On-line multiplication and division in real and complex bases

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***Abstract*—A positional numeration system is given by a base and by a set of digits. The base is a real or complex number β such that $|\beta| > 1$, and the digit set $\mathcal{A}$ is a finite set of real or complex digits (including 0). In this paper, we first formulate a generalized version of the on-line algorithms for multiplication and division of Trivedi and Ercegovac for the cases that β is any real or complex number, and digits are real or complex.**

We show that if $(\beta, \mathcal{A})$ satisfies the so-called (OL) Property, then on-line multiplication and division are feasible by the Trivedi-Ercegovac algorithms. For a real base β and alphabet $\mathcal{A}$ of contiguous integers, the system $(\beta, \mathcal{A})$ has the (OL) Property if $\#\mathcal{A} > |\beta|$.

Provided that addition and subtraction are realizable in parallel in the system $(\beta, \mathcal{A})$, our on-line algorithms for multiplication and division have linear time complexity.

Three examples are presented in detail: base $\beta = \frac{3+\sqrt{5}}{2}$ with alphabet $\mathcal{A} = \{-1, 0, 1\}$; base $\beta = 2i$ with alphabet $\mathcal{A} = \{-2, -1, 0, 1, 2\}$ (redundant Knuth numeration system); and base $\beta = -\frac{3}{2} + \imath\frac{\sqrt{3}}{2} = -1 + \omega$, where $\omega = \exp\frac{2i\pi}{3}$, with alphabet $\mathcal{A} = \{0, \pm 1, \pm\omega, \pm\omega^2\}$ (redundant Eisenstein numeration system).

***Index Terms*—on-line algorithm; numeration system;**

I. Introduction

A positional numeration system is given by a base and by a set of digits. The base is a real or complex number β such that $|\beta| > 1$, and the digit set $\mathcal{A}$ is a finite set of real or complex digits (including 0). The most studied numeration systems are of course the usual ones, where the base is a positive integer. But there have been also numerous studies, where the base is an irrational real number (the so-called *β-expansions*), a complex number, or a non-integer rational number, *etc*. A survey can be found in [6, Chapter 2].

On-line arithmetic, introduced in [14], is a mode of computation where operands and results flow through arithmetic units in a digit serial manner, starting with the most significant digit. To generate the first digit of the result, the first δ digits of the operands are required. The integer δ is called the delay of the algorithm. This technique allows for pipelining of different operations, such as addition, multiplication and division. It is also appropriate for the processing of real (or complex) numbers having infinite expansions: it is well known that when multiplying two real (or complex) numbers, only the left part of the result is significant. On-line arithmetic is used for special circuits such as in signal processing, and for very long precision arithmetic. An application to real-time control can be found in [1]. One of the interests of on-line computable functions is that they are continuous for the usual topology on the set of infinite sequences on a finite digit set. In order to be able to perform on-line computations, it is necessary to use a redundant numeration system, where a number may have more than one representation. A sufficient level of redundancy can also enable parallel addition and subtraction, which are used internally in the multiplication and division on-line algorithms.

On-line algorithms for multiplication and division in some complex numeration systems were given in [11], [9], and [7]. In this paper, we formulate a generalized version of the on-line algorithms for multiplication and division of Trivedi and Ercegovac [14], [2], for the cases that β is any real or complex number, and digits are real or complex.

Denote by $B(Z, \varepsilon)$ the ball of center Z and radius ε. Let us say that a pair $(\beta, \mathcal{A})$ has the (OL) Property if there exists a number $\varepsilon > 0$ and a bounded set I satisfying the following assumption: for every Z in $\cup_{T \in \beta I} B(T, \varepsilon)$ there exists a in $\mathcal{A}$ such that $B(Z, \varepsilon) \subset I + a$. We show that if $(\beta, \mathcal{A})$ has the (OL) Property and if 0 is in I, then on-line multiplication and division are feasible by the Trivedi-Ercegovac algorithms. For a real base β and an alphabet $\mathcal{A}$ of contiguous integers, the system $(\beta, \mathcal{A})$ has the (OL) Property if $\#\mathcal{A} > |\beta|$.

The key point of our algorithms is the specific choice of the function $\mathrm{Select}(W)$ performing selection of the digit to output. Since the definition of $\mathrm{Select}(W)$ uses just a reasonable approximation of W by a limited number of digits of W, it takes only constant time to evaluate it. In particular, we do not treat the real and the imaginary components separately.

The operation of division requires a preprocessing of the denominator, which we also discuss in this paper.

Provided that addition and subtraction are realizable in parallel in the system $(\beta, \mathcal{A})$ (see [4] for general results on this topic), our on-line algorithms for multiplication and division have linear time complexity.

Three examples are presented in full detail, wherein L denotes the number of fractional digits of W providing sufficient approximation of W within the on-line algorithm:
1) $\beta = \frac{3+\sqrt{5}}{2}$ and $\mathcal{A} = \{-1, 0, 1\}$: on-line multiplication is possible with delay $\delta = 4$ and with $L = 3$, on-line division with delay $\delta = 6$ and with $L = 9$.
2) $\beta = 2i$ and $\mathcal{A} = \{-2, -1, 0, 1, 2\}$ (redundant Knuth numeration system): on-line multiplication is possible with delay $\delta = 8$ and $L = 6$, and on-line division with delay $\delta = 11$ and $L = 11$.

3) $\beta = -\frac{3}{2} + \imath\frac{\sqrt{3}}{2} = -1 + \omega$, where $\omega = \exp\frac{2i\pi}{3}$ is the third root of unity, and $\mathcal{A} = \{0, \pm 1, \pm\omega, \pm\omega^2\}$ (redundant Eisenstein numeration system). We have two reasonable possibilities for the multiplication algorithm: $(\delta_{\min}, L) = (5, 7)$, with minimized delay δ, and $(\delta, L_{\min}) = (6, 6)$, with minimized parameter L; and similarly for the division algorithm: $(\delta_{\min}, L) = (7, 10)$, with minimized delay δ, and $(\delta, L_{\min}) = (10, 9)$, with minimized parameter L.

II. Algorithms of Trivedi and Ercegovac

The on-line multiplication and the on-line division algorithms we describe below are the same as the algorithms introduced by Trivedi and Ercegovac for computation in integer bases with a symmetric alphabet [14], [2]. Our modification for non-standard numeration systems for arbitrary base β (in general a complex number) and a digit set $\mathcal{A}$ (in general a finite set of complex numbers) stems from a specific choice of the function Select.

In the sequel, if $Z = \sum_{i=1}^{\infty} z_i\beta^{-i}$, we denote its partial sum by $Z_j = \sum_{i=1}^{j} z_i\beta^{-i}$; and let us put $A = \max\{|a| : a \in \mathcal{A}\}$.

A. On-line multiplication algorithm

The algorithm for on-line multiplication in a numeration system $(\beta, \mathcal{A})$ has one parameter – the delay $\delta \in \mathbb{N}$, which is specified later. The Select function here is called $\mathrm{Select_M}$.

