Rational base number systems for p-adic numbers

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Abstract. This paper deals with rational base number systems for p-adic numbers. We mainly focus on the system proposed by Akiyama, Frougny, and Sakarovitch in 2008, but we also show that this system is in some sense isomorphic to other ones. We identify the numbers with finite and eventually periodic representations and we also determine the number of representations of a given p-adic number.

1 Introduction

The field of p-adic numbers, denoted by $\mathbb{Q}_p$, is an extension of the field $\mathbb{Q}$ of rational numbers in a way complementary to the classical extension: the field $\mathbb{R}$ of real numbers. The letter p refers to a prime number p, and so there exist an infinite number of p-adic fields, each corresponding to one prime number. The topological structure of p-adic fields and of the field of real numbers are very different; in fact, one can find the topology of $\mathbb{Q}_p$ very nonintuitive. Nevertheless, it is still reasonable to be concerned with such an unusual construction since, due to the celebrated Ostrowski's theorem from 1918, the p-adic fields and the field of real numbers are in some sense all possible completions of $\mathbb{Q}$. Although Ostrowski's theorem clarified the full significance of p-adic numbers, they have been introduced and systematically studied earlier by Hensel; his first work [3] on this topic is from the year 1897.

There is a rich literature devoted to the p-adic analysis and number theory. Nice historical review and further references can be found in [7], very friendly and accessible introduction to p-adic numbers is [2]. Our aim is to propose a new non-standard way of how to represent p-adic numbers; the standard way is a representation in base p with digits in the alphabet $\mathcal{A}_p = \{0, 1, \ldots, p-1\}$. Every p-adic number has then a unique such representation in the form of a left infinite word over $\mathcal{A}_p$. Another representation was proposed in [5], where it is proved that the p-adic integers (certain subset of them) have a unique infinite representation as a product $\prod_{k=1}^{\infty}(1+b_n p^{r_n})$, where $1 \leq b_n \leq p-1, r_n \in \mathbb{N}$, and $r_{n+1} > r_n$. We will study a possibility of using rational base representation. Our starting point will be the number system introduced by Akiyama, Frougny, and Sakarovitch in [1]. They studied finite representations of the positive integers of the form of $\sum_{k=0}^{m} \frac{a_k}{q} \left(\frac{p}{q}\right)^k$, where $p > q \geq 1$ are coprime integers (p may not be prime!), $m \in \mathbb{N}$, and digits a_k are from the alphabet $\mathcal{A}_p = \{0, 1, \ldots, p-1\}$. We consider representations of the same form but we allow m to tend to infinity. Of course, such a series does not converge in $\mathbb{R}$, but, as we will see, it converges in $\mathbb{Q}_r$ if r is a prime factor of p. Hence, we get a new *non-standard* representation of a certain r-adic number; this representation turns out to be a natural generalization of the standard one.

We will find answers to questions usually connected with various numeration systems:

1. How many representations of a given number exist?
2. Which numbers have a finite representation (i.e., equal to ${}^{\omega}0w$ for some $w \in \mathcal{A}_p^*$)?
3. Which numbers have an (eventually or purely) periodic representation?

2 Fields of p-adic numbers

Definitions and results in this section are taken mainly from [2], where the reader can find also all proofs and further details.

As we mentioned above, the set of p-adic numbers $\mathbb{Q}_p$ is defined as a completion of $\mathbb{Q}$. Any completion must be defined with respect to some metric. Therefore we start with the definition of the *p-adic absolute value:*

Definition 1. *Let p be a prime number. The* p-adic valuation *on $\mathbb{Z}$ is the function $v_p : \mathbb{Z}\setminus\{0\} \to \mathbb{R}$ given by $n = p^{v_p(n)}n'$, with $p \nmid n'$. The extension to the set of rational numbers is as follows: for $x = \frac{a}{b} \in \mathbb{Q}$ we put $v_p(x) = v_p(a) - v_p(b)$.*

And, finally, the p-adic absolute value *on $\mathbb{Q}$ is defined by*

$$|x|_p = \begin{cases} 0 & \text{if } x = 0, \\ p^{-v_p(x)} & \text{otherwise.} \end{cases}$$

One can say that the valuation $v_p(x)$ measures the "divisibility" of x by p. To make it clearer, let us consider several examples: $v_p(p^n) = n$ and so $|p^n|_p = p^{-n}$ and p^n converges to 0; if q is a prime number different from p, then $v_p(q^n) = 0$ and $|q^n|_p = p^{-0} = 1$; if $x = p_1^{a_1} \cdots p_k^{a_k}$, where p_i are prime factors of x, then $|x|_{p_i} = p^{-a_i}$ and $|x|_q = 0$ for all other primes q.

Completion $\mathbb{Q}_p$ of $\mathbb{Q}$ with respect to $|\ |_p$ is then constructed in the usual way, i.e., by "adding all limits of Cauchy sequences in $\mathbb{Q}$", for details see [2, Chapter 3]. So, we have now these completions of $\mathbb{Q}$: the "classical" completion, i.e., the set of real numbers $\mathbb{R}$, and infinitely many completions $\mathbb{Q}_p$, where $p = 2, 3, 5, 7, 11, ...$ is prime. The following theorem states that we have constructed all possible completions, recalling that two absolute values are equivalent if they induce the same topology.

