MINIMAL WEIGHT EXPANSIONS IN SOME PISOT BASES

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ABSTRACT. For numeration systems representing real numbers and integers, which are based on Pisot numbers, we study expansions with signed digits which are minimal with respect to the absolute sum of digits. It is proved that these expansions are recognizable by a finite automaton if the base β is the root of a polynomial whose (integer) coefficients satisfy a certain condition (D). When β is the Golden Ratio or the Tribonacci number, the automaton is given explicitly.

1. INTRODUCTION

Let A be a set of (integer) digits and $x = x_1 \cdots x_n$ be a word in A^*. The *absolute sum of digits* of x is $\|x\| = \sum_{j=1}^{n} |x_j|$. The *Hamming weight* of x is the number of non-zero digits in x. Of course, when $A \subseteq \{-1, 0, 1\}$, the two definitions coincide.

Expansions of minimal weight in integer bases β have been studied extensively. When $\beta = 2$, it is known since Booth [5] and Reitwiesner [18] how to obtain such an expansion with the digit set $\{-1, 0, 1\}$. The well-known non-adjacent form (NAF) is a particular expansion of minimal weight with the property that the non-zero digits are isolated. It has many applications to cryptography, see in particular [15, 13, 16]. Other expansions of minimal weight in integer base are studied in [9, 11]. Ergodic properties of signed binary expansions are established in [6].

Recently, the investigation of minimal weight expansions has been extended to the Fibonacci numeration system by Heuberger [10], who gave an equivalent to the NAF.

In this paper, we study expansions in a real base $\beta > 1$ which is not an integer. These expansions have been introduced by Rényi [19] and studied initially by Parry [17]. Any number z in $[0, 1)$ has a so-called *greedy β-expansion* given by the β-transformation τ_β, which relies on a greedy algorithm: let $\tau_\beta(z) = \beta z - \lfloor \beta z \rfloor$ and define, for $j \geq 1$, $x_j = \lfloor \beta \tau_\beta^{j-1}(z) \rfloor$. Then $z = \sum_{j \geq 1} x_j \beta^{-j}$, where the x_j's are integer digits in the canonical alphabet $A_\beta = \{0, 1, \ldots, \lfloor \beta \rfloor\}$. We write $z = \raisebox{0.1ex}{.}x_1 x_2 \cdots$. By shifting, any non-negative real number has a β-expansion. If there exists a k such that $x_j = 0$ for all $j > k$, the expansion is said to be *finite* and we write $z = \raisebox{0.1ex}{.}x_1 x_2 \cdots x_k$.

Denote $\widetilde{A_\beta} = \{-\lfloor \beta \rfloor, \ldots, \lfloor \beta \rfloor\}$ the symmetrized alphabet. A word $x = x_1 \cdots x_n \in \widetilde{A_\beta}^*$ is said to be *β-heavy* if it is not minimal in weight for the absolute sum of digits function, more precisely, if there exists $y = y_\ell \cdots y_r \in \widetilde{A_\beta}^*$ with $\ell \leq 1, r \geq n$, such that

$$\sum_{j=1}^{n} x_j \beta^{-j} = \sum_{j=\ell}^{r} y_j \beta^{-j} \quad \text{and} \quad \sum_{j=\ell}^{r} |y_j| < \sum_{j=1}^{n} |x_j|.$$

We will say that the word y is *β-lighter* than x. A word $y_\ell \cdots y_0 \raisebox{0.1ex}{.} y_1 \cdots y_r$ is called a *signed β-expansion of minimal weight* of the real number z if $z = \sum_{j=\ell}^{r} y_j \beta^{-j}$ and if $y_\ell \cdots y_0 y_1 \cdots y_r$ is not β-heavy. We will identify $y_\ell \cdots y_0 \raisebox{0.1ex}{.} y_1 \cdots y_r$ and $y_\ell \cdots y_0 y_1 \cdots y_r$ whenever the position of the "decimal point" plays no role. A word $x_1 \cdots x_n \in \widetilde{A_\beta}^*$

is said to be *strictly β-heavy* if it is β-heavy and if $x_1 \cdots x_{n-1}$ and $x_2 \cdots x_n$ are not β-heavy.

In the following, we restrict our attention to bases β satisfying the (dominance) condition

$$\text{(D):} \quad \begin{array}{l} \beta > 1 \text{ and } P(\beta) = 0 \text{ for some polynomial} \\ P(X) = X^d - b_1 X^{d-1} - b_2 X^{d-2} - \cdots - b_d \in \mathbb{Z}[X] \text{ with } b_1 > \sum_{j=2}^d |b_j|, \end{array}$$

which was introduced in [2] in connection with questions related to finiteness properties of Pisot numbers, see [8, 12, 1]. Note that every polynomial $P(X)$ of this form has a root $\beta > 1$ except for the trivial cases $X-1$ and X^2-2X+1, and it can be shown that every such number β is necessarily a Pisot number, that is to say, an algebraic integer such that all the other roots of its minimal polynomial are in modulus less than one.

Example 1.1. Here are two classes of numbers β satisfying (D):

(1) If $1 = .t_1 t_2 \cdots t_m = \frac{t_1}{\beta} + \cdots + \frac{t_m}{\beta^m}$ with integers $t_1 \geq t_2 \geq \cdots \geq t_m \geq 1$, then the corresponding polynomial is $P(X) = (X^m - t_1 X^{m-1} - t_2 X^{m-2} \cdots - t_m)(X-1)$.
(2) If $1 = .t_1 t_2 \cdots t_m (t_{m+1})^\omega$ with $t_1 \geq t_2 \geq \cdots \geq t_m > t_{m+1} \geq 1$, then β is a root of $P(X) = (X^{m+1} - t_1 X^m - \cdots - t_{m+1}) - (X^m - t_1 X^{m-1} - \cdots - t_m)$.

Recall that the set of (greedy) β-expansions is recognizable by a finite automaton when β is a Pisot number [4]. In this work, we show that the set of β-heavy words is recognized by a finite automaton if β satisfies (D), and thus that the set of signed β-expansions of minimal weight is recognized by a finite automaton.

We then consider particular bases that have been extensively studied from various points of view. When β is the Golden Ratio, we construct a transducer that gives, for a strictly β-heavy word as input, a β-lighter word as output. From this transducer, we derive the minimal automaton recognizing the set of signed β-expansions of minimal weight. We give a transformation on some interval which provides a signed β-expansion of minimal weight of a given real number. By using this transformation, we show that the expected number of non-zero digits in an expansion of minimal weight of length n is $n/5$. Note that the corresponding value in base 2 expansions is $n/3$, see [3, 6]. We also extend the results to the representation of integers in the Fibonacci numeration system.

Finally, we obtain similar results for the case where β is the so-called *Tribonacci number*, that is to say the dominant root of the polynomial $X^3 - X^2 - X - 1$, and extend the results to the Tribonacci numeration system for the integers. In this case, the expected number of non-zero digits in a signed β-expansion of minimal weight of length n is $n\beta^3/(\beta^5+1) \approx 0.282\,n$.

We have put in the Appendix some proofs which are too long to be included in this extended abstract.

2. Redundancy automaton

All the automata considered in this paper process words from left to right, that is to say, most significant digit first.

Let $\beta > 1$ be a real number, $c \geq \lfloor \beta \rfloor$ a fixed integer, and

$$Z_\beta(c) = \Big\{ z_1 \cdots z_n \;\Big|\; n \geq 1,\ |z_j| \leq c,\ \sum_{j=1}^n z_j \beta^{-j} = 0 \Big\}.$$

We recall a result from [7].

Theorem 2.1. *If β is a Pisot number, then for every $c \geq \lfloor \beta \rfloor$ the set $Z_\beta(c)$ is recognized by a finite automaton.*

For convenience, we quickly explain the construction of the automaton $\mathcal{A}_\beta(c)$ which recognizes $Z_\beta(c)$. The states of $\mathcal{A}_\beta(c)$ are 0 and all $s \in \mathbb{Z}[\beta] \cap (-\frac{c}{\beta-1}, \frac{c}{\beta-1})$ which are accessible from 0 by paths consisting of transitions $s \xrightarrow{e} s'$ with $e \in \{-c, \dots, c\}$ such that $s' = \beta s + e$. The state 0 is both initial and terminal. When β is a Pisot number, then the set of states is finite.

Note that the automaton $\mathcal{A}_\beta(c)$ is symmetric, meaning that if $s \xrightarrow{e} s'$ is a transition, then $-s \xrightarrow{\bar{e}} -s'$ is also a transition, where $\bar{e}$ denotes the signed digit $(-e)$. The automaton $\mathcal{A}_\beta(c)$ is accessible and co-accessible.

