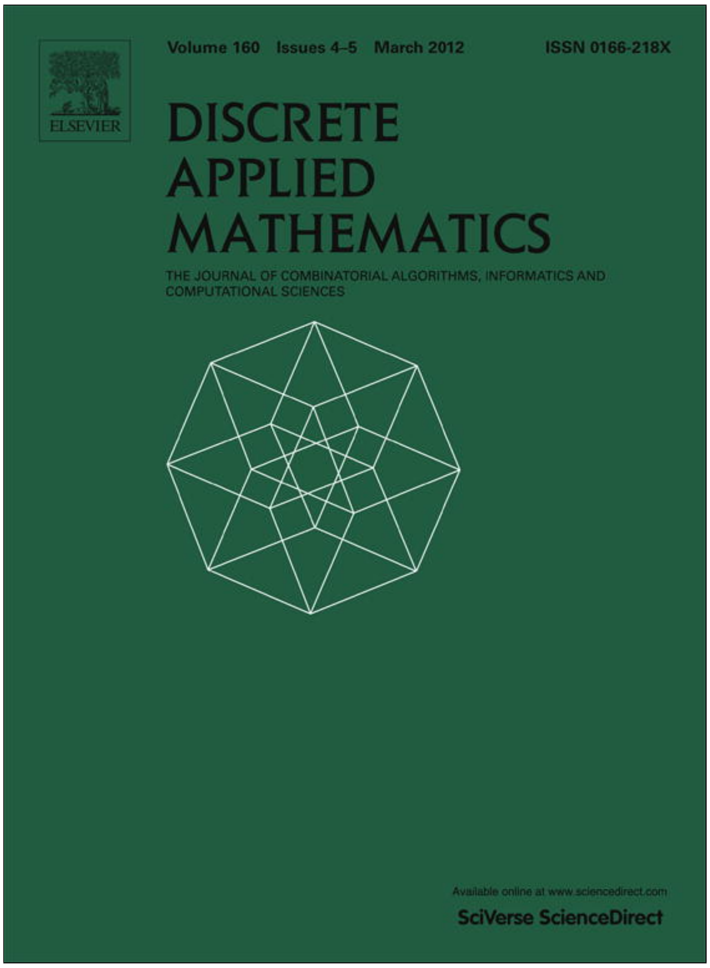
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Sturmian graphs and integer representations over numeration systems[☆]

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ABSTRACT

In this paper we consider a numeration system, originally due to Ostrowski, based on the continued fraction expansion of a real number α. We prove that this system has deep connections with the Sturmian graph associated with α. We provide several properties of the representations of the natural integers in this system. In particular, we prove that the set of lazy representations of the natural integers in this numeration system is regular if and only if the continued fraction expansion of α is eventually periodic. The main result of the paper is that for any number i the unique path weighted i in the Sturmian graph associated with α represents the lazy representation of i in the Ostrowski numeration system associated with α.

1. Introduction

The Ostrowski numeration system [15] is defined thanks to the continued fraction expansion of a real number, and it has been extensively studied, see for instance [4]. In the particular case of the golden ratio, it leads to the Fibonacci numeration system.

The purpose of this paper is to show that the Ostrowski numeration system associated with α has a deep connection with a structure defined in [8], the *Sturmian graph* associated with the continued fraction expansion of α. It is proved in [8] that these graphs turn out to be the underlying graphs of compact directed acyclic word graphs of central Sturmian words. The authors also show that, in analogy with the case of Sturmian words, these graphs converge to infinite ones. It has been proved that Sturmian graphs have a certain counting property. In particular, given an infinite Sturmian graph, it can "count" from 0 up to infinity, which means that, for any natural integer i, there exists in this Sturmian graph a unique path starting in the initial state and having weight i. Recent results on Sturmian graphs and Sturmian words and their generalizations can be found in [3].

The main result of the present paper is that for any natural integer i the unique path weighted i in the Sturmian graph associated with α represents the lazy representation of i in the Ostrowski numeration system associated with α.

To prove this, we first study the lazy representations in more general numeration systems defined by an increasing basis, such as the Fibonacci numeration system. Note that lazy β-expansions of real numbers have been studied in particular in [9,6], but to our knowledge, the results that we obtain on lazy representations of the natural integers are new. We prove that the set of lazy representations is regular if and only if the set of greedy representations is regular. Finally, we give an algorithm for finding the lazy representation of a positive integer N.

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From the previous results it can be deduced that the set of lazy representations of the natural integers in the Ostrowski numeration system associated with α is regular if and only if the continued fraction expansion of α is eventually periodic, that is to say, if α is a quadratic irrational.

In the particular case of Fibonacci Sturmian graphs, the link between the graph and the Fibonacci numeration system has been independently discovered. More precisely, in [18] it is proved that the structure of lengths of paths in the DAWGs of Fibonacci words is closely related to the Fibonacci numeration system and corresponds to a number-theoretic characterization of occurrences of subwords. Recently in [2] Ostrowski automata were introduced, that also give a very close relationship between Ostrowski numeration and graphs.

The paper is organized as follows. In the next section, we recall some basic notation on continued fraction expansions of a real number α and Sturmian words. The third section is devoted to finite and infinite Sturmian graphs, with particular regard to the description of the structure of infinite ones. In the fourth section we focus on the representations of the natural integers in numeration systems defined by a basis, and we prove some properties of greedy and lazy representations. Moreover, we give a new algorithm for finding the lazy representation of a positive integer N. In the fifth section we focus on the relationship between the continued fraction expansion of a real number α and numeration systems. In particular we prove some properties on the set of lazy expansions in the Ostrowski numeration system. In the sixth section we establish a deep connection between the set of lazy representations and the theory of *Sturmian graphs*. Finally, the last section contains some conclusions.

2. Continued fraction expansions and Sturmian words

The definitions given in this section are classical, and can be found in [1,5], [11, Chap. 10], [16], and [19], but for completeness we recall them below.

If α is a real number, we can expand α as a *simple continued fraction*

$$\alpha = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}$$

which is usually abbreviated as $\alpha = [a_0, a_1, a_2, a_3, \ldots]$, where a_0 is some integer ($a_0 \in \mathbb{Z}$) and all the other numbers a_i are positive integers.

The integers in the continued fraction expansion of a real number are called *partial quotients*.

The expansion may or may not terminate. If α is irrational, this representation is infinite and unique. If α is rational, there are two possible finite representations. Indeed, it is well known that $[a_0, \ldots, a_{s-1}, a_s, 1] = [a_0, \ldots, a_{s-1}, a_s + 1]$. In this paper, we only discuss the case where a_i is a positive integer for $i \geq 0$, i.e., α is greater than or equal to 1. Moreover, we will focus on infinite continued fraction expansions. Hence, in our paper α will be an irrational number.