The inputs of the algorithm are two strings

$$0\bullet x_1x_2\cdots x_\delta x_{\delta+1}x_{\delta+2}\cdots \quad \text{and} \quad 0\bullet y_1y_2\cdots y_\delta y_{\delta+1}y_{\delta+2}\cdots$$

with $x_i, y_i \in \mathcal{A}$ for all $i \in \mathbb{N}$, $x_i = y_i = 0$ for $i \leq \delta$, and such that $X = \sum_{i=1}^{\infty} x_i\beta^{-i}$ and $Y = \sum_{i=1}^{\infty} y_i\beta^{-i}$.

The output is a string $0\bullet p_1p_2p_3\cdots$ corresponding to a $(\beta, \mathcal{A})$-representation of the product $P = X.Y = \sum_{i=1}^{\infty} p_i\beta^{-i}$.

Set $W_0 = X_0 = Y_0 = p_0 = 0$. At the k-th step of the iteration ($k \geq 1$) compute:

$$W_k = \beta(W_{k-1} - p_{k-1}) + (x_kY_{k-1} + y_kX_k)\,, \quad (1)$$

$$p_k = \mathrm{Select_M}(W_k) \in \mathcal{A}\,. \quad (2)$$

Lemma II.1. *Definitions* (1) *and* (2) *imply that, for any* $k \geq 1$*:*

$$W_k = \beta^k(X_kY_k - P_{k-1})\,.$$

Moreover, if the sequence (W_k) *is bounded, then*

$$XY = \lim_{k\to+\infty} X_kY_k = \lim_{k\to+\infty} P_k = P\,.$$

B. On-line division algorithm

The algorithm for on-line division in $(\beta, \mathcal{A})$ numeration system has two parameters: the delay $\delta \in \mathbb{N}$ and the minimal value of the modulus of the denominator $D_{\min} > 0$. The Select function here is called $\mathrm{Select_D}$.

The inputs of the algorithm are two strings

$$0\bullet n_1n_2\cdots n_\delta n_{\delta+1}n_{\delta+2}\cdots \quad \text{and} \quad 0\bullet d_1d_2d_3\cdots$$

with $n_i, d_i \in \mathcal{A}$ for all $i \in \mathbb{N}$, with $n_i = 0$ for $i \leq \delta$, such that $N = \sum_{i=1}^{\infty} n_i\beta^{-i}$ is the numerator, $D = \sum_{i=1}^{\infty} d_i\beta^{-i}$ is the denominator and

$$|D_j| \geq D_{\min} \quad \text{for all } j \in \mathbb{N}, j \geq 1\,. \quad (3)$$

The output is a string $0\bullet q_1q_2q_3\cdots$ corresponding to a $(\beta, \mathcal{A})$-representation of the quotient $Q = N/D = \sum_{i=1}^{\infty} q_i\beta^{-i}$.

Set $W_0 = q_0 = Q_0 = 0$. Each k-th step of the iteration ($k \geq 1$) proceeds by calculating

$$W_k = \beta(W_{k-1} - q_{k-1}D_{k-1+\delta}) + \beta^{-\delta}(n_{k+\delta} - Q_{k-1}d_{k+\delta})\,, \quad (4)$$

$$q_k = \mathrm{Select_D}(W_k, D_{k+\delta}) \in \mathcal{A}\,. \quad (5)$$

Lemma II.2. *Definitions* (4) *and* (5) *imply that, for any* $k \geq 1$*:*

$$W_k = \beta^k(N_{k+\delta} - Q_{k-1}D_{k+\delta})\,.$$

Moreover, if the sequence (W_k) *is bounded, then*

$$Q = \lim_{k\to\infty} Q_k = \frac{N}{D}\,.$$

It is clear that the choice of the Select function is the crucial point for correctness of the algorithms, for both on-line multiplication and on-line division.

III. On-line multiplication and division in real and complex bases

In this section, we give a sufficient condition on $\beta \in \mathbb{C}$ and $\mathcal{A} \subset \mathbb{C}$, which guarantees that the numeration system $(\beta, \mathcal{A})$ allows on-line multiplication and division.

We fix the following notation: for $\varepsilon > 0$ and a set $\mathcal{T} \subset \mathbb{C}$, $\mathcal{T}^\varepsilon$ is the ε-neighbourhood of the set $\mathcal{T}$:

$$\mathcal{T}^\varepsilon = \bigcup_{T\in\mathcal{T}} B(T, \varepsilon)\,,$$

where $B(T, \varepsilon)$ is the ball of center T and radius ε.

For numbers $a, \beta \in \mathbb{C}$ and a set $\mathcal{T} \subset \mathbb{C}$, we denote

$$\mathcal{T} + a = \{T + a : T \in \mathcal{T}\} \text{ and } \beta\mathcal{T} = \{\beta T : T \in \mathcal{T}\}\,.$$

Definition III.1. A numeration system $(\beta, \mathcal{A})$ possesses the *(OL) Property* if there exist a number $\varepsilon > 0$ and a bounded set $I \subset \mathbb{C}$ satisfying the following assumption:

$$\forall\ Z \in (\beta I)^\varepsilon \quad \exists\ a \in \mathcal{A} \quad \text{such that} \quad B(Z, \varepsilon) \subset I + a\,. \quad (6)$$

Example III.2. The system defined by $\beta = \frac{3+\sqrt{5}}{2}$ and $\mathcal{A} = \{-1, 0, 1\}$ satisfies the (OL) Property with $\varepsilon = \frac{1}{2\beta(\beta+1)}$ and $I = [-(\frac{1}{2} + \varepsilon), \frac{1}{2} + \varepsilon]$. See Section VI-A for details.

The following lemma is a direct consequence of Def. III.1:

Lemma III.3. *Suppose that* $(\beta, \mathcal{A})$ *has the (OL) Property, and* $I \subset \mathbb{C}$ *and* $\varepsilon > 0$ *satisfy* (6)*. Then there exists a function* $\mathrm{Digit} : (\beta I)^\varepsilon \to \mathcal{A}$ *such that*

$$\mathrm{Digit}(V) = a \quad \Rightarrow \quad B(V, \varepsilon) \subset I + a\,. \quad (7)$$

When selecting the k^{th}-digit p_k (in the multiplication algorithm) or q_k (in the division algorithm), we do not want to evaluate the auxiliary variable W_k precisely, as it would be too costly. We shall use only a reasonable approximation by several most important digits of W_k.

Definition III.4. For $E > 0$, denote by Trunc_E a function $\mathbb{C} \to \mathbb{C}$ such that

$$|Z - \mathrm{Trunc}_E(Z)| < E \quad \text{for any } Z \in \mathbb{C}. \tag{8}$$

In the sequel, we use the Trunc_E function in the form of truncation of the less significant digits of the $(\beta, \mathcal{A})$-representation of the number $Z = \sum_{j=1}^{\infty} z_j \beta^{-j}$; namely $\mathrm{Trunc}_E(Z) = \sum_{j=1}^{L} z_j \beta^{-j}$ with $L \in \mathbb{N}$ such that $|\sum_{j=L+1}^{\infty} z_j \beta^{-j}| < E$ for any $z_j \in \mathcal{A}$.

A. Selection function for on-line multiplication

Definition III.5. Let $(\beta, \mathcal{A})$ be a numeration system with the (OL) Property, let $I \subset \mathbb{C}$ and $\varepsilon > 0$ satisfy (6), and let Digit be the function from Lemma III.3. The *selection function for multiplication* $\mathrm{Select_M} : (\beta I)^{\varepsilon/2} \to \mathcal{A}$ is defined by

$$\mathrm{Select_M}(U) = \mathrm{Digit}\big(\mathrm{Trunc}_{\varepsilon/2}(U)\big) \tag{9}$$

for any $U \in (\beta I)^{\varepsilon/2}$.