From this result one can define the *redundancy automaton* (or transducer) $\mathcal{R}_\beta(A)$ with respect to an alphabet A of integer digits. Write $A - A = \{-c, \dots, c\}$. Each transition $s \xrightarrow{e} s'$ of $\mathcal{A}_\beta(c)$ is replaced in $\mathcal{R}_\beta(A)$ by a set of transitions $s \xrightarrow{(a,b)} s'$, with $(a,b) \in A^2$ and $a - b = e$. As usual, we use the notation $s \xrightarrow{a|b} s'$.

Proposition 2.2. *The redundancy automaton $\mathcal{R}_\beta(A)$ recognizes the set*

$$\Big\{(x_1 \cdots x_n, y_1 \cdots y_n) \in A^* \times A^* \ \Big|\ n \geq 1,\ \sum_{j=1}^{n} x_j \beta^{-j} = \sum_{j=1}^{n} y_j \beta^{-j}\Big\}.$$

If β is a Pisot number, then $\mathcal{R}_\beta(A)$ is finite.

From the redundancy automaton $\mathcal{R}_\beta(A)$, one constructs another automaton $\mathcal{T}_\beta(A)$ with states of the form (s, δ), where s is a state of $\mathcal{R}_\beta(A)$ and $\delta \in \mathbb{Z}$. Transitions are of the form $(s, \delta) \xrightarrow{a|b} (s', \delta')$ if $s \xrightarrow{a|b} s'$ is a transition in $\mathcal{R}_\beta(A)$ and $\delta' = \delta + |b| - |a|$. The initial state is $(0,0)$, and terminal states are of the form $(0, \delta)$ with $\delta < 0$. Of course, this automaton $\mathcal{T}_\beta(A)$ is not finite.

Proposition 2.3. *The automaton $\mathcal{T}_\beta(A)$ recognizes the set*

$$\Big\{(x_1 \cdots x_n, y_1 \cdots y_n) \in A^* \times A^* \ \Big|\ n \geq 1, \sum_{j=1}^{n} x_j \beta^{-j} = \sum_{j=1}^{n} y_j \beta^{-j} \ \textit{and} \ \sum_{j=1}^{n} |y_j| < \sum_{j=1}^{n} |x_j|\Big\}.$$

3. The general case

From now on $A = \widetilde{A_\beta}$, thus $c = 2\lfloor \beta \rfloor$. For shortness, $\mathcal{A}_\beta(c)$ is denoted $\mathcal{A}_\beta$, $\mathcal{R}_\beta(A)$ is denoted $\mathcal{R}_\beta$, and $\mathcal{T}_\beta(A)$ is denoted $\mathcal{T}_\beta$. Furthermore, we will say that a word is heavy if it is β-heavy and the base β is clear.

The following result is shown implicitly in the proof of Theorem 4 in [2].

Proposition 3.1. *Let β satisfy (D), and $x_1 \cdots x_n \in \mathbb{Z}^*$ such that $|\sum_{j=1}^{n} x_j \beta^{-j}| < 1$. Then there exists a word $y_0 \cdots y_m \in \widetilde{A_\beta}^*$ such that $\sum_{j=0}^{m} y_j \beta^{-j} = \sum_{j=1}^{n} x_j \beta^{-j}$ and $\sum_{j=0}^{m} |y_j| \leq \sum_{j=1}^{n} |x_j|$.*

This result implies that, if a word w is heavy, then for any words u and v, uwv is heavy as well. So, to prove that the the set of heavy words is recognized by a finite automaton, it is sufficient to build a finite automaton that recognizes a set K of heavy words such that any heavy word contains a factor belonging to K.

Theorem 3.2. *If β satisfies (D), then the set of β-heavy words is recognized by a finite automaton.*

Proof. Let $x_1 \cdots x_n$ be a heavy word. Then there exists a word $y_\ell \cdots y_r \in \widetilde{A_\beta}^*$ with $\ell \le 1$ and $r \ge n$ such that $\sum_{j=1}^n x_j \beta^{-j} = \sum_{j=\ell}^r y_j \beta^{-j}$ and $\sum_{j=\ell}^r |y_j| < \sum_{j=1}^n |x_j|$. Set $x_j = 0$ for $j \in \{\ell, \ldots, r\} \setminus \{1, \ldots, n\}$. In the automaton $\mathcal{T}_\beta$, there is a path composed of transitions of the form $(s_{j-1}, \delta_{j-1}) \stackrel{x_j|y_j}{\longrightarrow} (s_j, \delta_j)$ for $\ell \le j \le r$ with $s_{\ell-1} = 0$, $\delta_{\ell-1} = 0$, $s_r = 0$, $\delta_r < 0$. Furthermore, we have $s_k = \sum_{j=\ell}^k (x_j - y_j)\beta^{k-j} = \sum_{j=k+1}^r (y_j - x_j)\beta^{k-j}$ for all k, $\ell \le k \le r$.

The rest of the proof consists in showing that only a finite part of the automaton $\mathcal{T}_\beta$ is needed, i.e., that only states (s, δ) with a bounded δ play a role, because a path of transitions as above passing by a δ of large absolute value implies that already a suffix or a prefix of $x_1 \cdots x_n$ is heavy.

Assume first $\delta_k \ge 2\lfloor\beta\rfloor p$ for some $k > p$, where p is the length of the shortest path from 0 to s_k in $\mathcal{A}_\beta$. Let $a_1 \cdots a_p \in \{-2\lfloor\beta\rfloor, \ldots, 2\lfloor\beta\rfloor\}^*$ be the label of such a path, thus $s_k = \sum_{j=1}^p a_j \beta^{p-j}$. Since $\delta_r < 0$, we have $\sum_{j=k+1}^r (|y_j| - |x_j|) < -2\lfloor\beta\rfloor p$. If we set $y'_{k-p+1} \cdots y'_r = (x_{k-p+1} - a_1) \cdots (x_k - a_p) y_{k+1} \cdots y_r$, then

$$\sum_{j=k-p+1}^{r} y'_j \beta^{-j} - \sum_{j=k-p+1}^{n} x_j \beta^{-j} = -\sum_{j=1}^{p} a_j \beta^{p-k-j} + \sum_{j=k+1}^{r} (y_j - x_j)\beta^{-j} = -s_k\beta^{-k} + s_k\beta^{-k} = 0$$

$$\text{and} \quad \sum_{j=k-p+1}^{r} |y'_j| - \sum_{j=k-p+1}^{n} |x_j| \le \sum_{j=1}^{k} |a_j| + \sum_{j=k+1}^{r} (|y_j| - |x_j|) < 2\lfloor\beta\rfloor p - 2\lfloor\beta\rfloor p = 0.$$

Note that y'_j need not be in $\widetilde{A_\beta}$. By Proposition 3.1, there exists $y'' = y''_{k-p} \cdots y''_m \in \widetilde{A_\beta}^*$ with $\sum_{j=k-p}^m y''_j \beta^{-j} = \sum_{j=k-p+1}^r y'_j \beta^{-j}$ and $\sum_{j=k-p}^m |y''_j| \le \sum_{j=k-p+1}^r |y'_j|$. Hence y'' is lighter than $x_{k-p+1} \cdots x_n$, and $x_1 \cdots x_n$ has a proper heavy suffix.

Let P_1 be the maximal length of a shortest path from 0 to s in $\mathcal{A}_\beta$. If we assume, w.l.o.g., that $y_\ell \cdots y_r$ is of minimal weight, then we have $\delta_j \le 2\lfloor\beta\rfloor P_1$ for all $j \le 0$. Therefore the condition $k > p$ in the preceding paragraph is satisfied if $\delta_k > 3\lfloor\beta\rfloor P_1$.

Assume now $\delta_k < -2\lfloor\beta\rfloor p$ for some $k < n - p$, where p is the length of the shortest path from s_k to 0 in $\mathcal{A}_\beta$, and let $a_1 \cdots a_p$ be the label of such a path, i.e., $\beta^p s_k + \sum_{j=1}^p a_j \beta^{p-j} = 0$. If we set $y'_\ell \cdots y'_{k+p} = y_\ell \cdots y_k (x_{k+1} - a_1) \cdots (x_{k+p} - a_p)$, then

$$\sum_{j=\ell}^{k+p} y'_j \beta^{-j} - \sum_{j=1}^{k+p} x_j \beta^{-j} = \sum_{j=\ell}^{k} (y_j - x_j)\beta^{-j} - \sum_{j=1}^{p} a_j \beta^{-k-j} = -s_k\beta^{-k} + s_k\beta^{-k} = 0$$

$$\text{and} \quad \sum_{j=\ell}^{k+p} |y'_j| - \sum_{j=1}^{k+p} |x_j| \le \sum_{j=\ell}^{k} (|y_j| - |x_j|) + \sum_{j=1}^{p} |a_j| \le \delta_k + 2\lfloor\beta\rfloor p < 0.$$

As above, Proposition 3.1 provides a word y'' which is lighter than $x_1 \cdots x_{k+p}$.