Theorem 2 (Ostrowski)**.** *Every non-trivial absolute value on $\mathbb{Q}$ is equivalent to the classical absolute value $|\ |$ or to one of the absolute values $|\ |_p$, where p is prime.*

The value of $|x|_p$ for all $x \in \mathbb{Q}$ is always equal to p^i for certain $i \in \mathbb{Z}$. This property is preserved even in $\mathbb{Q}_p \setminus \mathbb{Q}$:

Lemma 3. *Let $x \in \mathbb{Q}_p$. Then there exists $i \in \mathbb{Z}$ such that $|x|_p = p^i$.*

Hence, $|x|_p$ can attain only countably many values. Another crucial difference between p-adic absolute value and the classical one is that p-adic absolute value is ultrametric (also non-Archimedean):

Lemma 4. *The p-adic absolute value $|\ |_p$ is* ultrametric*, i.e., for all $x, y \in \mathbb{Q}_p$ the* strong triangular inequality *holds:*

$$|x + y|_p \leq \max\{|x|_p, |y|_p\}.$$

Now, having defined p-adic numbers we can proceed with the definition of the standard representation of p-adic numbers.

2.1 Standard representation of p-adic numbers

Standard and well studied way of how to represent p-adic numbers is the representation in the form of a power series in p.

Theorem 5. *Every $x \in \mathbb{Q}_p$ can be uniquely written as*

$$\begin{aligned} x &= a_{-k_0}p^{-k_0} + \cdots + a_0 + a_1 p + a_2 p^2 + \cdots + a_k p^k + \cdots \\ &= \sum_{k \geq -k_0} a_k p^k \end{aligned}$$

with $a_k \in \mathcal{A}_p$ and $-k_0 = v_p(x)$.

Definition 6. *The left infinite word* $\cdots a_2 a_1 a_0 \bullet a_{-1} \cdots a_{-k_0}, k_0 \in \mathbb{N}$, *over the alphabet* $\mathcal{A}_p$ *such that* $a_{-k_0} > 0$ *or* $k_0 = 0$ *and* $x = \sum_{k \geq -k_0} a_k p^k$ *is denoted by* $< x >_p$ *and called the* p-representation *of* x.

Of course, the infinite sum converges to x only with respect to the p-adic absolute value. There are several ways of how to calculate the word $< x >_p$. The most convenient for our purposes is the following algorithm defined for integers:

Algorithm 7. *Let* $x \in \mathbb{Z}$. *If* $x = 0$, *return the empty word* ϵ. *Otherwise, put* $s_0 = x$ *and for all* $i \in \mathbb{N}$ *define* s_{i+1} *and* $a_i \in \mathcal{A}_p$ *by*

$$s_i = p s_{i+1} + a_i.$$

Return $\mathbf{a} = \cdots a_2 a_1 a_0$.

The algorithm works only for integral x, but it can be easily modified for the rationals.

Definition 8. *Let* p *be a prime number, then the set of* p-adic integers *is* $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\}$.

Algorithm 9. *Let* $x = \frac{s}{t} \in \mathbb{Z}_p \cap \mathbb{Q}$. *Put* $s_0 = s$ *and for all* $i \in \mathbb{N}$ *define* s_{i+1} *and* $a_i \in \mathcal{A}_p$ *by*

$$\frac{s_i}{t} = p \frac{s_{i+1}}{t} + a_i.$$

Return $\mathbf{a} = \cdots a_2 a_1 a_0$.

If t and p are not mutually prime, i.e., $x \notin \mathbb{Z}_p \cap \mathbb{Q}$, multiply x by p until xp^ℓ can be written as $\frac{sp^\ell}{t'}$ with t' co-prime to p. Then apply the algorithm obtaining $< xp^\ell >_p = \cdots a_2 a_2 a_0$. Then, clearly, $< x >_p = \cdots a_{\ell+1} a_\ell \bullet a_{\ell-1} \cdots a_0$. Thus, there is no loss of generality.

Theorem 10. *Let* $x \in \mathbb{Q}_p$. *Then* $< x >_p$ *is*

1. *uniquely given,*
2. *finite if, and only if,* $x \in \mathbb{N}$,
3. *eventually periodic if, and only if,* $x \in \mathbb{Q}$.

3 Rational base number system

In this section we will study the rational base number system proposed by S. Akiyama, C. Frougny, and J. Sakarovitch in [1]. We will show that this system is a natural generalization of the standard way of representing p-adic numbers described in the previous section. In [1] the system is proposed as a new method to represent the non-negative integers in the form of a power series in $\frac{p}{q}$:

$$\sum_{k=0}^{n} \frac{a_k}{q} \left(\frac{p}{q}\right)^k,$$

where $p > q \geq 1$ are co-prime integers and digits a_i are from the alphabet $\mathcal{A}_p$. It is proved there that such a representation is unique and finite and that the language of all such representations is prefix-closed. In fact, it holds that if $w \in \mathcal{A}_p^*$ is a representation of an integer, then there exists at least one $a \in \mathcal{A}_p$ such that wa represents an integer as well. So, if $w = a_n a_{n-1} \cdots a_1 a_0$, we can study $\sum_{k=0}^{n} \frac{a_k}{q} \left(\frac{p}{q}\right)^{-k}$ and get a representation of a rational number. As we have said, w can be always prolonged by at least one letter and remain a representation of an integer. Doing this prolongation repetitively, n approaches to infinity and we can get even irrational numbers. Such infinite representations are then studied in [1] and they turn out to be very interesting and to relate to old and difficult problems of Number Theory; namely, Mahler's problem [6] and Josephus problem [9], [8].

We will take a different approach; we will study also infinite series but containing an infinite number of *positive* powers of $\frac{p}{q}$. It will naturally lead us to the p-adic numbers.

3.1 Modified division algorithm

In what follows we assume that $p > q \geq 1$ are co-prime positive integers (we do *not* assume that p is a prime number!). Let us consider the following algorithm introduced in [1] and named *modified division (MD) algorithm.*

Algorithm 11. *Let x be an integer. If $x = 0$, return the empty word ϵ. Otherwise, put $s_0 = x$ and for all $i \in \mathbb{N}$ define s_{i+1} and $a_i \in \mathcal{A}_p$ by $qs_i = ps_{i+1} + a_i$. Return $\mathbf{a} = \cdots a_2a_1a_0$.*

Clearly, for $q = 1$ we get Algorithm 7. As we will see, these two algorithms share a lot of properties. As well as Algorithm 7, the MD algorithm can be easily modified for rational $x = \frac{s}{t}$, s and t being the lowest terms.