The previous definition is correct only if $\mathrm{Trunc}_{\varepsilon/2}(U)$ belongs to the domain of the function Digit, i.e. $(\beta I)^{\varepsilon}$. Indeed, since $U \in (\beta I)^{\varepsilon/2}$ and $|U - \mathrm{Trunc}_{\varepsilon/2}(U)| < \varepsilon/2$, the value $\mathrm{Trunc}_{\varepsilon/2}(U)$ is in $(\beta I)^{\varepsilon}$, as needed.

Lemma III.6. *Let $U \in (\beta I)^{\varepsilon/2}$. Then $U - \mathrm{Select_M}(U) \in I$.*

Lemma III.7. *Let $(\beta, \mathcal{A})$ be a numeration system with the (OL) Property, let $I \subset \mathbb{C}$, $\varepsilon > 0$ satisfy (6), and let $\mathrm{Select_M}$ be the function (9) from Def. III.5. Then there exists $\delta \in \mathbb{N}$ such that, for any $U \in (\beta I)^{\varepsilon/2}$, any $x, y \in \mathcal{A}$, any $X = \sum_{i \geq \delta+1} x_i \beta^{-i}$ and $Y = \sum_{i \geq \delta+1} y_i \beta^{-i}$ with $x_i, y_i \in \mathcal{A}$,*

$U_{\mathrm{new}} = \beta\big(U - \mathrm{Select_M}(U)\big) + (yX + xY)$ belongs to $(\beta I)^{\varepsilon/2}$.

Proof. Find $\delta \in \mathbb{N}$ such that

$$\frac{1}{|\beta|^{\delta}} \frac{2A^2}{|\beta| - 1} < \varepsilon/2 \, . \tag{10}$$

Then $|yX + xY| < \frac{\varepsilon}{2}$, and, according to Lemma III.6, the value $\beta\big(U - \mathrm{Select_M}(U)\big) \in \beta I$. This concludes the proof. □

Theorem III.8. *Suppose that a numeration system $(\beta, \mathcal{A})$ has the (OL) Property, and $I \subset \mathbb{C}$ and $\varepsilon > 0$ satisfy (6). If $0 \in I$, then on-line multiplication in $(\beta, \mathcal{A})$ is performable by the Trivedi-Ercegovac algorithm.*

Proof. Since $W_0 = 0 \in I$, necessarily $W_0 \in (\beta I)^{\varepsilon/2}$. Lemma III.7 implies that $W_k \in (\beta I)^{\varepsilon/2}$ for any $k \in N$ as well, and thus the sequence (W_k) is bounded. The result follows from Lemma II.1. □

B. Selection function for on-line division

Suppose that the value $D_{\min} > 0$ is given. In this whole subsection, we assume that the numeration system $(\beta, \mathcal{A})$ has the (OL) Property, that $I \subset \mathbb{C}$, $\varepsilon > 0$ satisfy (6), and the divisor D satisfies (3).

The $\mathrm{Select_D}$ function in the Trivedi-Ercegovac algorithm for division has two variables: W_k and $D_{k+\delta}$. Again, we do not want to compute these values precisely. In order to determine a suitable level of approximation, find $\alpha > 0$ such that

$$\alpha\big(1 + |\beta| K + \varepsilon\big) \;<\; \tfrac{\varepsilon}{2} D_{\min}, \quad \text{where} \tag{11}$$

$$K \;=\; \max\{|z| : z \in I\} \, . \tag{12}$$

For specification of the function $\mathrm{Select_D}$ for division, we use the function Trunc_α from Def. III.4.

Definition III.9. Let a value $U \in \mathbb{C}$ and a divisor $D \in \mathbb{C}$ satisfy $U \in D(\beta I)^{\varepsilon/2}$, and let $\alpha > 0$ fulfill (11). The *selection function for division* is defined by

$$\mathrm{Select_D}(U, D) \;=\; \mathrm{Digit}\Big(\tfrac{V}{\Delta}\Big), \quad \text{where} \tag{13}$$

$$V = \mathrm{Trunc}_\alpha(U) \quad \text{and} \quad \Delta = \mathrm{Trunc}_\alpha(D) \, .$$

Let us stress that the domain of the function Digit is $(\beta I)^{\varepsilon}$. Thus Def. III.9 is correct only if V/Δ belongs to this domain; which is confirmed by the following lemma:

Lemma III.10. *For $U \in \mathbb{C}$, $D \in \mathbb{C}$ satisfying (3), and $\alpha > 0$ fulfilling (11), put $V = \mathrm{Trunc}_\alpha(U)$ and $\Delta = \mathrm{Trunc}_\alpha(D)$. Then*

$$U \in D(\beta I)^{\varepsilon/2} \quad \Longrightarrow \quad V \in \Delta(\beta I)^{\varepsilon} \, .$$

The following statement corresponds to the iterative step in the division algorithm.

Theorem III.11. *Let a numeration system $(\beta, \mathcal{A})$ have the (OL) Property, and let $I \in \mathbb{C}$, $\varepsilon > 0$ satisfy (6). If $0 \in I$, then on-line division in $(\beta, \mathcal{A})$ is performable by the Trivedi-Ercegovac algorithm.*

Proof. For the Trivedi-Ercegovac division algorithm, we use the function $\mathrm{Select_D}$ from Def. III.9. According to Lemma II.2, for correctness of the algorithm one has to show that the sequence (W_k) is bounded.

We prove by induction on the index $k \in \mathbb{N}$ that, for each $k \geq 0$, the value W_k satisfies $W_k \in D_{\delta+k}(\beta I)^{\varepsilon/2}$.

Let us set the delay δ such that:

$$\frac{A}{D_{\min}} \Big(1 + \tfrac{A}{|\beta| - 1} + K + \varepsilon\Big) < \tfrac{\varepsilon}{2}\, |\beta|^{\delta} \, , \tag{14}$$

with K from (12), and choose $\alpha > 0$ fulfilling (11).

With such selection of parameters, it can be shown that, for any $U, D, F, G \in \mathbb{C}$ with properties $U \in D(\beta I)^{\varepsilon/2}$, $|F| \leq A$ and $|G| \leq A\Big(1 + \frac{A}{|\beta| - 1}\Big)$, the numbers

$$U_{\mathrm{new}} = \beta(U - qD) + \tfrac{G}{\beta^{\delta}} \quad \text{and} \quad D_{\mathrm{new}} = D + \tfrac{F}{\beta^{\delta+1}} \, , \tag{15}$$

where $q = \mathrm{Select_D}(U, D)$, satisfy $U_{\mathrm{new}} \in D_{\mathrm{new}}(\beta I)^{\varepsilon/2}$.