Let P_2 be the maximal length of a shortest path from s to 0 in $\mathcal{A}_\beta$. If $\delta_j < -3\lfloor\beta\rfloor P_2$ for some $j \le n$, then we have some $k < n - P_2$ such that $\delta_k < -2\lfloor\beta\rfloor P_2$, and we obtain that $x_1 \cdots x_n$ contains a proper heavy prefix. Since $\delta_n \le \delta_j$ for all $j > n$, we obtain the same result if $\delta_j < -3\lfloor\beta\rfloor P_2$ for some $j > n$.

Note that in each state $(0, \delta)$ there is a loop labelled (a, a) for any $a \in \widetilde{A_\beta}$. We claim that the finite subautomaton $\mathcal{F}$ of $\mathcal{T}_\beta$ restricted to states (s, δ) with $-3\lfloor\beta\rfloor P_2 \leq \delta \leq 2\lfloor\beta\rfloor P_1$ recognizes as inputs all heavy words: take a heavy word w and suppose that it is not the input of a path in $\mathcal{F}$. It is the input of a path in $\mathcal{T}_\beta$, and so there exists on that path a state (s, η), with $\eta \notin \{-3\lfloor\beta\rfloor P_2, \ldots, 2\lfloor\beta\rfloor P_1\}$. As we have seen above, w has a proper prefix or a proper suffix w' which is heavy. The construction can be repeated on w', which is a shorter word, so it stops with a factor of w which is the input of a path in $\mathcal{F}$. Using the loops labelled by (a, a) in each state $(0, \delta)$, the word w is the input of a path in $\mathcal{F}$. □

We recall a general construction which provides Corollary 3.4.

Lemma 3.3. *Let $H \subset A^*$ and let $M = A^* \setminus A^*HA^*$. If H is rational, then so is M.*

Proof. Suppose that H is recognized by a finite automaton $\mathcal{H}$. Let P be the set of strict prefixes of H. We construct the minimal automaton $\mathcal{M}$ of M as follows. The set of states of $\mathcal{M}$ is the quotient $P/_\equiv$ where $p \equiv q$ if p and q arrive at the same set of states in $\mathcal{H}$. Since $\mathcal{H}$ is finite, $P/_\equiv$ is finite. Transitions are defined as follows. Let a be in A. There is a transition $p \xrightarrow{a} q$ if pa is in P and $q = [pa]_\equiv$, or if pa is not in P, $p = uv$ with v in P maximal in length, and $q = [v]_\equiv$. Every state is terminal. □

Corollary 3.4. *If β satisfies (D), then the set of signed β-expansions of minimal weight is recognized by a finite automaton.*

4. The Golden Ratio case

In this section we give explicit constructions for the case where β is the Golden Ratio $\frac{1+\sqrt{5}}{2}$. We have $1 = .11$, hence the condition of Example 1.1 (1) is satisfied, and $\widetilde{A_\beta} = \{-1, 0, 1\}$.

4.1. Signed β-expansions of minimal weight for $\beta = \frac{1+\sqrt{5}}{2}$.

Proposition 4.1. *Let $\beta = \frac{1+\sqrt{5}}{2}$. Then the set of strictly β-heavy words is*

$$H = 1(0100)^*1 \;\cup\; 1(0100)^*0101 \;\cup\; 1(00\bar{1}0)^*\bar{1} \;\cup\; 1(00\bar{1}0)^*0\bar{1}$$
$$\cup\; \bar{1}(0\bar{1}00)^*\bar{1} \;\cup\; \bar{1}(0\bar{1}00)^*0\bar{1}0\bar{1} \;\cup\; \bar{1}(0010)^*1 \;\cup\; \bar{1}(0010)^*01.$$

Proof. Every word $x \in H$ is the input of a path (with plain arrows) labelled by $x|y$ which is recognized by the automaton $\mathcal{S}_\beta$ in Figure 1, which is a part of $\mathcal{T}_\beta$. If the path starts in $(0, 0)$ and ends in $(0, -1)$, we obtain therefore immediately that x is heavy since y is lighter than x. Otherwise, extending the path by dashed arrows provides a word which is lighter than x.

For showing that all other heavy words contain a factor in H, we consider paths in $\mathcal{T}_\beta$ such that the input and the output contain no factor in H, since we may assume that the output is a signed β-expansion of minimal weight. The lexicographically maximal word for the input and the output is thus $1(0100)^\omega$, and it suffices to consider states (s, δ) with $|s| \leq 2 \times .1(0100)^\omega = 4/\sqrt{5} < 2$. The transitions in $\mathcal{T}_\beta$ are given by the redundancy automaton $\mathcal{R}_\beta$ restricted to states s with $|s| < 2$, see Figure 2.

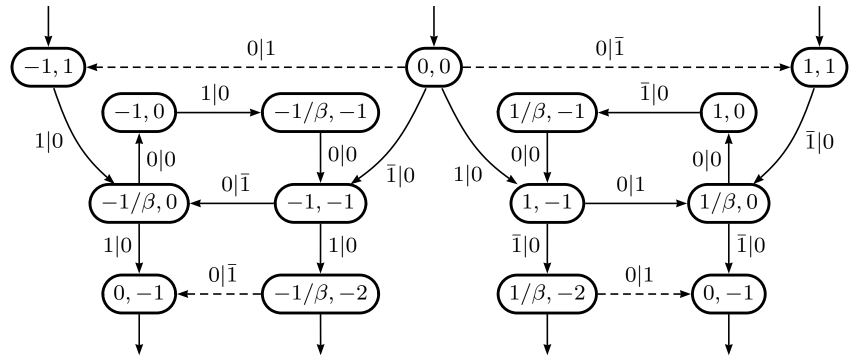

FIGURE 1. Automaton $\mathcal{S}_\beta$ of strictly heavy words for $\beta = \frac{1+\sqrt{5}}{2}$.

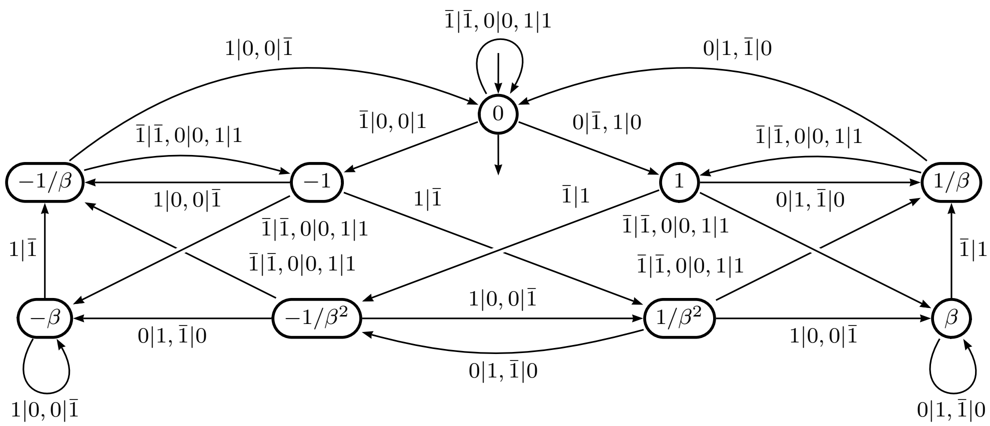

FIGURE 2. Redundancy automaton $\mathcal{R}_\beta$ with states $|s| < 2$ for $\beta = \frac{1+\sqrt{5}}{2}$.

A path in $\mathcal{T}_\beta$ corresponding to a strictly heavy word never passes a state $(0, \delta)$ with $\delta > 0$, since if there is a path of the form

$$(0,0) \stackrel{u|u'}{\longrightarrow} (0,\delta) \stackrel{v|v'}{\longrightarrow} (0,\delta')$$

with $\delta' < 0$, then there exists a path

$$(0,0) \stackrel{u|u}{\longrightarrow} (0,0) \stackrel{v|v'}{\longrightarrow} (0,\delta'-\delta)$$

where u and u', resp. v and v', are words of same length. Thus v is a heavy suffix of uv. Similarly, it is not necessary to consider non-trivial paths from $(0,0)$ to $(0,0)$.