Given the continued fraction expansion of α, it is possible to construct a sequence of rationals $\frac{P_i(\alpha)}{Q_i(\alpha)}$, called "convergents", that converge to α. They are defined by the following rules, with $P_i = P_i(\alpha)$ and $Q_i = Q_i(\alpha)$

$$\begin{array}{ll} P_0 = a_0 & Q_0 = 1 \\ P_1 = a_1 a_0 + 1 & Q_1 = a_1 \\ P_{i+1} = a_{i+1} P_i + P_{i-1} & Q_{i+1} = a_{i+1} Q_i + Q_{i-1}. \end{array} \tag{1}$$

Therefore, for $i \geq 0$, $P_i/Q_i = [a_0, \ldots, a_i]$.

Moreover, for any irrational number $\alpha = [a_0, a_1, a_2, \ldots] > 0$, it is possible to define inductively, for any $i \in \mathbb{N}$, $i \geq 0$, the following sequence of natural numbers $U_\alpha = (l_i)_{i \geq 0}$

$$\begin{cases} l_0 = 1 \\ l_1 = a_0 + 1 \\ l_{i+1} = a_i l_i + l_{i-1} \quad i \geq 1. \end{cases} \tag{2}$$

Remark 1. By induction, it is easy to prove that $l_{i+1} = P_i + Q_i$, for any $i \geq 0$.

Notice that under our assumptions a_i is a positive integer for $i \geq 0$ and hence the sequence $U_\alpha = (l_i)_{i \geq 0}$ is increasing. In the standard theory, instead, a_0 is allowed to be 0 and so $l_0 = l_1 = 1$.

Example 2. Let us consider the irrational number $\varphi = \frac{\sqrt{5}+1}{2}$. This number is well known as the *golden ratio* and has continued fraction expansion $\varphi = [1, 1, 1, 1, \ldots]$. The sequence $(l_i)_{i \geq 0}$ associated with this sequence is described in the following table and it is equal to the sequence of Fibonacci numbers $(F_i)_{i \geq 0}$.

i	0	1	2	3	4	5	6	7	8	…
a_i	1	1	1	1	1	1	1	1	1	…
P_i	1	2	3	5	8	13	21	34	55	…
Q_i	1	1	2	3	5	8	13	21	34	…
l_i	1	2	3	5	8	13	21	34	55	…

Simple continued fractions play a leading role in the construction of Sturmian words, that are aperiodic infinite words over a binary alphabet of minimal subword complexity, i.e., with exactly $n+1$ factors of length n.

Among the different definitions of Sturmian words, one is obtained by considering the intersections of a ray having an irrational slope $\alpha > 0$ with a square-lattice. The word obtained by coding each vertical intersection with an a, each horizontal intersection by a b and each corner with ab or ba is Sturmian. If the ray starts from the origin, the word obtained is called *characteristic*.

Another way of constructing characteristic Sturmian words is by applying the *standard method*. It consists in the inductive definition of the two sequences of words $\{A_n\}$ and $\{B_n\}$ obtained starting from the base case

$$\begin{cases} A_0 = a \\ B_0 = b \end{cases}$$

by applying the two *Rauzy rules* [17]

$$R_1 : \begin{cases} A_{n+1} = A_n \\ B_{n+1} = A_n B_n \end{cases} \qquad R_2 : \begin{cases} A_{n+1} = B_n A_n \\ B_{n+1} = B_n. \end{cases}$$

When each of the two rules is applied infinitely often, these two sequences converge to the same infinite word that is a characteristic word. Conversely, each characteristic word is obtained in this way.

Given a pair of words (A_n, B_n), we can associate to it its *directive sequence* (cf. [7]), that is the sequence of integers $[a_0, a_1, \ldots, a_s]$ such that $n = \sum_{i=0}^{s} a_i$, representing the fact that (A_n, B_n) is obtained by first applying rule R_1 to (A_0, B_0) a_0 consecutive times, rule R_2 to the resulting pair (A_1, B_1) a_1 consecutive times, and so on.

Words obtained by removing last two characters from A_n or B_n are called *central Sturmian words*.

If the directive sequence $[a_0, a_1, \ldots]$ is infinite, the infinite word to which A_n and B_n converge represents a ray having slope α, where α has $[a_0, a_1, \ldots]$ as its simple continued fraction expansion.

Given a pair (A_n, B_n) having directive sequence $[a_0, a_1, \ldots, a_s]$, it is possible to recursively define $\max\{|A_n|, |B_n|\}$ as the $(s+1)$-th element of the above defined sequence $(l_i)_{i\in\mathbb{N}}$.

Example 3. A well-known Sturmian word is the Fibonacci word

$$f = abaababaabaababaababaabaababaabaab\ldots.$$

It is associated to the directive sequence $[1, 1, 1, 1, \ldots]$. Let us show the first steps for the construction of f.

$$(a, b) \xrightarrow{R_1} (a, ab) \xrightarrow{R_2} (aba, ab) \xrightarrow{R_1} (aba, abaab) \xrightarrow{R_2} (abaababa, abaab) \xrightarrow{R_1}$$
$$(abaababa, abaababaabaab) \xrightarrow{R_2} (abaababaabaababaababa, abaababaabaab) \cdots.$$

As noticed above, $\max\{|A_n|, |B_n|\} = l_{s+1}$. The fact that $a_i = 1$ for any $i \geq 0$ implies that in this case $n = \sum_{i=0}^{s} a_i = \sum_{i=0}^{s} 1 = s+1$. This means that in the case of the Fibonacci word $\max(|A_n|, |B_n|) = l_n$ (the sequence (l_i) associated to the directive sequence $[1, 1, 1, 1, \ldots]$ is computed in Example 2).

If the directive sequence $[a_0, a_1, \ldots]$ is infinite, the infinite word to which A_n and B_n converge represents a ray having slope α, where α has $[a_0, a_1, \ldots]$ as its simple continued fraction expansion.

For references on Sturmian words and their geometric representation see [14, Chap. 2] and [12].

3. Sturmian graphs

First of all let us recall some definitions on Sturmian graphs that will be useful in the following (see [8] for further details).

A *weighted DAG* is a directed acyclic graph, where each arc is weighted by a real number. Arcs are usually represented by triples (p, c, q), that means that there exists an arc from state p to state q of weight c.

For any rational number $\frac{P}{Q} = [a_0, \ldots, a_s]$ with $\sum_{i=0}^{s} a_i \geq 2$, we inductively define a graph $G(\frac{P}{Q}) = G([a_0, \ldots, a_s])$ that we call the *Sturmian graph of* $\frac{P}{Q}$. This graph is a weighted DAG where weights are positive integers.