Algorithm 12 (MD algorithm)**.** *Let $x = \frac{s}{t}$, s being an integer and t a positive integer.*

(i) If $s = 0$, return the empty word $\mathbf{a} = \epsilon$.

(ii) If t is co-prime to p, put $s_0 = s$ and for all $i \in \mathbb{N}$ define s_{i+1} and $a_i \in \mathcal{A}_p$ by

$$\frac{qs_i}{t} = \frac{ps_{i+1}}{t} + a_i. \tag{1}$$

Return $\mathbf{a} = \cdots a_2a_1a_0$.

(iii) If t is not mutually prime with p, multiply $\frac{s}{t}$ by $\frac{p}{q}$ until $x\left(\frac{p}{q}\right)^{\ell}$ is of the form $\frac{s'}{t'}$, where t' is co-prime to p. Then apply the algorithm from (ii) returning $\mathbf{a}' = \cdots a_2'a_1'a_0'$. Return $\mathbf{a} = \cdots a_1a_0 \,\mathbf{.}\, a_{-1}a_{-2}\cdots a_{-\ell} = \cdots a_\ell' \,\mathbf{.}\, a_{\ell-1}'\cdots a_2'a_1'$.

Definition 13. *Let $x \in \mathbb{Q}$. The word $\mathbf{a}$ returned by the previous algorithm for x is said to be the* $\frac{1}{q}\frac{p}{q}$-expansion *of x and denoted by $< x >_{\frac{1}{q}\frac{p}{q}}$.*

Lemma 14. *Let $x = \frac{s}{t}$, where $s \neq 0$ and $t > 0$ is co-prime to p. Then for the sequence $(s_i)_{i\geq1}$ from the MD algorithm we have:*

(i) If $s > 0$ and $t = 1$, i.e., $x \in \mathbb{N}$, $(s_i)_{i\geq1}$ is eventually zero.

(ii) If $s > 0$ and $t > 1$, $(s_i)_{i\geq1}$ is either eventually zero or eventually negative.

(iii) If $s < 0$, $(s_i)_{i\geq1}$ is negative.

(iv) For all $i \in \mathbb{N}$, if $s_i < -\frac{p-1}{p-q}t$, then $s_i < s_{i+1}$.

(v) If $-\frac{p-1}{p-q}t \leq s_i < 0$, then $-\frac{p-1}{p-q}t \leq s_{i+1} < 0$.

(vi) $(s_i)_{i\geq1}$ is always bounded and eventually periodic.

(vii) $(s_i)_{i\geq1}$ is eventually zero (resp. eventually periodic) if, and only if, $\mathbf{a}$ is eventually zero (resp. eventually periodic).

The dynamics of the sequence $(s_i)_{i\geq0}$ is symbolically depicted in Figure 1; the interval $[-t\frac{p-1}{p-q}, 0]$ is a sort of attractor and this gives us the following bound for the length of the period of $\frac{1}{q}\frac{p}{q}$-expansions.

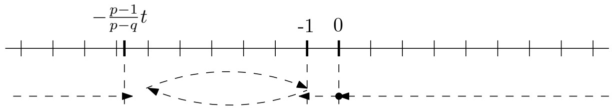

Figure 1: The dynamics of the sequence $(s_i)_{i\geq0}$ from the MD algorithm.

Corollary 15. *Let* $x = \frac{s}{t} \in \mathbb{Q}$*. Then the period of* $< x >_{\frac{1}{q}\frac{p}{q}}$ *is less than* $\lfloor \frac{p-1}{p-q} \rfloor t$*. Moreover, if* $< x >_{\frac{1}{q}\frac{p}{q}}$ *is purely periodic, then* $-1 \le s \le \lfloor \frac{p-1}{p-q} \rfloor t$*.*

Lemma 16. *Let* $x = \frac{s}{t} \in \mathbb{Q}$ *such that its* $\frac{1}{q}\frac{p}{q}$*-expansion* $< x >_{\frac{1}{q}\frac{p}{q}} = \cdots a_{-\ell+1}a_{-\ell}$*,* $\ell \in \mathbb{N}$*, is not finite (i.e., it is not eventually zero). Then*

$$\sum_{k=-\ell}^{\infty} \frac{a_k}{q} \left(\frac{p}{q}\right)^k$$

converges to x *with respect to the* r*-adic absolute value* $| \ |_r$ *if, and only if,* r *is a prime factor of* p*. Moreover, if* i *is the multiplicity*[1] *of* r *in* p*, then for all* $n \ge -\ell$ *we have*

$$\left| x - \sum_{k=-\ell}^{n} \frac{a_k}{q} \left(\frac{p}{q}\right)^k \right|_r \le r^{-i(n+1)}. \tag{2}$$

x	$< x >_{\frac{1}{q}\frac{p}{q}}$	$(s_i)_{i\ge0}$	abs. values
$p = 3, q = 2$			
5	2101	5, 3, 2, 1, 0, 0, ...	all
-5	$^{\omega}$2012	-5, -3, -2, -2, -2, ...	$\| \ \|_3$
11/4	201	11,6, 4, 0, 0, ...	all
11/8	$^{\omega}$1222	11, 2, -4, -8, -8, -8, ...	$\| \ \|_3$
11/5	$^{\omega}$(02)2112	11, 4, 1, -1, -4, -6, -4, -6, ...	$\| \ \|_3$
$p = 30, q = 11$			
5	11 25	5, 1, 0, 0, ...	all
-5	$^{\omega}$19 8 5	-5, -2, -1, -1,...	$\| \ \|_2, \| \ \|_3, \| \ \|_5$
11/7	$^{\omega}$(12 21 5) 23 13	11, 1, -5, -3, -6, -5, ...	$\| \ \|_2, \| \ \|_3, \| \ \|_5$

Table 1: Examples of $< x >_{\frac{1}{q}\frac{p}{q}}$. The last column contains the absolute values for which the $\frac{1}{q}\frac{p}{q}$-expansion from the second column converges to x (in terms of Lemma 16).