Now consider the outgoing transitions from the state $(1,0)$ which occur in $\mathcal{R}_\beta$.

$\stackrel{\bar{1}|1}{\longrightarrow}$: The only prolongation of $\stackrel{\bar{1}|1}{\longrightarrow}$ avoiding the factors 11 and $1\bar{1}$ in the input and ouput is $\stackrel{0|0}{\longrightarrow}$, but if we have a path

$$(0,0) \stackrel{u|u'}{\longrightarrow} (1,0) \stackrel{\bar{1}|1}{\longrightarrow} (-1/\beta^2, 0) \stackrel{0|0}{\longrightarrow} (-1/\beta, 0) \stackrel{v|v'}{\longrightarrow} (0,\delta),$$

with $\delta < 0$, then there exists as well a path

$$(0,0) \stackrel{u|u'}{\longrightarrow} (0,0) \stackrel{\bar{1}|0}{\longrightarrow} (-1,-1) \stackrel{0|\bar{1}}{\longrightarrow} (-1/\beta, 0) \stackrel{v|v'}{\longrightarrow} (0,\delta),$$

hence $\bar{1}0v$ is a heavy suffix of $u\bar{1}0v$. Therefore a path corresponding to a strictly heavy word cannot contain the transition $(1,0) \stackrel{\bar{1}|1}{\longrightarrow} (-1/\beta^2, 0)$.

- $\stackrel{0|1}{\to}$: Similarly, $(1,0) \stackrel{0|1}{\to} (1/\beta,1)$ is useless, since the prolongation $\stackrel{\bar{1}|0}{\to}$ leads to $(0,0)$ and $\stackrel{0|0}{\to}$ leads to $(1,1)$, which can be reached from $(0,0)$ by the transition $\stackrel{0|\bar{1}}{\to}$.
- $\stackrel{a|a}{\to}$: The only prolongation of $(1,0) \stackrel{0|0}{\to} (\beta,0)$ avoiding two consecutive non-zero digits is $\stackrel{\bar{1}|1}{\to} (1/\beta,0)$. Since we have a path $(0,0) \stackrel{0|\bar{1}}{\to} (1,1) \stackrel{\bar{1}|0}{\to} (1/\beta,0)$, this means that the outgoing transition $\stackrel{0|0}{\to}$ is useless. In general, it is not necessary to consider the transitions $\stackrel{1|1}{\to}$ and $\stackrel{\bar{1}|\bar{1}}{\to}$ if $\stackrel{0|0}{\to}$ is useless.
- $\stackrel{\bar{1}|0}{\to}$: The only outgoing transition from $(1,0)$ which can occur in a path in $\mathcal{T}_\beta$ corresponding to a strictly heavy word is $\stackrel{\bar{1}|0}{\to}$ (which is present in $\mathcal{S}_\beta$).

Every assertion made above is of course valid for the symmetric paths. For the other states in $\mathcal{S}_\beta$, a similar (but less involved) reasoning can be done to show that $\mathcal{S}_\beta$ contains all possible transitions for strictly heavy words.

Finally note that no word in H contains another word in H as a factor. Therefore H contains exactly the strictly heavy words. □

Using the construction given in the proof of Lemma 3.3, with $\mathcal{H}$ being the input automaton of $\mathcal{S}_\beta$, we obtain the following result.

Theorem 4.2. *If $\beta = \frac{1+\sqrt{5}}{2}$, then the set of signed β-expansions of minimal weight is recognized by the finite automaton $\mathcal{M}_\beta$ of Figure 3 where all states are terminal.*

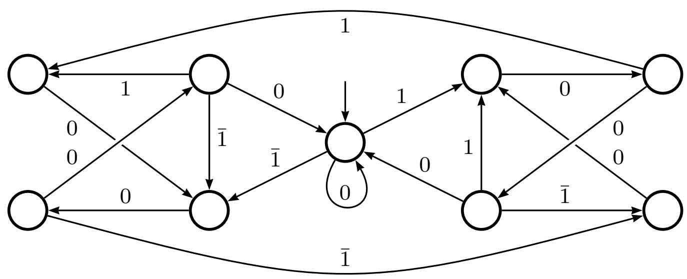

FIGURE 3. Automaton $\mathcal{M}_\beta$ of signed β-expansions of minimal weight, $\beta = \frac{1+\sqrt{5}}{2}$.

4.2. **Weight of the expansions.** In order to determine the expected weight of a signed β-expansion of minimal weight, we consider expansions of a particular shape, similar to the NAF in base 2.

Proposition 4.3. *If $\beta = \frac{\sqrt{5}+1}{2}$, then every $y \in \mathbb{Z}[\beta]$ has a unique expansion in $\widetilde{A_\beta}^*$ avoiding the factors 11, $1\bar{1}$, 101, $10\bar{1}$, $100\bar{1}$ and their opposites. This expansion is a signed β-expansion of minimal weight.*

Proof. To see that every $y \in \mathbb{Z}[\beta]$ has such an expansion, note first that it was shown in [8] that the (greedy) β-expansion of every (positive) $y \in \mathbb{Z}[\beta]$ is finite. Remark that the lexicographically maximal sequence satisfying the desired conditions is $(100)^\omega$ and that $\boldsymbol{.}(100)^\omega = \beta/2$. Therefore, we define a transformation

$$\tau:\ [-\beta/2,\beta/2) \to [-\beta/2,\beta/2), \qquad \tau(x) = \beta x - \lfloor x + 1/2 \rfloor,$$

and set $y_k = \lfloor \tau^{k-1}(y) + 1/2 \rfloor$ for $k \geq 1$ if $y \in [-\beta/2,\beta/2)$, i.e., $y = \boldsymbol{.}y_1 y_2 \cdots$. If $y_k = 1$ for some $k \geq 1$, then we obtain $\tau^k(y) \in \left[\frac{\beta}{2} - 1, \frac{\beta^2}{2} - 1\right) = \left[\frac{-1}{2\beta^2}, \frac{1}{2\beta}\right)$, hence

$y_{k+1} = 0$, $\tau^{k+1}(y) \in \big[\frac{-1}{2\beta}, \frac{1}{2}\big)$, hence $y_{k+2} = 0$, and $\tau^{k+2}(y) \in [-1/2, \beta/2)$, hence $y_{k+3} \in \{0, 1\}$. This shows that the given factors are avoided. A similar argument for $y_k = -1$ shows that the opposites are avoided as well. If $m \in \mathbb{Z}$ is chosen such that $y \in [-\beta^{m+1}/2, \beta^{m+1}/2)$, then an expansion of y is obtained by shifting the expansion of $y\beta^{-m}$. The expansion has minimal weight since it contains no strictly heavy factor.

If we choose $y_k = 0$ in case $\tau^{k-1}(y) > 1/2 = \bullet(010)^\omega$, then it is impossible to avoid the factors 11 and 101 in the following. If we choose $y_k = 1$ in case $\tau^{k-1}(y) < 1/2$, then $\beta\tau^{k-1}(y) - 1 < -1/(2\beta^2) = \bullet 0(00\bar{1})^\omega$, and thus it is impossible to avoid the factors $1\bar{1}$, $10\bar{1}$, $100\bar{1}$, $\bar{1}\bar{1}$ and $\bar{1}0\bar{1}$. Since $\tau^{k-1}(y) \neq 1/2$ for $y \in \mathbb{Z}[\beta]$ and similar relations hold for the opposites, the expansion is unique. $\square$

Remark. Heuberger [10] excluded (for the Fibonacci number system) the factor 1001 instead of $100\bar{1}$. Then the lexicographically maximal sequence is $(1000)^\omega$ and a similar reasoning can be done with the transformation $\tau(x) = \beta x - \big\lfloor \frac{\beta^2+1}{2\beta} x + \frac{1}{2} \big\rfloor$ on $\big[\frac{-\beta^2}{\beta^2+1}, \frac{\beta^2}{\beta^2+1}\big)$. Note that $\beta^2/(\beta^2+1) = \bullet(1000)^\omega$.