Sturmian Graphs have deep connections with Sturmian words. In fact, in [8] authors have shown that the Sturmian graph of a directive sequence $[a_0, \ldots, a_s]$, $G([a_0, \ldots, a_s])$ coincides with the CDAWG of the word w obtained by adding a \$ symbol to the longest central Sturmian word relative to directive sequence $[a_0, \ldots, a_s]$, where the label of each arc is replaced by the length of the factor it represents. Moreover, in that paper it has been proved that, analogously to Sturmian central words, the Sturmian graph $G([a_0, \ldots, a_s, 1])$ of directive sequence $[a_0, \ldots, a_s, 1]$ coincides with the one $G([a_0, \ldots, a_s+1])$ of directive sequence $[a_0, \ldots, a_s+1]$ and that $G([0, a_1, \ldots, a_s]) = G([a_1, \ldots, a_s])$.

If $a_0 = 0$ we set $G([a_0, a_1, \ldots, a_s]) = G([a_1, \ldots, a_s])$. Therefore in what follows we suppose that $a_0 \geq 1$.

The base case in the construction of the Sturmian graph is the graph $G([1, 1]) = G([2])$. It consists of only two states and two arcs, both going from state 1 to the final state F and having weights 1 and 2 respectively. It can be seen in Fig. 1.

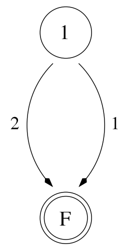

Fig. 1. Base case.

To give the inductive step, let us recall the definition of the sequence $U_\alpha = (l_i)_{i\in\mathbb{N}}$ given in (2):

$$\begin{cases} l_0 = 1 \\ l_1 = a_0 + 1 \\ l_{i+1} = a_i \cdot l_i + l_{i-1}. \end{cases}$$

Given the Sturmian graph of $[a_0, \ldots, a_s]$, $s \geq 0$, $\sum_{i=0}^{s} a_i \geq 2$, $G([a_0, \ldots, a_s])$, we define the Sturmian graph $G([a_0, \ldots, a_s, 1])$ in the following way. Each arc of maximal weight in $G([a_0, \ldots, a_s])$ (all of them end at the final state) is split in one arc of that weight minus 1 from the same outgoing state to a new state (the same for each arc) and two arcs from this new state towards the final one, one labeled 1 and the other labeled $l_s + 1$.

Moreover, if $a_s = 1$, then for each state of out-degree 2, except the new one, one must add a new outgoing arc labeled $l_s + 1$ towards the final state, with the exception of the new state that has already one such arc.

As $[a_0, a_1, \ldots, a_s, 1] = [a_0, a_1, \ldots, a_s + 1]$, the previously defined inductive step let us construct every Sturmian graph $G([a_0, \ldots, a_k])$, $k \geq 0$.

Let us give some examples. Fig. 2 shows graphs $G([3, 1])$ and $G([1, 1, 1, 1])$. The first one is obtained starting from $G([3])$, inductively built from the base case $G([2])$, being $G([3]) = G([2, 1])$. The second one is derived starting from $G([1, 1, 1])$ that, in turn, comes from the base case $G([1, 1])$.

In [8] the following proposition concerning the number of states of the Sturmian graph is proved.

Proposition 4. *The Sturmian graph* $G([a_0, \ldots, a_s, 1])$, $s \geq 0$, $n = \sum_{i=0}^{s} a_i + 1 \geq 2$, *contains exactly* n *states, among them* a_s *of out-degree* 2, *and* $3(n-1) - a_s$ *arcs.*

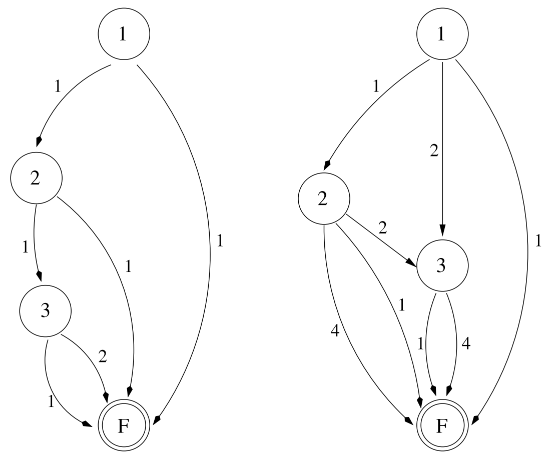

Fig. 2. Graphs $G([3, 1])$ and $G([1, 1, 1, 1])$.

Let us give, now, some definitions that will be useful in the following.

Definition 5. A DAG having a unique smallest state with respect to the order induced by the arcs is called *semi-normalized*. If it has also a unique greatest state, it is called *normalized*. The smallest state is called the *initial state* and the greatest is called the *final state*.

Note that any normalized DAG is also semi-normalized. Note also that any DAG can always be semi-normalized by adding at most one new state and can be normalized by adding at most two new states.

Now let us consider infinite Sturmian graphs. In [8] it has been proved that, given a real number α, the sequence of finite Sturmian graphs associated to the convergents to α converges, in turn, to a normalized infinite graph $G(\alpha)$, where the unique greatest state is called the "vanishing" state. All the other states reach the vanishing state with an arc of weight 1. By eliminating state and the arcs ingoing in it, we obtain a semi-normalized graph, that is called in [8] $G'(\alpha)$. In order to be self-contained, we give in this paper a direct definition of $G'(\alpha)$, that can be easily derived from the definition of $G(\alpha)$ in [8].

Definition 6. The *semi-normalized infinite Sturmian graph associated with* α, $G'(\alpha)$, is a weighted semi-normalized infinite graph where each state has outgoing degree 2. Moreover if we number each state, the initial one having number 0, the arcs are defined in the following way.

For any $s \geq 0$, let $b_s = \sum_{h=0}^{s} a_h$ and, for any $i \geq 0$, let $s(i)$ be the smallest integer such that $i < b_{s(i)}$. Then the state numbered i has an outgoing arc weighted $l_{s(i)}$ to state numbered $i+1$ and an outgoing arc weighted $l_{s(i)+1}$ to state numbered $1 + b_{s(i)}$.

In the following we give some examples of Sturmian graphs.

Example 7. Let us consider the sequence of Fibonacci numbers (see Example 2). The infinite Sturmian graph $G'(\varphi)$, where $\varphi = \frac{\sqrt{5}+1}{2}$ is the golden ratio, is represented in Fig. 3. It is called the semi-normalized Golden Graph (cf. [8]).

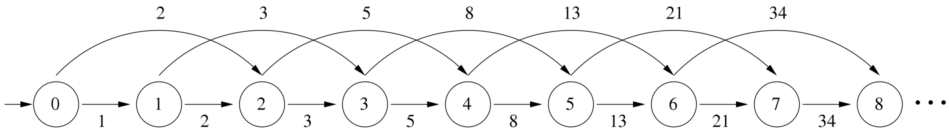

Fig. 3. The semi-normalized Golden Graph $G'(\varphi)$.