Now we apply (15) with $F = \frac{d_{\delta+k+1}}{\beta^k}$, $G = n_{k+1+\delta} - Q_k d_{k+1+\delta}$, $U = W_k$, $U_{\text{new}} = W_{k+1}$, $q = q_k$, $D = D_{k+\delta}$, $D_{\text{new}} = D_{k+\delta+1}$, and obtain the implication

$$W_k \in D_{\delta+k}(\beta I)^{\varepsilon/2} \quad \Longrightarrow \quad W_{k+1} \in D_{\delta+k+1}(\beta I)^{\varepsilon/2}.$$

The (OL) Property guarantees that the set I is bounded, and the values D_k are bounded by $\frac{A}{|\beta|-1}$ in modulus. Thus the sequence (W_k) is bounded too, as we wanted to demonstrate. □

IV. (OL) Property

A. Real bases and the (OL) Property

For a given base $\beta \in \mathbb{R}$, it is always possible to find a sufficiently large alphabet $\mathcal{A} \subset \mathbb{Z}$ such that the (OL) Property for the numeration system $(\beta, \mathcal{A})$ is fulfilled. But the challenge is to prove the (OL) Property (and find the appropriate set I and value $\varepsilon > 0$) for a given pair $(\beta, \mathcal{A})$, especially for alphabets of small size. In case of the real bases, we can provide a general solution to this question:

Lemma IV.1. *Let β be a real number with $|\beta| > 1$ and let $\mathcal{A} = \{m, m+1, \ldots, M-1, M\} \subset \mathbb{Z}$ with $m \le 0 \le M$. If $|\beta| < \#\mathcal{A} = M - m + 1$, then the numeration system $(\beta, \mathcal{A})$ has the (OL) Property. In particular:*

- *for $\beta > 1$, one of the pairs (I, ε) satisfying* (6) *is $I = [\lambda, \rho]$ and $\varepsilon > 0$ defined by*

$$\varepsilon = \frac{M - m + 1 - \beta}{2(\beta+1)}, \ \lambda = \frac{m + 2\varepsilon}{\beta - 1}, \ \ \rho = \frac{M - 2\varepsilon}{\beta - 1};$$

- *for $\beta < -1$, one of the pairs (I, ε) satisfying* (6) *is $I = [\lambda, \rho]$ and $\varepsilon > 0$ defined by*

$$\varepsilon = \frac{M - m + 1 + \beta}{2(1-\beta)}, \lambda = \frac{-M - 1}{1 - \beta}, \ \ \rho = \frac{1 - m}{1 - \beta}.$$

Remark IV.2. If $\beta < -1$, then the interval $I = [\lambda, \rho]$ in Lemma IV.1 always contains 0. The same is true if $\beta > 1$ and $m < 0 < M$. Thus, according to Theorems III.8 and III.11, the on-line algorithms work properly.

If $\beta > 1$ and $M = 0$, i.e., the alphabet consists of non-positive integers, then only non-positive numbers have a $(\beta, \mathcal{A})$-representation. Product or quotient of such numbers is positive, and thus without any $(\beta, \mathcal{A})$-representation. Therefore, no (on-line) algorithms for multiplication or division make sense in this case.

If $\beta > 1$ and $m = 0$, i.e., the alphabet consists of non-negative integers, then no interval $I \subset \mathbb{R}$ suitable for the (OL) Property contains 0. Nevertheless, even in this case the Trivedi-Ercegovac algorithm can be used. The Select function just has to be slightly modified as follows: Consider the interval $I = [\lambda, \rho]$ from Lemma IV.1. In particular, the left boundary of the interval is $\lambda = \frac{2\varepsilon}{\beta-1} > 0$. The $\mathrm{Select_M}$ function given by Def. III.5 has as its domain the interval $(\beta I)^{\varepsilon/2}$ with its left boundary $\beta\lambda - \frac{\varepsilon}{2}$. Put

$$\widetilde{\mathrm{Select}}_{\mathrm{M}}(U) = \begin{cases} 0 & \text{if } \ U < \beta\lambda - \frac{\varepsilon}{2}, \\ \mathrm{Select_M}(U) & \text{if } \ U \ge \beta\lambda - \frac{\varepsilon}{2}. \end{cases}$$

Using this extended $\widetilde{\mathrm{Select}}_{\mathrm{M}}$ function in the algorithm for multiplication (and analogously extended $\widetilde{\mathrm{Select}}_{\mathrm{D}}$ for division), starting with $W_0 = 0$, we get the digit 0 at the beginning on the output, and thus consequently $W_k \ge \beta W_{k-1}$. After several steps, W_k reaches the interval $(\beta I)^{\varepsilon/2}$. According to Lemma III.7 and to (15), the value of W_k then stays in $(\beta I)^{\varepsilon/2}$ in all further steps. Thus the sequence (W_k) is bounded, and the algorithms work properly.

Lemma IV.1 and Remark IV.2 imply the following:

Theorem IV.3. *Let β be a real number with $|\beta| > 1$ and $\mathcal{A} = \{m, m+1, \ldots, M-1, M\} \subset \mathbb{Z}$. Let us assume that $m \le 0 < M$ if $\beta > 1$, and $m \le 0 \le M$ if $\beta < -1$. If $|\beta| < \#\mathcal{A} = M - m + 1$, then multiplication and division in the numeration system $(\beta, \mathcal{A})$ are performable on-line by the Trivedi-Ercegovac algorithms.*

B. Complex bases and the (OL) Property

For complex bases $\beta \in \mathbb{C}$, it is always possible to find a sufficiently large alphabet $\mathcal{A} \subset \mathbb{C}$, so that the system $(\beta, \mathcal{A})$ has the (OL) Property. Nevertheless, it is an open question (without a general result so far) whether the (OL) Property is fulfilled for a given pair $(\beta, \mathcal{A})$.

We provide examples of the Penney [12] and Eisenstein numeration systems, working with (complex) alphabets of the smallest possible size allowing parallel addition: $\#\mathcal{A} = 5$ for the Penney base (Fig. 1), and $\#\mathcal{A} = 7$ for the Eisenstein base (Fig. 2) – see Section VI for more details.

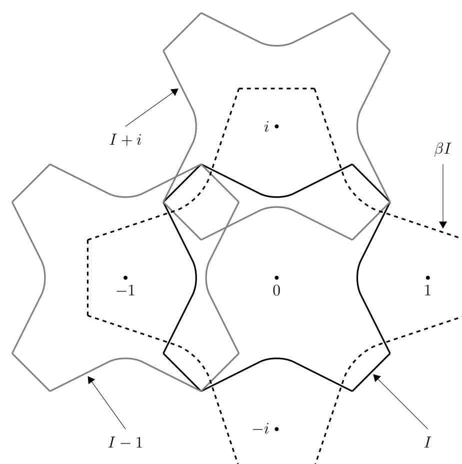

Fig. 1. Penney numeration system with base $\beta = -1 + \imath$ and alphabet $\mathcal{A} = \{0, \pm 1, \pm\imath\}$ fulfills the (OL) Property, due to the "star-shaped" set I illustrated hereby.