3.2 $\frac{1}{q}\frac{p}{q}$-expansions of the negative integers

The case of $\frac{1}{q}\frac{p}{q}$-expansions of the positive integers has been already studied in [1]. In the present subsection we will study the case of the negative integers.

Definition 17. *Let* $\mathbf{a} = \cdots a_{-\ell+1}a_{-\ell}$*,* $\ell \in \mathbb{N}$ *be an eventually periodic word over* $\mathcal{A}_p$*. The* evaluation map π *is defined by:*

$$\pi(\cdots a_{-\ell+1}a_{-\ell}) = x \quad \text{if, and only if, } < x >_{\frac{1}{q}\frac{p}{q}} = \cdots a_{-\ell+1}a_{-\ell}.$$

Lemma 18.

(i) If $\pi(\cdots a_2a_1a_0) \in \mathbb{Z}$*, then* $\pi(\cdots a_2a_1) \in \mathbb{Z}$*.*

(ii) If $x = \pi(\cdots a_2a_1a_0)$ *is a negative integer, then there exists* $a \in \mathcal{A}_p$ *such that* $\pi(\cdots a_2a_1a_0a)$ *is also a negative integer. Moreover,*

$$\min\{\pi(\cdots a_2a_1a_0a) \mid \pi(\cdots a_2a_1a_0a) \in \mathbb{Z}, a \in \mathcal{A}_p\} = \left\lceil x\frac{p}{q} \right\rceil \tag{3}$$

$$\max\{\pi(\cdots a_2a_1a_0a) \mid \pi(\cdots a_2a_1a_0a) \in \mathbb{Z}, a \in \mathcal{A}_p\} = \left\lfloor \frac{1}{q}(px + p - 1) \right\rfloor \tag{4}$$

[1] This means that i is the greatest integer such that r^i divides p.

In words, the set of $\frac{1}{q}\frac{p}{q}$-expansions of all negative integers is prefix-closed and all its elements are extendable to the right. Moreover, the $\frac{1}{q}\frac{p}{q}$-expansion of a negative integer is eventually periodic with period 1:

Proposition 19. *Let k be a positive integer. Denote $B = \left\lfloor \frac{p-1}{p-q} \right\rfloor$, then:*

(i) if $k \leq B$, then $< -k >_{\frac{1}{q}\frac{p}{q}} = {}^{\omega}b$ with $b = k(p-q)$,

(ii) otherwise, $< -k >_{\frac{1}{q}\frac{p}{q}} = {}^{\omega}bw$ with $w \in \mathcal{A}_p^+$ and $b = B(p-q)$.

3.2.1 Trees $T_{\frac{1}{q}\frac{p}{q}}$ and $\overline{T}_{\frac{1}{q}\frac{p}{q}}$

In [1] the language of $\frac{1}{q}\frac{p}{q}$-expansions of all positive integers is studied. It is proved there, among other properties of this language, that it is prefix-closed and extendable to the right. Thus, it is quite natural to represent the language as a tree with infinite branches. We first recall the results for the case of positive integers and then propose their analogues for the negative case.

Lemma 20. *Define the language*

$$L_{\frac{1}{q}\frac{p}{q}} = \{w \in \mathcal{A}_p^* \mid w \text{ is the } \frac{1}{q}\frac{p}{q}\text{-expansion of some } s \in \mathbb{N}\}.$$

The language $L_{\frac{1}{q}\frac{p}{q}}$ is prefix-closed, extendable to the right, and not context-free (if $q \neq 1$).

The proof is a direct consequence of the Pumping lemma and it can be found in [1].

Definition 21. *The tree $T_{\frac{1}{q}\frac{p}{q}}$ has the non-negative integers as nodes and the directed edges are labeled by letters from $\mathcal{A}_p$. Furthermore:*

(i) 0 is the root of the tree,

(ii) there is an edge from node n_1 to node n_2 with label a if $n_1 = \pi({}^{\omega}0w)$ for some $w \in L_{\frac{1}{q}\frac{p}{q}}$ and $n_2 = \pi({}^{\omega}0wa)$.

Tree $T_{\frac{1}{q}\frac{p}{q}}$ (surrounded by the dashed line) for $p = 3, q = 2$, is depicted in Figure 2.

It is reasonable to ask which non-negative integer is the least one with w of length n. Denote such an integer by G_n: sure $G_0 = 0$ and $G_1 = 1$. The children in the tree $T_{\frac{1}{q}\frac{p}{q}}$ of node n are given by the condition $\frac{1}{q}(pn + a) \in \mathbb{N}$, obviously, the least such integer is $\left\lceil \frac{p}{q}n \right\rceil$.

Lemma 22. *The least non-negative integer with $\frac{1}{q}\frac{p}{q}$-expansion of length $n \in \mathbb{N}$ is G_n, where*

$$G_0 = 0, \quad G_1 = 1, \quad G_{n+1} = \left\lceil \frac{p}{q} G_n \right\rceil.$$

We now propose equivalent objects for the negative integers. The language now reads

$$\overline{L}_{\frac{1}{q}\frac{p}{q}} = \{w \in \mathcal{A}_p^* \mid {}^{\omega}bw = < s >_{\frac{1}{q}\frac{p}{q}}, s \leq -B, \text{first letter of } w \neq b\}.$$

Clearly, the letter b is equal to $B(p-q)$. Using absolutely the same techniques, one can prove Lemma 20 even for the language $\overline{L}_{\frac{1}{q}\frac{p}{q}}$.