Example 8. Now consider the irrational number $\alpha = [1, 1, 2, 1, 2, 1, 2, \ldots] = \sqrt{3}$. Sequences $(l_i)_{i\geq 0}$, $(s(i))_{i\geq 0}$, $(b_{s(i)})_{i\geq 0}$ associated with this sequence are described in the following table.

i	0	1	2	3	4	5	6	7	...
a_i	1	1	2	1	2	1	2	1	...
b_i	1	2	4	5	7	8	10	11	...
l_i	1	2	3	8	11	30	41	112	...
$s(i)$	0	1	2	2	3	4	4	5	...
$b_{s(i)}$	1	2	4	4	5	7	7	8	...

Hence, the semi-normalized infinite Sturmian graph $G'(\alpha)$ is the one represented in Fig. 4.

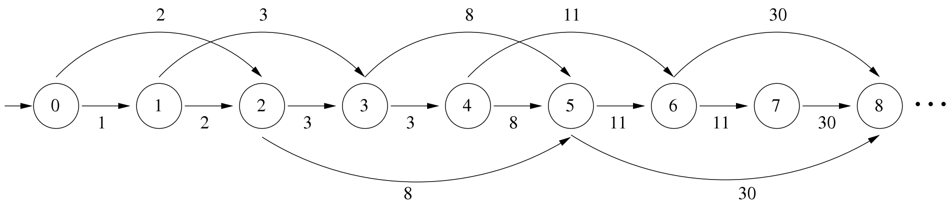

Fig. 4. The semi-normalized infinite Sturmian graph $G'([1, 1, 2, 1, 2, 1, 2, \ldots])$.

Remark 9. Notice that the graphs of Definition 6 are exactly the semi-normalized infinite Sturmian graphs defined in [8]. Indeed, given a real number α, both definitions lead to the same graph $G'(\alpha)$ that is the limit graph of the sequence $G'(\frac{P_n}{Q_n})$, $n \in \mathbb{N}$, where $\frac{P_n}{Q_n}$ is the sequence of convergents of α and $G'(\frac{P_n}{Q_n})$ is defined in [8, Remark 6].

3.1. Structure of infinite Sturmian graphs

In this subsection we want to analyze the structure of the semi-normalized infinite Sturmian graphs $G'(\alpha)$ (see Fig. 5).

We start with a proposition that characterizes the weights of arcs ingoing a given state.

Proposition 10. *For any $h \geq 1$, every arc ingoing the states in the set $S_h = \{b_{h-1} + 1, b_{h-1} + 2, \ldots, b_h\}$ has weight l_h and they are the unique arcs having this weight. If $h = 0$ then arcs ingoing the states in the set $S_0 = \{1, \ldots, b_h\}$ have weight l_h and they are the unique arcs having this weight.*

Proof. For any i in $\{b_{h-1}, b_{h-1}+1, \ldots, b_h - 1\}$ one has that $s(i) = h$, and, by definition, all arcs outgoing them having weight l_h reach the states $b_{h-1}+1, b_{h-1}+2, \ldots, b_h$. All other arcs having weight l_h are outgoing from states i such that $s(i) = h-1$, i.e. $b_{s(i)} = b_{h-1}$ and they all reach the state $b_{h-1}+1$, that is a state in S_h. The case when $h = 0$ is dealt with in an analogous way. □

Remark 11. For any $h \geq 0$ the cardinality of the set S_h is obviously a_h.

The following propositions, as well as being interesting in themselves, are important as they will also be useful in the final section where we will show the strong relationship between a particular set of representations of non-negative integers, called *lazy representations*, and the Sturmian graphs.

Proposition 12. *Let us consider an arc (i, j). If there exists a number $h \geq 0$ such that i is smaller than $b_{h-1}+1$ and j is greater than b_h then $i = b_{h-1}$ and $j = b_h + 1$.*

Proof. By Proposition 10, if $j > b_h$ then the arc (i, j) has a weight greater than or equal to l_{h+1}. If $i < b_{h-1}+1$ then $s(i) \leq h$ and, by definition, arc (i, j) has a weight either $l_{s(i)}$ or $l_{s(i)+1}$. The only possibility is that $s(i) = h$ and that the weight of (i, j) is l_{h+1}. But if $s(i) = h$ and $i < b_{h-1}+1$ then i must be equal to $i = b_{h-1}$ and, consequently, $j = b_h + 1$. □

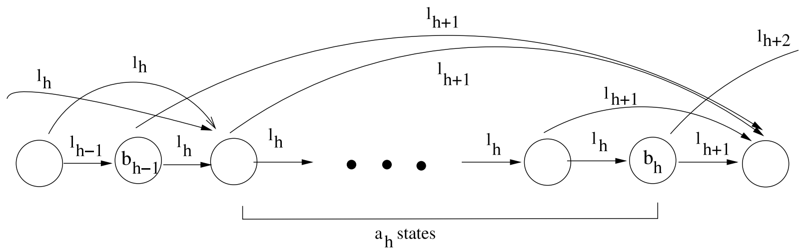

Fig. 5. The local structure of $G'(\alpha)$.

Proposition 13. *For any $h \geq 0$ there exist no paths in the graph having more than a_h arcs weighted by l_h. Moreover, if a path from the initial state reaches state b_h then it contains exactly a_h arcs weighted by l_h.*

Proof. The first arc of a path in the graph having the arcs weighted by l_h must reach a state in S_h, by Proposition 10. Every state i in S_h has exactly one arc weighted by l_h to $i+1$. Since, by Remark 11, the cardinality of the set S_h is a_h the thesis follows. □

Definition 14. Let us suppose to have an infinite graph G where the states are labeled by the integers greater than or equal to 0. Graph G can be weighted or non-weighted. We say that G is eventually periodic if there exist $\hat{n} \geq 0$ and an integer $p > 0$ called the *period* such that for any $n \geq \hat{n}$ one has that (n, j) is an arc of G if and only if $(n+p, j+p)$ is an arc of G.

Remark 15. Definition 14 states a property that leaves weights out of consideration.

Proposition 16. *$G'(\alpha)$ is eventually periodic if and only if the continued fraction expansion of α is eventually periodic.*

Proof. The number $\alpha = [a_0, a_1, \ldots]$ is eventually periodic if and only if there exists a number $\hat{h}$ and a number $p' > 0$ such that for any $h \geq \hat{h}$ one has $a_h = a_{h+p'}$. By the definition of $G'(\alpha)$ and by the previous definition it is easy to see that $G'(\alpha)$ is eventually periodic with period $p = \sum_{h=\hat{h}}^{\hat{h}+p'-1} a_h$ by considering the $\hat{n}$ in the previous definition to be equal to $\hat{n} = \sum_{h=0}^{\hat{h}} a_h$.