V. Parameters in on-line algorithms and time complexity

In this whole section, we assume a numeration system $(\beta, \mathcal{A})$ satisfying the (OL) Property. In order to use the on-line algorithms, we need to determine one parameter (δ) for multiplication, and two parameters (δ, $D_{\min}$) for division. The inequalities (10) and (14) provide formulae for δ, given the bounded set $I \subset \mathbb{C}$ and the parameter $\varepsilon > 0$, due to the (OL)

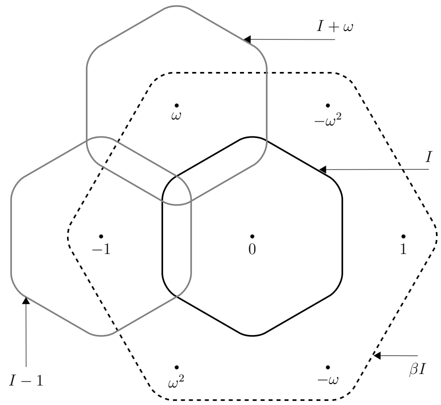

Fig. 2. Eisenstein numeration system with base $\beta = -1+\omega$ and alphabet $\mathcal{A} = \{0, \pm 1, \pm\omega, \pm\omega^2\}$, where ω is the third root of unity fulfills the (OL) Property, due to the "rounded hexagon" set I illustrated hereby, see Example VI-C.

Property; clearly, the bigger the value of ε, the smaller the delay δ can be. Determination of the value $D_{\min}$ is not so straightforward.

A. Preprocessing of divisor and $D_{\min}$

By "preprocessing of divisor", we mean a transformation of the divisor into the form $0{\bullet}d_1d_2d_3\cdots$ with $d_i \in \mathcal{A}$ satisfying

$$|D_j| = \Big|\sum_{i=1}^{j} d_i\beta^{-i}\Big| \geq D_{\min} \text{ for all } j \in \mathbb{N}, j \geq 1\,,$$

see (3). In particular, for $j=1$, we need $d_1 \neq 0$. Therefore, the transformation consists at least in shifting the fractional point to the most significant non-zero digit of the representation of the divisor, i.e., we multiply the divisor by a suitable power of β, and, after obtaining the result of the division, we must take this fact into account.

Let us denote

$$\mathcal{R} = \Big\{\Big|\sum_{i\geq 1} d_i\beta^{-i}\Big| \ : \ d_1 \neq 0, d_i \in \mathcal{A}\Big\}.$$

If $\inf\mathcal{R} > 0$, then one can put $D_{\min} = \inf\mathcal{R}$ into the on-line algorithm for division, and nothing else than shifting the fractional point is needed. In our further considerations about the parameter $D_{\min}$, the following notion plays a key role.

Definition V.1. Let $(\beta,\mathcal{A})$ be a numeration system. If $0 = \sum_{i\geq 1} z_i\beta^{-i}$, where $z_i \in \mathcal{A}$ for all $i \geq 1$ and $z_j \neq 0$ for at least one index j, then the string $z_1z_2z_3\cdots$ is called a *non-trivial $(\beta,\mathcal{A})$-representation of zero.*

The relation between representations of zero and $\mathcal{R}$ is obvious:

Lemma V.2. *$\inf\mathcal{R} = 0$ if and only if 0 has a non-trivial $(\beta,\mathcal{A})$-representation.*

In numeration systems having a non-trivial representation of zero, the determination of $D_{\min}$ and the divisor preprocessing are more laborious, and no general recipe applicable to all bases is available. The following lemma helps to identify such numeration systems with real bases (and the proof is due to a result of Rényi [13]):

Lemma V.3. *Let $\beta > 1$ and $\{-1,0,1\} \subset \mathcal{A} = \{m,\ldots,0,\ldots,M\} \subset \mathbb{Z}$. Then 0 has a non-trivial $(\beta,\mathcal{A})$-representation if and only if $\beta \leq \max\{M+1, -m+1\}$.*

Example V.4. If $\beta = 4$ and $\mathcal{A} = \{-2,-1,0,1,2\}$, then zero has only the trivial representation, and for $D_{\min}$ one can take $\frac{1}{12} = \min\mathcal{R} = 0{\bullet}1\bar{2}\,\bar{2}\,\bar{2}\cdots$, where $\bar{2}$ stands for the digit -2.

Remark V.5. If $\beta > 1$ and $\mathcal{A} = \{0,1,\ldots,M\}$, then zero has only a trivial representation. But this numeration system has another disadvantage: the operation of subtraction (needed for evaluation of W_k in both algorithms) is not doable in parallel.

Example V.6. In the numeration system with $\beta = 2$ and $\mathcal{A} = \{-1,0,1\}$, zero has two non-trivial representations, namely $0 = 0{\bullet}1\bar{1}\,\bar{1}\,\bar{1}\,\bar{1}\cdots = 0{\bullet}\bar{1}\,1\,1\,1\,1\cdots$, where $\bar{1}$ stands for the digit -1. Therefore, preprocessing is a bit more sophisticated. It is necessary to find a representation of the divisor such that $d_nd_{n-1} \neq 1\bar{1}$ and $d_nd_{n-1} \neq \bar{1}1$, where n is the maximal index such that $d_n \neq 0$. This can be achieved by replacing the leading pair of neighbouring digits $1\bar{1}$ with 01 or by replacing $\bar{1}1$ with $0\bar{1}$, and this procedure is repeated for as long as necessary. Finally, the fractional point is shifted to the first non-zero digit. For example:

$$0{\bullet}1\bar{1}\,\bar{1}\,\bar{1}\,0\bar{1} \mapsto 0{\bullet}01\,\bar{1}\,\bar{1}\,0\bar{1} \mapsto 0{\bullet}001\,\bar{1}\,0\bar{1} \mapsto 0{\bullet}00010\bar{1}\,,$$

and lastly, by shifting the fractional point, we get $0{\bullet}10\bar{1}$. The parameter $D_{\min}$ of the Trivedi-Ercegovac algorithm for division can be set to $D_{\min} = \frac{1}{4}$, since any divisor D after the described preprocessing satisfies

$$|D| \geq 0{\bullet}10\bar{1}\,\bar{1}\,\bar{1}\cdots = \tfrac{1}{2} - \tfrac{1}{8} - \tfrac{1}{16} - \cdots = \tfrac{1}{4} = D_{\min}\,.$$

We illustrate on two less trivial examples how to find $D_{\min}$ and how to perform preprocessing. In both those examples, the alphabet $\mathcal{A}$ consists of (possibly complex) units and zero, and so it is closed under multiplication. To shorten our list of preprocessing rules, let us adopt the following conventions:
1) instead of "If $w_1w_2\cdots w_k$ is a prefix of $\mathbf{d}$, replace it with $u_1u_2\cdots u_k$", we write "$w_1w_2\cdots w_k \longrightarrow u_1u_2\cdots u_k$";
2) the rule "$w_1w_2\cdots w_k \longrightarrow u_1u_2\cdots u_k$" is equivalent to the rule "$w'_1w'_2\cdots w'_k \longrightarrow u'_1u'_2\cdots u'_k$" if there exists $a \in \mathcal{A}\backslash\{0\}$ such that $w_j = aw'_j$ and $u_j = au'_j$ for all $j = 1,2,\ldots,k$.

In our list of preprocessing rules, we mention only one rule from each class of equivalence. Of course, each rule on the list preserves the value of the divisors, i.e., $\sum_{i=1}^{k} w_i\beta^i = \sum_{i=1}^{k} u_i\beta^i$, and sets $u_1 = 0$. In this convention, the list of preprocessing rules for base $\beta = 2$ and alphabet $\mathcal{A} = \{-1,0,1\}$ consists of one item only, namely the rule $1\bar{1} \longrightarrow 01$.