Since both languages have the same properties, $\overline{L}_{\frac{1}{q}\frac{p}{q}}$ can be also represented by a tree. Only difference is that the root of $\overline{T}_{\frac{1}{q}\frac{p}{q}}$ is $-B$. Tree $\overline{T}_{\frac{1}{q}\frac{p}{q}}$ (surrounded by the dotted line) for $p = 3, q = 2$ is depicted in Figure 2 as well.

Again, we can ask which integer $\leq -B$ is the least one with $\frac{1}{q}\frac{p}{q}$-expansion ${}^{\omega}bw$ with $w \in \overline{L}_{\frac{1}{q}\frac{p}{q}}$ of length n. Using the same reasoning as in the case of positive integers, we get:

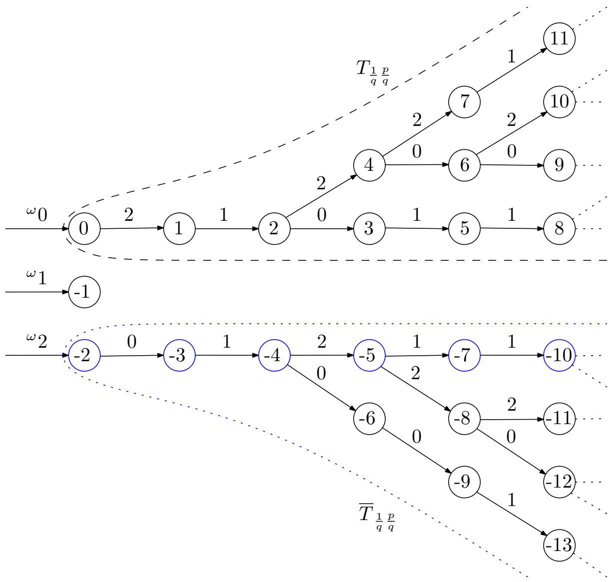

Figure 2: The graph containing $\frac{1}{q}\frac{p}{q}$-expansions of all integers for $p = 3, q = 2$. The dashed line surrounds tree $T_{\frac{1}{q}\frac{p}{q}}$ and the dotted line tree $\overline{T}_{\frac{1}{q}\frac{p}{q}}$.

Lemma 23. *The least negative integer with $\frac{1}{q}\frac{p}{q}$-expansion $^{\omega}bw$ with $b = B(p-q)$ and $w \in \overline{L}_{\frac{1}{q}\frac{p}{q}}$ of length $n \in \mathbb{N}$ is $\overline{G}_n$, where*

$$\overline{G}_0 = -B, \quad \overline{G}_{n+1} = \left\lceil \frac{p}{q}\overline{G}_n \right\rceil.$$

In Figure 2 we see that the trees $T_{\frac{1}{q}\frac{p}{q}}$ and $\overline{T}_{\frac{1}{q}\frac{p}{q}}$ are isomorphic in the case of $p = 3, q = 2$. But this is not true in general:

Proposition 24. *The trees $T_{\frac{1}{q}\frac{p}{q}}$ and $\overline{T}_{\frac{1}{q}\frac{p}{q}}$ are isomorphic if, and only if, $\frac{p-1}{p-q}$ is an integer.*

3.3 Finite $\frac{1}{q}\frac{p}{q}$-expansion

If x has a finite $\frac{1}{q}\frac{p}{q}$-expansion of length $m+1$, i.e.,

$$x = \sum_{k=0}^{m} \frac{a_k}{q}\left(\frac{p}{q}\right)^k,$$

then it equals $\frac{s}{q^{m+1}}$ for some $s \geq 1$. But not all numbers of this form have a finite $\frac{1}{q}\frac{p}{q}$-expansion, e.g., $x = 11/8 = 11/2^3$ from Table 1 has an eventually periodic representation $^{\omega}1222$. In order to better understand this, we introduce an alternative algorithm computing $\frac{1}{q}\frac{p}{q}$-expansion of the numbers of this form.

Algorithm 25. *Let $x = \frac{s}{q^m}$, s, m positive integers. Put $h_0 := s$ and h_{i+1} and $b_i \in \mathcal{A}_p$ define as follows: for $i = 0, 1, \ldots, m-1$ by*

$$\frac{h_i}{q^{m-(i+1)}} = p\frac{h_{i+1}}{q^{m-(i+1)}} + b_i,$$

for $i = m, m+1, \ldots$ by $qh_i = ph_{i+1} + b_i$. Return $\mathbf{b} = \cdots b_2 b_1 b_0$.

We have $\mathbf{b} =< x >_{\frac{1}{q}\frac{p}{q}}$. The alternative algorithm eventually coincides with the MD algorithm for an integer (after m steps we get the MD algorithm for integer h_m). Since we know that the $\frac{1}{q}\frac{p}{q}$-expansion of an integer is finite if, and only if, the integer is nonnegative, the representation of $x = \frac{s}{q^m}$ is finite if, and only if, h_m is nonnegative. This is the idea of the proof of the following result.

Proposition 26. *Let $q > 1$. Define for all positive integers m the set*

$$INF(m) := \left\{ i \,\middle|\, i > 0, \left\langle \frac{i}{q^m} \right\rangle_{\frac{1}{q}\frac{p}{q}} \text{ is infinite} \right\}.$$

Then

$$INF(1) = \emptyset \quad \text{and} \quad INF(m) = A(m) \cup B(m), \quad m = 2, 3, \ldots,$$

where

$$\begin{aligned} A(m) &= \left\{ -kp + aq^{m-1} \mid k > 1, a \in \mathcal{A}_p \right\} \cap \mathbb{N} \\ B(m) &= \left\{ pk + aq^{m-1} \mid k \in INF(m-1), a \in \mathcal{A}_p \right\}. \end{aligned}$$

The definition of INF(m) is rather tricky and, as one can see from the following examples, even the structure is very irregular.