Let $\tau(x) = \beta x - \lfloor x + 1/2 \rfloor$ as in the preceding proof and define sets $I_{000} = \big[\frac{-1}{2\beta^2}, \frac{1}{2\beta^2}\big)$, $I_{001} = \big[\frac{-1}{2\beta}, \frac{1}{2\beta}\big) \setminus I_{000}$, $I_{01} = \big[\frac{-1}{2}, \frac{1}{2}\big) \setminus (I_{000} \cup I_{001})$, $I_1 = \big[\frac{-\beta}{2}, \frac{-1}{2}\big) \cup \big[\frac{1}{2}, \frac{\beta}{2}\big)$, which form a partition of $\Omega = [-\beta/2, \beta/2)$. Define a sequence of random variables $(X_k)_{k\geq 0}$ by

$$\Pr[X_0 = j_0, \ldots, X_k = j_k] = \lambda(\{x \in \Omega : \, x \in I_{j_0}, \tau(x) \in I_{j_1}, \ldots, \tau^k(x) \in I_{j_k}\})/\beta$$

for all $j_0 \cdots j_k \in \{000, 001, 01, 1\}^{k+1}$, where λ denotes the Lebesgue measure. Since $\tau(I_j)$ is a union of sets I_i for all $j \in \{000, 001, 01, 1\}$, we have that $\tau^{-1}(I_j)$ is a union of disjoint sets of measure $\lambda(I_j)/\beta$, each of which is contained in one I_i. This implies

$$\begin{aligned}\Pr[X_k = j_k \mid X_{k-1} = j_{k-1}, \ldots, X_0 = j_0] &= \frac{\lambda\big(I_{j_0} \cap \cdots \cap \tau^{-(k-1)}(I_{j_{k-1}}) \cap \tau^{-k}(I_{j_k})\big)}{\lambda\big(I_{j_0} \cap \cdots \cap \tau^{-(k-1)}(I_{j_{k-1}})\big)} \\ &= \frac{\lambda\big(\tau^{-(k-1)}(I_{j_{k-1}}) \cap \tau^{-k}(I_{j_k})\big)}{\lambda\big(\tau^{-(k-1)}(I_{j_{k-1}})\big)} = \Pr[X_k = j_k \mid X_{k-1} = j_{k-1}]\end{aligned}$$

(whenever the first probability is defined), hence the sequence is a Markov chain. Since

$$\Pr[X_k = j_k \mid X_{k-1} = j_{k-1}] = \lambda\big(I_{j_{k-1}} \cap \tau^{-1}(I_{j_k})\big)/\lambda(I_{j_{k-1}}) = \Pr[X_1 = j_k \mid X_0 = j_{k-1}],$$

the Markov chain is homogeneous and its matrix of transition probabilities is

$$(\Pr[X_k = j \mid X_{k-1} = i])_{i,j \in \{000,001,01,1\}} = \begin{pmatrix} 1/\beta & 1/\beta^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2/\beta^2 & 1/\beta^3 & 0 & 0 \end{pmatrix}$$

The stationary distribution vector (given by the left eigenvector to the eigenvalue 1 of the matrix) is $(2/5, 1/5, 1/5, 1/5)$. Since the k-th digit of x is non-zero if and only if $T^{k-1}(x)$ is in I_1, the expected number of non-zero digits in a signed β-expansion of minimal weight of length n is given by $\sum_{k=0}^{n-1} \Pr[X_k = 1] = n/5 + \mathcal{O}(1)$.

4.3. **Fibonacci numeration system.** The reader is referred to [14, Chapter 7] for definitions on numeration systems defined by a sequence of integers. Recall that the linear numeration system canonically associated with the Golden Ratio is the Fibonacci (or Zeckendorf) numeration system defined by the sequence of Fibonacci numbers $F = (F_n)_{n\ge 0}$ with $F_n = F_{n-1} + F_{n-2}$, $F_0 = 1$ and $F_1 = 2$. Any non-negative integer $N < F_n$ can be represented as $N = \sum_{j=1}^{n} x_j F_{n-j}$ with the property that $x_1 \cdots x_n \in \{0,1\}^*$ does not contain the factor 11. We call a word $x_1 \cdots x_n \in \{-1,0,1\}^*$ *F-heavy* if there exists a word $y_\ell \cdots y_n \in \{-1,0,1\}^*$ with $\ell \le 1$ such that

$$\sum_{j=1}^{n} x_j F_{n-j} = \sum_{j=\ell}^{n} y_j F_{n-j} \quad \text{and} \quad \sum_{j=\ell}^{n} |y_j| < \sum_{j=1}^{n} |x_j|.$$

An important difference to the definition of β-heaviness is that the indices of y cannot exceed n. The properties *F-lighter* and *signed F-expansion of minimal weight* are defined similarly to the properties for β. Although the Fibonacci numeration system and β-expansions are slightly different, we obtain the following theorem.

Theorem 4.4. *The set of signed F-expansions of minimal weight is equal to the set of signed β-expansions of minimal weight for $\beta = \frac{\sqrt{5}+1}{2}$.*

Proof. Let $z_1 \cdots z_n \in \{-2,\ldots,2\}^*$ and set $y = \sum_{j=1}^{n} z_j \beta^{n-j}$, $N = \sum_{j=1}^{n} z_j F_{n-j}$. By using the equations $\beta^k = \beta^{k-1} + \beta^{k-2}$ and $F_k = F_{k-1} + F_{k-2}$, we obtain integers m_0 and m_1 such that $y = m_1\beta + m_0$ and $N = m_1 F_1 + m_0 F_0 = 2m_1 + m_0$. Clearly, $y = 0$ implies $m_1 = m_0 = 0$ and thus $N = 0$, but the converse is not true, since $N = 0$ only implies $m_0 = -2m_1$. This means that the redundancy automaton $\mathcal{R}_F$ contains $\mathcal{R}_\beta$ (and possibly some other states), and the main difference is that a sequence $(x_k, y_k)_{1\le k\le n}$ is accepted not only if it ends in state 0, but also if it ends in a state $m_1\beta - 2m_1 = -m_1/\beta^2$. It can be easily shown that the only possibilities for $m_1 \ne 0$ are ± 1 if $z_1 \cdots z_n \in \{-2,\ldots,2\}^*$.

By adding the transition $\xrightarrow{\bar{1}|1}$ from $(1,-1)$ to the terminal state $(1/\beta^2,-1)$ (and its opposite), it can be deduced that all β-heavy words are F-heavy as well (see the Appendix).

The proof that no other F-heavy words exist runs along the same lines as the proof for β-heavy words. We have to consider a redundancy automaton with states s satisfying $|s| \le 4/\sqrt{5} + 1/\beta^2 < 2 + 1/\beta^3$, but the states $|s| \ge \beta$ play no role, as in $\mathcal{S}_\beta$. This shows that the F-heavy words are exactly the β-heavy words. □

5. Tribonacci case

In this section, let β be the Tribonacci number, $\beta^3 = \beta^2 + \beta + 1$ ($\beta \approx 1.839$). Since $1 = .111$, the condition of Example 1.1 (1) is satisfied, and $\widetilde{A_\beta} = \{-1,0,1\}$.

5.1. **Heavy words.** It is convenient to determine first particular signed β-expansions of minimal weight. Then it suffices to consider these words as outputs of $\mathcal{S}_\beta$ since every heavy word can be transformed into a lighter word of this type.

Proposition 5.1. *If $\beta > 1$ is the Tribonacci number, then every $y \in \mathbb{Z}[\beta]$ has a unique expansion avoiding the factors 11, $1\bar{1}$, $10\bar{1}$ and their opposites. This expansion is of minimal weight.*

The lexicographically maximal word avoiding these factors is $(10)^\omega$, and $.(10)^\omega = \beta^2/(\beta^2+1)$. The expansion in Proposition 5.1 is provided by the transformation

$$\tau:\ \Big[\frac{-\beta^2}{\beta^2+1}, \frac{\beta^2}{\beta^2+1}\Big) \to \Big[\frac{-\beta^2}{\beta^2+1}, \frac{\beta^2}{\beta^2+1}\Big), \qquad \tau(x) = \beta x - \Big\lfloor \frac{\beta^2+1}{2\beta} x + \frac{1}{2} \Big\rfloor.$$