Example V.7. Let $\beta = \frac{1+\sqrt{5}}{2}$ and $\mathcal{A} = \{-1,0,1\}$. Since $\beta^2 - \beta - 1 = 0$, zero has the representation $0 = 0{\bullet}1\,\bar{1}\,\bar{1}$.
On $D = 0{\bullet}d_1d_2d_3\cdots$, we use three preprocessing rules:
1) $10\bar{1} \longrightarrow 010$, 2) $1\bar{1}0 \longrightarrow 001$, 3) $1\bar{1}\bar{1} \longrightarrow 000$.

If $d_1 \neq 0$ and the rules 1) - 3) cannot be applied to the string $\mathbf{d} = d_1 d_2 d_3 \cdots$, then $|D| \geq D_{\min} = \frac{1}{\beta^5}$. This can be shown by inspecting all the possible triplets $d_1 d_2 d_3$.

Example V.8. Let $\beta = -1 + \omega$, where $\omega = \exp \frac{2i\pi}{3}$ is the third root of unity, and consider the alphabet $\mathcal{A} = \{0, \pm 1, \pm\omega, \pm\omega^2\}$. Firstly, we show that

$$D_{\max} = \max\{|0{\bullet}d_1 d_2 d_3 \cdots| \ : d_i \in \mathcal{A}\} = \tfrac{1}{2}\sqrt{7}\,. \qquad (16)$$

Since $|x\beta + y| \leq |\beta - 1| = \sqrt{7}$ for any $x, y \in \mathcal{A}$, we have $|0{\bullet}d_1 d_2 d_3 \cdots| \leq \sqrt{7} \sum_{k \geq 1} |\beta|^{-2k} = \frac{1}{2}\sqrt{7}\,$.

We use 9 equivalence classes of preprocessing rules (where $\overline{\omega}$ denotes $-\omega$, and $\overline{\omega^2}$ stands for $-\omega^2$):

- From the equality $\beta + (1 - \omega) = 0$, we get basic rules: A) $1\,1 \longrightarrow 0\,\omega$, B) $1\,\overline{\omega} \longrightarrow 0\,\overline{1}$; and
- by suitable combinations of these basic rules, we obtain: C) $1\,0\,\omega \longrightarrow 0\,\omega\,1$, D) $1\,0\,\overline{\omega^2} \longrightarrow 0\,\overline{1}\,\overline{\omega}$, E) $1\,\overline{\omega^2}\,\omega \longrightarrow 0\,\omega\,\omega^2$, F) $1\,\overline{\omega^2}\,\overline{\omega^2} \longrightarrow 0\,\omega\,\overline{\omega}$, G) $1\,0\,\overline{1} \longrightarrow 0\,\omega\,\overline{\omega}$, H) $1\,\omega^2\,\overline{1} \longrightarrow 0\,\overline{1}\,\overline{\omega}$, and I) $1\,\omega^2\,\omega \longrightarrow 0\,\overline{1}\,1$.

If $D = 0{\bullet}d_1 d_2 d_3 \cdots$ with $d_1 \neq 0$, and the rules A) - I) cannot be applied to the string $\mathbf{d} = d_1 d_2 d_3 \cdots$, then

$$|D| \geq D_{\min} = \frac{\sqrt{3}(6 - \sqrt{7})}{18}\,. \qquad (17)$$

Without loss of generality, we can assume $d_1 = 1$. By exploring all possible triplets $1 d_2 d_3$ to which no rules can be applied, we see that $|0{\bullet}d_1 d_2 d_3| \geq \frac{\sqrt{3}}{3}$. Therefore, $|D| \geq \frac{\sqrt{3}}{3} - \frac{1}{|\beta|^3} D_{\max}$, which, together with (16), proves (17).

B. Time complexity

The time complexity of both (multiplication and division) algorithms depends on the number of steps needed to compute the auxiliary value W_k and the k-th output digit by the function Select. If both tasks can be performed in constant time, then the time complexity (and also memory usage) of computing the first n most significant digits of the result is $\mathcal{O}(n)$.

The (OL) Property of a numeration system $(\beta, \mathcal{A})$ forces the system to be redundant. For real bases, it implies $\#\mathcal{A} > |\beta|$. Lemma IV.1 says that $\#\mathcal{A} > |\beta|$ is also a sufficient condition for (OL) Property. Nevertheless, $\#\mathcal{A} > |\beta|$ does not guarantee that in $(\beta, \mathcal{A})$ addition and subtraction are doable in constant time. Usually, the alphabet has to be extended further. For example, both systems $(\frac{1+\sqrt{5}}{2}, \{0, 1\})$ and $(\frac{1+\sqrt{5}}{2}, \{\overline{1}, 0, 1\})$ have the (OL) Property, but parallel addition is possible only in the second one. The question of the minimal size of $\mathcal{A}$ for parallel addition is treated in general in [5].

We focus here on the evaluation of the function Select. Let us restrict to the on-line multiplication (as the on-line division is then quite analogous). The function Digit assigns a digit to $\mathrm{Trunc}_{\varepsilon/2}(U)$. The value $\varepsilon/2$ determines the number L of fractional digits that we take into consideration. Finding such L consists in solving the inequality

$$\tfrac{A}{|\beta|^{L+1}} + \tfrac{A}{|\beta|^{L+2}} + \tfrac{A}{|\beta|^{L+3}} + \cdots = \tfrac{A}{|\beta|^{L}(|\beta| - 1)} < \tfrac{\varepsilon}{2}\,. \qquad (18)$$

The value $\mathrm{Trunc}_{\varepsilon/2}(U)$ in the course of the algorithm belongs to the domain of the function Digit, i.e., to the bounded set $(\beta I)^{\varepsilon}$. We would like to limit also the number of digits before the fractional point of U. Let us denote

$$\mathcal{S} = \{x_{-n} \cdots x_L \in \mathcal{A}^* \ : \ x_{-n} \neq 0\,, \sum_{i=L}^{-n} x_i \beta^{-i} \in (\beta I)^{\varepsilon}\}\,.$$

If $\mathcal{S}$ is finite, one can create a table consisting of all elements of $\mathcal{S}$ together with the Digit values assigned to them. Therefore, the evaluation of the function $\mathrm{Select}_{\mathrm{M}}$ can be made in constant time in each iterative step of the on-line algorithm.

If the set $\mathcal{S}$ is not finite, one must modify a representation of U after each iterative step of the algorithm, so as not to allow any representation of U with unbounded index of the leading coefficient. In fact, we apply the same list of rewriting rules as for the divisor preprocessing (only without shifting the fractional point). We can summarize as follows:

Lemma V.9. *Let us assume a numeration system $(\beta, \mathcal{A})$ with a finite list of (preprocessing) rules and $D_{\min} > 0$ such that any $D = 0{\bullet}d_1 d_2 d_3 \cdots$ with $d_1 \neq 0$ on which no rule of the list can be applied has modulus $|D| \geq D_{\min}$. Then the set $\mathcal{S}' = \{s \in \mathcal{S} :$ no rules can be applied to $s\}$ is finite.*

We can conclude this section with the following statement: if a numeration system $(\beta, \mathcal{A})$ with the (OL) Property allows parallel addition, parallel subtraction, and preprocessing of divisors (into the form (3)), then the time complexity of algorithms for on-line multiplication and division is $\mathcal{O}(n)$.