Conversely, if $G'(\alpha)$ is eventually periodic then there exists $\hat{n} \geq 0$ and an integer p such that for any $n \geq \hat{n}$ one has that (n, j) is an arc of $G'(\alpha)$ if and only if $(n+p, j+p)$ is an arc of $G'(\alpha)$.

First of all we notice, by the very definition, that each state i in $G'(\alpha)$ has exactly two outgoing arcs, one having ending state $i+1$ and the other $j = b_h + 1$, for some number h. Therefore if (i, j) and $(i+1, j')$ are two arcs of $G'(\alpha)$ such that $j \neq i+1, j' \neq i+2$ and $j \neq j'$ then the unique possibility is that $i+1$ is equal to a b_{h-1} for some $h > 0$. Moreover this fact characterizes the states of the form b_{h-1} in the sense that if (i, j) and $(i+1, j')$ are two arcs of $G'(\alpha)$ such that $j = j'$ then $i+1$ is not of the form b_{h-1} for some $h > 0$.

Let us suppose now that h is such that $b_{h-1} - 1 \geq \hat{n}$. Since the graph has period p, the same reasoning can be followed using $(i+p, j+p)$ and $(i+p+1, j'+p)$ and obtaining that $i+1+p$ is equal to a $b_{h'-1}$, for some number h' where $b_{h'-1} = b_{h-1} + p$.

For any state k with $b_{h-1} < k < b_h - 1$ we have that *if* (k, j) and $(k+1, j')$ are two arcs of $G'(\alpha)$ such that $j \neq k+1, j' \neq k+2$ then $j = j'$. And, clearly, when $k = b_h - 1$ if (k, j) and $(k+1, j')$ are two arcs of $G'(\alpha)$ such that $j \neq k+1$ and $j' \neq k+2$ *then* $j \neq j'$. Since the graph has period p, the same happens replacing h by h'. This, in turn, implies that $a_h = a_{h'}$. We can iterate this argument and obtain that $a_{h+1} = a_{h'+1}, a_{h+2} = a_{h'+2}, \ldots, a_{h+s} = a_{h'+s}$ until $a_{h+s} = a_{h'}$. Therefore the development in the continued fraction of $\alpha = [a_0, a_1, \ldots]$ is eventually periodic and the periodic part has period s. □

4. Numeration systems and lazy representations of integers

This section is devoted to the representations of non-negative integers in numeration systems defined by an increasing basis. In particular, we focus on greedy and lazy representations and we give a new algorithm for finding the lazy representation of a positive integer N.

Classically speaking, a numeration system is defined by a pair composed of either a *base* or a *basis*, which is an increasing sequence of numbers, and of an alphabet of digits. Standard numeration systems, such as the binary and the decimal ones, are represented in the first manner, i.e., through a base, while we are interested in the second one, i.e., through a basis. The reader may consult the survey [14, Chapt. 7]. More formally, we give the following definitions.

Definition 17. Let $U = (u_i)_{i\geq 0}$ be an increasing sequence of integers with $u_0 = 1$, the *basis*. A U-representation of a non-negative integer N is a word $d_k \cdots d_0$ where the digits d_i, $0 \leq i \leq k$, are integers, such that $N = \sum_{i=0}^{k} d_i u_i$.

Set $c_i = \lceil \frac{u_{i+1}}{u_i} \rceil - 1$. The U-representation $d_k \cdots d_0$ is said to be *legal* if for any i, $0 \leq i \leq k$, one has that $0 \leq d_i \leq c_i$. The set $A = \{c \in \mathbb{N} \mid \exists i, 0 \leq c \leq c_i\}$ is the *canonical alphabet*.

By convention, the representation of 0 is the empty word ϵ.

Even if it is a very natural concept, the definition of legal U-representation is new. The concept of legal U-representation represents an essential requisite of our theory, because it allows us to link lazy representations with Sturmian graphs.

In the following we will prove several properties of this kind of representation.

Example 18. Let $F = (F_n)_{n\geq 0} = 1, 2, 3, 5, 8, 13, 21, 34, \ldots$ be the sequence of Fibonacci numbers obtained in the following way

$$\begin{cases} F_0 = 1 \\ F_1 = 2 \\ F_{n+1} = F_n + F_{n-1} \quad n \geq 1. \end{cases}$$

The canonical alphabet is equal to $A = \{0, 1\}$. It is the well-known Fibonacci numeration system. An F-representation of the number 31 is 1010010. Another representation is 1001110. By definition, every F-representation of a non-negative integer N over the alphabet $A = \{0, 1\}$ is legal.

Among all possible U-representations of a given non-negative integer N, one can be distinguished and it is known as the *greedy* (or *normal*) U-representation of N.

Definition 19. A *greedy* (or *normal*) U-representation of a given non-negative integer N is the word $d_k \cdots d_0$, where the most significant digit $d_k > 0$ and $d_j \geq 0$ for $0 \leq j < k$, and satisfying for each i, $0 \leq i \leq k$,

$$d_i u_i + \cdots + d_0 u_0 < u_{i+1}.$$

Greedy representations are so-called because they can be obtained through the following greedy algorithm.

Greedy algorithm.

Given integers m and p let us denote by $q(m, p)$ and $r(m, p)$ the quotient and the remainder of the Euclidean division of m by p. Let $k \geq 0$ be such that $u_k \leq N < u_{k+1}$ and let $d_k = q(N, u_k)$ and $r_k = r(N, u_k)$, and, for $i = k-1, \ldots, 0$, $d_i = q(r_{i+1}, u_i)$ and $r_i = r(r_{i+1}, u_i)$. Then $N = d_k u_k + \cdots + d_0 u_0$.

Remark 20. A greedy U-representation is always legal.

Example 21. If we consider the Fibonacci numeration system of Example 18, the greedy F-representation of the number 31 is 1010010.

From now on we will consider in this paper only legal representations. Therefore, unless explicitly mentioned, when we say "U-representation" we mean "legal U-representation".

The proof of the following result can be found in [14, Proposition 7.3.4].

Proposition 22. *The greedy U-representation of an integer is the greatest in the radix order of all the U-representations of that integer with non-null most significant digit.*

Remark 23. Note that the order between natural numbers is given by the radix order between their greedy U-representations.

Starting from the statement of Proposition 22 we consider another peculiar U-representation, that is called the *lazy U-representation* of a natural number N.

Definition 24. A word $e_k \cdots e_0$, with the most significant digit $e_k > 0$, is the *lazy U-representation* of a natural number N if it is the smallest legal U-representation of N in the radix order.

Example 25. If we consider the Fibonacci numeration system of previous examples, the lazy F-representation of the number 31 is 111110.

Definition 26. Let $w = d_k \cdots d_0$ be a U-representation. Denote $\underline{d_i} = c_i - d_i$, and by extension, $\underline{w} = \underline{d_k} \cdots \underline{d_0}$ the *complement* of w.