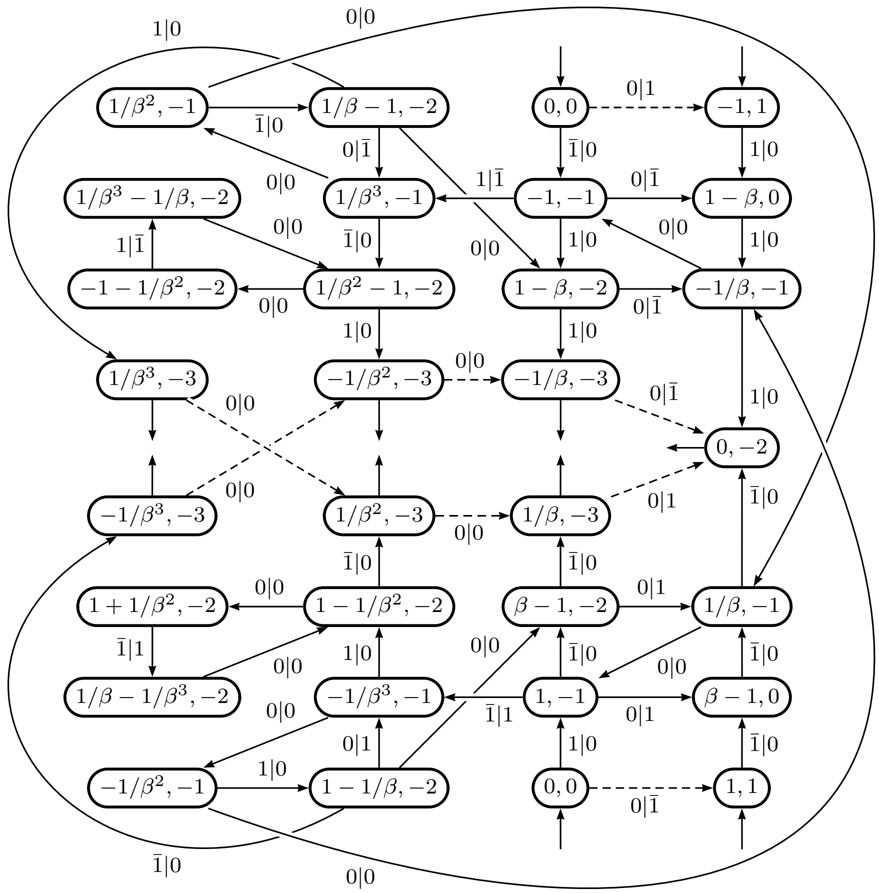

FIGURE 4. Automaton $\mathcal{S}_\beta$ of strictly heavy words for the Tribonacci number.

The proofs of Proposition 5.1 and Theorem 5.2 are similar to the proofs for the Golden Ratio case and therefore omitted (see the Appendix).

Theorem 5.2. *If β is the Tribonacci number, then the strictly β-heavy words are exactly the inputs of the automaton $\mathcal{S}_\beta$ in Figure 4. The signed β-expansions of minimal weight are given by the automaton $\mathcal{M}_\beta$ in Figure 5 where all states are terminal.*

5.2. **Weight of the expansions.** As for the Golden Ratio, we determine the expected number of non-zero digits by an appropriate Markov chain. A particular signed β-expansion of minimal weight is given by Proposition 5.1. Therefore we define sets $I_{00} = \big[\frac{-1}{\beta^2+1}, \frac{1}{\beta^2+1}\big)$, $I_{01} = \big[\frac{-\beta}{\beta^2+1}, \frac{\beta}{\beta^2+1}\big) \setminus I_{00}$, $I_1 = \big[\frac{-\beta^2}{\beta^2+1}, \frac{-\beta}{\beta^2+1}\big) \cup \big[\frac{\beta}{\beta^2+1}, \frac{\beta^2}{\beta^2+1}\big)$ and obtain a Markov chain with transition probabilities

$$(\Pr[X_k = j \mid X_{k-1} = i])_{i,j \in \{00,01,1\}} = \begin{pmatrix} 1/\beta & 1-1/\beta & 0 \\ 0 & 0 & 1 \\ 1-1/\beta^2 & 1/\beta^2 & 0 \end{pmatrix}$$

The stationary distribution vector of the Markov chain is $\big(\frac{\beta^3+\beta^2}{\beta^5+1}, \frac{\beta^3}{\beta^5+1}, \frac{\beta^3}{\beta^5+1}\big)$, hence the expected number of non-zero digits in a signed β-expansion of minimal weight of length n is asymptotically $n\beta^3/(\beta^5+1)$ (with $\beta^3/(\beta^5+1) = \mathbf{.}(0011010100)^\omega \approx 0.28219$).

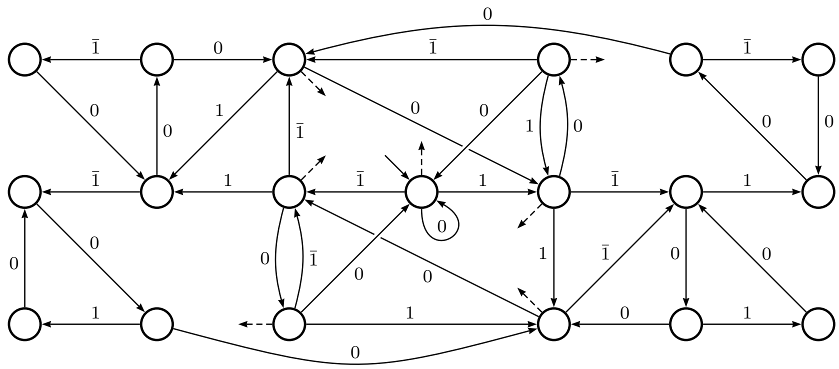

FIGURE 5. Automaton $\mathcal{M}_\beta$ of signed β-expansions of minimal weight for the Tribonacci number.

5.3. **Tribonacci numeration system.** The linear numeration system canonically associated with the Tribonacci number is the Tribonacci numeration system defined by the sequence $T = (T_n)_{n\geq 0}$ with $T_0 = 1$, $T_1 = 2$, $T_2 = 4$ and

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}.$$

Any non-negative integer $N < T_n$ has a representation $N = \sum_{j=1}^n x_j T_{n-j}$ with the property that $x_1 \cdots x_n \in \{0,1\}^*$ does not contain the factor 111. The properties T-*heavy*, T-*lighter* and *signed* T-*expansion of minimal weight* are defined analogously to the Fibonacci numeration system.

Theorem 5.3. *The signed* T*-expansions of minimal weight are recognized by the automaton in Figure 5 where only the states with a dashed outgoing arrow are terminal.*

Proof. Similarly to the Golden Ratio case, two words representing the same number as a β-expansion with the Tribonacci number β, represent the same number in the Tribonacci numeration system. More precisely, if $y = \sum_{j=1}^n z_j \beta^{n-j}$ and $N = \sum_{j=1}^n z_j T_{n-j}$ for some word $z_1 \cdots z_n \in \{-2, \ldots, 2\}^*$, then we have integers m_0, m_1, m_2 such that $y = m_2\beta^2 + m_1\beta + m_0$ and $N = m_2T_2 + m_1T_1 + m_0T_0 = 4m_2 + 2m_1 + m_0$. Therefore we have $N = 0$ if and only if $m_0 = -2m_0'$ and $m_1 = -2m_2 + m_0'$, hence

$$y = m_2(\beta^2 - 2\beta) + m_0'(\beta - 2) = m_2/\beta^2 + m_0'/\beta^3 \quad \text{for some integers } m_0', m_2.$$

It can be shown that all β-heavy words are T-heavy as well (see the Appendix). Therefore the signed T-expansions of minimal weight are a subset of the words which are recognized by $\mathcal{M}_\beta$ in Figure 5. The shortest words ending in states without dashed outgoing arrow are T-heavy because of the following T-lighter words (or their opposites).

$$(1)\bar{1}1 \to (1)0\bar{1}, \quad (1)\bar{1}10 \to (1)0\bar{1}0, \quad (1)\bar{1}10\bar{1} \to (1)0\bar{1}00$$

The difference of the corresponding β-expansions is either $\pm 1/\beta^2$, $\pm 1/\beta^3$ or $\pm(1/\beta^2 + 1/\beta^3)$. By construction (cf. the proof of Lemma 3.3), all other words ending in these states are T-heavy as well.

The proof that no other words are T-heavy runs again along the same lines as for β-heavy words. $\square$

References

[1] S. Akiyama, On the boundary of self affine tilings generated by Pisot numbers, *J. Math. Soc. Japan* **54** (2002), 283–308.

[2] S. Akiyama, H. Rao and W. Steiner, A certain finiteness property of Pisot number systems, *J. Number Theory* **107** (2004), 135–160.

[3] S. Arno and F. S. Wheeler, Signed digit representations of minimal Hamming weight, *IEEE Trans. Comput.* **42** (1993), 1007–1010.

[4] A. Bertrand, Développements en base de Pisot et répartition modulo 1, *C. R. Acad. Sci. Paris, Sér. A* **285** (1977), 419–421.