Example 27. *Let $p = 3, q = 2$. Then $INF(1) = \emptyset$ and*

$$\begin{aligned} INF(2) &= \{-3 + 2 * 2\} = \{1\}, \quad \text{indeed } \frac{1}{4} \text{ has an infinite representation,} \\ INF(3) &= \{-6 + 2 * 4, -3 + 1 * 4, -3 + 2 * 4\} \cup \{1 * 3 + 0 * 4, 1 * 3 + 1 * 4, 1 * 3 + 2 * 4\} \\ &= \{2, 1, 5, 3, 7, 11\}, \\ INF(4) &= \{1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 25, 29, 31, 33, 37, 41, 49\}. \end{aligned}$$

3.4 $\frac{1}{q}\frac{p}{q}$-representation of r-adic numbers

Within this subsection letters $r, r_1, r_2, \ldots$ will stand for prime numbers, p will be then a general integer greater than one. So far, we have been concerned with representation of rational numbers in $\mathbb{Q}_r$ in the form of

$$\sum \frac{a_k}{q} \left(\frac{p}{q} \right)^k \tag{5}$$

with $p > q \geq 1$ co-prime integers. We have shown that there exists at least one such representation for all rational numbers, namely the $\frac{1}{q}\frac{p}{q}$-expansion obtained by the MD algorithm, provided that r is a prime factor of p. Is this representation the only one of this type? Does it exist even for non-rational r-adic numbers?

Definition 28. *A left infinite word $\cdots a_{-\ell_0+1} a_{-\ell_0}, \ell_0 \in \mathbb{N}$, over $\mathcal{A}_p$ is a* $\frac{1}{q}\frac{p}{q}$-representation *of $x \in \mathbb{Q}_r$ if $a_{-\ell_0} > 0$ or $\ell_0 = 0$ and*

$$x = \sum_{k=-\ell_0}^{\infty} \frac{a_k}{q} \left(\frac{p}{q} \right)^k$$

with respect to $| \, |_r$.

Theorem 29. *Let r be a prime factor of p with multiplicity i and let $x \in \mathbb{Z}_r$.*

(i) If p is not a power of r, then there exist uncountably many $\frac{1}{q}\frac{p}{q}$-representations $\mathbf{a} = \cdots a_2 a_1 a_0$ such that for all $n \in \mathbb{N}$:

$$\left| x - \sum_{k=0}^{n} \frac{a_k}{q} \left(\frac{p}{q} \right)^k \right|_r \leq r^{-(n+1)i}. \tag{6}$$

(ii) If p is a power of r, there exists a unique $\frac{1}{q}\frac{p}{q}$-representation satisfying (6).

Moreover, it holds that any $\frac{1}{q}\frac{p}{q}$-representation must satisfy (6).

Theorem 30 provides an answer to the question on uniqueness of a representation and also characterizes all representations of $x \in \mathbb{Q}_r$ which converge to x with respect to $|\ |_r$. We have seen that for rational x, which is an element of $\mathbb{Q}_r$ for all r prime, the $\frac{1}{q}\frac{p}{q}$-expansion $< x >_{\frac{1}{q}\frac{p}{q}}$ converges with respect to all absolute values $|\ |_r, r$ a prime factor of p. So it seems reasonable to study $\frac{1}{q}\frac{p}{q}$-representations which represent a rational x in $\mathbb{Q}_r$ for all r from any nonempty subset of prime factors of p.

Theorem 30. *Let $r_1, \dots, r_k$ be all prime factors of p. Let $\{i_1, \dots, i_\ell\}$ be a subset of $\{1, \dots, k\}$ and let $x = \frac{s}{t} \in \mathbb{Q}$. Then:*

- *(i) If $\ell < k$, there exist uncountably many $\frac{1}{q}\frac{p}{q}$-representations of x converging with respect to $|\ |_{r_{i_j}}, j = 1, \dots, \ell$.*
- *(ii) There exists a unique $\frac{1}{q}\frac{p}{q}$-representation of x converging with respect to all $|\ |_{r_j}, j = 1, \dots, k$, namely, the $\frac{1}{q}\frac{p}{q}$-expansion.*

Example 31. *Let $p = 30, q = 11, r_{i_1} = 2, r_{i_2} = 3$. The following are aperiodic $\frac{1}{q}\frac{p}{q}$-representations of 1 in both fields $\mathbb{Q}_2$ and $\mathbb{Q}_3$:*

$$\cdots 27\ 24\ 24\ 29\ 26\ 29\ 27\ 25\ 25\ 24\ 28\ 24\ 28\ 27\ 29$$
$$\cdots 20\ 22\ 21\ 22\ 22\ 22\ 19\ 18\ 18\ 19\ 23\ 18\ 22\ 22\ 23 \quad .$$

3.5 Periodicity

In the preceding section we have answered the first two of the three questions asked in the introduction of this chapter: see Theorem 29 and Proposition 26. In this subsection we will find an answer to the last one. It turns out that the $\frac{1}{q}\frac{p}{q}$-expansion $< x >_{\frac{1}{q}\frac{p}{q}}$ of a rational x plays an important role between all $\frac{1}{q}\frac{p}{q}$-representations of x not only because it is the only one which converges in all $\mathbb{Q}_r, r$ a prime factor of p, but also because it is the only one which is eventually periodic:

Proposition 32. *Let $x \in \mathbb{Q}_{r_i}, r_i > 1$ a prime factor of $p = r_1^{\ell_1} \cdots r_k^{\ell_k}$. Then the $\frac{1}{q}\frac{p}{q}$-representation $\mathbf{a}$ of x is eventually periodic if, and only if, $x \in \mathbb{Q}$ and $\mathbf{a} =< x >_{\frac{1}{q}\frac{p}{q}}$.*

4 Relation with other numeration systems

4.1 $\frac{p}{q}$-representations

So far, we have studied $\frac{1}{q}\frac{p}{q}$-representations of the form of power series in $\frac{p}{q}$ with digits divided by q, i.e., of the form of $\sum \frac{a_k}{q}\left(\frac{p}{q}\right)^k$. What if we do not require the digits to be divided by q? The resulting representation would even more resemble usual positional numeration systems. The answer is that almost nothing changes for such a system. To show this, let us start with the MD algorithm.

