tiles T_0, T_{01} and T_{011}, see [11, 16] or [15, Chapter 7]. Obviously our partition of the Rauzy fractal is a refinement of the classical division, with $T_0 = \Omega_{SL} \cup \Omega_{ML} \cup \Omega_{LL}$, $T_{01} = \Omega_{LM}$, and $T_{011} = \Omega_{LS}$.

Thus the domain exchange ρ defined on the Rauzy fractal (see Theorem 7.4.4 in [15]) is just the following:

$$T_0 = \Omega_{SL} \cup \Omega_{ML} \cup \Omega_{LL} \overset{\rho}{\mapsto} \Omega_{LS} \cup \Omega_{LM} \cup \Omega_{LL}$$

$$T_{01} = \Omega_{LM} \overset{\rho}{\mapsto} \Omega_{ML}$$

$$T_{011} = \Omega_{LS} \overset{\rho}{\mapsto} \Omega_{SL}.$$

From Proposition 5 one obtains the following result.

Proposition 6 *In the Rauzy fractal we have the following relations*

(i) $\Omega_{ML} = \Omega_{LM} + \alpha^{-1} + \alpha^{-2}$
(ii) $\Omega_{SL} = \Omega_{LS} + \alpha^{-1}$
(iii) $\Omega_{LL} = \alpha\Omega_{LS} + 1 = \alpha^2\Omega_{LM} + \alpha + 1.$

Proof (i). Let b'_n be in Ω_{ML}. Then the β-expansion of b_n is of the form $w10(000)^q$. Using that the word $10(000)^q$ has the same value in base β (or α) as the word $(011)^q01{\bullet}11$, we obtain that b'_n belongs to $\Omega_{LM} + \alpha^{-1} + \alpha^{-2}$.

Conversely, let b'_p be in Ω_{LM}. Then the β-expansion of b_p is of the form $v01$. The word $v01{\bullet}11$ has same value as $v10$. If $v10$ is already in normal form, it is an element of Ω_{ML}. If not, it is of the form $u(011)^k10$, with k maximal, and its normal form is $u10(000)^k$, thus the result follows.

(ii) The proof is similar.
(iii) It follows easily from (i) and (ii), and from the fact that $\Omega_{LL} = \alpha\Omega_{SL} = \alpha^2\Omega_{ML}$. □

Case 2. Symmetry of order 7: $\beta^3 = 2\beta^2 + \beta - 1$

The substitution σ_β allows only the following configurations (respectively Voronoi cells): LL, LS, SL, SM and ML.

Let $\mathcal{M}_1 = 2(01)^*$ and $\mathcal{M}_2 = 2(01)^*0$. The set $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$ is the set of maximal words of each length in the radix order, i.e., the set of prefixes of $d_\beta(1)$.

Proposition 7 *In each of the following assertions (i) and (ii) are equivalent. For* $n \geqslant 1$

6.1 (i) b_n *is* LL*; (ii)* $\langle b_n \rangle_\beta$ *ends in* $w1$ *where* $w \notin \mathcal{M}_2$.

6.2 (i) b_n *is* LS*; (ii)* $\langle b_n \rangle_\beta$ *ends in a word* $w \in \mathcal{M}_1$.

6.3 (i) b_n *is* SL*; (ii)* $\langle b_n \rangle_\beta$ *ends in* $w0$ *where* $w \notin \mathcal{M}_1$ *or by* $(0)^{2q+1}$, $q \in \mathbb{N}^*$.

6.4 (i) b_n *is* SM*; (ii)* $\langle b_n \rangle_\beta$ *ends in a word* $w \in \mathcal{M}_2$.

6.5 (i) b_n *is* ML*; (ii)* $\langle b_n \rangle_\beta$ *ends in* $(0)^{2q+2}$, $q \in \mathbb{N}$.

Proof. It is similar to the proof of Proposition 5. □

References

1. P. Arnoux et G. Rauzy, Représentation géométrique de suites de complexité $2n+1$. *Bull. Soc. Math. France* **119** (1991), 199–215.
2. A. Bertrand, Développements en base de Pisot et répartition modulo 1, *C. R. Acad. Sc. Paris*, Série A, t.285, (1977) 419–421.
3. Č. Burdik, Ch. Frougny, J. P. Gazeau, R. Krejcar, Beta-integers as natural counting systems for quasicrystal, *J. of Physics A: Math. Gen.* **31** (1998) 6449–6472.
4. A. Elkharrat, Etude et application des beta-réseaux aux structures apériodiques, Ph. D. Thesis, Université Paris 7, novembre 2004.
5. S. Fabre, Substitutions et β-systèmes de numération, *Theor. Comp. Sci.* **137** (1995) 219–236.

6. Ch. Frougny, J. P. Gazeau, R. Krejcar, Additive and multiplicative properties of point sets based on beta-integers, *Theor. Comp. Sci.* **303** (2003) 491-516.
7. Ch. Frougny, Z. Masáková, E. Pelantová, Complexity of infinite words associated with beta-expansions, *RAIRO-Inf. Theor. Appl.* **38** (2004) 163–185. Corrigendum, *RAIRO-Inf. Theor. Appl.* **38** (2004) 269–271.
8. J. C. Lagarias, Meyer's concept of quasicrystal and quasiregular sets, *Commun. Math. Phys.* **179** (1996) 365–376.
9. J. C. Lagarias, Geometric models for quasicrystals: I. Delone sets of finite type, *Discrete Comput. Geom.* **21** (1999) 161–191.
10. M. Lothaire, *Algebraic combinatorics on words*, Cambridge University Press (2002).
11. A. Messaoudi, Frontière du fractal de Rauzy et système de numération complexe, *Acta Arithmetica* **95** (2000) 195–224.
12. Y. Meyer, *Algebraic numbers and harmonic analysis*, North-Holland (1972).
13. Y. Meyer, Quasicrystals, Diophantine approximation and algebraic numbers, in *Beyond Quasicrystals*, F. Axel, D. Gratias (Eds), Les Editions de Physique, Springer (1995).
14. W. Parry, On the β-expansions of real numbers, *Acta Math. Acad. Sci. Hungar.* **11** (1960) 401–416.
15. N. Pytheas Fogg, *Substitutions in Dynamics, Arithmetics and Combinatorics*, Lecture Notes In Mathematics **1794**, Springer (2002).
16. G. Rauzy, Nombres algébriques et substitutions, *Bull. Soc. Math. France* **110** (1982) 147–178.
17. A. Rényi, Representations for real numbers and their ergodic properties, *Acta Math. Acad. Sci. Hung.* **8** (1957) 477–493.
18. K. Schmidt, On periodic expansions of Pisot numbers and Salem numbers, *Bull. London Math. Soc.* **12** (1980) 269–278.
19. W. Thurston, *Groups, tilings, and finite state automata.* AMS Colloquium Lecture Notes, Boulder, (1989).