[5] A. D. Booth, A signed binary multiplication technique, *Q. J. Mech. Appl. Math.* **4** (1951), 236–240.

[6] K. Dajani, C. Kraaikamp and P. Liardet, Ergodic properties of signed binary expansions, *Discrete Contin. Dyn. Syst.* **15** (2006), 87–119.

[7] Ch. Frougny, Representation of numbers and finite automata, *Math. Syst. Theory* **25** (1992), 37–60.

[8] Ch. Frougny and B. Solomyak, Finite beta-expansions, *Ergod. Theory Dyn. Syst.* **12** (1992), 713–723.

[9] P. Grabner and C. Heuberger, On the number of optimal base 2 representations of integers, *Des. Codes Cryptography* **40** (2006), 25–39.

[10] C. Heuberger, Minimal expansions in redundant number systems: Fibonacci bases and greedy algorithms, *Period. Math. Hung.* **49** (2004), 65–89.

[11] C. Heuberger and H. Prodinger, Analysis of alternative digit sets for nonadjacent representations, *Monatsh. Math.* **147** (2006), 219–248.

[12] M. Hollander, Linear numeration systems, finite beta-expansions and discrete spectrum of substitution dynamical systems, PhD Thesis, University of Washington, 1996.

[13] M. Joye and C. Tymen, Compact encoding of non-adjacent forms with applications to elliptic curve cryptography, in *Public Key Cryptography*, L.N.C.S. **1992** (2001), 353–364.

[14] M. Lothaire, *Algebraic Combinatorics on Words*, Cambridge University Press, 2002.

[15] F. Morain and J. Olivos, Speeding up the computations on an elliptic curve using addition-subtraction chains, *RAIRO, Inform. Théor. Appl.* **24** (1990), 531–543.

[16] J.A. Muir and D.R. Stinson, Minimality and other properties of the width-w nonadjacent form, *Math. Comput.* **75** (2005), 369–384.

[17] W. Parry, On the β-expansions of real numbers, *Acta Math. Acad. Sci. Hung.* **11** (1960), 401–416.

[18] G. W. Reitwiesner, Binary arithmetic, *Advances in Computers*, vol. 1, Academic Press, New York, 1960, 231–308.

[19] A. Rényi, Representations for real numbers and their ergodic properties, *Acta Math. Acad. Sci. Hung.* **8** (1957), 477–493.

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Appendix

Lemma 5.4. *If β satisfies (D), then β is a Pisot number.*

Proof. It is easy to see (e.g. by multiplying both sides with $\beta - 1$) that

$$1 = \bullet(b_1 - 1)(b_1 + b_2 - 1)\cdots(b_1 + b_2 + \cdots + b_{d-1} - 1)(b_1 + b_2 + \cdots + b_d - 1)^\omega$$

is an expansion of 1 in base β with non-negative digits. We have

$(X^d - b_1 X^{d-1} - b_2 X^{d-2} \cdots - b_{d-1} X - b_d) = (X - \beta)(X^{d-1} + r_2 X^{d-2} + \cdots + r_{d-1} X + r_d)$

with $r_j = \bullet b_j b_{j+1} \cdots b_d$, and

$$\begin{aligned}\sum_{j=2}^{d} |r_j| &= \sum_{j=2}^{d} \left| \sum_{i=j}^{d} \frac{b_i}{\beta^{i-j+1}} \right| \\ &\le \sum_{j=2}^{d} \sum_{i=j}^{d} \frac{|b_i|}{\beta^{i-j+1}} = \bullet(|b_2| + \cdots + |b_d|)(|b_3| + \cdots + |b_d|)\cdots(|b_{d-1}| + |b_d|)(|b_d|) \\ &\le \bullet(b_1 - 1)(b_1 - |b_2| - 1)\cdots(b_1 - |b_2| - \cdots - |b_{d-1}| - 1) \\ &\le \bullet(b_1 - 1)(b_1 + b_2 - 1)\cdots(b_1 + b_2 + \cdots + b_{d-1} - 1) \\ &\le 1\end{aligned}$$

We have equality everywhere if and only if $b_j \le 0$ for all $j \in \{2, 3, \ldots, d\}$ and $b_1 = \sum_{j=2}^{d} |b_j| + 1$. In this case,

$$1 = \bullet t_1 t_2 \cdots t_{d-1} = \bullet(b_1 - 1)(b_1 + b_2 - 1)\cdots(b_1 + b_2 + \cdots + b_{d-1} - 1)$$

with $t_1 \ge t_2 \ge \cdots \ge t_{d-1}$, and it is well known that β is a Pisot number.

If $\sum_{j=2}^{d} |r_j| < 1$, then

$$|x^{d-1}| > \sum_{j=2}^{d} |r_j||x|^{d-1} \ge \sum_{j=2}^{d} |r_j||x|^{d-j} \ge |r_2 x^{d-2} + \cdots + r_{d-1} x + r_d|$$

for all x with $|x| \ge 1$. Therefore, all conjugates of β lie inside the unit circle. □

Lemma 5.5. *If a word is β-heavy for $\beta = \frac{\sqrt{5}+1}{2}$, then it is F-heavy.*

Proof. Let $x = x_1 \cdots x_n \in \widetilde{A_\beta}^*$ be β-heavy. Then the following situations can occur:

- If x contains a factor 11 (and $\bar{1}\bar{1}$ respectively), then $0x$ contains a factor 011 or $\bar{1}11$ (and $0\bar{1}\bar{1}$ or $1\bar{1}\bar{1}$ respectively), hence x is F-heavy.
- If x contains a factor $1\bar{1}$ (and $\bar{1}1$ respectively), then x contains either a factor $1\bar{1}0$ or $\bar{1}10$, or the x_j are alternately 1 and $\bar{1}$. Since $F_1 - F_0 = F_0$, x is F-heavy.
- If z contains no two consecutive non-zero digits and $z_\ell \cdots z_r$, $1 \le \ell \le r < n$, is a strictly β-heavy factor of z, then $0z_\ell \cdots z_r 0$ is a factor of $0z$. $\mathcal{S}_\beta$ provides a β-lighter word of the same length as $0z_\ell \cdots z_r 0$, hence z is F-heavy.
- If z contains no two consecutive non-zero digits and $z_\ell \cdots z_n$, $1 \le \ell \le n$, is a strictly β-heavy factor of z, then $\mathcal{S}_\beta$ provides a word which is F-lighter than $0z_\ell \cdots z_n$ if we add the transition $(1, -1) \xrightarrow{\bar{1}|1} (1/\beta^2, -1)$ and its opposite, since $(\pm 1/\beta^2, -1)$ are terminal states. □

Proof of Proposition 5.1. The proof of the existence and the uniqueness of an expansion avoiding the factors 11, $1\bar{1}$, $10\bar{1}$ and their opposites is exactly as the proof of Proposition 4.3. It remains to show that the weight of these expansions is minimal.

Suppose that there exist a path in $\mathcal{T}_\beta$, where the input is a strictly heavy word avoiding the given factors (prolongated possibly by zeros) and the output is a lighter word (which we may assume of minimal weight). All factors of the input are lexicographically less than $(10)^\omega$, and all factors of the output are lexicographically less than $11(010)^\omega$, since we can exclude the heavy words given by Figure 4. Therefore it is sufficient to consider states in $\mathcal{T}_\beta$ with $|s| < \bullet(10)^\omega + \bullet 11(010)^\omega < \beta$.

Suppose that the first transition is $(0,0) \stackrel{1|0}{\to} (1,-1)$. A possible prolongation is

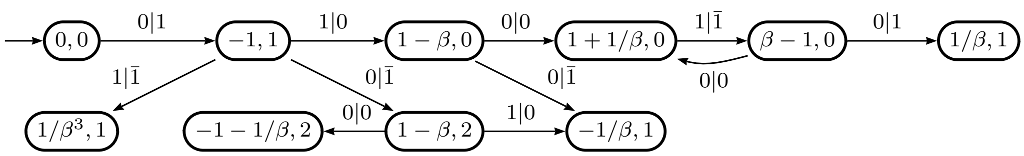
$$(0,0) \stackrel{1|0}{\to} (1,-1) \stackrel{0|1}{\to} (\beta-1,0) \stackrel{0|0}{\to} (1+1/\beta,0) \stackrel{\bar{1}|1}{\to} (\beta-1,0),$$

but this cannot provide a strictly heavy input because of the path $(0,0) \stackrel{0|\bar{1}}{\to} (1,1) \stackrel{\bar{1}|0}{\to} (\beta-1,0)$. The only possible branching in the above path provides

$$(0,0) \stackrel{1|0}{\to} (1,-1) \stackrel{0|1}{\to} (\beta-1,0) \stackrel{0|1}{\to} (1/\beta,1) \stackrel{a|a}{\to} (1,1)$$

with $a \in \{0,\bar{1}\}$. If $a=0$, then we have the shortcut $(0,0) \stackrel{0|\bar{1}}{\to} (1,1)$. If $a=\bar{1}$ and $011\bar{1}v$ is lighter than $100\bar{1}u$, then $\bar{2}v$ is lighter than $\bar{1}u$ (and $\bar{2}v$ can be transformed into a word on $\widetilde{A_\beta}^*$ by Proposition 3.1), hence $100\bar{1}u$ is not strictly heavy.