For $k \geq 0$ set $C_k = \sum_{i=0}^{k} c_i u_i$.

By using the previous definition, we can link the greedy and lazy U-representations of a natural number N.

Proposition 27. *A U-representation w of a number N, $u_k \leq N < u_{k+1}$, is greedy if and only if its complement $\underline{w}$ is the lazy U-representation of the number $N' = C_k - N$, up to eliminating all the initial zeros.*

Proof. Let $w = d_k \cdots d_0$ be the greedy U-representation of N. Then $\underline{w} = e_k \cdots e_0$ is a U-representation of $N' = C_k - N$, with $e_i = c_i - d_i$ for $0 \leq i \leq k$. First suppose that $e_k > 0$. If $e_k \cdots e_0$ is not the lazy representation of N', then there exists another representation of N', $f_k \cdots f_0$, smaller than $e_k \cdots e_0$ in the radix order. Assume that there exists j, $0 \leq j \leq k$, such that $f_j < e_j$ and $e_k \cdots e_{j+1} = f_k \cdots f_{j+1}$. From $f_j u_j + \cdots + f_0 u_0 = e_j u_j + \cdots + e_0 u_0$ and $f_j \leq e_j - 1$, it follows that $e_{j-1} u_{j-1} + \cdots + e_0 u_0 \leq -u_j + f_{j-1} u_{j-1} + \cdots + f_0 u_0$. Taking the complement, one gets $d_{j-1} u_{j-1} + \cdots + d_0 u_0 \geq C_{j-1} + u_j - (f_{j-1} u_{j-1} + \cdots + f_0 u_0) \geq u_j$ since $f_{j-1} u_{j-1} + \cdots + f_0 u_0 \leq C_{j-1}$. This contradicts the fact that $d_k \cdots d_0$ is greedy.

If $e_k = 0, e_{k-1} = 0, \ldots, e_{n+1} = 0$, and $e_n > 0$, then $e_n \cdots e_0$ is the lazy representation of N'. □

Proposition 27 allows us to characterize the lazy U-representation of a natural number N.

Corollary 28. *A U-representation $e_k \cdots e_0$ of a number N is lazy if and only if for each i, $0 \leq i \leq k$, $e_i u_i + \cdots + e_0 u_0 > C_i - u_{i+1}$.*

Let us denote by m_i the greatest in the radix order of greedy U-representations of length i. Clearly m_i is the greedy representation of the integer $u_i - 1$. Recall that $m_0 = \varepsilon$. Denote by $M(U) = \{m_i \mid i \geq 0\}$.

Proposition 29. (1) *A U-representation $w = d_k \cdots d_0$ of a natural number N is greedy if and only if for any i, $0 \leq i \leq k$, $d_i \cdots d_0 \leq m_{i+1}$ (in the radix order).*

(2) *A U-representation $w = d_k \cdots d_0$ of a natural number N is lazy if and only if for any i, $0 \leq i \leq k$, $\underline{d_i} \cdots \underline{d_0} \leq m_{i+1}$ (in the radix order).*

Proof.

(1) Since, by definition, m_{i+1} is the greatest in the radix order of greedy representations of length $i+1$, the proof follows trivially from Remark 23, because the order between natural numbers is given by the radix order between their greedy U-representations.

(2) By Proposition 27, a U-representation $w = d_k \cdots d_0$ of a natural number N is lazy if and only if its complement $\underline{w} = \underline{d_k} \cdots \underline{d_0}$ is the greedy U-representation of the number $N' = C_k - N$. The thesis follows then by item 1. □

A direct consequence of Proposition 29 is the following result.

Corollary 30. *For each $i \geq 0$ the number $u_i - 1$ has a unique legal U-representation.*

Now we give an algorithm computing the lazy U-representation of a positive integer N.

Lazy algorithm:

Let $k = k(N)$ be the integer such that $C_{k-1} < N \leq C_k$. This ensures that the length of the lazy U-representation of N is $k+1$.

Compute a U-representation $d_k \cdots d_0$ of $N' = C_k - N$ by the following algorithm:

Let $d_k = q(N', u_k)$ and $r_k = r(N', u_k)$, and, for $i = k-1, \ldots, 0$, $d_i = q(r_{i+1}, u_i)$ and $r_i = r(r_{i+1}, u_i)$. Then $d_k \cdots d_0$ is a greedy U-representation of N' with possibly initial zeros.

By Proposition 27, the lazy U-representation of N is $\underline{d_k} \cdots \underline{d_0}$.

Denote by $\mathsf{Greedy}(U)$ and by $\mathsf{Lazy}(U)$ the sets of greedy and lazy U-representations of the non-negative integers.

The regularity of the set $\mathsf{Greedy}(U)$ has been extensively studied. In particular, the following result is known, see [10, Prop. 2.3.51, Prop. 2.6.4].

Proposition 31. *The set $\mathsf{Greedy}(U)$ is regular if and only if the set $M(U)$ of greatest U-representations in the radix order is regular.*

Proposition 29 Item 2 implies that the characterization of regularity in the lazy case works as in the greedy case:

Proposition 32. *The set $\mathsf{Lazy}(U)$ is regular if and only if the set $\mathsf{Greedy}(U)$ is regular if and only if the set $M(U)$ of greatest U-representations in the radix order is regular.*

5. Ostrowski numeration system and lazy representations

In this section we are interested in the relationship between the continued fraction expansion of a real number α and numeration systems. Let us go into the details by first recalling a numeration system, originally due to Ostrowski, which is based on continued fractions, see [1, p. 106] and [4]. This numeration system, called the Ostrowski numeration system, can be viewed as a generalization of the Fibonacci numeration system.

Definition 33. The sequence $(Q_i(\alpha))_{i\geq 0}$, defined in (1), of the denominators of the convergents of the infinite simple continued fraction of the irrational $\alpha = [a_0, a_1, a_2, \dots] > 0$ forms the basis of the *Ostrowski numeration system based on* α.

Proposition 34. *Let $\alpha = [a_0, a_1, a_2, \dots] > 0$. The sequence $U_\alpha = (l_i)_{i\geq 0}$ associated in (2) with α is identical to the sequence $(Q_i)_{i\geq 0} = (Q_i(\beta))_{i\geq 0}$ defined in (1) for the number $\beta = [b_0, b_1, b_2, b_3, \dots] = [0, a_0 + 1, a_1, a_2, \dots]$.*

Proof. By definition we have that

- $l_0 = 1$ and $Q_0 = 1$,
- $l_1 = a_0 + 1$ and $Q_1 = b_1$,
- $l_2 = a_1 l_1 + l_0 = a_1(a_0 + 1) + 1$ and $Q_2 = b_2 Q_1 + Q_0 = b_2 b_1 + 1$,

and so on. Therefore we can set $b_0 = 0$, $b_1 = a_0 + 1$, $b_2 = a_1, \dots$, and then $\beta = [0, a_0 + 1, a_1, a_2, \dots]$. □

It is easy to verify that there exists a relation between β and α.