VI. Examples

A. Base $\beta = \frac{3+\sqrt{5}}{2}$ and alphabet $\mathcal{A} = \{-1, 0, 1\}$

For illustration how the on-line algorithms for multiplication and division work, we take a well-studied numeration system, with base $\beta = \frac{3+\sqrt{5}}{2}$ and alphabet $\mathcal{A} = \{-1, 0, 1\}$. Let us list the most important properties of this system:

- The base $\beta = \frac{3+\sqrt{5}}{2} = \left(\frac{1+\sqrt{5}}{2}\right)^2$ is a quadratic Pisot unit with minimal polynomial $f(t) = t^2 - 3t + 1$.
- The numeration system with base $\beta = \frac{3+\sqrt{5}}{2}$ and digit set $\mathcal{A} = \{-1, 0, 1\}$ allows parallel addition [5].
- By Lemma V.3, zero has only the trivial $(\beta, \mathcal{A})$-representation, and
$$D_{\min} = 0{\bullet}\,1\,\overline{1}\,\overline{1}\,\overline{1}\,\cdots = \frac{1}{\beta} - \sum_{j=2}^{\infty} \frac{1}{\beta^j} = \frac{1}{\beta^2}\,. \qquad (19)$$
Thus the sign of the first non-zero digit in a representation determines the sign of the represented number. Preprocessing of divisor is just shifting of the fractional point.
- If $D = 0{\bullet}d_1 d_2 d_3 \cdots$ is a $(\beta, \mathcal{A})$-representation, then
$$-\frac{1}{\beta - 1} \leq D \leq \frac{1}{\beta - 1} = D_{\max}.$$
- By Lemma IV.1, $(\beta, \mathcal{A})$ has the (OL) Property with
$$\varepsilon = \frac{1}{2\beta(\beta + 1)} > 0 \quad \text{and} \quad I = [-\rho, \rho]\,, \quad \text{where} \quad \rho = \tfrac{1}{2} + \varepsilon\,.$$

- By Lemma III.3, the function $\text{Digit} : [-\rho - \varepsilon, \rho + \varepsilon] \to \{\overline{1}, 0, 1\}$ is defined by

$$\text{Digit}(V) = \begin{cases} 1 & \text{if} \quad V > \frac{1}{2}, \\ -1 & \text{if} \quad V < -\frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

1) On-line multiplication in base $\beta = \frac{3+\sqrt{5}}{2}$ and alphabet $\mathcal{A} = \{-1, 0, 1\}$: For on-line multiplication, the delay δ according to (10) has to satisfy $\frac{2}{\beta^\delta(\beta-1)} < \frac{1}{4\beta(\beta+1)}$, and the smallest such delay is $\delta = 4$.

It remains to find an easy way of evaluating the function $\text{Select}_\text{M}(W) = \text{Digit}(\text{Trunc}_{\varepsilon/2}(W))$. By Def. III.5, its domain is $(\beta I)^{\varepsilon/2} = [-\beta\rho - \frac{1}{2}\varepsilon, \beta\rho + \frac{1}{2}\varepsilon]$, which ensures that:

- Any $Z \in (\beta I)^{\varepsilon/2}$ has a $(\beta, \mathcal{A})$-representation of the form $Z = z_{-1}z_0{\bullet}z_1 z_2 \cdots$.
- This $Z = z_{-1}z_0{\bullet}z_1z_2\cdots$ fulfills $|Z - V| < \frac{1}{2}\varepsilon$ for the $(\beta, \mathcal{A})$-representation $V = z_{-1}z_0{\bullet}z_1z_2z_3z_4$.
- The value $V = z_{-1}z_0{\bullet}z_1z_2z_3z_4 > \frac{1}{2}$ if and only if

$$z_{-1}z_0z_1z_2z_3 \succ 0\,1\,\overline{1}\,\overline{1}\,0 \quad \text{or} \quad \big(z_{-1}z_0z_1z_2 = 0\,0\,1\,1 \quad \text{and} \quad z_3 \neq \overline{1}\big), \tag{20}$$

where $\succ$ denotes the lexicographic order on strings.

This can be proved by inspection of all possibilities, and using the fact that the lexicographically greatest infinite expansion of 1 is $1 = 0{\bullet}21111\cdots$, and from the symmetry of $\mathcal{A}$.

Therefore, the evaluation of the function Select_M can be done via a finite table of values; and the statements above imply that such table has 3^5 elements. But the lexicographic order relations in (20) enable us to provide a more effective means of Select_M evaluation.

In summary, on-line multiplication is possible by the Trivedi-Ercegovac algorithm with delay $\delta = 4$, and with linear time complexity. The number of digits we need to evaluate for W within the algorithm is $L = 3$ behind the fractional point, and another 2 digits before the fractional point.

2) On-line division in base $\beta = \frac{3+\sqrt{5}}{2}$ and alphabet $\mathcal{A} = \{-1, 0, 1\}$: To determine the algorithm for on-line division, we have to specify two parameters: δ and D_min. We put $D_\text{min} = \frac{1}{\beta^2}$ (cf. (19)). To find the delay δ, we may again follow the general formula (14), and obtain $\delta = 7$. By a more elaborated calculation, using more precise estimates (specific to this numeration system) in the formulas, the delay can be further optimized to $\delta = 6$, with the number $L = 9$ of fractional digits to evaluate in expressions of W and D.

B. Knuth numeration system

D. E. Knuth showed in 1955 [8] that in the numeration system with base $\beta = 2i$ and alphabet $\mathcal{C} = \{0, 1, 2, 3\}$, any complex number Z has a representation of the form $Z = \sum_{k \geq n} z_k \beta^{-k}$, where $n \in \mathbb{Z}$ and $z_k \in \{0, 1, 2, 3\}$. In this numeration system, almost all complex numbers have a unique representation. We consider a redundant system with the same base and a symmetric alphabet $\mathcal{A} = \{-2, -1, 0, 1, 2\}$. Let us list some relevant properties of this system:

- In $(\beta, \mathcal{A})$, parallel addition is possible, see [3].
- The system $(\beta, \mathcal{A})$ has the (OL) Property, as the oblong I with vertices $\pm\frac{5}{9} \pm i\frac{11}{9}$ and $\varepsilon = \frac{1}{18}$ satisfy (6).
- The function Digit is defined by

$$\text{Digit}(V) = \begin{cases} 2 & \text{if} \quad \Re(V) > \frac{3}{2}, \\ 1 & \text{if} \quad \Re(V) \in (\frac{1}{2}, \frac{3}{2}], \\ 0 & \text{if} \quad \Re(V) \in [-\frac{1}{2}, \frac{1}{2}], \\ -1 & \text{if} \quad \Re(V) \in [-\frac{3}{2}, -\frac{1}{2}), \\ -2 & \text{if} \quad \Re(V) < -\frac{3}{2}. \end{cases}$$

- A number $Z = \sum_{k=1}^{\infty} z_k \beta^{-k}$ with $z_k \in \mathcal{A}$ can be decomposed into real and imaginary part as follows:

$$Z = \sum_{k=1}^{\infty} z_{2k}(-4)^{-k} + 2i \sum_{k=1}^{\infty} z_{2k-1}(-4)^{-k}.$$

So the real and imaginary parts (each separately) are represented in real numeration system with base -4 and alphabet $\{-2, \ldots, 2\}$, with only the trivial representation of 0.
- $D_\text{min} = \frac{1}{6}$. It follows from the fact that if $z_1 \neq 0$ then

$$|Z| \geq |\Im(Z)| = 2\Big|\sum_{k=1}^{\infty} \frac{z_{2k-1}}{(-4)^k}\Big| \geq \frac{2}{12} = 2{\cdot}(0{\bullet}\overline{1}\,2\,\overline{2}\,2\,\overline{2}\cdots).$$

Using the parameters ε, D_min and the oblong I mentioned above, and using the general formulas (10) and (18) for on-line multiplication, we obtain the delay $\delta = 9$ and the number $L = 7$ of fractional digits of W to evaluate. However, by a more elaborate calculation, we can decrease these parameters further down to $\delta = 8$ and $L = 6$.