The key step in the MD algorithm, which determines the form of the obtained representation, is this:

$$\frac{qs_i}{t} = \frac{ps_{i+1}}{t} + a_i, \quad a_i \in \mathcal{A}_p.$$

But if we replace this relation with

$$\frac{qs_i}{t} = \frac{ps_{i+1}}{t} + qa_i, \quad a_i \in \mathcal{A}_p, \tag{7}$$

and if we still assume $p > q \geq 1, t > 0$ mutually prime integers, the resulting representation will be again a power series in $\frac{p}{q}$ but *without* q dividing the digits, i.e., a sum of the type of $\sum a_k \left(\frac{p}{q}\right)^k$. Using the same reasoning as for the MD algorithm we can prove that the sum, if infinite, converges to x with respect to $|\ |_r$ if, and only if, r is a prime factor of p. As this is a perfect analogue of $\frac{1}{q}\frac{p}{q}$-expansion, let us call it $\frac{p}{q}$-*expansion* and denote by $< x >_{\frac{p}{q}}$.

Having this analogue of $\frac{1}{q}\frac{p}{q}$-expansion, we can go even further and define an analogue of the more general notion of $\frac{1}{q}\frac{p}{q}$-representation: for $x \in \mathbb{Q}_r$, any word $\mathbf{a} = \cdots a_{-k_0+1}a_{-k_0}, k_0 \in \mathbb{N}$, over $\mathcal{A}_p$, such that

$$x = \sum_{k=-k_0}^{\infty} a_k \left(\frac{p}{q}\right)^k$$

with respect to $|\ |_r$, is said to be a $\frac{p}{q}$-*representation* of x in $\mathbb{Q}_r$.

As for the questions on periodicity and the number of representations, it is again true that there is almost no difference between $\frac{1}{q}\frac{p}{q}$- and $\frac{p}{q}$-representations.

Theorem 33. *There exists a finite right sequential transducer $\mathcal{C}$ converting $\frac{p}{q}$-representation of any $x \in \mathbb{Z}_r, r$ prime factor of p, to its $\frac{1}{q}\frac{p}{q}$-representation; the inverse of $\mathcal{C}$ is also a finite right sequential transducer.*

The convertor $\mathcal{C}$ for $p = 3$ and $q = 2$ is in Figure 3.

This theorem says that there is a one-to-one mapping between the sets of all $\frac{p}{q}$- and $\frac{1}{q}\frac{p}{q}$-representations of a given $x \in \mathbb{Z}_r$. This mapping, moreover, preserves eventual periodicity, meaning that $\frac{p}{q}$-expansions are mapped to $\frac{1}{q}\frac{p}{q}$-expansions:

Corollary 34. *A $\frac{1}{q}\frac{p}{q}$-representation is eventually periodic if, and only if, the output by $\mathcal{C}$ is eventually periodic.*

This is not a surprising result as it is still true that only rational numbers can have an eventually periodic $\frac{p}{q}$-representation.

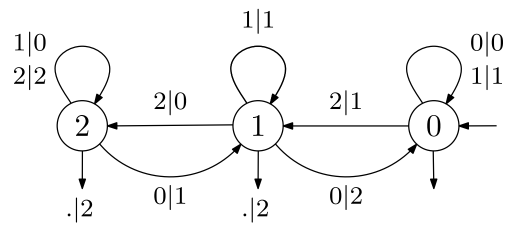

Figure 3: Converter from $\frac{p}{q}$-representations to $\frac{1}{q}\frac{p}{q}$-representations for $p = 3, q = 2$.

4.2 Conversion from the integer base system

As we have seen, there exists a finite convertor between $\frac{p}{q}$- and $\frac{1}{q}\frac{p}{q}$-representation. Another natural question is whether there exists a convertor of p-representations in integer base p to $\frac{1}{q}\frac{p}{q}$-representations. The answer is positive, but the convertor is not finite but is realized by an infinite on-line algorithm. If there was a finite convertor, it would convert non-regular language $\mathrm{L}_{\frac{1}{q}\frac{p}{q}}$ to $\mathcal{A}_p^*$ which is not possible.

Definition 35. *Let $\varphi : {}^{\mathbb{N}}\mathcal{A} \to {}^{\mathbb{N}}\mathcal{B}, \mathcal{A}, \mathcal{B}$ some alphabets. Then φ is a (right)* on-line function *if for any $N \geq 0$ there exists $\phi_N : \mathcal{A}^{N+1} \to \mathcal{B}$ such that $b_N = \phi_N(a_N \cdots a_0)$ for any $\cdots a_1a_0 \in {}^{\mathbb{N}}\mathcal{A}$ such that $\cdots b_1b_0 = \varphi(\cdots a_1a_0)$.*

It holds in general that any on-line function is Lipschitz[2] and so uniformly continuous. Obviously, the following algorithm is on-line.

Algorithm 36.
Input: $\cdots a_1 a_0 \in {}^{\mathbb{N}}\mathcal{A}_p$
Output: $\cdots b_1 b_0 \in {}^{\mathbb{N}}\mathcal{A}_p$
The rewriting rule is defined by: $z_0 = 0, i = 0$ *and*

$$(z_i, i) \xrightarrow{a_i|b_i} (z_{i+1}, i+1),$$

with $a_i, b_i \in \mathcal{A}_p$ *such that*

$$a_i q^i + z_i = \frac{b_i}{q} + \frac{p}{q} z_{i+1}.$$

Clearly, z_i is always nonnegative and uniquely given.