Proposition 35. *If α is greater than or equal to 1 then $\beta = \frac{1}{\alpha+1}$.*

Proof. By definition we have that

- $b_0 = 0 = \lfloor \frac{1}{\alpha+1} \rfloor$, because $\alpha \geq 1$,
- $b_1 = \lfloor \frac{1}{\beta} \rfloor = \lfloor \alpha + 1 \rfloor = \lfloor \alpha \rfloor + 1 = a_0 + 1$,
- $b_2 = \lfloor \frac{1}{\frac{1}{\beta} - \lfloor \frac{1}{\beta} \rfloor} \rfloor = \lfloor \frac{1}{\alpha + 1 - (a_0+1)} \rfloor = \lfloor \frac{1}{\alpha - a_0} \rfloor = a_1$,

and so on. Therefore the result is proved. □

In what follows, $\alpha = [a_0, a_1, \dots]$ is greater than 1 and thus the sequence $U_\alpha = (l_i)_{i\geq 0}$ is increasing. In view of Definition 33 and Proposition 34, the Ostrowski numeration system based on $\beta = \frac{1}{\alpha+1}$ and the numeration system with basis U_α, in the sense of Definition 17, coincide. So we call the numeration system with basis U_α the *Ostrowski numeration system associated with* α.

By Definition 17, a U_α-representation $d_k \cdots d_0$ is legal if $d_i \leq a_i$, for any i such that $0 \leq i \leq k$, since in this case $c_i = a_i = \lceil \frac{l_{i+1}}{l_i} \rceil - 1$, thus the canonical alphabet is $A = \{a \in \mathbb{N} \mid \exists i, 0 \leq a \leq a_i\}$.

The Ostrowski numeration system associated with the golden ratio φ is the Fibonacci numeration system defined in Example 18. It is folklore that a U_φ-representation of an integer is greedy if and only if it does not contain any factor of the form 11, and is lazy if and only if it does not contain any factor of the form 00.

We now extend this property to U_α-representations for any $\alpha > 1$. The greedy case is classical, see [1].

Proposition 36. (1) *A U_α-representation $w = d_k \cdots d_0$ is greedy if and only if it contains no factor $d_i d_{i-1}$, $1 \leq i \leq k$, with $d_i = a_i$ and $d_{i-1} > 0$.*
(2) *A U_α-representation $w = d_k \cdots d_0$ is lazy if and only if it contains no factor $d_i d_{i-1}$, $1 \leq i \leq k$, with $d_i = 0$ and $d_{i-1} < a_{i-1}$.*

Proof.

(1) The proof is trivial.
(2) By Proposition 27, $w = d_k \cdots d_0$ is lazy if and only if $\underline{w} = \underline{d_k} \cdots \underline{d_0}$ is greedy. By the previous item, $\underline{w}$ is greedy if and only if it contains no factor $\underline{d_i d_{i-1}}$, $1 \leq i \leq k$, with $\underline{d_i} = a_i$ and $\underline{d_{i-1}} = h$, with $0 < h \leq a_{i-1}$. Since $\underline{a_i} = 0$ and $\underline{h} = d_{i-1} < a_{i-1}$, the proof follows. □

Now we give a characterization of m_{i+1}, the greatest in the radix order of greedy U_α-representations of length $i + 1$.

Lemma 37. *For any $i \geq 0$, $m_{i+1} = a_i 0 a_{i-2} 0 \cdots a_2 0 a_0$ if i is even, and $m_{i+1} = a_i 0 a_{i-2} 0 \cdots a_1 0$ if i is odd.*

The sequence $(a_i)_{i\geq 0}$ is *eventually periodic* if there exist integers $m \geq 0$ and $p \geq 1$ such that $a_{i+p} = a_i$ for $i \geq m$. It is a classical result that the sequence $(a_i)_{i\geq 0}$ is eventually periodic if and only if α is a quadratic irrational.

The regularity of the set of the greedy U_α-representations has been already studied. In particular, it is proved in [20,13] that the set of greedy expansions in the Ostrowski numeration system associated with $\alpha > 1$ is regular if and only if the sequence $(a_i)_{i\geq 0}$ is eventually periodic.

Lemma 38. *The set $M(U_\alpha)$ is regular if and only if the sequence $(a_i)_{i\geq 0}$ is eventually periodic.*

Proof. If $(a_i)_{i\geq 0}$ is eventually periodic then the canonical alphabet of digits A is finite, and the set $M(U_\alpha)$ is a regular set of A^* by Lemma 37. The converse follows from Lemma 37. □

The next proposition follows from Proposition 32 and Lemma 38.

Proposition 39. *The sets of greedy expansions and of lazy expansions in the Ostrowski numeration system associated with $\alpha > 1$ are regular if and only if the sequence $(a_i)_{i\geq 0}$ is eventually periodic if and only if α is a quadratic irrational.*

6. Sturmian graphs and lazy representations

The goal of this section is to establish a deep connection between the set of lazy representations and the set of paths in a well defined *Sturmian graph*.

In the theory of combinatorics on words, it often happens that some properties are first proved for Fibonacci words and then they are extended to all Sturmian words. In fact, for Fibonacci words the link between the set of representations of a non-negative integer in a numeration system and the DAWG of a word has been already discovered. More precisely, in [18], the simple structure of DAWGs of Fibonacci words is used to give in many cases simplified alternative proofs and a new interpretation of several well-known properties of Fibonacci word. In particular, it is proved that the lengths of paths in the DAWGs of Fibonacci words are closely related to the Fibonacci numeration system. This property leads to a number-theoretic characterization of occurrences of any subwords.

We start by recalling a classical definition.

Definition 40. Let G be a weighted graph, the weight of a path in G is the sum of the weights of all arcs in the path.

In [8], Epifanio et al. have proved several properties on finite and infinite Sturmian graphs. Among them, an important result concerns a *counting property* of Sturmian graphs. In the infinite case, it can be stated in the following way.

Definition 41. An infinite semi-normalized weighted DAG G' has the $(h, +\infty)$-*counting property*, or, in short, counts from h to $+\infty$, if any non-empty path starting in the initial state has weight in the range $h \cdots + \infty$ and for any i, $i \geq h$, there exists a unique path that starts in the initial state and has weight i.

Starting from this definition, at the end of the proof of Proposition 35 in [8] the following result has been proved concerning the counting property of infinite Sturmian graphs, that is very useful in the next.