For on-line division, with $K = \max\{|z| : z \in I\} = \frac{\sqrt{146}}{9}$, the general formula (14) provides a delay $\delta = 11$ and the number $L = 11$ of fractional digits of W and D to evaluate.

In summary, the Knuth numeration system enables on-line multiplication and division, with linear time complexity. The preprocessing of divisor is just a shift of the fractional point. We have another 3 digits of W (and $\frac{W}{D}$) before the fractional point to evaluate for on-line multiplication (and division).

C. Eisenstein numeration system

The Eisenstein numeration system has a complex base $\beta = -1+\omega$, where $\omega = \exp\frac{2i\pi}{3}$ is the third root of unity ($\omega^3 = 1$).

It is known that this base β with the alphabet $\mathcal{C} = \{0, 1, 2\}$ forms a numeration system, in which any complex number has a $(\beta, \mathcal{C})$-representation. The same property is true also for other alphabets of cardinality $\#\mathcal{C} = 3$, for example $\mathcal{C} = \{0, 1, -\omega\}$. This follows from Theorem 3.2 in [10].

Nevertheless, we choose to work with a larger, redundant (complex) alphabet $\mathcal{A}$ of size $\#\mathcal{A} = 7$:

$$\mathcal{A} = \{0, \pm 1, \pm\omega, \pm\omega^2\}.$$

This numeration system $(\beta, \mathcal{A})$ has favorable properties:

- $\mathcal{A}$ is both symmetric and closed under multiplication;
- the numeration system $(\beta, \mathcal{A})$ enables parallel addition (a separate result, to be published), and $\#\mathcal{A} = 7$ is the

minimal size of alphabet allowing parallel addition for the Eisenstein base (as proved in [5]);
- there are non-trivial representations of zero in $(\beta, \mathcal{A})$, nevertheless, the preprocessing of divisor D for on-line division is possible (see Example V.8), and ensures that

$$\frac{\sqrt{3}(6-\sqrt{7})}{18} = D_{\min} \leq |D| \leq D_{\max} = \frac{\sqrt{7}}{2}.$$

Due to these properties, the Eisenstein numeration system with alphabet $\mathcal{A} = \{0, \pm 1, \pm\omega, \pm\omega^2\}$ allows on-line multiplication and division with linear time complexity, as shown below.

1) On-line property of the Eisenstein numeration system: For each digit $a \in \mathcal{A}$, we denote the set

$$H_a = \{Z \in \mathbb{C} : |Z - a| \leq |Z - b| \text{ for all } b \in \mathcal{A}, b \neq a\}.$$

The sets H_a for $a \neq 0$ are unbounded, while the set H_0 is the regular hexagon with center in point zero and vertices $\pm\omega^k(\frac{1}{2} + \imath\frac{\sqrt{3}}{6})$. It can be easily verified that $r = \frac{\sqrt{3}}{6}$ is the maximum possible value $r > 0$ such that

$$(\beta H_0)^r \subset \bigcup_{a \in \mathcal{A}} (H_0 + a).$$

We work with the following Digit function:

$$\mathrm{Digit}(V) = a \quad \Rightarrow \quad V \in H_a.$$

Using the parameter $r = \frac{\sqrt{3}}{6}$, we can set $\varepsilon > 0$ as $\varepsilon = \frac{r}{|\beta|+1} = \frac{3-\sqrt{3}}{12}$, and thereby fulfill the (OL) Property with the set $I = (H_0)^\varepsilon$, see Fig.IV-B. Nevertheless, we modify our approach, in order to optimize values for the delay δ and the number L of fractional digits of arguments to evaluate in the function(s) Select.

2) On-line multiplication in the Eisenstein numeration system: We consider two parameters $\mu, \nu > 0$ such that

$$\sqrt{3}\mu + \nu = |\beta|\mu + \nu \leq r = \frac{\sqrt{3}}{6}. \qquad (21)$$

The selection function for multiplication is defined by

$$\begin{aligned} \mathrm{Select}_{\mathrm{M}} &: (\beta H_0)^r \mapsto \mathcal{A} \\ \mathrm{Select}_{\mathrm{M}}(W) &= \mathrm{Digit}(\mathrm{Trunc}_{\mu/2}(W)), \text{ implying} \\ W - \mathrm{Select}_{\mathrm{M}}(W) &\in (H_0)^\mu \quad \text{for any} \quad W \in (\beta H_0)^r. \end{aligned}$$

From the formulas and requirements above, we find two reasonable combinations of the parameters L and δ in the algorithm of on-line multiplication in Eisenstein numeration system:

- $(\delta_{\min}, L) = (5, 7)$, with minimized delay δ; and
- $(\delta, L_{\min}) = (6, 6)$, where the parameter L is minimized.

3) On-line division in the Eisenstein numeration system: When specifying the algorithm for on-line division, we use again the general formula (14). The Trunc function provides partial evaluations $V = \mathrm{Trunc}_\alpha(W)$ and $\Delta = \mathrm{Trunc}_\alpha(D)$, where the parameter $\alpha > 0$ is set so that $\left|\frac{W}{D} - \frac{V}{\Delta}\right| \leq \frac{\mu}{2}$.

It can be shown that

$$\frac{W}{D} \in (\beta H_0)^\nu \quad \text{implies} \quad \frac{W_{\mathrm{new}}}{D_{\mathrm{new}}} \in (\beta H_0)^\nu$$

provided that parameters $\mu, \nu > 0$ fulfill the inequality (21). The inequalities translating the relations between parameters μ, ν and the desired results δ and L are somewhat more laborious here than in the case of on-line multiplication, but we can obtain two reasonable combinations of the parameters L and δ in the on-line division algorithm for the Eisenstein numeration system:

- $(\delta_{\min}, L) = (7, 10)$, where the delay δ is minimized; and
- $(\delta, L_{\min}) = (10, 9)$, where the parameter L is minimized.

VII. Conclusion

The algorithms of Trivedi and Ercegovac for on-line multiplication and division, originally introduced for integer numeration systems (β, A), can be extended to real or complex systems as well, provided that $(\beta, \mathcal{A})$ has the (OL) Property. Investigating the (OL) Property and defining the preprocessing rules for a given system $(\beta, \mathcal{A})$ remains an open problem, particularly if we want to use a digit set $\mathcal{A}$ minimal in size.

Acknowledgment

The second author acknowledges support of ANR/FWF project "FAN". The third and fourth authors acknowledge support of GAČR 13-03538S, the first author support of SGS 11/162/OHK4/3T/14.

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