Lemma 37. *Let r be a prime factor of p and let $\cdots a_1 a_0 \in {}^{\mathbb{N}}\mathcal{A}_p$ such that*

$$x = \sum_{i=0}^{\infty} a_i p^i \in \mathbb{Z}_r.$$

Then for the output $\cdots b_1 b_0 \in {}^{\mathbb{N}}\mathcal{A}_p$ of Algorithm 36 we have

$$x = \sum_{i=0}^{\infty} \frac{b_i}{q} \left(\frac{p}{q}\right)^i \in \mathbb{Z}_r.$$

Lemma 38. *If the input of Algorithm 36 is finite (i.e., eventually zero), then the output is finite as well.*

The input is finite if it is a representation of a non-negative integer in base p. This implies that the lemma cannot be reversed since, as we know, there are finite outputs obtained for infinite inputs (a trivial example is the representations of $\frac{p}{q}$).

4.3 Negative base number systems

So far we have mentioned three representations: two with rational base, namely $\frac{1}{q}\frac{p}{q}$- and $\frac{p}{q}$-representations and one integer base p-representation. For each of these one can define its negative base version, let us denote them as $\frac{1}{q}(-\frac{p}{q})$- ,$(-\frac{p}{q})$- and $(-p)$-representations. They are representations of the form of

$$\sum \frac{a_i}{q}\left(-\frac{p}{q}\right)^i, \quad \sum a_i \left(-\frac{p}{q}\right)^i, \quad \text{and} \sum a_i(-p)^i,$$

respectively. As in the positive case, there is an analogue of the MD algorithm computing representations (expansions, more precisely) of rational numbers. The key step for $\frac{1}{q}(-\frac{p}{q})$-representations reads

$$q\frac{s_i}{t} = -p\frac{s_{i+1}}{t} + a_i$$

and for $(-\frac{p}{q})$-representations

$$q\frac{s_i}{t} = -p\frac{s_{i+1}}{t} + a_i q.$$

The $(-p)$-representations are then obtained for $q = 1$.

Of course, in these negative base systems even some negative rational numbers can have finite representation, but in some sense the negative base systems are still isomorphic to their positive

[2] For the usual distance on ${}^{\mathbb{N}}\mathcal{A}$: for $\mathbf{a}, \mathbf{b} \in {}^{\mathbb{N}}\mathcal{A}$ we define $\mathrm{d}(\mathbf{a}, \mathbf{b}) = 2^{-i}$ with $i = \min\{j \in \mathbb{N} \mid a_j \neq b_j\}$.

base analogues. It is because that there are finite convertors between them as in the case of $\frac{1}{q}\frac{p}{q}$- and $\frac{p}{q}$-representations (Theorem 33). Possibilities of conversions between all mentioned number systems are visualized in Figure 4. Since in the negative base number systems even the negative integers have a unique finite representation, these systems are *canonical number systems* [4]. This means exactly that each $x \in \mathbb{Z}$ has a unique finite representation.

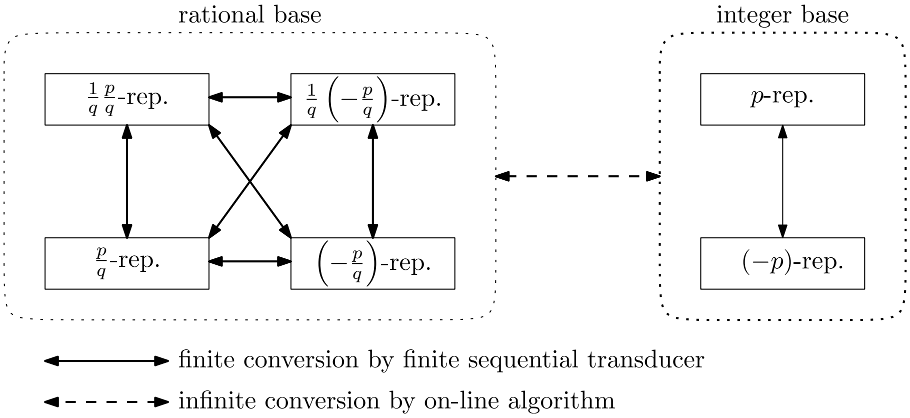

Figure 4: Converters between rational and integer base number systems.

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References

[1] S. Akiyama, C. Frougny, and J. Sakarovitch. Powers of rationals modulo 1 and rational base number systems. *Israel J. Math*, 168:53–91, 2008.

[2] F. Q. Gouvêa. *p-adic numbers: an introduction.* Universitext. Springer, 1997.

[3] K. Hensel. Über eine neue Begründung der Theorie der algebraischen Zahlen. *Jahresber. Deutsch. Math. Verein*, 6:83–88, 1987.

[4] I. Kátai and J. Szabó. Canonical number systems for complex integers. *Acta Sci. Math. (Szeged)*, 37:255–260, 1975.

[5] A. Knopfmacher and J. Knopfmacher. A product expansion in p-adic and other non-Archimedean fields. *Proc. Amer. Math. Soc.*, 104(4):1031–1035, 1988.

[6] K. Mahler. An unsolved problem on the powers of 3/2. *J. Austral. Math. Soc.*, 8:313–321, 1968.

[7] M. R. Murty. *Introduction to p-adic analytic number theory.* American Mathematical Society, 2002.

[8] A. Odlyzko and H. Wilf. Functional iteration and the josephus problem. *Glasgow Mathematical Journal*, 33:235–240, 1991.

[9] W. J. Robinson. The Josephus problem. *Math. Gazette*, 44:47–52, 1960.