Now consider the transition $(0,0) \stackrel{0|1}{\to} (-1,1)$ and look at



The branch ending in $(-1-1/\beta,2)$ has no prolongation since $1\bar{1}0\bar{1}$ is heavy. The other branches need not be continued by the same reasons as in the preceding paragraph. All other transitions can be excluded easily, and the Proposition is proved. □

Proof of Theorem 5.2. We proceed similarly to the proof of Theorem 4.2. Suppose that we have a strictly heavy word, which is not recognized by $\mathcal{S}_\beta$ in Figure 4. Therefore there exists a path in $\mathcal{T}_\beta$ with this word as input and a word of minimal weight as output. By Proposition 5.1, we may assume that the output avoids the factors 11, $1\bar{1}$, $10\bar{1}$ and their opposites. The factors of the input are lexicographically less than $11(010)^\omega$. Therefore, it is sufficient to consider states (s,δ) with $|s| < \beta$, as in the previous proof.

We can exclude $|s| = 1+1/\beta$, $\delta = 0$, since the only possible prolongations are $(1+1/\beta,0) \stackrel{\bar{1}|1}{\to} (1-1/\beta^3,0) \stackrel{0|0}{\to} (1+1/\beta,0)$ and $(1+1/\beta,0) \stackrel{\bar{1}|1}{\to} (1-1/\beta^3,0) \stackrel{\bar{1}|0}{\to} (1/\beta,-1)$. In the second case, the incoming transition must be $\stackrel{0|0}{\to}$, hence we have the shortcut $(0,0) \stackrel{0|\bar{1}}{\to} \stackrel{\bar{1}|0}{\to} \stackrel{\bar{1}|0}{\to} (1/\beta,-1)$.

It can be easily verified that no transitions from the states $(-1,1)$, $(1-\beta,0)$, $(-1/\beta,-1)$, $(-1,-1)$, $(1-\beta,-2)$, $(1/\beta^3,-1)$, $(1/\beta^3-1/\beta,-2)$ (and their opposites) than those which are present in the transducer of Figure 4 can occur. For the other states, consider the following facts:

- $(-1/\beta^2,-1) \xrightarrow{\bar{1}|\bar{1}} (1/\beta,-1)$ is cut short by $(0,0) \xrightarrow{1|0} \big(\xrightarrow{0|1}\xrightarrow{\bar{1}|0}\xrightarrow{0|0} \big)^* \xrightarrow{0|1}\xrightarrow{\bar{1}|0} (1/\beta,-1)$.
- Since $1(0\bar{1}0)^*0\bar{1}\bar{1}$ is heavy, we have no transition from $(1/\beta{-}1,-2)$ with input $\bar{1}$.
- Since $1(0\bar{1}0)^*\bar{1}\bar{1}$ is heavy, we have no transition from $(1/\beta^2{-}1,-2)$ with input $\bar{1}$. $(1/\beta^2-1,-1) \xrightarrow{0|\bar{1}} (-1/\beta^2,-1)$ is cut short by $(0,0) \xrightarrow{1|0}\xrightarrow{\bar{1}|1} \big(\xrightarrow{0|0}\xrightarrow{1|0}\xrightarrow{0|1} \big)^* \xrightarrow{0|0} (-1/\beta^2,-1)$.
- $(-1{-}1/\beta^2,-2) \xrightarrow{1|0} (-1{-}1/\beta^2{-}1/\beta^4,-3)$ must be followed by the transition $\xrightarrow{1|\bar{1}}$ because $-\beta-1/\beta-1/\beta^3=-2-1/\beta$, but this means that the input contains a (heavy) factor $\bar{1}(010)^*011$. Similarly, $(-1{-}1/\beta^2,-2) \xrightarrow{0|\bar{1}} (-1{-}1/\beta^2{-}1/\beta^4,-1)$ is impossible because the output must not contain $\bar{1}\bar{1}$.
- Since $\bar{1}(010)^*011$ is heavy, we have no transition from $(1/\beta^3-1/\beta,-2)$ with input 1. □

Lemma 5.6. *If a word is β-heavy for the Tribonacci number β, then it is T-heavy.*

Proof. Let $x = x_1\cdots x_n \in \widetilde{A_\beta}^*$ be a β-heavy word. The following situations can occur:

- If x contains the factor 111 (and $\bar{1}\bar{1}\bar{1}$ respectively), then $0x$ contains 0111 or $\bar{1}111$ (and $0\bar{1}\bar{1}\bar{1}$ or $1\bar{1}\bar{1}\bar{1}$ respectively), hence x is T-heavy.
- ¿From now on, suppose that x does not contain the factors 111 and $\bar{1}\bar{1}\bar{1}$. Let $x_\ell\cdots x_r$, $1\le\ell\le r\le n$, be a strictly β-heavy factor of x. If the corresponding path in $\mathcal{S}_\beta$ runs from $(0,0)$ to $(0,-2)$, then the output $y_\ell\cdots y_r$ provides immediately a T-lighter word. If the path runs from $(-1,1)$ to $(0,-2)$, then $x_\ell x_{\ell+1}=11$, hence $x_{\ell-1}\in\{0,\bar{1}\}$ and $x_1\cdots x_{\ell-2}(x_{\ell-1}+1)y_\ell\cdots y_r x_{r+1}\cdots x_n$ is a T-lighter word.
- Suppose that the path ends in $(1/\beta^i,-3)$ and $n-r<i$. Then we obtain a T-lighter word by adding the following transitions. For $i=1$ and $n=r$, add $(\beta-1,-2) \xrightarrow{\bar{1}|1} (1/\beta-1,-2)$. Since $1/\beta-1=-1/\beta^2-1/\beta^3$, this state is terminal. For $i=2$, $n=r-1$, add $(1/\beta^2,-3) \xrightarrow{0|1} (1/\beta-1,-2)$. For $i=3$, $n=r-2$, add $(1/\beta^2,-3) \xrightarrow{0|1} (1/\beta-1,-2)$, $(1/\beta^2,-3) \xrightarrow{\bar{1}|0} (1/\beta-1,-4)$ and $(1-\beta,-2) \xrightarrow{1|\bar{1}} (1/\beta-1,-2)$. In the other cases, nothing has to be done.
- Suppose that the path ends in $(1/\beta^i,-3)$ and $n-r\ge i$. If $x_{r+i}\in\{0,\bar{1}\}$, then we obtain immediately a T-lighter word. If $x_{r+i}=1$, then we obtain a lighter word $x_1\cdots x_{\ell-2}y_{\ell-1}\cdots y_{r-1}0x_{r+1}\cdots x_{r+i-1}2x_{r+i+1}\cdots x_n$ (on the alphabet $\{-1,0,1,2\}$).

  We distinguish two cases. If $x_{r+i-1}\ne1$ (or $i=1$) and $n-r-i\ge3$, then we add $1\bar{2}001$ at the appropriate place such that the 2 vanishes. Now we have a immediately a T-lighter word if $x_{r+i+3}\ne1$. If $x_{r+i-1}=1$ and $n-r-i\ge1$, then we add $1\bar{1}\bar{1}\bar{1}$ such that 12 becomes 01. (Note that $x_{r+i-2}\ne1$ since we have exluded the factor 111.) Hence we have a T-lighter word if $x_{r+i+1}\ne\bar{1}$.

  Both operations do not increase the weight of the word and create at most one new 2 (or $\bar{2}$) at a position j which is closer to the end of the word. Therefore we can iterate this procedure until we have found a word on the alphabet $\{\bar{1},0,1\}$, or until $n-j<3$ in case that 2 is not preceded by 1 (or that $\bar{2}$ is not preceded by $\bar{1}$), $n=j$ else. In every case, we obtain easily a word avoiding 2 and $\bar{2}$ without increasing the weight. □