Theorem 42. *For any positive irrational α, $G'(\alpha)$ can count from 0 up to infinity.*

Moreover, we can prove the following result that characterizes the path weights of states in $G'(\alpha)$. Here, in order to be self-contained, we give a proof that depends on the definition of $G'(\alpha)$ and on the results of Section 3.1. We could have given a different proof depending on the convergence results of Sturmian graphs given in [8].

Proposition 43. *For any state $i > 1$ in $G'(\alpha)$ let b_s be the maximum non-negative integer such that $i > b_s$ (cf. Definition 6). The maximum weight of the paths from the initial state ending in i is*

$$g(i) = \sum_{j=0}^{s} a_j l_j + (i - b_s) l_{s+1}.$$

The only paths from the initial state ending in states $i = 0$, $i = 1$ respectively, have weights 0, 1 respectively.

Proof. By Proposition 10, the maximum weight of the paths from the initial state ending in i is equal to the maximum weight of the paths from the initial state ending in $i - 1$ plus l_s. The proof by induction of the proposition is, at this point, straightforward. □

Corollary 44. *For any state i in $G'(\alpha)$, all paths from the initial state ending in i are weighted N, where N is such that $g(i-1)+1 \leq N \leq g(i)$ and, conversely, if N is such that $g(i-1)+1 \leq N \leq g(i)$ then N is the weight of a path ending in i.*

Proof. The first part of the corollary is an immediate consequence of the previous theorem. Conversely, since $G'(\alpha)$ counts from 0 to $+\infty$ then every natural number N must be a weight of a path in $G'(\alpha)$ that begins from the initial state. Therefore if N is such that $g(i-1)+1 \leq N \leq g(i)$ then the unique path having weight N from the initial state by previous theorem cannot end in any state $j \leq i - 1$ and, so, it must end in i. □

Example 45. Let us consider the irrational number $\alpha = [1, 1, 2, 1, 2, 1, 2, \ldots] = \sqrt{3}$ and its semi-normalized infinite Sturmian graph $G'(\alpha)$ represented in Fig. 4.

The following table represents, for any state i in $G'(\alpha)$, the values b_s and s of Proposition 43, as well as the minimum, $\min_i$, and the maximum, $\max_i$, non-negative integers among the weights of all paths from the initial state ingoing in i.

i	0	1	2	3	4	5	6	7	…
b_s			1	2	2	4	5	5	…
s			0	1	1	2	3	3	…
$\min_i$	0	1	2	4	7	10	18	29	…
$\max_i$	0	1	3	6	9	17	28	39	…

Before coming to the main result of the paper, we define the correspondence between paths and representations.

Definition 46. Let $G'(\alpha)$ be the semi-normalized Sturmian graph associated to α and U_α be the Ostrowsky numeration system associated with α. For any $N \in \mathbb{N}$, we say that the U_α representation $d_k \cdots d_0$ of N *corresponds* to the unique path weighted N in $G'(\alpha)$ if, for any $i \geq 0$, d_i represents the number of consecutive arcs labeled Q_i in the path.

The next theorem establishes a deep connection between the set of lazy representations and the theory of Sturmian Graphs. This theorem, and in particular its claim, that shows that the sequence involved in the proof of the theorem satisfies a condition for being lazy, represents the core of this paper.

Theorem 47. *Let N be a non-negative integer. A U_α-representation of N is lazy if and only if it corresponds to the unique path weighted N in the semi-normalized Sturmian graph $G'(\alpha)$.*

Proof. By Theorem 42 we know that for any non-negative integer N there is in the semi-normalized Sturmian graph $G'(\alpha)$ a unique path of weight N from the initial state. By definition and by Proposition 13 the sequence of weights in this unique path gives a legal U_α-representation of N. We claim that this sequence satisfies the condition of the second item of Proposition 36. This implies that the sequence is lazy and vice versa.

Proof of the claim: if there is a factor of the form $d_h d_{h-1}$ with $d_h = 0$ then in the corresponding path there exists one arc (i, j) that satisfy the hypothesis of Proposition 12. Therefore, by Proposition 12, $i = b_{h-1}$ and, by Proposition 13 this path has exactly a_{h-1} arcs having weight l_{h-1}. This, in turn, implies that $d_{h-1} = a_{h-1}$. □

More precisely, the importance of this result lies on the fact that for any number i, we can connect the unique path weighted i in the Sturmian graph associated with α and the lazy representation of i in the Ostrowski numeration system associated with α.

Example 48. The lazy U_φ-representation of 5 is equal to 110. Recall that in this case $U_\varphi = \{1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots\}$. This means that 110 gives the decomposition $5 = 0 \cdot l_0 + 1 \cdot l_1 + 1 \cdot l_2 = 2 + 3$.

On the other hand, given the golden graph $G'(\varphi)$ of Fig. 3, there exists a unique path starting from 0 with weight 5. It is labeled 2, 3.

Example 49. For $\alpha = [1, 1, 2, 1, 2, 1, 2, \ldots] = \sqrt{3}$ the lazy U_α-representation of 7 is equal to 201. Recall that in this case $U_\alpha = \{1, 2, 3, 8, 11, 30, 41, 112, \ldots\}$. Then 201 gives the decomposition $7 = 1 + 2 \cdot 3$.

On the other hand, given the graph $G'(\alpha)$ of Fig. 4, there exists a unique path starting from 0 with weight 7. It is labeled 1, 3, 3, that exactly corresponds to the lazy U_α-representation of $7 = 1 + 2 \cdot 3$.

The next corollary is an immediate consequence of the previous theorem, of Propositions 16 and 39.

Corollary 50. *The set of all the U_α-representations that correspond to paths in $G'(\alpha)$ is regular if and only if $G'(\alpha)$ is eventually periodic.*

7. Conclusions

This paper contains a neat and natural theory on lazy representations in Ostrowski numeration system based on the continued fraction expansion of a real number α. This theory has a deep connection with the Sturmian graph associated with α, even better the study of Sturmian graphs gave us the idea to formalize and to develop this theory. Indeed, the set of lazy representations is naturally linked with the set of paths in the Sturmian graph associated with α. In fact, for any non-negative integer N, the unique path weighted N in the semi-normalized Sturmian graph gives the lazy representation of N. Moreover, we have proved several other properties of the representations of integers in the Ostrowski numeration system. In particular, the set of greedy and lazy representations of real numbers is regular if and only if the continued fraction expansion of α is eventually periodic.

It would be interesting to deepen this theory in order to prove other properties of the representations in the Ostrowski numeration system based on the continued fraction expansion of a real number α through the Sturmian graph associated with α. For example, Ostrowski numeration systems can be applied to real numbers; we are investigating the possibility to relate the obtained representations with Sturmian graphs. Moreover, it would be nice to find algorithms for performing the elementary arithmetic operations. Another question is to relate the Ostrowski odometer – that performs the addition of 1 – to the Sturmian graph.

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